9/2006

Raport Badawczy
Research Report

RB/48/2006

On sharp minima in vector optimization with applications to multicriteria linear problems

E. Bednarczuk

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 8373578

fax: (+48) (22) 8372772

Kierownik Pracowni zgłaszający pracę:
Prof. dr hab. inż. Kazimierz Malanowski

Warszawa 2006

On sharp minima in vector optimization with applications to multicriteria linear problems.

Ewa Bednarczuk
Systems Research Institute of the PAS, Warsaw, Poland

Abstract

In the paper we discuss the notions of local and global sharp and weak sharp solutions to vector optimization problems. As an application we provide sufficient conditions for stability of solutions in perturbed problems and we specialize these conditions to linear multicriteria problems.

Keywords: vector optimization, sharp solutions, weak sharp solutions, stability

1 Introduction

Let Y and Y be normed spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. We consider vector optimization problems of the form

$$(VOP) \qquad \begin{array}{c} \mathcal{K} - \min f(x) \\ \text{subject to } x \in A \end{array}$$

where $f : X \to Y$, $A \subset Y$ is a feasible set. By $E \subset Y$ we denote the set of all global efficient points of (VOP), i.e., $\alpha \in E$ if $(f(A) - \alpha) \cap (-\mathcal{K}) = \{0\}$. The set $S \subset X$ defined as $S = A \cap f^{-1}(E)$ is the set of all global solutions to (VOP).

A point $x_0 \in A$, $f(x_0) = \alpha$, is a *local solution* to (VOP) if there exists $\rho > 0$ such that

$$(f(A \cap B(x_0, \rho)) - \alpha) \cap (-\mathcal{K}) = \{0\}.$$

$\alpha \in f(A)$ is a *local efficient point* if there exists $\rho > 0$ such that

$$(f(A) \cap B(\alpha, \rho) - \alpha) \cap (-\mathcal{K}) = \{0\}.$$

If $f(x_0) \in f(A)$ is a local efficient point to (VOP) and f is continuous at x_0, then x_0 is local solution to (VOP). Moreover, if $f(A)$ is a convex set, then the sets of local and global efficient points coincide.

We discuss the notions of sharp and weak sharp solutions to (VOP). In Section 2 local notions are presented and their basic properties are proved. In Section 3 global sharp and weak sharp solutions are discussed.

The equivalence between local and global notions is investigated under standard convexity assumptions. In Section 4 sharp and weak sharp solutions are exploited to derive general stability results for perturbed problems. Finally, in Section 5, the general results are applied to formulate stability conditions for linear multicriteria optimization problems.

2 Local sharp and weak sharp solutions

In scalar optimization, the role of weak sharp minima and their relationships to the existence of error bounds and stability is widely recognized, see e.g. [1, 2, 3, 5, 6]. In Section 4 we use vector weak sharp minima to derive stability results.

By $B(x_0, r)$ we denote the open ball of centre x_0 and radius r, by B_X and B_Y we denote open unit balls in X and Y, respectively. Moreover, for any set $C \subset X$, $d(x, C) = \inf\{\|x - c\| \ : \ c \in C\}$.

For any $\alpha \in E$, put

$$S_\alpha := \{x \in S : \ f(x) = \alpha\}.$$

Definition 2.1 *We say that $x_0 \in A$, $f(x_0) = \alpha$, is a* local α-weak sharp solution *to (VOP) if there exist constants $\tau > 0$ and $\rho > 0$ such that*

$$f(x) - f(x_0) \notin \tau d(x, S_\alpha) B_Y - \mathcal{K} \quad \text{for all} \quad x \in A \cap B(x_0, \rho) \quad x \notin S_\alpha. \quad (1)$$

Any local α-weak sharp solution $x_0 \in A$ is a local solution to (VOP) since by (1),

$$f(x) - f(x_0) \notin -\mathcal{K}, \quad \text{for} \quad x \in A \cap B(x_0, \rho) \quad f(x) \neq f(x_0)$$

which says that x_0 is a local solution to (VOP).

A function f is locally Lipschitz around $x_0 \in X$ if there exist constants $L_f > 0$ and $\rho > 0$ such that

$$f(x) - f(x') \in L_f \|x - x'\| B_Y \quad \text{for} \quad x, x' \in B(x_0, \rho).$$

The following proposition relates weak sharp solution to well-posedness of (VOP) (see also [11, 12, 13]. It improves Proposition 4.3 of [4] in the sense that here we do not assume that int$\mathcal{K} \neq \emptyset$. On the other hand, the multifunction $\mathcal{E}^\alpha$ defined below is slightly different from that used in Proposition 4.3 of [4].

Proposition 2.1 *Let $x_0 \in A$, $f(x_0) = \alpha$.*

(i) If x_0 is a local α-weak sharp solution to (VOP) with constants $\tau > 0$ and $\rho > 0$, then

$$f(x) - f(x_0) \notin \tau d(x, S_\alpha) B_Y \quad x \in A \cap B(x_0, \rho) \quad x \notin S_\alpha. \quad (2)$$

(ii) x_0 is a local α-weak sharp solution to (VOP) with constants $\tau > 0$ and $\rho > 0$ if and only if the following condition holds:

(**C**) *there exist* $\varepsilon_0 > 0$ *and* $\rho > 0$ *such that for each* $0 < \varepsilon < \varepsilon_0$

$$\mathcal{E}^\alpha(\varepsilon)\cap B(x_0,\rho) := A\cap B(x_0,\rho)\cap f^{-1}(\alpha+\varepsilon B_Y-\mathcal{K}) \subset S_\alpha+\varepsilon\frac{1}{\tau}B_Y.$$

Proof. (i) immediate.

To prove (ii), suppose on the contrary that (C) does not hold, i.e., there exist sequences $\varepsilon_n \to 0^+$ and $(x_n) \subset A$, $x_n \to x_0$ satisfying

$$f(x_n) \in \alpha + \varepsilon_n B_Y - \mathcal{K}, \quad \text{for} \quad n \geq 1,$$

and $d(x_n, S_\alpha) \geq \varepsilon_n \frac{1}{\tau}$. Hence, $x_n \notin S_\alpha$ and for $n \geq 1$

$$f(x_n) = \alpha + \varepsilon_n b_n - k_n, \quad \text{where} \quad b_n \in B_Y, \quad k_n \in \mathcal{K}.$$

Therefore,

$$f(x_n) = \alpha + \frac{\varepsilon_n}{\tau d(x_n, S_\alpha)}\tau d(x_n, S_\alpha) b_n - k_n.$$

Clearly,

$$\bar{b}_n := \frac{\varepsilon_n}{\tau d(x_n, S_\alpha)} b_n \in B_Y,$$

and

$$f(x_n) = \alpha + \tau d(x_n, S_\alpha)\bar{b}_n - k_n \in \alpha + \tau d(x_n, S_\alpha) B_Y - \mathcal{K}.$$

Suppose now that $x_0 \in S_\alpha$ is not a local α-weak sharp solution with constant $\tau > 0$. There exists a sequence $(x_n) \subset A \setminus S_\alpha$, $x_n \to x_0$, such that

$$f(x_n) - \alpha \in \tau d(x_n, S_\alpha) B_Y - \mathcal{K},$$

and one can choose $\tau_n < \tau$ such that

$$f(x_n) - \alpha \in \tau_n d(x_n, S_\alpha) b_n - k_n, \quad \text{where} \quad b_n \in B_Y, \quad k_n \in \mathcal{K}.$$

