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Well-posedness and Lipschitzness of solutions in vector optimization

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1 Introduction

The role of well-posedness in scalar optimization problems is widely recognized. The notion of a well-posed problem and its generalizations play an important role in model building, in numerical problem solving, and in investigating stability of solutions.
Nowadays, vector optimization (or multiple objective optimization) is gaining momentum in the development of its theory and applications. It has its origin primarily in economics. Recently, multiple objective techniques enter also in solving engineering design problems.
Different approaches to well-posedness in vector optimization are scattered in the literature. Since the behaviour of minimizing sequences seems to be crucial from the point of view of applications we choose the approach to well posedness in vector optimization via convergence of minimizing sequences so as to encompass the non-uniqueness and noncompactness of solution sets.
In the present paper we investigate the concept of strict and strong solutions to vector optimization problems. When applied to scalar optimization problems, these concepts both reduce to the concept of weak sharp minima due to Polyak [12] and investigated by many authors, eg. Studniarski and Ward [16], Burke and Deng [10], Burke and Ferris [9]. It is known that strict solutions play an important role in deriving conditions for Hőlder calmness in scalar optimization (see e.g. [8]). In Theorem 6.1 we prove calmness of solutions to parametric vector optimization problems at points which are strict and strong.
In the class of well-posed problems we study conditions ensuring Lipschitz and/or Hőlder continuity of efficient solutions to parametric vector optimization problems. We prove that in the case where calmness of the solution set-valued mapping S at some solution x_0 is of interest it is enough to assume that the solution set is simnltaneously strict and strong around x_0.

2 Preliminaries

Let $Y = (Y, \|\cdot\|)$ be a normed linear space with the open unit ball B_Y and let $\mathcal{K} \subset Y$ be a closed convex convex pointed cone in Y. Let $A \subset Y$ be a subset of Y. An element $y \in A$ is *minimal* , $y \in \mathrm{Min}(A, \mathcal{K})$ iff $(A - y) \cap (-\mathcal{K}) = \{0\}$. An element $y \in A$ is a *local minimum*, $y \in \mathrm{LMin}(A, \mathcal{K})$ iff there is a neighbourhood V of y $(A - y) \cap (-\mathcal{K}) \cap V = \{y\}$.
Let $U = (U, \|\cdot\|)$ be a normed space with the open unit ball B_U.. A set-valued mapping $\Gamma : U \rightrightarrows Y$, is

locally upper Lipschitz at u_0 (see Robinson [15], Aubin,Ekeland [1]) if there are positive numbers L and r such that $\Gamma(u) \subset \Gamma(u_0) + L\|u - u_0\| B_U$ for $u \in u_0 + rB_U$

locally Lipschitz at u_0 if there are positive numbers L and r such that $\Gamma(u_1) \subset \Gamma(u_2) + L\|u_1 - u_2\| B_U$ for $u_1, u_2 \in u_0 + rB_U$

locally upper Hőlder of order m at u_0 if there are positive numbers L and r such that $\Gamma(u) \subset \Gamma(u_0) + L\|u - u_0\|^m B_U$ for $u \in u_0 + rB_U$

locally Hőlder of order m at u_0 if there are positive numbers L and r such that $\Gamma(u_1) \subset \Gamma(u_2) + L\|u_1 - u_2\|^m B_U$ for $u_1, u_2 \in u_0 + rB_U$

calm of order q or **Hőlder calm of order** q at $(u_0, y_0) \in \mathrm{graph}\Gamma$ if there exist constants $L > 0$, $r > 0$, and $t > 0$ such that $\Gamma(u) \cap (x_0 + rB_Y) \subset \Gamma(u_0) + L\|u - u_0\|$ for $\|u - u_0\| < t$.

lower Lipschitz at $(u_0, y_0) \in \mathit{graph}\Gamma$, if there exist constants $L > 0$ and $t > 0$ such that $(y_0 + L\|u - u_0\| B_Y) \cap \Gamma(u) \neq \emptyset$ for $\|u - u_0\| < t$.

3 Well-posedness of vector optimization problems

Let $X = (X, \|\cdot\|)$ be a normed space with the open unit ball B_X. Vector optimization problem

$$\begin{aligned} &\mathcal{K} - \min \ f_0(x) \\ &\text{subject to } x \in A_0, \end{aligned} \qquad (P_0)$$

consists in finding the set $\mathrm{Min}(f_0, A_0, \mathcal{K}) = \mathrm{Min}(f_0(A_0)|\mathcal{K})$ called the *minimal* (or *efficient*) point set of (P_0), and the *solution set* $S(f_0, A_0, \mathcal{K}) = \{x \in A_0 \mid f_0(x) \in \mathrm{Min}(f_0, A_0, \mathcal{K})\}$, where $f_0 : X \to Y$ is a mapping and $A_0 \subset X$ is a subset of X .

