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Proper Efficiency via Tradeoff directions. Convex Case

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Proper efficiency via tradeoff directions. Convex case

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Abstract We investigate tradeoff directions defined as efficient points of the tangent cone. We give characterization of existence of nonzero tradeoff directions.

Key words: efficient points, tangent cones, tradeoff directions

1 Introduction

Different types of efficiency can be expressed via different cones. The corresponding results are spread out in the literature, see eg [3], [7], [2]. Our aim here is to locate properly efficient points via tradeoff directions which are, roughly speaking, efficient points of the tangent cone. However, it is worth noticing that tradeoff directions have little in common with tradeoff ratios defined for pairs of objective and expressing maximal possible gain in one objective due to unit decrease of a second objective.

Let $(Y, \|\cdot\|)$ be a normed space and let $\Omega \subset Y$ be a closed convex pointed cone. We denote $\Omega^{\geq} := \Omega \setminus \{0\}$.

Let $A \subset Y$ be a subset of Y. The set $P(A)$, of *Pareto efficient points of* A is defined as

$$P(A) = \{y \in A \mid A \cap (y + \Omega^{\geq}) = \emptyset\}.$$

$P(A)$ is called the Pareto set. The set $GP(A)$, of *properly Pareto efficient points of* A is defined as

$$GP(A) = \{y \in A \mid A \cap (y + C^{\geq}) = \emptyset \text{ for some convex cone } C \text{ with } \Omega^{\geq} \subset \mathrm{int} C\}.$$

Let $y \in A$. Vector $v \in Y$ is tangent to A at y, $v \in T_A(y)$, if

$$\exists \ t_n > 0 \ \ t_n \downarrow 0 \ \ v_n \to v \ \text{ such that } y + t_n v_n \in A.$$

The cone of all tangent directions of A at y, $T_A(y)$, is called the Bouligand, or tangent, cone of A at y.

By $D_A(y)$ we denote the set of all feasible directions of A at $y \in A$, ie.,

$$D_A(y) = \{d \in Y \mid y + td \in A \text{ for some } t > 0\}.$$

We have $T_A(y) \subset \mathrm{cl} D_A(y)$.
If A is convex, the tangent cone $T_A(y)$ is convex and

$$T_A(y) = \mathrm{cl}\ \mathrm{cone}(A - y) = \mathrm{cl} D_A(y).$$

Definition 1.1 *([4]) Let $y \in P(A)$. The cone $P(T_A(y))$ of all Pareto points of $T_A(y)$ is called the cone of all* tradeoff directions *and is denoted by $PT_A(y) = P(T_A(y))$.*

Remark 1.1 *Let us note that if A is convex and $y \in A \setminus P(A)$, then $PT_A(y) = \emptyset$. Indeed, if $y \notin P(A)$, here exists an $a \in A$ such that $a - y \in \Omega^{\geq}$. On the other hand, since A is convex, $a - y \in D_A(y) \subset T_A(y)$, and y cannot be Pareto efficient.*
It is a direct consequence of the fact that $T_A(y)$ is a cone that

$$PT_A(y) \neq \emptyset \Leftrightarrow 0 \in PT_A(y).$$

2 Basic facts and definitions

We start with the following characterizations of efficiency and proper efficiency. A similar result was mentioned in [4]. We recall that $\Theta \subset \Omega$ is a base of Ω if $0 \notin \mathrm{cl}\Theta$, and $\Omega = \mathrm{cone}(\Theta)$.

Theorem 2.1 *Let $\Omega \subset Y$ be a closed convex pointed cone in Y. Let $A \subset Y$ be a convex subset of Y. We have*

(i) $y \in P(A)$ if and only if $D_A(y) \cap \Omega^{\geq} = \emptyset$,

(ii) if $\mathrm{int}\Omega \neq \emptyset$, and $y \in P(A)$, then $T_A(y) \cap \mathrm{int}\Omega = \emptyset$,

(iii) Let Θ be a base of Ω. $y \in GP(A)$ if and only if $T_A(y) \cap \Omega^{\geq} = \emptyset$.

Proof. (i). $\rightarrow$ On the contrary, suppose that there exists a nonzero vector $d \in D_A(y) \cap \Omega$. By definition, $y' = y + td \in A$ for some $t > 0$. This yields $y' - y \in \Omega \setminus \{0\}$ which contradicts the efficiency of y.
$\leftarrow$ On the contrary, suppose that $y \notin P(A)$, ie., there is an $y' \in A \cap (y + \Omega)$, $y' \neq y$. This entails that $y' - y \in D_A(y) \cap \Omega$.