Take $\varepsilon_n := \tau_n d(x_n, S_\alpha) \to 0$. Hence, for any given $\rho > 0$ and all n sufficiently large

$$x_n \in A \cap B(x_0, \rho) \cap f^{-1}(\alpha + \varepsilon_n B_Y - \mathcal{K})$$

while $d(x_n, S_\alpha) = \frac{1}{\tau_n}\varepsilon_n > \frac{1}{\tau}\varepsilon_n$ which contradicts condition (C). ■

Condition (C) of Proposition 2.1 (ii) can be rephrased as follows. The set-valued mapping $\mathcal{E}^\alpha : R_+ \rightrightarrows X$ defined as

$$\mathcal{E}^\alpha(\varepsilon) := A \cap f^{-1}(\alpha + \varepsilon B_Y - \mathcal{K})$$

is calm at any $(0, x_0) \in \operatorname{graph} \mathcal{E}$, with constants $\rho > 0$ and $\frac{1}{\tau} > 0$, where a set-valued mapping $\Gamma : X \rightrightarrows Y$ is *calm* at $(x_0, y_0) \in \operatorname{graph} \Gamma$ if there exist constants $\rho > 0$, $L > 0$ and $t > 0$ such that

$$\Gamma(x) \cap (y_0 + \rho B_Y) \subset \Gamma(x_0) + L\|x - x_0\| B_Y \quad \text{for} \quad x \in B(x_0, t).$$

Moreover, $\mathcal{E}^\alpha(0) = S_\alpha$. For similar results see e.g. Proposition 4.1 and Proposition 4.3 of [4].

Below we define local sharp solutions to (VOP).

Definition 2.2 *We say that $x_0 \in A$, $f(x_0) = \alpha$, is a* local sharp solution *to (VOP) if there exist constants $\tau > 0$ and $\rho > 0$ such that*

$$f(x) - f(x_0) \notin \tau\|x - x_0\|B_Y - \mathcal{K} \quad for \quad x \in A \cap B(x_0, \rho) \quad x \notin S_\alpha. \quad (3)$$

Clearly, each local sharp solution x_0, $f(x_0) = \alpha$, is a local α-weak sharp solution. Let us note that condition (3) does not imply that x_0 is a locally unique solution to (VOP), whereas the condition

$$f(x) - f(x_0) \notin \tau\|x - x_0\|B_Y - \mathcal{K} \quad \text{for} \quad x \in A \cap B(x_0, \rho) \quad x \neq x_0.$$

(see e.g. [10]) implies that x_0 is a locally unique solution to (VOP) in the sense that $f(x) \neq f(x_0)$ for all $x \in B(x_0, \rho)$.

We say that $\alpha \in E$ is a *local strict efficient point of order 1* to (VOP) if there exist constants $\gamma > 0$ and $\rho > 0$ such that

$$f(x)-f(x_0) \notin \gamma\|f(x)-f(x_0)\|B_Y-\mathcal{K} \quad \text{for} \quad x \in A\cap B(x_0, \rho), \quad x \notin S_\alpha. \quad (4)$$

Proposition 2.2 *If f is locally Lipschitz around x_0 with constant L_f and $x_0 \in A$, $f(x_0) = \alpha$, is a local sharp solution with constant $\tau > 0$, then*

$$\tau \leq L_f,$$

and α is a local strict efficient point of order 1 with constant $\frac{\tau}{L_f}$.

Proof. Since f is locally Lipschitz around x_0, there exist constants $L_f > 0$ and $\rho > 0$ such that

$$\|f(x) - f(x_0)\| \leq L_f\|x - x_0\| \quad \text{for} \quad x \in B(x_0, \rho). \quad (5)$$

If it were $\tau > L_f$, then it would be

$$f(x) - f(x_0) \notin L_f\|x - x_0\|B_Y - \mathcal{K} \quad \text{for} \quad x \in A \cap B(x_0, \rho), \quad x \notin S_\alpha$$

and consequently

$$f(x) - f(x_0) \notin L_f\|x - x_0\|B_Y$$

contradictory to (5). Moreover, by (5),

$$f(x) - f(x_0) \notin \frac{\tau}{L_f}\|f(x) - f(x_0)\|B_Y - \mathcal{K} \quad \text{for} \quad x \in A \cap B(x_0, \rho), \quad x \notin S_\alpha$$

which proves that α is a local strictly efficient point of order 1. ∎

We define directional differentiability of f at x_0 in the direction u via the contingent derivative

$$f'(x_0; u) = \lim_{(t,v)\to(0^+,u)} \frac{f(x_0 + tv) - f(x_0)}{t}$$

and we say that f is directionally differentiable at x_0 if f is directionally differentiable at x_0 in any direction $v \in X$.

The following proposition provides sufficient conditions for sharp solutions in terms of contingent directional derivatives. Put

$$\tilde{S}_\alpha = \{x \in S \ : \ f(x) = f(x_0) \quad x \neq x_0\}.$$

Proposition 2.3 *Let X be a finite dimensional space. Let f be directionally differentiable at $x_0 \in A$, $f(x_0) = \alpha$. If, for any tangent direction $0 \neq v \in T_{A \setminus \bar{S}_\alpha}(x_0)$*

$$f'(x_0; v) \notin \tau \bar{B}_Y - \mathcal{K},$$

where $\bar{B}_Y$ stands for the closure of B_Y, then $x_0 \in A$ is a local sharp solution to (VOP) with constant $\tau > 0$.

Conversely, if $x_0 \in A$ is a local sharp solution with constant $\tau > 0$, then for any tangent direction $v \in T_{A \setminus \bar{S}_\alpha}(x_0)$, $v \neq 0$,

$$f'(x_0; v) \notin \tau B_Y - \mathcal{K}.$$

Proof. Suppose that x_0 is not a local sharp solution with constant $\tau > 0$. For each $n \geq 1$ there exists $x_n \in A \cap B(x_0, \frac{1}{n})$, $x_n \notin S_\alpha$, $x_n \to x_0$, such that

$$f(x_n) - f(x_0) \in \tau \|x_n - x_0\| B_Y - \mathcal{K}.$$

Putting $v_n := \frac{x_n - x_0}{\|x_n - x_0\|}$ we get $v_n \to v \in T_{A \setminus \bar{S}_\alpha}(x_0)$, $v \neq 0$, and

$$\frac{f(x_n) - f(x_0)}{\|x_n - x_0\|} \in \tau B_Y - \mathcal{K}, \quad \text{i.e.} \quad f'(x_0; v) \in \tau \bar{B}_Y - \mathcal{K}.$$

Conversely, suppose that x_0 is a local sharp solution to (VOP). There exist constans $\tau > 0$ and $\rho > 0$ such that

$$f(x) - f(x_0) \notin \tau \|x - x_0\| B_Y - \mathcal{K} \quad \text{for} \quad x \in A \cap B(x_0, \rho), \quad x \notin S_\alpha.$$

Take any $x_n \subset A \setminus S_\alpha$, $x_n \to 0$ and put $v_n := \frac{x_n - x_0}{\|x_n - x_0\|}$ and $t_n = \|x_n - x_0\|$. Then $v_n \to v \in T_{A \setminus \bar{S}_\alpha}(x_0)$ and

$$\frac{f(x_0 + t_n v_n) - f(x_0)}{t_n} \notin \tau B_Y - \mathcal{K}.$$

Hence, $f'(x_0; v) \notin \tau B_Y - \mathcal{K}$. ■

Corollary 2.1 *Let X be a finite-dimensional space and let f be directionally differentiable at $x_0 \in A$. Then, x_0 is a local sharp solution to (VOP) if and only if for any $v \in T_{A \setminus \bar{S}_\alpha}$, $v \neq 0$,*

$$f'(x_0; v) \notin -\mathcal{K}.$$

Proof. 'If' part of the proof is the same as the 'if' part of the proof of Proposition 2.3 with $\tau = \frac{1}{n}$.

