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Ewa Bednarczuk

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 8373578

fax: (+48) (22) 8372772

Pracę zgłosił: prof. dr hab. inż. Kazimierz Malanowski

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Stability Analysis for Parametric Vector Optimization Problems

Ewa M.Bednarczuk
Systems Research Institute
Polish Academy of Sciences
01-447 Warsaw, Newelska 6

Preface

We study stability of minimal points and solutions to parametric (or perturbed) vector optimization problems in the framework of real topological vector spaces and, if necessary, normed spaces. Because of particular importance of finite-dimensional problems, called multicriteria optimization problems, which model various real-life phenomena, a special attention is paid to the finite-dimensional case. Since one can hardly expect the sets of minimal points and solutions to be singletons, set-valued mappings are natural tools for our studies.

Vector optimization problems can be stated as follows. Let X be a topological space and let Y be a topological vector space ordered by a closed convex pointed cone $\mathcal{K} \subset Y$. Vector optimization problem

$$\begin{array}{ll} \mathcal{K} - \min\ f_0(x) & \\ \text{subject to } x \in A_0\,, & (P_0) \end{array}$$

where $f : X \to Y$ is a mapping, and $A_0 \subset X$ is a subset of X, relies on finding the set $\mathrm{Min}(f_0, A_0, \mathcal{K}) = \{y \in f_0(A_0) \mid f_0(A_0) \cap (y - \mathcal{K}) = \{y\}\}$ called the **Pareto** or **minimal point** set of (P_0), and the **solution set** $S(f_0, A_0, \mathcal{K}) = \{x \in A_0 \mid f_0(x) \in \mathrm{Min}(f_0, A_0, \mathcal{K})\}$. We often refer to problem (P_0) as the **original problem** or **unperturbed one**. The space X is the **argument** space and Y is the **outcome** space.

Let U be a topological space. We embed the problem (P_0) into a family (P_u) of vector optimization problems parametrised by a parameter $u \in U$,

$$\begin{array}{ll} \mathcal{K} - \min\ f(u, x) & \\ \text{subject to } x \in A(u)\,, & (P_u) \end{array}$$

where $f : U \times X \to Y$ is the parametrised objective function and $A : U \rightrightarrows Y$, is the feasible set multifunction, (P_0) corresponds to a parameter value u_0. The performance multifunction $\mathcal{M} : U \rightrightarrows Y$, is defined as $\mathcal{M}(u) = \text{Min}(f(u,\cdot), A(u), \mathcal{K})$, and the solution multifunction $\mathcal{S} : U \rightrightarrows Y$, is given as $\mathcal{S}(u) = S(f(u,\cdot), A(u), \mathcal{K})$, and $f : U \times X \to Y$, $A(u) \subset X$.

Our aim is to study continuity properties of $\mathcal{M}$ and $\mathcal{S}$ as functions of the parameter u. Continuous behaviour of solutions as functions of parameters is of crucial importance in many aspects of the theory of vector optimization as well as in applications(correct formulation of the model and/or approximation) and numerical solution of the problem in question.
We investigate continuity in the sense of Hausdorff and Hőlder of the multivalued mappings of minimal points $\mathcal{M}(u)$ and solutions $\mathcal{S}(u)$ as functions of the parameter u under possibly weak assumptions. We attempt to avoid as much as possible compactness assumptions which are frequently over-used (see eg [83]).
It is a specific feature of vector optimization that the outcome space is equipped with a partial order generated by a cone the properties of which are important for stability analysis. In many spaces cones of nonnegative elements have empty interiors and because of this we derive stability results for cones with possibly empty interior. This kind of results are specific for vector optimization and do not have their counterpart in scalar optimization.
We introduce two new concepts: the notion of containment(with some variants for cones with empty interiors),[16], and the notion of strict minimality, [12].

