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Abstract

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In the present paper we propose the definition of the criticality for vector-valued functions based on the concept of quasi-relative interior. This allows to make the concept of criticality operational for vector optimization problems where the interior of the order generating cone has empty interior. Basing on the introduced concept we prove necessary optimality conditions for closed convex pointed cones and cone-convex vector-valued functions as well as for closed convex pointed generating cones and general directionally differentiable vector-valued mappings.

O punktach krytycznych odwzorowań o wartościach wektorowych

W pracy zaproponowana jest definicja punktów krytycznych dla odwzorowań o wartościach wektorowych. Definicja ta wykorzystuje pojęcie quasi relatywnego wnętrza. Pozwala ona rozważać punkty krytyczne dla problemów optymalizacji, w których stożek generujący porządek ma puste wnętrze oraz niepuste quasi relatywne wnętrze. Własność tę ma między innymi stożek elementów nieujemnych w przestrzeni funkcji całkowalnych z kwadratem.

Udowodnione są warunki konieczne optymalności dla dwóch typów zadań optymalizacji wektorowej:

- ze stożkowo wypukłymi funkcjami o wartościach wektorowych z porządkiem zdefiniowanym przez wypukły ostry stożek domknięty ,

- z kierunkowo różniczkowalnymi funkcjami o wartościach wektorowych z porządkiem zdefiniowanym przez wypukły ostry domknięty stożek generujący przestrzeń.

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1 Introduction

The concept of criticality is central to the study of optimization problems. For a directionally differentiable mapping $f : X \to Y$ defined on a normed space X and taking values in a topological vector space Y ordered by a closed convex cone $K \subset Y$, int K $\neq \emptyset$ we say that $x_0 \in X$ is a critical point (c.f. Smale [18, 19, 20, 21]) of f if

$$f'(x_0; d) \notin -\text{int}\, K \quad \text{for any direction } d \in X,$$

where $f'(x_0; d)$ is a directional derivative at x_0 in the direction $d \in X$,

$$f'(x_0; d) := \lim_{t \downarrow 0} \frac{f(x_0 + td) - f(x_0)}{t}.$$

In the present paper we discuss the idea of replacing the existing concept of criticality by the following one:
x_0 is a critical point of a mapping $f : X \to Y$ if

$$f'(x_0; d) \notin -\text{qri}\,(K), \quad \text{for any } d \in X.$$

where qri (K) is a quasi relative interior of K as defined by Borwein and Lewis ([11], Def.2.3).

2 Quasi interiors

The definition of criticality cited above applies only to cones K with nonempty topological interiors. However, in many important cases this is too strong a requirement. For instance, let $p \in [1, +\infty)$. In the space

$$\ell^p = \{x = (\xi)_{i \in \mathbb{N}} \mid \xi_i \in \mathbb{N},\ i \in \mathbb{R} \ \sum_{i=1}^{\infty} |\xi_i|^p < +\infty\}$$

with the norm $\|x\|_{\ell^p} = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}$ the natural ordering cone

$$K_{\ell^p} = \{x \in \ell^p \mid \xi_i \geq 0 \quad i \in \mathbb{N}\}$$

has empty topological interior. Also, if for a nonempty subset $\Omega \subset \mathbb{R}^n$ we consider the space $L^p(\Omega)$ of all p-th power Lebesgue integrable functions, i.e.

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \mid \int_{\Omega} |f(x)|^p \, dx < +\infty\}$$

with the norm $\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p \, dx\right)^{1/p}$, the natural ordering cone

$$K_{L^p(\Omega)} = \{f \in L^p \mid f(x) \geq 0 \text{ almost everywhere on } \Omega\}$$

has empty topological interior.

First attempts to define quasi interiors of cones in locally convex topological vector spaces Y are made by R.E.Fullerton and C.C.Braunschweiger [13]. Namely, for a convex cone $K \subset Y$, a point $x \in K$, $x \neq 0$, is a *quasi- interior point* of K if the set $P_x = K \cap (x - K)$ generates the space Y, i.e. if $Y = \text{claff}(K)$, where cl$(\cdot)$ is a topological closure of a set in Y and aff(A) is the smallest linear subspace generated by A. Ponstein [9] and Aliprantis & Tourky [1] introduced the concept of *relative interior* to investigate duality theory for abstract optimization problems and Borwein and Lewis [11] introduced the concept of *quasi-relative interior* to derive a generalization of Fenchel duality theorem. This concept was also used by Jeyakumar and Wolkowicz [8], Bot, Csetnek and Wanka [3] in the formulation of the generalized Slater constraint qualification condition and regularity conditions. A survey of interiority notions is given by Zalinescu [24] and by Borwein and Goebel [10]. The terminology is partially set up in the book by Holmes [7]. Separation theorems involving sets with quasi-relative interiors are proved in [4, 5, 3].