To complete the proof, assume that there exists $v \in T_{A \setminus \bar{S}_\alpha}$, $v \neq 0$, such that

$$f'(x_0; v) = k_0 \in -\mathcal{K}.$$

The remaining part of the proof follows the lines of the second part of the proof of Theorem 4.1 of [10]. ■

Now we discuss the relationships between local sharp solutions and local Henig proper solutions.

We say that $\alpha \in E$ is a *local Henig proper efficient point* to (VOP) if there exists a closed convex cone $\Omega \subset Y$, $\operatorname{int}\Omega \neq \emptyset$, such that $\mathcal{K} \setminus \{0\} \subset \operatorname{int}\Omega$ and $\rho > 0$ such that

$$(f(x) - \alpha) \cap (-\Omega) = \{0\} \quad \text{for} \quad x \in A \cap B(x_0, \rho) \setminus S_\alpha.$$

Moreover, $x_0 \in S$, $f(x_0) = \alpha$, is a *local Henig proper solution* to (VOP) if α is a local Henig proper efficient point to (VOP).

Proposition 2.4 *Let $\mathcal{K}$ be a based cone with a compact base Θ.*

(i) $\alpha \in E$ is a local Henig proper efficient point to (VOP) if and only if α is a local strict efficient point of order 1.

(ii) *Let f be locally Lipschitz around x_0. If $x_0 \in A$ is a local sharp solution to (VOP), then x_0 is a local Henig proper solution.*

Proof. (i). Suppose that α is not a local strict efficient point to (VOP). There exists $x_n \in A \setminus S_\alpha$, $x_n \to x_0$ such that

$$f(x_n) - \alpha \in \frac{1}{n}\|f(x_n) - f(x_0)\|B_Y - \mathcal{K},$$

i.e., there exist $\lambda_n > 0$ and $\theta_n \in \Theta$ such that

$$f(x_n) - f(x_0) = \frac{1}{n}\|f(x_n) - f(x_0)\|b_n - \lambda_n\theta_n, \quad \text{where} \quad b_n \in B_Y. \tag{6}$$

Hence,

$$\frac{f(x_n) - f(x_0)}{\|f(x_n) - f(x_0)\|} = \frac{1}{n}b_n - \frac{\lambda_n}{\|f(x_n) - f(x_0)\|}\theta_n.$$

Hence, since Θ is bounded, $\|\theta_n\| \leq M$ for some constant $M > 0$ and

$$1 \leq \frac{1}{n} + \frac{\lambda_n}{\|f(x_n) - f(x_0)\|}M$$

and consequently,

$$\frac{\lambda_n}{\|f(x_n) - f(x_0)\|} \geq \frac{1}{2M}.$$

This proves that $\frac{\|f(x_n)-f(x_0)\|}{\lambda_n} \leq 2M$.

Thus, $\varepsilon_n := \frac{1}{n}\frac{\|f(x_n)-f(x_0)\|}{\lambda_n} \to 0$ and

$$f(x_n) - f(x_0) = -\lambda_n(\varepsilon_n(-b_n) + \theta_n)$$

which proves that α is not a local Henig proper efficient point.

Suppose now that α is not a local Henig proper efficient point. Hence, there exists $x_n \in A \setminus S_\alpha$, $x_n \to x_0$ such that

$$f(x_n) - \alpha \in -\operatorname{cone}(\frac{1}{n}B_Y + \Theta),$$

i.e., there exist $\lambda_n > 0$ and $\theta_n \in \Theta$ such that

$$f(x_n) - f(x_0) = \frac{\lambda_n}{n}b_n - \lambda_n\theta_n, \quad \text{where} \quad b_n \in B_Y. \tag{7}$$

Hence,

$$\frac{f(x_n) - f(x_0)}{\lambda_n} = \frac{1}{n} b_n - \theta_n,$$

and since Θ is compact, we can assume that $\theta_n \to \theta_0 \in \Theta$, $\theta_0 \neq 0$ and

$$v_n := \frac{f(x_n) - f(x_0)}{\lambda_n} \to -\theta_0.$$

This proves that there exists $M > 0$ such that $\frac{\|f(x_n)-f(x_0)\|}{\lambda_n} \geq M$ and consequently

$$\frac{\lambda_n}{\|f(x_n) - f(x_0)\|} \leq \frac{1}{M}.$$

Hence, $\varepsilon_n := \frac{1}{n} \frac{\lambda_n}{\|f(x_n)-f(x_0)\|} \to 0$ and by (7),

$$f(x_n) - \alpha = \varepsilon_n \|f(x_n) - f(x_0)\| b_n - k_n, \quad \text{where} \quad k_n \in \mathcal{K}.$$

This proves that α is not a local strict efficient point.

(ii). If $x_0 \in A$, $f(x_0) = \alpha$, is a local sharp solution to (VOP), then by Proposition 2.2, α is a local strict efficient point of order 1, and by part *(ii)* , α is a local Henig efficient point to (VOP).

■

3 Global sharp and weak sharp solutions

Let $\alpha \in E$.

Definition 3.1 *[4] We say that $x_0 \in A$, $f(x_0) = \alpha$, is a* global α-weak sharp solution *to (VOP) if there exists a constant $\tau > 0$ such that*

$$f(x) - \alpha \notin \tau d(x, S_\alpha) B_Y - \mathcal{K} \quad \text{for all} \quad x \in A \setminus S_\alpha. \tag{8}$$

Any global α-weak sharp solution $x_0 \in A$ is a solution to (VOP). Indeed, if x_0 is a global α-weak sharp solution to (VOP), then

$$f(x) - f(x_0) \notin -\mathcal{K}, \quad \text{for} \quad f(x) \neq f(x_0)$$

which shows that x_0 is a solution to (VOP).

If $\operatorname{int} \mathcal{K} \neq \emptyset$ we consider also *weak solutions to* (VOP). $x_0 \in A$ is weak solution to (VOP), $x_0 \in WS$, if $(f(A) - f(x_0) \cap (-\operatorname{int} \mathcal{K}) = \emptyset$. It is easy to show that if $x_0 \in A$ is a weak solution to (VOP) and not a solution, then x_0 cannot be a global α-weak sharp solution to (VOP). Indeed, if $x_0 \in WS \setminus S$, then there exists an $x \in A$ such that

$$f(x) - f(x_0) \in -\mathcal{K} \setminus \operatorname{int} \mathcal{K}$$

and hence

$$f(x) - f(x_0) \in \tau d(x, S) B_Y - \mathcal{K}.$$

Each global α-weak sharp solution is a local weak sharp solution. The converse implications will be proved in Proposition 3.3 below.