The point $x_0 \in A_0$ is called a *local minimal solution* of (P_0), $x_0 \in \mathrm{LS}(A_0, f_0, \mathcal{K})$, iff there is a neighbourhood V of x_0 such that $(f_0(A_0 \cap V) - f_0(x_0)) \cap (-\mathcal{K}) = \{0\}$. In other words, $x_0 \in \mathrm{LS}(A_0, f_0, \mathcal{K})$ iff there is no $x \in A_0 \cap V$, $x \neq x_0$, such that $f_0(x) - f_0(x_0) \in -\mathcal{K}$.
In the sequel we often refer to problem (P_0) as the *original problem* or the *unperturbed problem*. The space X is called the *decision* space and Y is called the *outcome* space.

Definition 3.1 *Let $\varepsilon \in Y$. The problem (P_0) is* **upper Lipschitz well posed** *if*

$$\Pi(\varepsilon) = A_0 \cap f_0^{-1}(Min(f_0, A_0, \mathcal{K}) + \varepsilon - \mathcal{K})$$

$$\Pi(\varepsilon) = \bigcup_{\eta \in Min(f_0, A_0, \mathcal{K}))} A_0 \cap f_0^{-1}(\eta + \varepsilon - \mathcal{K})$$

is locally upper Lipschitz at 0.

Definition 3.2 *Let $\eta \in Min(f_0, A_0, \mathcal{K})$. Let $\varepsilon \in Y$. The problem (P_0) is $\eta-$* **upper Lipschitz well posed** *if*

$$\Pi^{\eta}(\varepsilon) = A_0 \cap f_0^{-1}(\eta + \varepsilon - \mathcal{K})$$

is locally upper Lipschitz at 0.

Note that $\Pi(\varepsilon) = \emptyset$ for $\varepsilon \in -\mathcal{K}$. Π is locally upper Lipschitz at 0 if

$$\Pi(\varepsilon) = A_0 \cap f_0^{-1}(Min(f_0, A_0, \mathcal{K}) + \varepsilon - \mathcal{K}) \subset Min(f_0, A_0, \mathcal{K}) + \|\varepsilon\| B_X$$

for $\|\varepsilon\| \leq r_0$.

4 Lipschitzness of solutions to perturbed vector optimization problems

Let U be the space of parameters. In the sequel we assume that U is a normed space. We embed the problem (P_0) into a family (P_u) of vector optimization problems parametrised by a parameter $u \in U$,

$$\begin{array}{ll} \mathcal{K} - \min\ f(u, x) & \\ \text{subject to } x \in A(u), & \qquad (P_u) \end{array}$$

where $f : U \times X \to Y$ is the parametrised objective function and $\mathcal{A} : U \rightrightarrows X$, is the feasible set multifunction. The problem (P_0) corresponds to a parameter value u_0, $f_0 = f(u_0, \cdot)$, $\mathcal{A}(u_0) = A_0$. The *performance* multifunction $\mathcal{P} : U \rightrightarrows Y$ is defined as $\mathcal{P}(u) = \mathrm{Min}(f(u, \cdot), \mathcal{A}(u), \mathcal{K})$ and the *solution* multifunction $\mathcal{S} : U \rightrightarrows Y$ is defined as $\mathcal{S}(u) = \ S(f(u, \cdot), \mathcal{A}(u), \mathcal{K})$.

In the theorem below we prove local upper lipschitzness of the solution set-valued mapping $\mathcal{S}$ at a given u_0 for a family of parametric problems of the form

$$\begin{array}{ll}\mathcal{K}-\min\ f_0(x) & \\ \text{subject to } x\in A(u)\,, & \qquad (P_u)\end{array}$$

in the class of the original problems (P_0) being upper Lipschitz well posed.

Theorem 4.1 *Let X, Y and U be normed spaces. Let $\mathcal{K}\subset Y$ be a closed convex pointed cone in Y, $int\mathcal{K}\neq\emptyset$. If*

(i) (P_0) is upper Lipschitz well posed,

(ii) f_0 is Lipschitz on X, $\mathcal{A}$ is locally upper Lipschitz at u_0,

(iii) $\mathcal{P}$ is locally upper Lipschitz at u_0,

then $\mathcal{S}$ is locally upper Lipschitz at u_0.