The containment property (CP), defined in topological vector spaces, is introduced to study upper semicontinuities (in the sense of Hausdorff) of minimal points, [11, 16]. It is a variant of the domination property (DP), which appears frequently in the context of stability of solutions to parametric vector optimization problems. Although it is not a commonly adopted view point, the domination property may be accepted as a solution concept which generalizes the standard concept of a solution to scalar optimization problem. In consequence, the containment property (CP) may also be seen as a solution concept in vector optimization. To investigate more deeply this aspect we interpret the containment property as a generalization of the concept of the set of $\phi-$local solutions appearing in the

context of Lipschitz continuity of solutions to scalar optimization problems. Under mild assumptions the containment property imply that the set weakly minimal points equals the set of minimal points. This equality, in turn, is a typical ingradient of standard finite-dimensional sufficient conditions for upper semicontinuity of minimal points.
To study Hőlder upper continuity of minimal points we define the rate of containment of a set with respect to a cone, which is a real-valued function of a scalar argument, see [14, 15]. The rate of growth of this function influence decisively the rate of Hőlder continuity of minimal points, [15].

Strictly minimal points are introduced to study lower semicontinuities (lower Hausdorff, lower Hőlder) of minimal points [20, 13]. The definition of a strictly minimal point is given in topological vector spaces and it is a generalization of the notion of a super efficient point in the sense of Borwein and Zhuang defined in normed spaces. We discuss strict minimality in vector optimization by proving that it is a vector counterpart of the concept of $\phi-$ local solution to scalar optimization problem.

Theory of vector optimization may be considered as an abstract study of optimization problems with mappings taking values in the outcome space equipped with a partial order structure. As such, it contains many concepts and results which generalize and/or have their counterparts in scalar optimization. The very definition of the set of minimal points of vector optimization problem in the outcome space may serve as an example here. This is a counterpart of the optimal value of scalar optimization problem. Another example is the concept of well-posed optimization problem. In subsequent developments we often compare our results and considerations with the corresponding approaches in scalar optimization. For instance, we define several classes of well-posed vector optimization problems by generalizing the concept of scalar minimizing sequence and in these classes we investigate continuity of solutions. For scalar optimization problems, the existing approaches and results on well-posedness are extensively discussed in the monograph by Dontchev and Zolezzi [33].

Convergence and rates of convergence of solutions to perturbed optimization problems is one of crucial topics of stability analysis in optimization both from theoretical and numerical points of view. For scalar optimization it was investigated by many authors see eg., [72], [32], [47], [78], [55], [81], [59], [60], [82], [2], and many others. An exhaustive survey of current state of research is given in the recent monograph by Bonnans and Shapiro [26]. In vector optimization the results on Lipschitz continuity of solutions are not so numerous, and concern some classes of problems, for linear case see eg.,[28], [29], [30], for convex case see eg., [25], [31].

The organization of the material is as follows. In Chapter 2 we investigate upper Hausdorff continuity of the multivalued mapping M, $M(u) = \text{Min}(\Gamma(u)|\mathcal{K})$ assigning to a given parameter value u from a topological space U the set of minimal points of the set $\Gamma(u) \subset Y$ with respect to cone $\mathcal{K} \subset Y$, where for any subset A of a topological vector space Y the set of minimal points is defined as $\text{Min}(A|\mathcal{K}) = \{y \in A \mid A \cap (y - \mathcal{K}) = \{y\}\}$, and $\Gamma : U \rightrightarrows Y$, is a given multivalued mapping. The main tool which allows us to obtain the general result is the containment property (CP). Some infinite-dimensional examples are discussed. A special attention is paid to the containment property (CP) in finite-dimensional case, when $Y = R^m$.

In Chapter 3 we discuss upper Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the rate of containment δ which is a one-variable nondecreasing function, defined for a given set A and the order generating cone $\mathcal{K}$. The assumption of sufficiently fast growth rate of this function appears to be the crucial assumption for all upper Hőlder stability results of Chapter 3.

In Chapter 4 we apply the results obtained in Chapters 2 and 3 to derive conditions for upper Hausdorff and upper Hőlder stability of minimal points to parametric vector optimization problems by taking $\Gamma(u) = f(u, A(u))$. Moreover, we introduce the concept of $\Phi-$ strong solutions to vector optimization problem (P_0), which is a generalization of the concept of a $\phi-$local minimizer to scalar optimization problem, the latter being introduced by Attouch and

Wets [6].

In Chapter 5 we investigate the lower continuity and lower Hőlder continuity of the minimal point multivalued mapping M. To this aim we introduce the notion of strict minimality mentioned above and the rate of strict minimality. In Section 5.5 we apply the results obtained in Chapter 5 to parametric vector optimization problems and we derive sufficient conditions for lower and lower Hőlder continuity of Pareto point multivalued mapping $\mathcal{M}$. An important tool here is the notion of $\Phi-$ strict solution to vector optimization problem introduced in Section 6.1. This notion can be interpreted as another possible generalization of the concept of $\phi-$local minimizer.