Moreover, if $x_0 \in A$ is a global α-weak sharp solution, then x_0 is a local α-weak sharp solution. The converse implications will be proved in Proposition 3.3 below.

A function $f : X \to Y$ satisfies the *Lipschitz condition* on A with constant $L_f > 0$ if

$$\|f(x) - f(x')\| \leq L_f \|x - x'\| \quad \text{for all} \quad x, x' \in A$$

Proposition 3.1 *Let $x_0 \in A$ be a global α-weak sharp solution to (VOP) with constant $\tau > 0$.*

(i) If f is Lipschitz on A with constant L_f, then

$$\tau \leq L_f.$$

(ii) The following condition holds:

(**C1**) *there exists $\varepsilon_0 > 0$ such that for each $0 \leq \varepsilon \leq \varepsilon_0$*

$$A \cap f^{-1}(\alpha + \varepsilon B_Y - \mathcal{K}) \subset S_\alpha + \varepsilon \frac{1}{\tau} B_Y.$$

Proof. (i) By assumption,

$$f(x) - \alpha \notin \tau d(x, S_\alpha) B_Y - \mathcal{K} \quad \text{for} \quad x \in A \setminus S_\alpha.$$

Suppose on the contrary that $\tau > L_f$. Take any $x \in A \setminus S_\alpha$ and $L_f, \sigma < \tau$. One can choose $\varepsilon > 0$ and $\bar{x} \in S_\alpha$ such that

$$d(x, \bar{x}) < d(x, S_\alpha) + \varepsilon, \quad 0 < d(x, \bar{x}) - \varepsilon$$

and

$$L_f < \sigma_1 < \tau, \quad \text{where} \quad \sigma_1 := \sigma \frac{d(x, \bar{x}) - \varepsilon}{d(x, \bar{x})}.$$

(ii) Suppose on the contrary that $(C1)$ does not hold, i.e., there exist sequences $\varepsilon_n \to 0^+$ and $(x_n) \subset A$ such that

$$f(x_n) \in \alpha + \varepsilon_n B_Y - \mathcal{K}, \quad n \geq 1,$$

and $d(x_n, S_\alpha) > \varepsilon_n \frac{1}{\tau}$. Hence, for $n \geq 1$, $x_n \notin S_\alpha$, $\tau d(x_n, S_\alpha) > \varepsilon_n$ and

$$f(x_n) \in \alpha + \tau d(x_n, S_\alpha) B_Y - \mathcal{K},$$

which contradicts the fact that x_0 is a global α-weak sharp solution to (VOP). ■

Condition $(C1)$ of Proposition 3.2 (ii) can be interpreted in the following way: the set-valued mapping $\mathcal{E}^\alpha : R_+ \rightrightarrows X$ defined in Section 2,

$$\mathcal{E}^\alpha(\varepsilon) = A \cap f^{-1}(\alpha + \varepsilon B_Y - \mathcal{K})$$

is upper Lipschitz at any $0 \in \operatorname{dom} \mathcal{E}$, with constant $\frac{1}{\tau} > 0$, where a set-valued mapping $\Gamma : X \rightrightarrows Y$ is *upper Lipschitz* at $x_0 \in \operatorname{dom} \Gamma$ if there exist constants $L > 0$ and $t > 0$ such that

$$\Gamma(x) \subset \Gamma(x_0) + L\|x - x_0\| B_Y \quad \text{for} \quad x \in B(x_0, t).$$

Now we define global sharp solutions to (VOP).

Definition 3.2 *We say that $x_0 \in A$, $f(x_0) = \alpha$, is a* global sharp solution *to (VOP) if there exist a constant $\tau > 0$ such that*

$$f(x) - f(x_0) \notin \tau \|x - x_0\| B_Y - \mathcal{K} \quad for \quad x \in A \setminus S_\alpha. \tag{9}$$

If $x_0 \in A$, $f(x_0) = \alpha$, is a sharp solution, then x_0 is a global α-weak sharp solution to (VOP).

Recall that $\alpha \in E$ is a *global strict efficient point of order 1 to* (VOP) ([4]) if there exists a constant $\gamma > 0$ such that

$$f(x) - \alpha \notin \gamma \|f(x) - \alpha\| B_Y - \mathcal{K} \quad \text{for} \quad x \in A \quad f(x) \neq \alpha.$$

If f is Lipschitz on A with constant $L_f > 0$ and x_0 is an α-weak sharp solution with constant $\tau > 0$, then

$$\|f(x) - \alpha\| \leq L_f \|x - x'\| \leq L_f d(x, S_\alpha) \quad \text{for all} \quad x \in A \setminus S_\alpha \quad x' \in S_\alpha$$

and

$$f(x) - \alpha \notin \frac{\tau}{L_f} \|f(x) - \alpha\| B_Y - \mathcal{K} \quad \text{for} \quad x \in A \setminus S_\alpha, \tag{10}$$

i.e.

$$f(x) - \alpha \notin \frac{\tau}{L_f} \|f(x) - \alpha\| B_Y - \mathcal{K} \quad \text{for} \quad x \in A, \quad f(x) \neq \alpha. \tag{11}$$

This means that $\alpha \in E$ is a *global strict* efficient point of order 1 with the rate $\frac{\tau}{L_f}$.

In this way we proved the following proposition.

Proposition 3.2 *Let f be Lipschitz on A with constant L_f. If x_0 is a global α-weak sharp solution to (VOP) with constant $\tau > 0$, then $\alpha \in E$ is a strict efficient point of order 1 with the rate $\frac{\tau}{L_f}$.*

Since each global sharp solution $x_0 \in A$, $f(x_0) = \alpha$, is a global α-weak sharp solution, we get the following corollary.

Corollary 3.1 *Let f be Lipschitz on A with constant L_f. If $x_0 \in A$, $f(x_0) = \alpha$, is a global sharp solution to (VOP) with constant $\tau > 0$, then $\alpha \in E$ is a strict efficient point of order 1 with the rate $\frac{\tau}{L_f}$.*

Recall that f is a $\mathcal{K}$*-convex* function if for any $\lambda \in [0, 1]$ and $x, x' \in X$

$$f(\lambda x + (1 - \lambda)x') \in \lambda f(x) + (1 - \lambda) f(x') - \mathcal{K}.$$

Note that if A is convex and f is $\mathcal{K}$-convex, the sets S_α, $\alpha \in E$, are convex. Indeed, for any $x, x' \in S_\alpha$

$$f(\lambda x + (1 - \lambda)x') \in \alpha - \mathcal{K},$$

and it must be $f(\lambda x + (1 - \lambda)x') = \alpha$ since $\alpha \in E$.

Proposition 3.3 *Let A be convex and let f be $\mathcal{K}$-convex. Let $\alpha \in E$ and $x_0 \in S_\alpha$.*

(i) x_0 is a local sharp solution iff x_0 is a global sharp solution.