Proof. There exists L_1 such that

$$\mathcal{A}(u)\subset\mathcal{A}(u_0)+L_1\|u-u_0\|B_X=A_0+L_1\|u-u_0\|B_X, \tag{1}$$

for $u\in u_0+r_1B_U$. By Lipschitz well-posedness of (P_0), we have

$$A_0\cap f_0^{-1}\{\mathrm{Min}(f_0,A_0,\mathcal{K})+\varepsilon-\mathcal{K}\}\subset\mathcal{S}(u_0)+L\|\varepsilon\|B_X$$

for any $\varepsilon\in r_0B_Y$. Assuming that $\varepsilon=O(\|u-u_0\|)$, i.e., $\|\varepsilon\|\leq\alpha\|u-u_0\|$ and

$$A_0\cap f_0^{-1}\{\mathrm{Min}(f_0,A_0,\mathcal{K})+\alpha\|u-u_0\|B_Y-\mathcal{K}\}\subset\mathcal{S}(u_0)+L\alpha\|u-u_0\|B_X. \tag{2}$$

By (1), and (2),

$$A_0\ \subset[\mathcal{S}(u_0)+L\alpha\|u-u_0\|B_X]\cup[X\setminus f_0^{-1}\{\mathrm{Min}(f_0,A_0,\mathcal{K})+\alpha\|u-u_0\|B_Y-\mathcal{K}\}],$$

and

$$\begin{aligned}\mathcal{A}(u)\ &\subset A_0+L_1\|u-u_0\|B_X\\ &\subset[\mathcal{A}(u_0)+L\alpha\|u-u_0\|B_X+L_1\|u-u_0\|B_X]\\ &\cup[X\setminus f_0^{-1}\{\mathrm{Min}(f_0,A_0,\mathcal{K})+\alpha\|u-u_0\|B_Y-\mathcal{K}\}+L_1\|u-u_0\|B_X].\end{aligned} \tag{3}$$

Since f_0 is Lipschitz on X, there exists L_3 such that for any $x_1,x_2\in X$

$$f_0(x_1)\subset f_0(x_2)+L_3\|x_1-x_2\|B_Y.$$

Hence,

$$\begin{aligned}&f_0[X\setminus f_0^{-1}\{\mathrm{Min}(f_0,A_0,\mathcal{K})+\alpha\|u-u_0\|B_Y-\mathcal{K}\}+L\|u-u_0\|B_X]\\ &\subset f_0[X\setminus f_0^{-1}\{\mathrm{Min}(f_0,A_0,\mathcal{K})+\alpha\|u-u_0\|B_Y-\mathcal{K}\}]+L_3L\|u-u_0\|B_Y,\end{aligned}$$

and

$$f_0[X \setminus f_0^{-1}\{\mathrm{Min}(f_0, A_0, \mathcal{K}) + \alpha\|u - u_0\|B_Y - \mathcal{K}\}] + L_3L\|u - u_0\|B_Y$$
$$\subset Y \setminus [\mathrm{Min}(f_0, A_0, \mathcal{K}) + \alpha\|u - u_0\|B_Y - \mathcal{K}] + L_3L\|u - u_0\|B_Y.$$

Assuming that $LL_3 < \alpha$ we get

$$Y \setminus [\mathrm{Min}(f_0, A_0, \mathcal{K}) + \alpha\|u - u_0\|B_Y - \mathcal{K}] + L_3L\|u - u_0\|B_Y$$
$$\subset Y \setminus [\mathrm{Min}(f_0, A_0, \mathcal{K}) + L_3L\|u - u_0\|B_Y - \mathcal{K}],$$

and

$$f_0^{-1}[Y \setminus \mathrm{Min}(f_0, A_0, \mathcal{K}) + L_3L\|u - u_0\|B_Y - \mathcal{K}] \subset X \setminus f_0^{-1}[\mathrm{Min}(f_0, A_0, \mathcal{K}) + L_3L\|u - u_0\|B_Y - \mathcal{K}]$$

Hence, by (3),

$$\mathcal{A}(u) \subset A_0 + L\|u - u_0\|B_X$$
$$\subset [\mathcal{S}(u_0) + L\alpha\|u - u_0\|B_X + L\|u - u_0\|B_X] \cup [X \setminus f_0^{-1}[\mathrm{Min}(f_0, A_0, \mathcal{K}) + L_3L\|u - u_0\|B_Y - \mathcal{K}]]$$

By *(iii)*, for any $x \in \mathcal{S}(u)$, $u \in u_0 + r_1B_U$

$$f(x) \subset \mathrm{Min}(f_0, A_0, \mathcal{K}) + L_2\|u - u_0\|B_Y,$$

and under the assumption that $L_2 \leq LL_3$ we get

$$\mathcal{S}(u) \subset \mathcal{S}(u_0) + (L\alpha + LL_3)\|u - u_0\|B_X$$

□

5 Strict and strong solutions to vector optimization problems

In this section we recall the notions of strict and strong solutions to vector optimization problem (P_0).