In Chapter 6 we propose several definitions of a well-posed vector optimization problem. All these definitions are based on properties of $\varepsilon-$solutions to vector optimization problems. For well-posed vector optimization problems we prove upper Hausdorff continuity of solution multivalued mapping S, $S(u) = S(f(u,\cdot), A(u), \mathcal{K})$.

1

Preliminaries

The general framework of our developments are Hausdorff vector (or linear) topological spaces over the scalar field R of real numbers. A topological vector space is any linear space Y equipped with a topology which is **compatible** with the linear space structure, that is, both linear space operations $(y_1, y_2) \to y_1 + y_2$, $y_1, y_2 \in Y$, and $(r, y) \to ry$, $r \in R$, $y \in Y$, are continuous on their domains, $Y \times Y$ and $R \times Y$, respectively. A topological space is **Hausdorff** (or separated) if any two distinct points have disjoint neighbourhoods.

Proposition 1.0.1 *Let Y be a topological vector space. For each $a \in Y$ the translation $f : f(x) = x + a$ is a homeomorphism of Y onto itself. In particular, if U is a base of neighbourhoods of the origin, $a + \mathcal{U}$ is a base of neighbourhoods of a.*

It is a consequence of Proposition 1.0.1 that the topological structure of Y is determined by a base of neighbourhoods of the origin. Further consequences of continuity of the linear space operations are given in the following proposition.

Proposition 1.0.2 *If $\mathcal{U}$ is a base of neighbourhoods of the origin, then for each $U \in \mathcal{U}$,*

(i) U is absorbing, ie., for any $y \in Y$ there is some $\bar{\lambda} > 0$ such that $\lambda y \in A$ for any $0 \leq \lambda \leq \bar{\lambda}$,

(ii) there exists a balanced neighbourhood V, $V \subset U$, ie., for all $v \in V$, $\lambda v \in V$ whenever $|\lambda| \leq 1$,

(iii) there exists $W \in \mathcal{U}$ such that $W + W \subset U$.

If $\mathcal{U}$ is a base of neighbourhoods in a topological vector space Y, then Y is a Hausdorff space if and only if $\bigcap_{U \in \mathcal{U}} U = \{0\}$.
Let $A \subset Y$ be a subset of Y. We say that A is **convex** if, for all $x \in A$ and $y \in A$, $\lambda x + (1-\lambda)y \in A$ whenever $0 \leq \lambda \leq 1$. By a **locally convex space** we mean a topological vector space possesing a base of convex neighbourhoods of the origin.

Proposition 1.0.3 *A locally convex space Y has a base $\mathcal{U}$ of neighbourhoods of the origin with the following properties:*

(i) if $U \in \mathcal{U}$, $V \in \mathcal{U}$, there is a $W \in \mathcal{U}$ with $W \subset U \cap V$;

(ii) if $U \in \mathcal{U}$ and $\lambda \neq 0$, $\lambda U \in \mathcal{U}$;

(iii) each $U \in \mathcal{U}$ is absolutely convex (ie, U is balanced and convex).

A locally convex space topology can be described in terms of seminorms. A nonnegative finite real-valued function p defined on Y is a seminorm if, for any $x, y \in Y$ and $\lambda \in R$

(i) $p(\lambda y) = |\lambda| p(x)$,

(ii) $p(x+y) \leq p(x) + p(y)$.

If p is a seminorm on Y, then for each $\alpha > 0$ the sets $\{y \in Y \mid p(y) < \alpha\}$ and $\{y \in Y \mid p(y) \leq \alpha\}$ are absolutely convex and absorbent. To each absolutely convex and absorbent subset U of Y corresponds a seminorm p, defined as $p(y) = \inf\{\lambda \mid \lambda > 0, \quad y \in \lambda U\}$, which is continuous at the origin when U is $0-$neighbourhood (p is continuous if and only if it is continuous at 0). By this, in every locally convex space there exists a family of continuous seminorms. The converse is specified in the following theorem.