(ii). Suppose that $d \in T_A(y) \cap \mathrm{int}\Omega$. By definition of the tangent cone, there exist $\lambda_i > 0$, $\lambda_i \downarrow 0$, and $d_i \to d$ such that $y_i = y + \lambda_i d_i \in A$. Hence, $d_i \in D_A(y) \cap \Omega$, and, by 2.1 (i), $y \notin P(A)$.

(iii). $\rightarrow$ By definition, $y \in GP(A)$ if and only if there exists a closed convex cone $C \subset Y$, int(C) $\neq \emptyset$, $\Omega^{\geq}\mathrm{int}C$, such that y is Pareto with respect to C. Or

equivalently, by (i), $D_A(y) \cap \mathrm{int}C = \emptyset$. In turn, the latter holds if and only if $T_A(y) \cap \mathrm{int}C = \mathrm{cl}D_A(y) \cap \mathrm{int}C = \emptyset$, and consequently, $T_A(y) \cap \Omega^{\geq} = \emptyset$.
$\leftarrow$ Assume that $T_A(y) \cap \Omega^{\geq} = \emptyset$. Let $\Theta \subset \Omega$ be a base of Ω. For each $\theta \in \Theta$ there exists a ball B_θ such that

$$T_A(y) \cap \bigcup_{\theta \in \Theta} (\theta + B_\theta) = \emptyset \quad \text{and consequently} \quad T_A(y) \cap \mathrm{cone}(\bigcup_{\theta \in \Theta} (\theta + B_\theta))^{\geq} = \emptyset.$$

The cone $C = \mathrm{cone}(\bigcup_{\theta \in \Theta} (\theta + B_\theta))$ has a nonempty interior. By (ii), $y \in P(A)$ with respect to C.

$\square$

To see that the converse to (ii) does not hold it is enough to take $A = \{(x, y) \mid x \geq 0 \;\; y \leq 0\}$ and $\Omega = \{(x, y) \mid x \geq 0 \;\; y \geq 0\}$. Here $P(A) = \emptyset$, while $T_y(A) \cap \mathrm{int}\Omega = \emptyset$.

Theorem 2.2 *$PT_A(y) \neq \emptyset$ if and only if $y \in GP(A)$.*

Proof. $\leftarrow$ Let $y \in GP(A)$. By Theorem2.1 (iii), $T_A(y) \cap \Omega^{\geq} = \emptyset$, which is true if and only if $0 \in PT_A(y)$.
$\rightarrow$ Let $PT_A(y) \neq \emptyset$. This is true if and only if $0 \in PT_A(y)$. By definition, $T_A(y) \cap \Omega^{\geq} = \emptyset$. By 2.1 (ii), $y \in GP(A)$.

$\square$

Theorem 2.3 *$y \in GP(A)$ if and only if $0 \in GPT_A(y)$.*

Proof. Follows directly from Theorem 2.1, (iii).

$\square$

Definition 2.1 *Let $y \in P(A)$. A direction $d \in T_A(y)$ is a* proper tradeoff direction *of A at y if $d \in GPT_A(y)$.*

Remark 2.1 *$d \in GPT_A(y)$ if and only if $T_{T_A(y)}(d) \cap \Omega^{\geq} = \emptyset$.*

Example 2.1 *Let us consider an ice-cream cone (or a Lorentz cone)*

$$C^k = \{y \in R^k \mid \begin{pmatrix} u \\ t \end{pmatrix}, \quad u \in R^{k-1}, \; t \in R \;\; \|u\| \leq t\}.$$

Some of his nonzero boundary points are not properly efficient with respect to R^k_+. For instance, in C^3 the rays $(-t, 0, t)$ and $(0, -t, t)$ are not properly efficient.

Proposition 2.1 *Let Ω be a closed convex pointed ordering cone. Let K be a closed convex cone such that $K \cap \Omega^{\geq} = \emptyset$. If $K \subset -\Omega$, then $\{0\} = P(K)$.*

Proof. Suppose that there exists $d \in K$, $d \neq 0$, $d \in P(K)$. That is, $(K - d) \cap (\Omega)^{\geq} = \emptyset$, and consequently, $d \notin -\Omega$.

□

Theorem 2.4 *Let $\Omega \subset Y$ be a closed convex pointed ordering cone in Y. Let $A \subset Y$ be a closed convex cone in Y. Let $y \in GP(A)$. If $T_A(y) \subset \Omega$, then $GP(A) = \{0\}$.*

Proof. Follows directly from Proposition 2.1.