We say that $\phi : R_+ \to R_+$ is an *admissible function* if ϕ is nondecreasing and $\lim_{t\to 0} \phi(t) = 0$.

Definition 5.1 *[2] We say that $x_0 \in S(f_0, A_0, \mathcal{K})$ is ϕ–*strict *if there is an $r > 0$ such that for each $x \in A_0 \cap (x_0 + rB)$, $x \neq x_0$,*

$$(f_0(x) - f_0(x_0)) \cap (\phi(\|x - x_0\|)B_Y - \mathcal{K}) = \emptyset,$$

where ϕ is an admissible function.

Definition 5.2 *We say that $x_0 \in S(f_0, A_0, \mathcal{K})$ is* strict of order m *if there is an $r > 0$ such that for each $x \in A_0 \cap (x_0 + rB)$, $x \neq x_0$,*

$$(f_0(x) - f_0(x_0)) \cap (\alpha\|x - x_0\|^m B_Y - \mathcal{K}) = \emptyset.$$

Definition 5.3 *[2] Let int$\mathcal{K} \neq \emptyset$. We say that a solution $x_0 \in S(f_0, A_0, \mathcal{K})$ is* $\phi-$strong *if there is an $r > 0$ such that for each $x \in A_0 \cap (x_0 + rB_X)$ there exists $s_x \in S(f_0, A_0, \mathcal{K}) \cap (x_0 + rB_X)$*

$$f_0(x) - f_0(s_x) - \phi(\|x - s_x\|)B_Y \subset \mathcal{K},$$

where ϕ is an admissible function.

Definition 5.4 *[2, 13, 14] Let int$\mathcal{K} \neq \emptyset$. We say that a solution $x_0 \in S(f_0, A_0, \mathcal{K})$* is strong of order m *if there are constants $r > 0$ and $\alpha > 0$ such that for each $x \in A_0 \cap (x_0 + rB_X)$ there exists $s_x \in S(f_0, A_0, \mathcal{K}) \cap (x_0 + rB_X)$*

$$f_0(x) - f_0(s_x) - \alpha\|x - s_x\|^m B_Y \subset \mathcal{K}.$$

If $m = 1$ we say that the solution set is strong.

6 Hőlder calmness of solutions to parametric vector optimization problems

In this section we prove calmness of order 1/2 of solutions to parametric vector optimization problems at points which are simultaneously strict of order 2 and strong. Similar results for scalar optimization problems were obtained by Bonnans and Shapiro [8], sec.4.4.2. In finite dimensional spaces, weak sharp minima of order 2 were investigated by Ioffe and Shapiro [11].
To prove our theorem we need one more definition.

Definition 6.1 *We say that the set $\bar{S} \subset S(f_0, A_0, \mathcal{K})$ is a* set of strict minima of order *m of the problem (P_0) if there is $\alpha > 0$ and $r > 0$ such that for any $\bar{x} \in \bar{S}$ and any $x \in A_0 \cap (\bar{S} + rB_X)$, $x \notin \bar{S}$, we have*

$$[f_0(x) - f_0(\bar{x})] \cap [\alpha dist(x, \bar{S})^m B_Y - \mathcal{K}] = \emptyset,$$

where, for any subset $C \subset X$, we put $dist(x, C) = \inf\{\|x - c\| \mid c \in C\}$. If $m = 1$ we say that the solution set $\bar{S}$ is strong.

Now we are in a position to prove our main result. To fix notations let us recall that any function $f_0 : X \to Y$ is locally Lipschitz around x_0 if there exist constants $L_1 > 0$ and $r_2 > 0$ such that for any x_1, x_2, $\|x_1 - x_0\| < r_2$, $\|x_2 - x_0\| < r_2$, we have

$$\|f_0(x_1) - f_0(x_2)\| \leq L_1\|x_1 - x_2\|.$$

Theorem 6.1 *Let $X = (X, \|\cdot\|)$, $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, int$\mathcal{K} \neq \emptyset$. Assume that there exists $r_1 > 0$ such that $\bar{S} = S(f_0, A_0, \mathcal{K}) \cap (x_0 + r_1 B_X)$ is a set of strict minimal solutions of order 2 to (P_0). If* *(i)* $f_0 : X \to Y$ *is Lipschitz locally at* x_0,