Theorem 1.0.1 (Robertson,Robertson[68]) *Given a set Q of seminorms on a vector space Y there is a coarsest topology on Y compatible with algebraic structure in which every seminorm in Q is continuous. Under this topology Y is a locally convex space and a base of closed neighbourhoods is formed by the sets*

$$\{y \in Y \mid \sup_{1 \leq i \leq n} p_i(y) \leq \varepsilon\} \quad \varepsilon > 0, \quad p_i \in Q.$$

Proposition 1.0.4 (Robertson,Robertson[68]) *Under the topology determined by the set Q of seminorms, Y is Hausdorff if and only if for each non-zero $y \in Y$ there is $p \in Q$ with $p(y) > 0$.*

In consequence, if Y is not Hausdorff, there are nonzero y in Y with $p(y) = 0$, and all such y form a subspace $N = \bigcap_{U \in \mathcal{U}} U$. Hence, if a locally convex space Y is not Hausdorff it can be converted into Hausdorff space by identification of elements whose difference lies in N (see Robertson, Robertson [68], Ch.V, suppl.2). On the other hand, the assumption that the space is Hausdorff is crucial in some fundamental constructions.
Let Y^* be the topological dual of Y ie, the space of all continuous functionals defined on Y. We have the following proposition.

Proposition 1.0.5 (Holmes, [42], Cor.11.E) *Let Y be a Hausdorff locally convex space. Then Y^* separates points.*

1.1 Cones in topological vector spaces

In this section we collect basic facts about convex cones which will be used in subsequent sections. A subset $\mathcal{K} \subset Y$ of a vector space Y is a **cone** if

$$y \in \mathcal{K}, \text{ and } \lambda \geq 0 \Rightarrow \lambda y \in \mathcal{K}.$$

By definition, each nonempty cone contains the origin 0 of the space Y and $\{0\}$ is the trivial cone. A convex cone is a cone which is a convex subset of Y. A cone $\mathcal{K}$ is **pointed** if $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.
We use the following definition of a base of a cone.

Definition 1.1.1 *Let Y be a Hausdorff topological vector space and let $\{0\} \neq \mathcal{K} \subset Y$ be a convex cone in Y. A nonempty convex subset $\Theta \subset \mathcal{K}$ of $\mathcal{K}$ is a* **base** *for $\mathcal{K}$ if $0 \notin cl\Theta$, $\mathcal{K} = \bigcup\{\lambda\Theta \mid \lambda \geq 0\}$.*

In recent publications this is the most frequently used definition. A based cone is necessarily pointed and convex.

Example 1.1.1 *Let $Y = R^2$, $\mathcal{K} = R^2_+$. The set $\Theta = \mathcal{K} \cap \{(y_1, y_2) \mid -y_1 + 2 \leq y_2 \leq -y_1 + 4\}$ is a base of $\mathcal{K}$. Indeed, $0 \notin cl\Theta$, and each*

$0 \neq k \in \mathcal{K}$ can be represented as $(k_1, k_2) = \lambda(y_1, y_2)$, where $\lambda > 0$ and $(y_1, y_2) \in \Theta$. It is enough to take any λ such that

$$-\frac{k_1}{\lambda} + 2 \leq \frac{k_2}{\lambda} \leq -\frac{k_1}{\lambda} + 4,$$

ie., any λ satisfying $(k_1 + k_2)/4 \leq \lambda \leq (k_1 + k_2)/2$. Hence, $\mathcal{K} = \bigcup\{\lambda\Theta \mid \lambda \geq 0\}$. This example shows that there is no uniqueness of the representation of $k \in \mathcal{K}$.

Conditions ensuring uniqueness of the representation are given in the following proposition.

Proposition 1.1.1 (Peressini[65],Jahn [44]) *Let Y be a vector space. Let $\mathcal{K} \subset Y$ be a convex cone in Y and let $\Theta \subset \mathcal{K}$ be a nonempty convex subset of $\mathcal{K}$. The following conditions are equivalent:*

(i) each nonzero element $y \in \mathcal{K}$ has a unique representation of the form $y = \lambda\theta$, where $\lambda > 0$, $\theta \in \Theta$,

(ii) $\mathcal{K} = \bigcup\{\lambda\Theta : \lambda \geq 0\}$ and the smallest linear manifold in Y containing Θ does not contain 0.