(ii) Let X be reflexive. Let A be closed and f continuous on A. x_0 is a local α-weak sharp solution iff x_0 is a global α-weak sharp solution.

Proof. Note that by the assumptions, S_α is closed and convex.

(i). If x_0 a global sharp solution, then clearly x_0 is a local sharp solution. To see the converse, suppose on the contrary that x_0 is not a global sharp solution with constant $\tau > 0$. Then, there exists $x \in A \setminus S_\alpha$ such that

$$f(x) - f(x_0) \in \tau\|x - x_0\|B_Y - \mathcal{K}.$$

Let $\lambda \in [0,1]$. By the convexity assumptions,

$$f(\lambda x + (1-\lambda)x_0) \in \lambda f(x) + (1-\lambda)f(x_0) - \mathcal{K}$$

and consequently

$$f(\lambda x + (1-\lambda)x_0) - f(x_0) \in \lambda(f(x) - f(x_0)) - \mathcal{K} \subset \lambda\tau\|x - x_0\|B_Y - \mathcal{K}.$$

Thus, for any $\rho > 0$ there is $\lambda \in [0,1]$ such that $\lambda x + (1-\lambda)x_0 \in B(x_0, \rho)$ and

$$f(\lambda x + (1-\lambda)x_0) - f(x_0) \in \tau\|x - x_0\|B_Y - \mathcal{K}.$$

which contradics the fact that x_0 is a local sharp solution.

(ii). If x_0 is a global α-weak sharp solution, then x_0 is a local α-weak sharp solution. To see that converse, suppose on the contrary that x_0 is not a global α-weak sharp solution with constant τ. There exist $x \in A \setminus S_\alpha$ such that

$$f(x) - f(x_0) \in \tau d(x, S_\alpha)B_Y - \mathcal{K}.$$

Moreover, there exists $\bar{x} \in S_\alpha$ such that $d(x, \bar{x}) = d(x, S_\alpha)$. Let $\lambda \in (0,1]$. By convexity assumptions, $\lambda x + (1-\lambda)\bar{x} \in A \setminus S_\alpha$, and

$$f(\lambda x + (1-\lambda)\bar{x}) \in \lambda f(x) + (1-\lambda)f(\bar{x}) - \mathcal{K}.$$

Hence, for any $\rho > 0$ one can choose $\lambda \in [0,1]$ such that $x + (1-\lambda)\bar{x} \in A \cap B(S_\alpha, \rho)$ and

$$f(\lambda x + (1-\lambda)\bar{x}) - f(\bar{x}) \in \lambda(f(x) - f(\bar{x})) - \mathcal{K} \subset \tau d(x, S_\alpha)B_Y - \mathcal{K}$$

which contradicts the fact that x_0 is a local α-weak sharp solution. ■

Recall that $\alpha \in E$ is a *global Henig proper efficient point* if there exists a closed convex pointed cone $\Omega \subset Y$, $\operatorname{int}\Omega \neq \emptyset$, such that $\mathcal{K} \setminus \{0\} \subset \operatorname{int}\Omega$ and

$$(f(x) - \alpha) \cap (-\Omega) = \{0\} \quad \text{for all } x \in A \setminus S_\alpha$$

Moreover, $x_0 \in A$, $f(x_0) = \alpha$, is a *global Henig proper solution* to (VOP) if

$$(f(x) - f(x_0)) \cap (-\Omega) = \{0\} \quad \text{for } \; x \in A \setminus S_\alpha,$$

i.e. $x_0 \in S_\alpha$ is a global Henig proper solution if and only if α is a global Henig proper efficient point.

Clearly, each global Henig proper solution is a local Henig proper solution. As in Proposition 3.3, we can prove that if A is convex and f is $\mathcal{K}$-convex, then each local Henig proper solution (efficient point) is a global Henig proper solution (efficient point).

By this and by Proposition 3.3 we obtain the following global counterpart of Proposition 2.4.

Proposition 3.4 *$\mathcal{K}$ be a closed convex pointed cone with a compact base Θ. Let A be convex and let f be $\mathcal{K}$-convex.*

(i) $\alpha \in E$ is a global Henig proper efficient point if and only if α is a global strict efficient point of order 1.

(ii) Let f be Lipschitz on A. If $x_0 \in S_\alpha$ is a global sharp solution, then x_0 is a global Henig proper solution to (VOP).

Proof. Follows immediately from Proposition 3.3 and Proposition 2.4. ■

4 Lipschitz continuities of efficient points

Consider now the parametric problem of the form

$$(VOP)_u \qquad \begin{array}{c} \mathcal{K} - \min f(x) \\ \text{subject to } x \in A(u), \end{array}$$

where the parameter u belongs to a normed space U.

By $E(u)$ and $S(u)$ we denote the set of efficient points and the solution set to $(VOP)_u$, respectively.

The behaviour of the sets $A(u)$ around a given u_0 is characterised by the behaviour of the feasible set-valued mapping $F : U \rightrightarrows X$ defined as

$$F(u) = A(u), \quad F(u_0) = A.$$

In this section we formulate global results concerning stability properties of the whole sets $E(u)$ and $S(u)$ near u_0. Local results which refer to the behaviour of $E(u)$ and $S(u)$ near u_0 around a given $x_0 \in S$ will be given elsewhere (see also [4]). For other types of convergence of efficient points see e.g. [11, 12].

In Section 3 we gave the definition of the upper Lipschitz set-valued mapping. Now we recall that a set-valued mapping $\Gamma : U \rightrightarrows X$ is

lower Lipschitz at $(u_0, x_0) \in \operatorname{graph}\Gamma$ if there exist constants $L > 0$ and $t > 0$ such that

$$x_0 \subset \Gamma(u) + L\|u - u_0\|B_Y \quad \text{for} \quad u \in B(u_0, t),$$

lower Lipschitz at $u_0 \in \operatorname{dom}\Gamma$ if there exist constants $L > 0$ and $t > 0$ such that

$$\Gamma(u_0) \subset \Gamma(u) + L\|u - u_0\|B_Y \quad \text{for} \quad u \in B(u_0, t),$$

Lipschitz around $u_0 \in \operatorname{dom}\Gamma$ if there exist constants $L > 0$ and $t > 0$ such that

$$\Gamma(u) \subset \Gamma(u') + L\|u - u'\|B_Y \quad \text{for} \quad u, u' \in B(u_0, t),$$

We start with conditions ensuring lower Lipschitness of E and S. Recall that the *domination property* (DP) holds for (VOP) if for any $x \in A$ there exists $\bar{x} \in S$ such that $f(\bar{x}) \in f(x) - \mathcal{K}$, i.e. each feasible point x is dominated by an element $\bar{x} \in S$. Let us note that when $f : X \to R$ this property is authomatically satisfied provided that the solution set $S \neq \emptyset$.