(ii) $\mathcal{A} : U \to X$ *is calm and Lipschitz lower semicontinuous at* $(u_0, x_0) \in$ *graph*$\mathcal{A}$,

(iii) $x_0 \in S(f_0, A_0, \mathcal{K})$ *is strong with constants* $\alpha > 0$ *and* $r_2 > 0$,

then $\mathcal{S}$ *is Hőlder calm at* (u_0, x_0) *of order* $\frac{1}{2}$ *in the sense that for a constant* $L > 0$, *a neighbourhood* V *of* x_0 *and any* $x(u) \in \mathcal{S}(u) \cap V$ *we have*

$$dist(x(u), \mathcal{S}(u_0)) \leq L\|u - u_0\|^{\frac{1}{2}}$$

for all u *in some neighbourhood* U_0 *of* u_0.

Proof. By the calmness of $\mathcal{A}$, at $(u_0, x_0) \in$ graph$\mathcal{A}$ there is an L_0, $r_0 > 0$ and $t_0 > 0$ satisfying

$$\mathcal{A}(u) \cap (x_0 + r_0 B_X) \subset \mathcal{A}(u_0) + L_0\|u - u_0\| B_X \,,$$

for $\|u - u_0\| < t_0$. Without losing generality we can assume that $r_0 + t_0 < r_1$. Put $r = \min\{r_0, r_2\}$. For each $x(u) \in \mathcal{A}(u) \cap (x_0 + rB_X)$ there is $z(u) \in \mathcal{A}(u_0)$ such that

$$\|z(u) - x(u)\| \leq L_0\|u - u_0\| \,.$$

Without loss of generality we can assume that $\mathcal{S}(u) \cap (x_0 + rB_X) \neq \emptyset$ for all u, $\|u - u_0\| < t$, $t > 0$. Take any $x(u) \in \mathcal{S}(u) \cap (x_0 + rB_X)$. There exists $z(u) \in \mathcal{A}(u_0)$ such that

$$\|x(u) - z(u)\| \leq L_0\|u - u_0\| \,.$$

By the local lipschitzness of f_0 around x_0,

$$\|f_0(z(u)) - f_0(x(u))\| \leq L_1\|z(u) - x(u)\| \leq L_1 L_0\|u - u_0\| \,.$$

Since by *(iii)*, $x_0 \in S(f_0, A_0, \mathcal{K})$ is strong, and $z(u) \in A_0 \cap (x_0 + rB_X)$ there exists $\bar{z}(u) \in S(f_0, A_0, \mathcal{K}) \cap (x_0 + rB_X)$ such that

$$f_0(\bar{z}(u)) = f_0(z(u)) - k_u, \quad k_u \in \mathcal{K} \;\; k_u + \alpha\|z(u) - \bar{z}(u)\| B_Y \subset \mathcal{K} \,.$$

By the lower Lipschitz continuity of $\mathcal{A}$ there exist $L_3 > 0$, $t_1 > 0$, and $\bar{x}(u) \in \mathcal{A}(u)$ such that

$$\|\bar{x}(u) - \bar{z}(u)\| \leq L_3\|u - u_0\|,$$

for $\|u - u_0\| \leq t_1$. Now we show that

$$\|f_0(\bar{z}(u)) - f_0(z(u))\| < \frac{L_0(L_1 + L_3)}{\alpha}\|u - u_0\| \,.$$

Indeed, by the local lipschitzness of f_0 around x_0,

$$\|f_0(\bar{x}(u)) - f_0(\bar{z}(u))\| \leq L_1 \|\bar{x}(u) - \bar{z}(u)\| \leq L_1 L_3 \|u - u_0\|,$$

and hence,

$$\begin{aligned} f_0(\bar{x}(u)) - f_0(x(u)) &= [f_0(\bar{x}(u) - f_0(\bar{z}(u))] + [f_0(\bar{z}(u)) - f_0(z(u))] + [f_0(z(u)) - f_0(x(u))] \\ &= -k_u + w(u), \end{aligned}$$

where

$$w(u) = [f_0(\bar{x}(u) - f_0(\bar{z}(u))] + [f_0(z(u)) - f_0(x(u))] \quad \text{and} \quad \|w(u)\| \leq L_1(L_3 + L_0)\|u - u_0\|.$$