Proof. If (i) holds, then $\mathcal{K} = \bigcup\{\lambda\Theta \mid \lambda \geq 0\}$. The smallest linear manifold containing Θ is $L = \{\mu\theta + (1-\mu)\theta' \mid \theta, \theta' \in \Theta \;\; \mu \in R\}$. If it were $0 \in L$, there would be $\mu_0 > 1$ and $\theta_0, \theta_0' \in \Theta$ such that $\mu_0\theta_0 = (\mu_0 - 1)\theta_0'$ contradictory to (i).
To show uniqueness in (i), suppose on the contrary that $\lambda\theta = \lambda'\theta'$ for $\theta, \theta' \in \Theta$, and positive reals λ, λ', $\lambda \neq \lambda'$. Then

$$0 = \frac{1}{\lambda - \lambda'}\{\lambda\theta - \lambda'\theta'\} \in L,$$

contradictory to (ii).

□

In some textbooks and monographs the base of a cone is defined as a nonempty convex subset of a cone satisfying condition (i) of Proposition 1.1.1 (see eg. [44],[46],[65]). If Θ satisfies condition (i) of Proposition 1.1.1, then $0 \notin \Theta$.

As we see from Example 1.1.1, Definition 1.1.1 do not assure uniqueness of the representation of elements of the cone through elements of the base. However, we have the following result.

Proposition 1.1.2 *Let Y be a locally convex Hausdorff topological vector space and let $\mathcal{K} \subset Y$ be a convex cone in Y with a base Θ. There exists another base Θ_1 of $\mathcal{K}$ such that $\Theta_1 = f^{-1}(1) \cap \mathcal{K}$, where f is a continuous linear functional defined on Y.*

Proof. Since $0 \notin \mathrm{cl}\Theta$, there exists a convex $0-$neighbourhood V in Y such that $V \cap \mathrm{cl}\Theta = \emptyset$. By separation arguments (see eg Holmes [42], Th.11.E,12.F), there exists a continuous functional f defined on Y such that $f(\theta) > 0$ for $\theta \in \Theta$. Now, by putting $\Theta_1 = f^{-1}(1) \cap \mathcal{K}$ we obtain a base of $\mathcal{K}$.

□

Let us note that the base Θ_1 constructed in the proof of Proposition 1.1.2 satisfies condition (i) of Proposition 1.1.1.

Let Y^* be the topological dual of a topological vector space Y. The subset $\mathcal{K}^* \subset Y^*$ of Y^* is called the **dual** cone of $\mathcal{K}$ if

$$\mathcal{K}^* = \{f \in Y^* \mid f(y) \geq 0 \text{ for all } y \in \mathcal{K}\}.$$

The dual cone is nonempty and weakly$-*-$closed. To see the latter suppose that f_λ is a net of functionals from $\mathcal{K}^*$ converging weakly$-*-$ to f. Then $f_\lambda(y)$ converges to $f(y)$ for all $y \in Y$; in particular, $f_\lambda(k)$ converges to $f(k)$ for any $k \in \mathcal{K}$, which entails $f(k) \geq 0$ for all $k \in \mathcal{K}$ since $f_\lambda(k) \geq 0$, for all λ and all $k \in \mathcal{K}$.

For any subset $A \subset Y$ of a topological vector space Y the **polar** $A^o \subset Y^*$ of A is defined as

$$A^o = \{f \in Y^* \mid f(a) \leq 1 \text{ for all } a \in A\}.$$

The polar set is nonempty since $0 \in A^o$ and weakly$-*-$closed. We have $\mathcal{K}^* = -\mathcal{K}^o$.
In the same way we define the polar set $A^o \subset Y$ for any set $A \subset Y^*$, ie.,

$$A^o = \{y \in Y \mid f(y) \leq 1 \text{ for all } f \in A\}.$$

The **bipolar** set $A^{oo} \subset Y$ of $A \subset Y$ (see [42], p.67) is

$$A^{oo} = \{y \in Y \mid f(y) \leq 1 \text{ for all } f \in A^o\}.$$

Theorem 1.1.1 **([42],Th.12.C)** *Let A be a subset of a locally convex space. Then*

$$A^{oo} = cl((conv\{0 \cup A\})$$

A topological linear space Y is said to be a **Mackey space** (see [46]) if $B^o \subset Y$ is a 0–neighbourhood in Y whenever $B \subset Y^*$ is a convex and weakly– $*$ –compact subset of Y.