□

3 Main results

In this section we prove necessary and sufficient conditions for a tangent direction to be a tradeoff direction. We start with necessary conditions. To proceed we need some existence results for efficient points. We start by recalling the following existence result.

Theorem 3.1 (Th.6.3 of [5]) *Let S be a nonempty subset of a partially ordered topological linear space Y with a closed ordering cone Ω. If the set S has a compact section, there exists at least one efficient element of S.*

Lemma 3.1 *Let $\Omega \subset Y$ be a closed convex cone in Y with a compact base Θ. Let $A \subset Y$ be a closed bounded subset of Y. Then $P(A) \neq \emptyset$.*

Proof. We start by showing that for any $y \in A$ the section $A_y = (y + \Omega) \cap A$ is compact. Indeed, take any sequence $z_n \in A_y$. We have

$$z_n = y + \lambda_n \theta_n . \tag{1}$$

Since A is bounded and Θ is compact, the sequence $\{\lambda_n\}$ is bounded. Hence, there exist subsequences $\theta_{n_k} \to \theta_0 \in \Theta$, and $\lambda_{n_k} \to \lambda_0$. By closedness of A, the sequence $\{z_{n_k}\}$ converges to $z_0 \in A$. This proves that each section A_y of A is compact. By Theorem 3.1, (see also Borwein [1] , and Luc [7], Cor.3.8), $P(A) \neq \emptyset$.

□

Theorem 3.2 *Let $\Omega \subset Y$ be a closed convex cone in Y with a compact base Θ. Let $A \subset Y$ be a closed convex subset of Y, and $y \in GP(A)$.*
If $d \in T_A(y)$ is a tradeoff direction, ie., $d \in PT_A(y)$, then there is a representation $d = \lim_n d_n$, $h_n d_n = y_n - y$, with $h_n \downarrow 0$, $y_n \in P(A)$.

Proof. Let $d \in T_A(y)$. Suppose on the contrary that for each representation $d = \lim_n d_n$, $h_n d_n = y_n - y$, where $h_n \downarrow 0$, and $y_n \in A$ we have $y_n \notin P(A)$. This means that

$$(A - y_n) \cap \mathrm{cone}(\Theta) \neq \emptyset .$$

There exists $a_n \in A$, $\theta_n \in \Theta$, $\lambda_n > 0$ such that

$$a_n - y_n = \lambda_n \theta_n , \tag{2}$$

and

$$a_n - y - h_n d_n = \lambda_n \theta_n . \tag{3}$$

Dividing (3) by λ_n we get

$$\frac{a_n - y}{\lambda_n} - \frac{h_n}{\lambda_n} d_n = \theta_n . \tag{4}$$

Firstly, if $h_n \leq \lambda_n$, then $\frac{h_n}{\lambda_n} \leq 1$, and, since, without losing generality, we can assume that $\theta_n \to \theta_0 \neq 0 \in \Theta$, by (4), we get

$$\frac{a_n - y}{\lambda_n} \to d_1 \in T_A(y) .$$

If $\frac{h_n}{\lambda_n} \to 0$, then $d_1 \in \Omega$, and $y \notin GP(A)$, contradictory to the assumption.
If $\frac{h_n}{\lambda_n} \to \lambda_0 \neq 0$, then, by passing to the limit in (4), we get

$$d_1 - \lambda_0 d = \theta_0 \in \Omega ,$$

and $d \notin PT_A(y)$, which is a contradiction.

Secondly, if $h_n \geq \lambda_n$, then dividing (3) by h_n we get

$$\frac{a_n - y}{h_n} - d_n = \frac{\lambda_n}{h_n} \theta_n ,$$

and, by passing to the limit, we obtain

$$\lim_{n \to +\infty} \frac{a_n - y}{h_n} = d_1 \in T_A(y) .$$

If $\frac{\lambda_n}{h_n} \to 0$, then $d_1 = d$. This means that any sequence $a_n \in A$, $a_n \in y_n + \Omega$ leads to the same tangent direction d in the sense that for any $a_n \in [A - y_n] \cap \Omega$ we have

$$\frac{a_n - y}{h_n} \to d .$$

Since $\lambda_n \leq h_n$ and h_n tends to zero, each set $(A - y_n) \cap \Omega$ is closed and bounded. Hence, by Lemma 3.1, there exists $\bar{a}_n \in P([A - y_n] \cap \Omega)$. Clearly, $\bar{a}_n \in P(A)$, and the direction d can be represented as $d = \lim_n v_n$, $h_n v_n = \bar{a}_n - y$, contrary to the assumption.