Theorem 4.1 *Let $f : X \to Y$ be Lipschitz on X with constant $L_f > 0$ and $x_0 \in S_\alpha$, $\alpha \in E$. Assume that*

(i) the set valued mapping $F : U \rightrightarrows X$ is Lipschitz at $u_0 \in \operatorname{dom} F$ with constants $L_c > 0$, $t > 0$,

(ii) (DP) holds for all $(VOP)_u$, $u \in B(u_0, t)$,

(iii) all $x_0 \in S(f, A)$ are global sharp solutions to (VOP) with the same constant $\tau > 0$, i.e. for any $\alpha \in E(f, A)$ and $x_0 \in S(f, A)$

$$f(x) - f(x_0) \notin \tau \|x - x_0\| B_Y - \mathcal{K} \quad \text{for} \quad x \in A \setminus S_\alpha.$$

Then $\mathcal{E}$ is lower Lipschitz at $u_0 \in \operatorname{dom} \mathcal{E}$, i.e.,

$$E(f, A) \in E(f, A(u)) + (L_f L_c + \frac{2L_f^2 L_c}{\tau})\|u - u_0\| B_Y \quad \text{for} \quad u \in B(u_0, t).$$

Moreover, if instead of (iii) we assume that

(iv) all $\bar{z} \in S(f, A(u))$ for $u \in B(u_0, t)$ are global sharp solutions to (P_u) with the same constant $\tau > 0$, i.e. for any $\eta \in E(f, A(u))$

$$f(z) - f(\bar{z}) \notin \tau \|z - \bar{z}\| B_Y - \mathcal{K} \quad \text{for} \quad z \in A(u) \setminus S_\eta(u).$$

then $\mathcal{S}$ is lower Lipschitz at $u_0 \in \operatorname{dom} \mathcal{S}$. Precisely,

$$S(f, A) \subset S(f, A(u)) + (\frac{2L_f L_c}{\tau} + L_c)\|u - u_0\| B_Y \quad \text{for} \quad u \in B(u_0, t).$$

Proof. We start by proving the lower Lipschitz continuity of $\mathcal{S}$. Let us notice first that by (ii), $S(f, A(u)) \neq \emptyset$ for $u \in B(u_0, t)$ and thus $u_0 \in \operatorname{int} \operatorname{dom} \mathcal{S}$. Take any $x_0 \in S(f, A)$ and $u \in B(u_0, t)$. By (i), there is $z \in A(u)$ such that

$$\|x_0 - z\| \leq L_c \|u - u_0\|.$$

If $z \in S(f, A(u))$, the conclusion follows. Otherwise, by (DP), there exists $\bar{z} \in S(f, A(u))$ such that $f(\bar{z}) \in f(z) - \mathcal{K}$ and $f(z) \neq f(\bar{z})$. If $\|z - \bar{z}\| \leq \frac{2L_c L_f}{\tau} \|u - u_0\|$, then

$$\|x_0 - \bar{z}\| \leq (L_c + \frac{2L_c L_f}{\tau})\|u - u_0\|$$

and the conclusion follows. So, assume that

$$\|z - \bar{z}\| > \frac{2L_c L_f}{\tau} \|u - u_0\|. \tag{12}$$

By (iv), $\bar{z} \in S(f, A(u))$ is a global sharp solution to (VOP_u). Since $f(z) \neq f(\bar{z})$ we have

$$f(z) - f(\bar{z}) \notin \tau \|z - \bar{z}\| B_Y - \mathcal{K}.$$

By (i), there exists $x \in A$ such that

$$\|\bar{z} - x\| \leq L_a \|u - u_0\|.$$

By the Lipschitzness of f,

$$\|f(\bar{z}) - f(x)\| \leq L_f L_a \|u - u_0\| \quad \text{and} \quad \|f(z) - f(x_0)\| \leq L_f L_a \|u - u_0\|$$

and hence, in view of (12),

$$\begin{aligned}\|f(x_0)-f(x)\| &\geq \|f(z)-f(\bar{z})\| - \|f(x)-f(\bar{z})\| - \|f(z)-f(x_0)\| \\ &\geq \tau\|z-\bar{z}\| - 2L_aL_f\|u-u_0\| > 0\end{aligned}$$

which proves that $f(x) \neq f(x_0)$. Hence, since x_0 is a global sharp solution to (VOP),

$$f(x)-f(x_0) \notin \tau\|x-x_0\|B_Y - \mathcal{K}. \tag{13}$$

On the other hand,

$$\begin{aligned}f(x)-f(x_0) &= (f(x)-f(\bar{z})) + (f(\bar{z})-f(z)) + (f(z)-f(x_0)) \\ &\in 2L_fL_c\|u-u_0\|B_Y - \mathcal{K}\end{aligned} \tag{14}$$

By (13) and (14),

$$\|x-x_0\| \leq \frac{2L_fL_c}{\tau}\|u-u_0\|.$$

Consequently

$$\|x_0-\bar{z}\| \leq \|x_0-x\| + \|x-\bar{z}\| \leq (L_a + \frac{2L_fL_a}{\tau})\|u-u_0\|.$$

which proves the assertion.

To prove that $\mathcal{E}$ is lower Lipschitz at $u_0 \in \operatorname{dom}\mathcal{E}$ take any $\alpha \in E(f,A)$ and $u \in B(u_0,t)$. There exists $\bar{x} \in S(f,A)$ such that $f(\bar{x}) = \alpha$. By (i), there exists $z \in A(u)$ such that

$$\|\bar{x}-z\| \leq L_c\|u-u_0\|$$

and by the Lipschitzness of f,

$$\|f(\bar{x})-f(z)\| \leq L_fL_c\|u-u_0\|.$$

If $z \in S(f,A(u))$, then $f(z) \in E(f,A(u))$ and the conclusion follows. Otherwise, there exists $\bar{z} \in S(f,A(u))$ such that $f(\bar{z}) \in f(z)-\mathcal{K}$ and $f(\bar{z}) \neq f(z)$.

By (i), there exists $x \in A$ such that

$$\|x-\bar{z}\| \leq L_c\|u-u_0\| \quad \text{and} \quad \|f(x)-f(\bar{z})\| \leq L_fL_c\|u-u_0\|.$$

If $f(x) = f(\bar{x})$, then the conclusion follows. If $f(x) \neq f(\bar{x})$, by (iii)

$$f(x)-f(\bar{x}) \notin \frac{\tau}{L_f}\|f(x)-f(\bar{x})\|B_Y - \mathcal{K}.$$

On the other hand, as before

$$\begin{aligned}f(x)-f(\bar{x}) &= (f(x)-f(\bar{z})) + (f(\bar{z})-f(z)) + (f(z)-f(\bar{x})) \\ &\in 2L_fL_a\|u-u_0\|B_Y - \mathcal{K}.\end{aligned}$$

This proves that

$$\|f(x)-f(\bar{x})\| \leq \frac{2L_aL_f^2}{\tau}\|u-u_0\|$$

and consequently

$$\begin{aligned}\|f(\bar{x})-f(\bar{z})\| &\leq \|f(\bar{x}-f(x)\| + \|f(x)-f(\bar{z})\| \\ &\leq (L_fL_a + \tfrac{2L_f^2L_a}{\tau})\|u-u_0\|\end{aligned}$$

which proves the assertion. ■

Theorem 4.2 *Let $f : X \to Y$ be Lipschitz on X with constant $L_f > 0$. Assume that*

(i) the set valued mapping $F : U \rightrightarrows Y$ is Lipschitz at $u_0 \in \operatorname{dom} F$ with constants $L_c > 0$ and $t > 0$,

(ii) (DP) holds for (VOP),

(iii) all $x \in S(u)$, $u \in B(u_0, t)$, are sharp with constant $\tau > 0$.