Theorem 1.1.2 (Jameson[46], Th 3.8.6) *Let $\mathcal{K} \subset Y$ be a convex cone in a locally convex topological linear space Y. Then*

(i) if $\mathcal{K}$ has an interior point, then $\mathcal{K}^*$ has a weakly– $*$ –compact base,

(ii) if Y is a Mackey space, $\mathcal{K}$ is closed and $\mathcal{K}^*$ has a weakly– $*$ –compact base, then $\mathcal{K}$ has an interior point.

Proof. (i). Let $e \in \text{int}\mathcal{K}$ and let

$$\Theta = \{f \in \mathcal{K}^* \mid f(e) = 1\}.$$

Θ is a base of $\mathcal{K}^*$. Now $\mathcal{K} - e$ is a 0–neighbourhood in Y, and hence $(\mathcal{K} - e)^*$ is weakly– $*$ – compact. The result follows since Θ is a weakly– $*$ –closed subset of $(\mathcal{K} - e)^*$.
(ii). Suppose that Θ is a weakly– $*$ –compact base of $\mathcal{K}^*$. There is an element y_0 of Y such that $f(y_0) \geq 1$ for $f \in \Theta$. Since Y is a Mackey space, Θ^o is a 0–neighbourhood in Y. For $y \in \Theta^o$ and $f \in \Theta$, $f(y_0 + y) \geq 0$, so $y_0 + y \in \mathcal{K}^{**} = \mathcal{K}$. Hence, $y_0 + \Theta^o \subset \mathcal{K}$.

□

In the following example we show the cone $\mathcal{K}$ with empty interior such that $\mathcal{K}^*$ have a bounded and closed base in the norm topology.

Example 1.1.2 *([46],p.123) Let $Y = c_0$ be the space of real sequences converging to zero with the usual cone c_0^+ of nonnegative elements. Then c_0^+ has no interior points, and $(c_0^+)^*$ is the usual* *nonnegative cone* ℓ_1^+ *in* ℓ_1. *The set of sequences* $\{\xi_n\} \subset (c_0^+)^*$ *such that* $\sum \xi_n = 1$ *is a base for* $(c_0^+)^*$ *that is bounded and closed in the norm topology.*

The set

$$\mathcal{K}^{*i} = \{f \in \mathcal{K}^* \mid f(y) > 0 \text{ for all } y \in \mathcal{K} \setminus \{0\}\}$$

is called the **quasi-interior** of $\mathcal{K}^*$. Note that $\mathcal{K}^{*i}$ may be empty.

Example 1.1.3 ([65], Ex.3.7b, p.27) *Let* $Y = B[a,b]$, *be the set of all bounded, real-valued functions on the interval* $< a, b >$ *and*

$$\mathcal{K} = \{f \in B[a,b] \mid f(y) \geq 0 \text{ for all } y \in < a, b >\}.$$

The quasi-interior of $\mathcal{K}$ *is empty.*

Necessary and sufficient conditions for $\mathcal{K}^{*i}$ to be nonempty were given by Dauer and Gallagher in [34].

Proposition 1.1.3 (Dauer and Gallagher,[34]) *Let* Y *be a topological vector space and let* $\mathcal{K}$ *be a convex cone in* Y. *Then* $\mathcal{K}^{*i}$ *is nonempty if and only if there exists an open convex subset* U *in* Y *satisfying*

(i) $0 \notin U$,

(ii) $\mathcal{K} \subset cone(U) = \bigcup\{\lambda U \mid \lambda \geq 0\}$.

Proof. If $\mathcal{K}^{*i}$ is nonempty, the set $U = \{y \in Y \mid f(y) > 0\}$, $f \in \mathcal{K}^{*i}$, satisfies (i), (ii).
Let U be a subset of Y satisfying (i), (ii). Since $0 \notin U$, by separation arguments (see [79], p.58), there exists $f \in Y^*$ such that $f(0) < f(u)$ for $u \in U$. Thus, $f(u) > 0$ for all $u \in U$. From (ii) it follows that $f \in \mathcal{K}^{*i}$.