If it were

$$\|k_u\| > \frac{L_1^2(L_3 + L_0)}{\alpha}\|u - u_0\|,$$

then

$$L_1\|\bar{z}(u) - z(u)\| > \frac{L_1^2(L_3 + L_1)}{\alpha}\|u - u_0\|$$

and

$$\alpha\|\bar{z}(u) - z(u)\| > L_1(L_3 + L_1)\|u - u_0\|.$$

Then it would be $w(u) \in \alpha\|\bar{z}(u) - z(u)\|B_X$ which would contradict the minimality of $x(u)$, since it would imply that

$$k_u + w(u) \in \mathcal{K}.$$

This proves that

$$\|f_0(\bar{z}(u)) - f_0(z(u))\| \leq \frac{L_1^2(L_0 + L_3)}{\alpha}\|u - u_0\|,$$

or

$$f_0(z(u)) - f_0(\bar{z}(u)) \in \frac{L_1^2(L_0 + L_3)}{\alpha}\|u - u_0\|B_Y\,.$$

Observe now that $\|\bar{z}(u) - x_0\| < r$ and hence $\bar{z}(u) \in \bar{S}$. By the strict local minimality of $\bar{S}$

$$f_0(z(u)) - f_0(\bar{z}(u)) \notin L_2\mathrm{dist}(z(u), \bar{S})^2 B_Y - \mathcal{K}\,.$$

Finally,

$$\frac{L_1^2(L_0 + L_3)}{\alpha}\|u - u_0\|B_Y \not\subset L_2\mathrm{dist}(z(u), \bar{S})^2 B_Y - \mathcal{K}\,,$$

and consequently

$$\frac{L_1^2(L_0 + L_3)}{\alpha}\|u - u_0\|B_Y \not\subset L_2\mathrm{dist}(z(u), \bar{S})^2 B_Y\,,$$

which means that

$$\operatorname{dist}(z(u),\bar{S})^2 \leq \frac{L_1^2(L_0+L_3)}{\alpha L_2}\|u-u_0\|$$

or

$$\operatorname{dist}(z(u),\bar{S}) \leq \sqrt{\frac{L_1^2(L_0+L_3)}{\alpha L_2}}\|u-u_0\|^{\frac{1}{2}}.$$

Finally,

$$\operatorname{dist}(x(u),\bar{S}) \leq \|x(u)-z(u)\| + \operatorname{dist}(z(u),\bar{S}) \leq \left(L_0+\sqrt{\frac{L_1^2(L_0+L_3)}{\alpha L_2}}\right)\|u-u_0\|^{\frac{1}{2}}.$$

□

In the theorem below we prove general Hőlder calmness of the solution set-valued mapping $\mathcal{S}$ of order $\min\{p, \frac{p}{2m}\}$, around (u_0, x_0), whenever the order of continuity of the set-valued mapping $\mathcal{A}$ is $p \geq 1$, x_0 is strong of order $m \geq 1$ and the solution set $S(f_0, A_0, \mathcal{K})$ is strict of order 2 in some neighbourhood of x_0.

Theorem 6.2 *Let $X = (X, \|\cdot\|)$, $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $int\mathcal{K} \neq \emptyset$. Assume that there exists $r_1 > 0$ such that $\bar{S} = S(f_0, A_0, \mathcal{K}) \cap (x_0 + r_1 B_X)$ is a set of strict minimal solutions of order 2 to (P_0). If*

(i) $f_0 : X \to Y$ is Lipschitz locally at x_0,

(ii) $\mathcal{A} : U \to X$ is calm of order $p > 1$ and Lipschitz lower semicontinuous of order $p \geq 1$ at $(u_0, x_0) \in graph\mathcal{A}$,

(iii) $x_0 \in S(f_0, A_0, \mathcal{K})$ is strong of order $m \geq 1$ with constants $\alpha > 0$ and $r_2 > 0$,

then $\mathcal{S}$ is Hőlder calm at (u_0, x_0) of order $\min\{p, \frac{p}{2m}\}$ in the sense that for a constant $L > 0$, a neighbourhood V of x_0 and any $x(u) \in \mathcal{S}(u) \cap V$ we have

$$dist(x(u), \mathcal{S}(u_0)) \leq L\|u-u_0\|^{\min\{p,\frac{p}{2m}}$$

for all u in some neighbourhood U_0 of u_0.