Then

S is upper Lipschitz at $u_0 \in \operatorname{dom} S$, i.e.,

$$S(u) \subset S + (L_c + \frac{2L_c L_f}{\tau})\|u - u_0\| B_X \quad \text{for} \quad u \in B(u_0, t)$$

E is upper Lipschitz at $u_0 \in \operatorname{dom} E$, i.e.,

$$E(u) \subset E + (L_f L_c + \frac{2L_c L_f^2}{\tau})\|u - u_0\| B_Y \quad \text{for} \quad u \in B(u_0, t).$$

Proof. Let $\bar{z} \in S(u)$, $u \in B(u_0, t)$. By the upper Lipschitzness of F, there exists $x \in A$ such that

$$\|x - \bar{z}\| \leq L_c \|u - u_0\|$$

and

$$\|f(x) - f(\bar{z})\| \leq L_f L_c \|u - u_0\|.$$

By (DP), there exists $\bar{x} \in S$ such that $f(\bar{x}) \in f(x) - \mathcal{K}$. By lower Lipschitzness of F, there exists $z \in F(u)$ such that

$$\|\bar{x} - z\| \leq L_c \|u - u_0\|,$$

and consequently, $\|f(\bar{x}) - f(z)\| \leq L_f L_c \|u - u_0\|$. Hence,

$$f(z) - f(\bar{z}) \in 2L_f L_c \|u - u_0\| B_Y - \mathcal{K}.$$

On the other hand, since $\bar{z} \in S(u)$ is a sharp solution to $(VOP)_u$

$$f(z) - f(\bar{z}) \notin \tau \|z - \bar{z}\| B_Y - \mathcal{K} \tag{15}$$

and $\|z - \bar{z}\| \leq \frac{2L_f L_c}{\tau} \|u - u_0\|$. Hence,

$$\|\bar{z} - \bar{x}\| \leq \|\bar{z} - z\| + \|z - \bar{x}\| \leq (L_c + \frac{2L_f L_c}{\tau})\|u - u_0\|.$$

To see the second assertion, since $\bar{x} \in S(u)$ is a sharp solution to $(VOP)_u$,

$$f(z) - f(\bar{z}) \notin \frac{\tau}{L_f} \|f(z) - f(\bar{z})\| B_Y - \mathcal{K}$$

and

$$f(\bar{x}) - f(\bar{z}) = (f(\bar{x}) - f(z)) + (f(z) - f(\bar{z})) \in (L_f L_c + \frac{2L_f^2 L_c}{\tau})\|u - u_0\| B_Y.$$

■

Theorem 4.3 *Let $f : X \to Y$ be Lipschitz on X with constant $L_f > 0$. Assume that*

(i) the set valued mapping $F : U \rightrightarrows Y$ is Lipschitz at $u_0 \in \operatorname{dom} F$ with constants $L_c > 0$ and $t > 0$,

(ii) (DP) holds for $(VOP)_u$, $u \in B(u_0, t)$,

(iii) for $u \in B(u_0, t)$ and $\alpha \in E(u)$ the solutions $x \in S(u)$ to $(VOP)_u$, are α-weakly sharp with constant $\tau > 0$.

Then S is lower Lipschitz at $u_0 \in \operatorname{dom} S$. Precisely, for any $u \in B(u_0, t)$,

$$S \subset S(u) + (3L_f L_c + \frac{2L_c L_f}{\tau})\|u - u_0\|B_X.$$

Proof. Let $\bar{x} \in S$. By Proposition 3.2 and Theorem ??, there exists $\bar{z} \in S(u)$, $u \in B(u_0, t)$ such that

$$f(\bar{x}) - f(\bar{z}) \in (L_c L_f + \frac{2L_c L_f^2}{\tau})\|u - u_0\|B_Y.$$

By the lower Lipschitzness of F, there exists $z \in A(u)$ such that

$$\|\bar{x} - z\| \leq L_c \|u - u_0\|$$

and by the Lipschitzness of F,

$$\|f(\bar{x}) - f(z)\| \leq L_f L_c \|u - u_0\|.$$

Hence,

$$f(z) - f(\bar{z}) = (f(z) - f(\bar{x})) + (f(\bar{x}) - f(\bar{z}) \in (2L_f L_c + \frac{2L_c L_f^2}{\tau})\|u - u_0\|B_Y.$$

On the other hand, since $\bar{z} \in S(u)$, $f(\bar{z}) = \alpha$, is a global α-weak sharp solution,

$$f(z) - f(\bar{z}) \notin \tau d(z, S_\alpha(u))B_Y - \mathcal{K},$$

where $S_\alpha(u) = \{x \in S(u) \ : \ f(x) = \alpha\}$, and consequently,

$$d(\bar{x}, S(u)) \leq d(\bar{x}, S_\alpha(u)) \leq d(\bar{x}, z) + d(z, S_\alpha(u)) \leq 3L_f L_c + \frac{2L_c L_f^2}{\tau})\|u - u_0\|.$$

■

Theorem 4.4 *Let $f : X \to Y$ be Lipschitz on X with constant $L_f > 0$. Assume that*

(i) the set valued mapping $F : U \rightrightarrows Y$ is Lipschitz at $u_0 \in \operatorname{dom} F$ with constants $L_c > 0$ and $t > 0$,

(ii) (DP) holds for $(VOP)_u$, $u \in B(u_0, t)$,

(iii) for any $\alpha \in E$, the solutions $x \in S_\alpha$ to (VOP), are α-weakly sharp with constant $\tau > 0$.

Then S is upper Lipschitz at $u_0 \in \operatorname{dom} S$, i.e. for any $u \in B(u_0, t)$,

$$S(u) \subset S + (3L_f L_c + \frac{2L_c^2 L_f}{\tau})\|u - u_0\|B_X.$$

Proof. Let $\bar{z} \in S(u)$, $u \in U_0$. By Proposition 3.2 and Theorem 4.2, there exists $\bar{x} \in S$ such that

$$f(\bar{z}) - f(\bar{x}) \in (L_c L_f + \frac{2L_c L_f^2}{\tau})\|u - u_0\| B_Y.$$

By the upper Lipschitzness of F, there exists $x \in A$ such that

$$\|\bar{z} - x\| \leq L_c \|u - u_0\|$$

and

$$\|f(\bar{z}) - f(x)\| \leq L_f L_c \|u - u_0\|.$$

Hence,

$$f(x) - f(\bar{x}) = (f(x) - f(\bar{z})) + (f(\bar{z}) - f(\bar{z}) \in (2L_f L_c + \frac{2L_c L_f^2}{\tau})\|u - u_0\| B_Y.$$

On the other hand, since $\bar{x} \in S_\alpha$, is a global α-weak sharp solution,

$$f(x) - f(\bar{x}) \notin \tau d(x, S_\alpha(u)) B_Y - \mathcal{K},$$

and consequently,

$$d(\bar{z}, S(u)) \leq d(\bar{z}, S_\alpha(u)) \leq d(\bar{z}, x) + d(x, S_\alpha(u)) \leq 3L_f L_c + \frac{2L_c L_f^2}{\tau})\|u - u_0\|.$$