□

By Proposition 1.1.3, for any convex cone $\mathcal{K}$ in a locally convex space Y, $\mathcal{K}^{*i}$ is nonempty if and only if $\mathcal{K}$ is based. If Y is separable and $\mathcal{K}$ is closed convex and pointed, then $\mathcal{K}^{*i}$ is nonempty (see [50], Thm 2.1).

Let $A \subset Y$ be a subset of a linear space Y. The set

$$\mathrm{cor}(A) = \{a \in A \mid \forall y \in Y \ \exists \bar{\lambda} > 0 \ \text{with} \ a + \lambda y \in A \ \text{for} \ 0 \leq \lambda \leq \bar{\lambda}\}$$

is called the **algebraic interior** or the **core** of A. For any cone $\mathcal{K} \subset Y$ in a linear vector space, $cor\mathcal{K} \neq \emptyset$, implies that $\mathcal{K}$ is **reproducing,** ie., $\mathcal{K} - \mathcal{K} = Y$, (see Lemma 1.13 of [44]).

Theorem 1.1.3 (cf. [44], Lemmas 1.25, 1.26) *Let $\mathcal{K}$ be a closed convex cone in a topological vector space Y with $\mathcal{K}^* \neq \{0\}$. Then*

(i) $cor\mathcal{K} \subset \{y \in Y \mid f(y) > 0 \ \ for\ all\ f \in \mathcal{K}^* \setminus \{0\}\} \stackrel{def}{=} \mathcal{K}^i$.

(ii) If Y^* separates points of Y (ie., for any two different points $y_1, y_2 \in Y$ there exists $f \in Y^*$ such that $f(y_1) \neq f(y_2)$) and $\mathcal{K}^{*i} \neq \emptyset$, then $cor\mathcal{K}^* \subset \{f \in \mathcal{K}^* \mid f(y) > 0 \ \ for\ all\ y \in \mathcal{K} \setminus \{0\}\} = \mathcal{K}^{*i}$.

Proof. (i) Let $k \in \mathrm{cor}\mathcal{K}$. Thus, $k \in \mathcal{K}$ and for any $y \in Y$ there exists $\bar{\lambda} > 0$ with $k + \lambda y \in \mathcal{K}$ for $0 \leq \lambda \leq \bar{\lambda}$. Hence, for any $f \in \mathcal{K}^* \setminus \{0\}$, $f(k + \lambda y) \geq 0$ for any $0 \leq \lambda \leq \bar{\lambda}$. Since $f \in \mathcal{K}^* \setminus \{0\}$, there exists $y_0 \in Y$ with $f(y_0) < 0$ and we get $f(k) \geq -\bar{\lambda} f(y_0) > 0$. Hence, $f(k) > 0$.

(ii) Let $f \in \mathrm{cor}\mathcal{K}^*$. Thus, $f \in \mathcal{K}^*$ and for any $g \in Y^*$ there exists $\bar{\lambda} > 0$ with $f + \lambda g \in \mathcal{K}^*$ for $0 \leq \lambda \leq \bar{\lambda}$. Hence, $(f + \lambda g)y \geq 0$ for any $y \in \mathcal{K}$, and any $0 \leq \lambda \leq \bar{\lambda}$. By taking any $g_0 \in Y^*$ with $g_0(y) < 0$ we get $f(y) \geq -\bar{\lambda} g_0(y) > 0$. Hence, $f(y) > 0$.

□

When $\mathcal{K}^* = \{0\}$ Theorem 1.1.3 is not true; to see this it is enough to take $\mathcal{K} = Y$. As shown in [44], Lemma 1.27, in any linear vector space $\mathcal{K}^*$ is pointed whenever $cor\mathcal{K} \neq \emptyset$. It follows from Theorem 1.1.3 that then $\mathcal{K}^*$ is based.

Proposition 1.1.4 *Let Y be a locally convex topological vector space and let $\mathcal{K}$ be a closed convex cone in Y. If $cor\mathcal{K} \neq \emptyset$, and $\mathcal{K}^*$ is non-trivial, then $\mathcal{K}^*$ has a base.*

Proof. Let $y_0 \in cor\mathcal{K}$. Then the set

$$\Theta^* = \{f \in \mathcal{K}^* \mid f(y_0) = 1\}$$

is a base of $\mathcal{K}^*$. Θ^* is convex, weakly$-*-$closed, $0 \notin w - * - cl\Theta^*$. Moreover, for any $0 \neq f \in \mathcal{K}^*$, we have $f(y_0) = \lambda_f \neq 0$, and $f/\lambda_f \in \Theta^*$.