Proof. By the calmness of $\mathcal{A}$, at $(u_0, x_0) \in \operatorname{graph}\mathcal{A}$ there is an L_0, $r_0 > 0$ and $t_0 > 0$ satisfying

$$\mathcal{A}(u) \cap (x_0 + r_0 B_X) \subset \mathcal{A}(u_0) + L_0\|u-u_0\|^p B_X,$$

for $\|u - u_0\| < t_0$. Without losing generality we can assume that $r_0 + t_0 < r_1$. Put $r = \min\{r_0, r_2\}$. For each $x(u) \in \mathcal{A}(u) \cap (x_0 + rB_X)$ there is $z(u) \in \mathcal{A}(u_0)$ such that

$$\|z(u) - x(u)\| \leq L_0 \|u - u_0\|^p .$$

Without loss of generality we can assume that $\mathcal{S}(u) \cap (x_0 + rB_X) \neq \emptyset$ for all u, $\|u - u_0\| < t$, $t > 0$. Take any $x(u) \in \mathcal{S}(u) \cap (x_0 + rB_X)$. There exists $z(u) \in \mathcal{A}(u_0)$ such that

$$\|x(u) - z(u)\| \leq L_0 \|u - u_0\|^p .$$

By the local lipschitzness of f_0 around x_0,

$$\|f_0(z(u)) - f_0(x(u))\| \leq L_1 \|z(u) - x(u)\| \leq L_1 L_0 \|u - u_0\|^p .$$

Since by *(iii)*, $x_0 \in S(f_0, A_0, \mathcal{K})$ is strong of order m, and $z(u) \in A_0 \cap (x_0 + rB_X)$ there exists $\bar{z}(u) \in S(f_0, A_0, \mathcal{K}) \cap (x_0 + rB_X)$ such that

$$f_0(\bar{z}(u)) = f_0(z(u)) - k_u, \quad k_u \in \mathcal{K} \;\; k_u + \alpha \|z(u) - \bar{z}(u)\|^m B_Y \subset \mathcal{K} .$$

By the lower Lipschitz continuity of $\mathcal{A}$ there exist $L_3 > 0$, $t_1 > 0$, and $\bar{x}(u) \in \mathcal{A}(u)$ such that

$$\|\bar{x}(u) - \bar{z}(u)\| \leq L_3 \|u - u_0\|^p ,$$

for $\|u - u_0\| \leq t_1$. Now we show that

$$\|f_0(\bar{z}(u)) - f_0(z(u))\| < \frac{L_0(L_1 + L_3)}{\sqrt[m]{\alpha}} \|u - u_0\|^{p/m} .$$

Indeed, by the local lipschitzness of f_0 around x_0,

$$\|f_0(\bar{x}(u)) - f_0(\bar{z}(u))\| \leq L_1 \|\bar{x}(u) - \bar{z}(u)\| \leq L_1 L_3 \|u - u_0\|^p ,$$

and hence,

$$\begin{aligned} f_0(\bar{x}(u)) - f_0(x(u)) &= [f_0(\bar{x}(u) - f_0(\bar{z}(u))] + [f_0(\bar{z}(u)) - f_0(z(u))] + [f_0(z(u)) - f_0(x(u))] \\ &= -k_u + w(u), \end{aligned}$$

where

$$w(u) = [f_0(\bar{x}(u) - f_0(\bar{z}(u))] + [f_0(z(u)) - f_0(x(u))] \quad \text{and} \quad \|w(u)\| \leq L_1(L_3 + L_0)\|u - u_0\|^p .$$

Assume that $L_1(L_3 + L_0) \leq (L_1(L_0 + L_3))^m$. If it were

$$\|k_u\| > \frac{L_1^2(L_3 + L_0)}{\sqrt[m]{\alpha}} \|u - u_0\|^{p/m} ,$$

then

$$L_1 \|\bar{z}(u) - z(u)\| > \frac{L_1^2(L_3 + L_1)}{\sqrt[m]{\alpha}} \|u - u_0\|^{p/m}$$

and

$$\alpha\|\bar{z}(u) - z(u)\|^m > (L_1(L_3 + L_1))^m \|u - u_0\|^p.$$

Then it would be $w(u) \in \alpha\|\bar{z}(u) - z(u)\|^m B_X$ which would contradict the minimality of $x(u)$, since it would imply that

$$k_u + w(u) \in \mathcal{K}.$$

This proves that

$$\|f_0(\bar{z}(u)) - f_0(z(u))\| \leq \frac{L_1^2(L_0 + L_3)}{\sqrt[m]{\alpha}} \|u - u_0\|^{p/m},$$

or

$$f_0(z(u)) - f_0(\bar{z}(u)) \in \frac{L_1^2(L_0 + L_3)}{\sqrt[m]{\alpha}} \|u - u_0\|^{p/m} B_Y .$$