If $\frac{\lambda_n}{h_n} \to \lambda_0 \neq 0$, then

$$d_1 - d = \lambda_0 \theta_0 \in \Omega\,,$$

and $d \notin PT_A(y)$, which is a contradiction.

□

In the next theorem we express tradeoff directions as limit points of sequences of directions $\{d_n\}$ with $y + h_n d_n \in GP(A)$. To this aim we use the existence result of properly efficient points proved by X.D.H.Truong [8].
We recall that Y is a *weakly complete* space if every fundamental sequence in the weak topology has a weak limit. The spaces R^n, ℓ^p, $L^p_{[0,1]}$, $1 < p < +\infty$, are weakly complete. Following Krasnoselskii [6] we say that a cone $K \subset Y$ of a topological vector space Y is *completely regular* if any bounded increasing net contained in K and in a complete subset of E has a limit. It can be shown that in a weakly complete space Y complete regularity of a cone $K \subset Y$ is equivalent to the normality of K. A subset A of Y is $\Omega-$*bounded* if there exists a bounded subset $M \subset Y$ of Y such that $A \subset M - \Omega$.
The following result holds.

Theorem 3.3 ([8]) *Suppose that Y is a weakly complete Banach space, and $\Omega \subset Y$ is a completely regular cone with a closed base. Let $A \subset Y$ be closed and $\Omega-$bounded subset of Y. Then $GP(A) \neq \emptyset$.*

With this result we can formulate the following variant of Theorem 3.2.

Theorem 3.4 *Let Y be a weakly complete Banach space. Let $\Omega \subset Y$ be a closed convex cone in Y which is completely regular and posseses a compact base Θ. Let $A \subset Y$ be a closed convex subset of Y, and $y \in GP(A)$.
If $d \in T_A(y)$ is a tradeoff direction, ie., $d \in PT_A(y)$, then there is a representation $d = \lim_n d_n$, $h_n d_n = y_n - y$, with $h_n \downarrow 0$, $y_n \in GP(A)$.*

Proof. Let $d \in T_A(y)$. Suppose on the contrary that for each representation $d = \lim_n d_n$, $h_n d_n = y_n - y$, where $h_n \downarrow 0$, and $y_n \in A$ we have $y_n \notin GP(A)$. This means that

$$(A - y_n) \cap \operatorname{cone}(\Theta + \frac{1}{n}B) \neq \emptyset\,,$$

where B is the unit ball in Y. There exists $a_n \in A$, $\theta_n \in \Theta$, $\lambda_n > 0$, and $b_n \in B$ such that

$$a_n - y_n = \lambda_n(\theta_n + \frac{1}{n}b_n)\,, \tag{5}$$

and

$$a_n - y - h_n d_n = \lambda_n \theta_n + \lambda_n \frac{1}{n} b_n\,. \tag{6}$$

Dividing (6) by λ_n we get

$$\frac{a_n - y}{\lambda_n} - \frac{h_n}{\lambda_n} d_n = \theta_n + \frac{1}{n} b_n . \tag{7}$$

Firstly, if $h_n \leq \lambda_n$, then $\frac{h_n}{\lambda_n} \leq 1$, and, since, without losing generality, we can assume that $\theta_n \to \theta_0 \neq 0 \in \Theta$, by (7), we get

$$\frac{a_n - y}{\lambda_n} \to d_1 \in T_A(y) .$$

If $\frac{h_n}{\lambda_n} \to 0$, then $d_1 \in \Omega$, and $y \notin GP(A)$, contradictory to the assumption.
If $\frac{h_n}{\lambda_n} \to \lambda_0 \neq 0$, then, by passing to the limit in (5), we get

$$d_1 - \lambda_0 d = \theta_0 \in \Omega ,$$

and $d \notin PT_A(y)$, which is a contradiction.