■

Remark 4.1 *By examining the proof of Theorem 4.3 we observe that the lower Lipschitzness of E follows immediately from the assumptions. Similarily, the assumptions of Theorem 4.4 yield the upper Lipschitzness of E. Moreover, for the proofs of Theorem 4.3 and Theorem 4.4 instead of the α-weak sharp condition the following (weaker) condition*

$$f(x) - f(\bar{x}) \notin \tau d(x, S_\alpha) B_Y$$

for $\bar{x} \in S_\alpha$ *(* $\bar{x} \in S_\alpha(u)$*) is sufficient. This observation will be used in Section 5.*

5 Linear multicriteria problems

Let us consider now the linear multicriteria problem of the form

$$(LMP) \qquad \begin{array}{c} \mathcal{K} - \min Cx \\ \text{subject to } x \in A, \end{array}$$

where C is an $k \times n$ matrix, $k < n$, $A \subset R^n$ is a nonempty polyhedral convex set and $\mathcal{K} \subset R^k$ is a polyhedral cone.

The parametric linear multicriteria problem takes the form

$$(LMP)_u \qquad \begin{array}{c} \mathcal{K} - \min Cx \\ \text{subject to } x \in A(u), \end{array}$$

where $A(u) \subset R^n$ is a nonempty polyhedral convex set.

Basing on Proposition 3.4, each solution to (LMP) is sharp. The following fact holds true.

Proposition 5.1 *Let $x_0 \in A$ be a solution to (LMP). x_0 is a global Henig proper solution iff x_0 is a global sharp solution to (VOP).*

Proof. In view of Proposition 3.4 , we need to show that any Henig proper solution x_0 to (LMP) is a global sharp solution.

Suppose that x_0 is not a sharp solution. For each $n \geq 1$ there exists $x_n \in A \setminus S_\alpha$ such that

$$C(x_n - x_0) = \frac{1}{n}\|x_n - x_0\|b_n - \lambda_n\theta_n,$$

for some $b_n \in B_Y$, $\theta_n \in \Theta$ and $\lambda_n > 0$. Since $v_n := \frac{x_n - x_0}{\|x_n - x_0\|} \to v \neq 0$, it must be

$$\frac{\lambda_n}{\|x_n - x_0\|} \to \lambda_0 \neq 0$$

and by putting $\varepsilon_n := \frac{1}{n}\frac{\|x_n - x_0\|}{\lambda_n} \to 0$ we get

$$C(x_n - x_0) = -\lambda_n(\varepsilon_n b_n + \theta_n)$$

which contradicts the fact that x_0 is a global Henig proper efficient solution. ■

In linear vector optimization each solution is a global Henig proper solution.

By Theorem 2.2 of [9], if A is convex and f is $\mathcal{K}$-convex, the domination property (DP) holds for $(V0P)$ if and and only if $E \neq \emptyset$.

Basing on Theorem 4.2 we obtain the following stability result.

Theorem 5.1 *Assume that $\alpha \in E$ and*

(i) the set valued mapping $F : U \rightrightarrows Y$ is Lipschitz at $u_0 \in \operatorname{dom} F$ with constants $L_c > 0$ and $t > 0$,

(ii) $E(u) \neq \emptyset$ for $u \in B(u_0, t)$.

Then E to $(LMP)_u$ is lower Lipschitz at $(u_0, \alpha) \in \operatorname{graph} E$ and S to $(LMP)_u$ is lower Lipschitz at $(u_0, x_0) \in \operatorname{dom} S$.

Moreover, $x_0 \in S$ if and only if there exists strictly positive vector a of weights $a = [a_1, ..., a_k]$, $a_i > 0$, such that x_0 solves the linear programming problem

$$(LP) \qquad \begin{array}{l} \min \sum_{i=1}^{k} a_i C_i x_i \\ \text{subject to } x \in A, \end{array}$$

where C_i are rows of the matrix C.

Moreover, according to Theorem 5.4 of [8], there exist finitely many vectors $a(r) = [a_1^r, , , a_k^r]$, $1 \leq r \leq p$, $a_j^r > 0$ for $1 \leq j \leq k$, $1 \leq r \leq p$, $\sum_{j=1}^{k} a_j^r \leq 1$, $1 \leq r \leq p$ such that the solution set S can be represented as

$$S = \sum_{r=1}^{p} S^r, \quad \text{where} \quad S^r = \arg min\{a(r)^T C(x) \ : \ x \in A\}.$$

Hence, by the the same arguments as in [7], there exists a constant $\tau > 0$ such that for any $\alpha \in E$

$$f(x) - \alpha \notin \tau d(x, S_\alpha) B_Y.$$

According to Remark 4.1, together with Theorem 4.4 we obtain the following result.

Theorem 5.2 *Assume that*

(i) the set valued mapping $F : U \rightrightarrows Y$ *is Lipschitz at* $u_0 \in \operatorname{dom} F$ *with constants* $L_c > 0$ *and* $t > 0$,

(ii) $E(u) \neq \emptyset$ *for* $u \in B(u_0, t)$.

Then E *to* $(LMP)_u$ *is upper Lipschitz and* S *to* $(LMP)_u$ *is upper Lipschitz at* $u_0 \in \operatorname{dom} S$.

References

[1] H. Attouch, R.J.-B. Wets, Quantitative stability of variational systems II. A framework for nonlinear conditionning, SIAM J. Optim., vol.3(1993), 359-381

[2] A.Auslander, J.-P.Crouzeix, Global regularity conditions, Math.of O.R., vol.13(1988),243-253

[3] D.Azé, J.-N.Corvellec, On the sensitivity analysis of Hoffmann constants for systems of linear inequalities, SIAM J. Optim. vol.12(2002), 913-927

[4] E.M.Bednarczuk, Weak sharp efficiency and growth condition for vector-valued functions with applications, Optimization vol.53(2004) no.5-6, pp.455-474

[5] F.Bonnans, A.D. Ioffe, Quadratic growth and stability in convex programming problems with multiple solutions, J.Convex Analysis vol.2(1995), 41-57

[6] J.V.Burke, S.Deng, Weak Sharp Minima Revisited. Part I: Basic Theory, Control and Cybernetics, vol. 31(2002),439-471

[7] S.Deng, X.Q.Yang, Weak sharp minima in multicriteria linear programming, SIAM J.Optim. vol.15(2004), 456-460

[8] M.Henig, Proper efficiency with respect to cones, JOTA vol.36(1982), 387-407

[9] M.Henig, The domination property in multicriteria optimization, IMAA, vol.114(1986), 7-17

[10] B.Jiménez, V.Novo, First and second order sufficient conditions for strict minimality in nonsmooth vector optimization, JMAA, vol.284(2003), 496-510

[11] E.Miglierina, E.Molho, Well-posedness and convexity in vector optimization, Math.Meth.Op.Res., vol.58(2003), 375-385

[12] E.Miglierina, E.Molho, Convergence of minimal sets in convex vector optimization, to appear in SIAM J.Opt.

[13] A.Zaffaroni, Degrees of efficiency and degrees of minimality, SIAM J.Control and Optim., vol.42(2003), 1071-1086