Observe now that $\|\bar{z}(u) - x_0\| < r$ and hence $\bar{z}(u) \in \bar{S}$. By the strict local minimality of $\bar{S}$

$$f_0(z(u)) - f_0(\bar{z}(u)) \notin L_2 \mathrm{dist}(z(u), \bar{S})^2 B_Y - \mathcal{K}.$$

Finally,

$$\frac{L_1^2(L_0 + L_3)}{\sqrt[m]{\alpha}} \|u - u_0\|^{p/m} B_Y \not\subset L_2 \mathrm{dist}(z(u), \bar{S})^2 B_Y - \mathcal{K},$$

and consequently

$$\frac{L_1^2(L_0 + L_3)}{\sqrt[m]{\alpha}} \|u - u_0\|^{p/m} B_Y \not\subset L_2 \mathrm{dist}(z(u), \bar{S})^2 B_Y ,$$

which means that

$$\mathrm{dist}(z(u), \bar{S})^2 \leq \frac{L_1^2(L_0 + L_3)}{\sqrt[m]{\alpha} L_2} \|u - u_0\|^{p/m}$$

or

$$\mathrm{dist}(z(u), \bar{S}) \leq \sqrt{\frac{L_1^2(L_0 + L_3)}{\sqrt[m]{\alpha} L_2}} \|u - u_0\|^{\frac{p}{2m}} .$$

Finally,

$$\mathrm{dist}(x(u), \bar{S}) \leq \|x(u) - z(u)\| + \mathrm{dist}(z(u), \bar{S}) \leq \left(L_0 + \sqrt{\frac{L_1^2(L_0 + L_3)}{\sqrt[m]{\alpha} L_2}} \right) \|u - u_0\|^{\min\{p, \frac{p}{2m}\}} .$$

□

Note, in particular that in the case when the solution set is strict of order 2 around x_0 and x_0 is strong of order 2, then the solution set-valued mapping is calm at (u_0, x_0) of order 1/4 which differs from the scalar case.

References

[1] Aubin J.-P., Ekeland I., Applied Nonlinear Analysis, Wiley, New York 1984

[2] Bednarczuk E.M., Stability Analysis for Parametric Vector Optimization Problems, submitted do Disertationes Mathemaiticae

[3] Bednarczuk E., On lower Lipschitz continuity of minimal points Discussiones Mathematicae, Diffferential Inclusion, Control and Optimization, 20(2000), 245-255

[4] Bednarczuk E., Upper Hőlder continuity of minimal points, Journal on Convex Analysis vol.9(2002),no.2 327-338

[5] Bednarczuk E., Hőlder-like properties of minimal points in vector optimization, Control and Cybernetics vol.31(2002) no.3 423-438

[6] Bednarczuk E., Some stability results for vector optimization problems in partially ordered topological vector spaces, Proceedings of the First World Congress of Nonlinear Analysts, Tampa, Florida, August 19-26 1992, 2371-2382, ed V.Lakshimikantham, Walter de Gruyter, Berlin, New York 1996

[7] Bednarczuk E., An approach to well-posedness in vector optimization: consequences to stability, Control and Cybernetics 23(1994), 107-122

[8] Bonnans J.F., Shapiro A., Pertnrbation Analysis of Optimization Problems, Springer, New York, 2000

[9] Burke J.V., Ferris M.C., Weak sharp minima in mathematical programming, SIAM J. on Control and Optimization, vol.71(1995), 179-194

[10] Burke J.V., Deng S., Weak sharp minima revisited, PartI: Basic theory, Control and Cybernetics

[11] Ioffe A.D., Bonnans F., Quadratic growth and stability in convex programming problems with multiple solutions,, Journal on Convex Analysis, vol.2(1995), 41-57

[12] Polyak B.T., Sharp minima. Institute of Control Sciences. Lecture Notes, Moscow 1979, presented at the IIASA Workshop on Generalized Lagrangians and Their Applications, IIASA, Laxenburg, Austria, 1979

[13] Jiménez B., Strict efficiency in vector optimization, Journal of Mathematical Analysis and Applications, vol.264(2002), 264-284

[14] Jiménez B., Strict minimality conditions in nondifferentiable multiobjective programming, JOTA, vol.116(2003), 99-116

[15] Robinson S.M., Some continuity properties of polyhedral multifunctions, Mathematical Programming Study, vol.14(1981), 206-214

[16] Studniarski M., Ward D.E., Weak sharp minima: characterization and sufficient conditions, SIAM J. on Control and Optimization, vol.38(1999), 219-236