Secondly, if $h_n \geq \lambda_n$, then dividing (6) by h_n we get

$$\frac{a_n - y}{h_n} - d_n = \frac{\lambda_n}{h_n} \theta_n + \frac{\lambda_n}{h_n} \frac{1}{n} b_n ,$$

and, by passing to the limit, we obtain

$$\lim_{n \to +\infty} \frac{a_n - y}{h_n} = d_1 \in T_A(y) .$$

If $\frac{\lambda_n}{h_n} \to 0$, then $d_1 = d$. This means that any sequence $a_n \in A$, $a_n \in y_n + \Omega$ leads to the same tangent direction d in the sense that for any $a_n \in [A - y_n] \cap \Omega$ we have

$$\frac{a_n - y}{h_n} \to d .$$

Since $\lambda_n \leq h_n$ and h_n tends to zero, each set $(A - y_n) \cap \Omega$ is closed and bounded. Hence, by Theorem 3.3, there exists $\bar{a}_n \in GP([A - y_n] \cap \Omega)$. Since A is convex, $\bar{a}_n \in GP(A)$, and the direction d can be represented as $d = \lim_n v_n$, $h_n v_n = \bar{a}_n - y$, contrary to the assumption.
If $\frac{\lambda_n}{h_n} \to \lambda_0 \neq 0$, then

$$d_1 - d = \lambda_0 \theta_0 \in \Omega ,$$

and $d \notin PT_A(y)$, which is a contradiction.

$\square$

In finite-dimensional space R^n Theorem 3.4 takes the following form.

Corollary 3.1 *Let $\Omega \subset R^n$ be a closed convex pointed normal cone in R^n. Let $A \subset R^n$ be a closed convex subset of R^n, and $y \in GP(A)$.*
If $d \in T_A(y)$ is a tradeoff direction, ie., $d \in PT_A(y)$, then there is a representation $d = \lim_n d_n$, $h_n d_n = y_n - y$, with $h_n \downarrow 0$, $y_n \in GP(A)$.

We prove the following sufficient conditions for a direction $d \in T_A(y)$ to be a tradeoff.

Theorem 3.5 *Let $\Omega \subset Y$ be a closed convex pointed cone in Y. Let $A \subset Y$ be a closed convex subset of Y. Let $y \in GP(A)$, and $d \in T_A(y)$. If in each representation of d as the limit $d = \lim_{n\to+\infty} d_n$, where $h_n d_n = y_n - y$, with $h_n \downarrow 0$, and $y_n \in A$, we have $y_n \in GP(A)$, then $d \in PT_A(y)$.*

Proof. Suppose on the contrary that $d \in T_A(y)$ is not efficient. There exists $d_1 \in T_A(y)$, $d_1 \neq d$, such that

$$d_1 - d \in \Omega . \tag{8}$$

By definition $d_1 = \lim_n d_n^1$, $h_n^1 d_n^1 = y_n^1 - y$, $d = \lim_n d_n$, $h_n d_n = y_n - y$, $y_n, y_n^1 \in A$, $h_n, h_n^1 \downarrow 0$.
Let C be a closed convex cone, int$C \neq \emptyset$, $\Omega \subset \text{int}C$. By (8), for all n sufficiently large

$$d_n^1 - d_n \in \text{int}C ,$$

and

$$\frac{y_n^1 - y}{h_n^1} - \frac{y_n - y}{h_n} \in C .$$

If $h_n^1 \geq h_n$, then, by passing eventually to some $\bar{y}_n^1 \in A$, we get

$$\frac{\bar{y}_n^1 - y}{h_n} - \frac{y_n - y}{h_n} \in C ,$$

and consequently

$$\bar{y}_n^1 - y_n \in C ,$$

which proves that y_n is not properly efficient.
If $h_n \geq h_n^1$, then, by passing eventually to some $\bar{y}_n \in A$, we get

$$\frac{y_n^1 - y}{h_n^1} - \frac{\bar{y}_n - y}{h_n^1} \in C ,$$

and consequently

$$y_n^1 - \bar{y}_n \in C ,$$

which proves that $\bar{y}_n$ is not properly efficient.

□

References

[1] Borwein J.M., On the existence of Pareto efficient points, Math.Oper.Res. vol.8, 64-73

[2] Borwein J.M., Zhuang D., Super efficiency in vector optimization, Transactions of the AMS, 338(1993), 105-122

[3] Guerraggio A., Molho E., Zaffaroni A., On the notion of proper efficiency in vector optimization, JOTA, 82(1994), 1-21

[4] Henig M., Buchanan J.T., Tradeoff directions in multiobjective optimization problems, mathenatical Programming (1997), 357-374

[5] Jahn J., Mathematical Vector Optimization in Partially Ordered Linear Spaces, Verlag Peter Lang 1986

[6] Krasnoselskii M.A., Positive Solutions of Operator Equations, Noordhoff, Groningen, Holland 1964

[7] Luc D.T., Theory of Vector Optimization, Springer Verlag, Berlin-Heidelberg 1989

[8] X.D.H.Truong, Existence and density results for proper efficiency in cone compact sets, JOTA, 111(2001), 173-194