□

1.2 Basic minimality concepts.

Let Y be a topological vector space and let $\mathcal{K} \subset Y$ be a convex cone in Y. The **order relation** in Y associated with $\mathcal{K}$ is the relation $\leq$ defined as

$$y_1 \leq y_2 \Leftrightarrow y_1 - y_2 \in \mathcal{K}.$$

The relation $\leq$ is reflexive and transitive. The relation $\leq$ is anti-symmetric if and only if $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.
Let $A \subset Y$ be a subset of Y. An element $y \in A$ is a **minimal point** of A with respect to $\mathcal{K}$ if $(y - \mathcal{K}) \cap A = \{y\}$. By $\mathrm{Min}(A|\mathcal{K})$ we denote the set of all minimal points of A with respect to $\mathcal{K}$. When $\mathrm{int}\mathcal{K} \neq \emptyset$, we say that an element $y \in A$ is **weakly minimal** if $(y - \mathrm{int}\mathcal{K}) \cap A = \emptyset$. By $\mathrm{WMin}(A|\mathcal{K})$ we denote the set of all minimal points of A with respect to $\mathcal{K}$. We say that the domination property, (DP), holds for A if $A \subset Min(A|\mathcal{K}) + \mathcal{K}$.

1.3 Continuity of set-valued mappings

Let U be a topological space (space of parameters) and let Y be a Hausdorff topological vector space. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y.
For any multivalued mapping $\Gamma : U \rightrightarrows Y$, we define its domain as

$$\mathrm{dom}\Gamma = \{u \in U \mid \Gamma(u) \neq \emptyset\}.$$

A multivalued mapping $\Gamma : U \rightrightarrows Y$, is **upper Hausdorff continuous** at u_0 if for every $0-$neighbourhood W in Y there exists a neighbourhood U_0 of u_0 such that $\Gamma(u) \subset \Gamma(u_0) + W$ for $u \in U_0$. Γ is **lower continuous** at u_0 if for any $0-$neighbourhood W and any $y_0 \in \Gamma(u_0)$ there exists a neighbourhood U_0 of u_0 such that $(y_0 + W) \cap \Gamma(u) \neq \emptyset$ for all $u \in U_0$. Γ is **lower Hausdorff continuous** at u_0 if it is uniformly lower continuous on $\Gamma(u_0)$, ie., for any $0-$neighbourhood W there exists a neighbourhood U_0 of u_0 such that $\Gamma(u) \subset \Gamma(u_0) + W$ for all $u \in U_0$.
Following Nikodem [62] we define $\mathcal{K}-$ Hausdorff continuitites. We say that Γ is **$\mathcal{K}-$Hausdorff upper continuous** at u_0 if $\Gamma_K = \Gamma + \mathcal{K}$ is upper Hausdorff continuous at u_0, ie., for every $0-$neighbourhood W there exists a neighbourhood U_0 of u_0 such that $\Gamma(u) \subset \Gamma(u_0) + W + \mathcal{K}$ for $u \in U_0$. We say that Γ is **$\mathcal{K}-$Hausdorff lower continuous** at u_0 if Γ_K is lower Hausdorff continuous at u_0, ie., for every $0-$neighbourhood W there exists a neighbourhood U_0 of u_0 such that $\Gamma(u_0) \subset \Gamma(u) + W + \mathcal{K}$ for $u \in U_0$. Following [63] we say that Γ is **$\mathcal{K}-$lower continuous** at u_0 if Γ_K is lower continuous at u_0, ie., for every $y_0 \in \Gamma(u_0)$ and every $0-$neighbourhood W there exists a neighbourhood U_0 of u_0 such that $\Gamma(u) \cap (y_0 + W - \mathcal{K}) \neq \emptyset$ for $u \in U_0$.

Bibliographical note Classical textbooks on topological vector spaces are eg Alexiewicz [1], Schaefer [80], Robertson and Robertson [68], Peressini [65].
Ordered topological vector spaces are main subject of the monograph of Jameson [46].
Presentations of different aspects of the theory of multivalued mappings can be found in monographs by Berge [24], Aubin and Frankowska [9], Kuratowski [51].

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