2.1 Algebraic concepts

Let X be a linear space and $A \subset X$ be a nonempty subset of X. The *algebraic interior (core)*, (Holmes [7], p.7) of A with respect to a nonempty subset B of X, $\operatorname{cor}_B(A)$ is

$$\begin{aligned}\operatorname{cor}_B(A) : &= \{a \in A \ : \ \forall_{b\in B} \ \exists_{0<\delta<1} \ \forall_{0<\lambda\le\delta} \ a+\lambda b \in A\} \\ &= \{a \in A \ : \ \forall_{b\in B} \ \exists_{x\in(a,b)} \ [a,x] \subset A\},\end{aligned}$$

where $[a,x] := \{z \in X \ : \ z = \lambda a + (1-\lambda)x, 0 \le \lambda \le 1\}$. Two special cases are distinguished. The algebraic interior of A with respect to X is called *the algebraic interior* and is denoted $\operatorname{cor}(A)$, i.e. $\operatorname{cor}(A) := \operatorname{cor}_X(A)$. Let B is the smallest affine subspace containing A, i.e. B is the affine hull of A, $B = \operatorname{aff}(A)$, where $\operatorname{aff}(A) := x + \operatorname{span}(A-A)$, for any fixed $x \in A$. The set $\operatorname{cor}_{\operatorname{aff}(A)}(A)$ is called *intrinsic core*, $\operatorname{icr}(A) := \operatorname{cor}_{\operatorname{aff}(A)}(A)$, i.e.

$$\begin{aligned}\operatorname{icr}(A) : &= \{a \in A \ : \ \forall_{z\in\operatorname{aff}(A)} \ \exists_{0<\delta<1} \ \forall_{0<\lambda\le\delta} \ a+\lambda z \in A\} \\ &= \{a \in A \ : \ \forall_{z\in\operatorname{aff}(A)} \ \exists_{x\in(a,z)} \ [-x,x] \subset A\}.\end{aligned}$$

In particular, when $A \subset X$ is convex,

$$\operatorname{icr}(A) = \{a \in A \ : \forall_{z\in A\setminus\{a\}} \ \exists_{x\in A} \ a \in (z,x)\}.$$

For convex sets $A \subset X$ the concept of intrinsic interior coincides with the concept of *pseudo relative interior*, $\operatorname{pri}(A)$, where,

$$\operatorname{pri}(A) := \{x \in A \ : \ \operatorname{cone}(A-x) \ \text{ is a linear subspace of } \ X\}.$$

The following lemma holds.

Lemma 2.1 (Borwein & Goebel [10]). *Let X be a linear space. For any nonempty and convex subset A of X*

$$pri\ (A) = icr(A).$$

Proof. Let $a \in \operatorname{icr}(A)$ and let a nonzero $v \in \operatorname{cone}\ (A-a)$ be given, i.e. $x = \lambda v + a \in A \setminus \{a\}$ for some $\lambda > 0$. Hence, there exists $\mu > 0$ such that

$$(1-\mu)(\lambda v + a) + \mu a = (\mu - 1)\lambda(-v) + a \in A,$$

and $-v \in \operatorname{cone}(A-a)$ which proves that $a \in \operatorname{pri}\ (A)$.

Let $a \in \operatorname{pri}\ (A)$. Take any $x \in A$, $x \neq a$. Then $x - a \in \operatorname{cone}(A-a)$ and $a - x \in \operatorname{cone}(A-a)$. Hence, there is $\lambda > 0$ such that $a - x = \lambda(z-a)$, where $z \in A$. This gives $(\mu-1)(a-x) + a \in A$, where $\mu = 1 + \frac{1}{\lambda} > 1$, or $\mu a + (1-\mu)x \in A$. □

Let $A \subset X$ be a nonempty convex subset of X. Then

$$\operatorname{span}(A-a) = \operatorname{cone}(A-A) \ \text{ for any } \ a \in A. \tag{1}$$

To see this, let $a \in A$. Let $0 \neq x \in \text{cone}(A - A)$. Then $x = \lambda(a_1 - a_2)$, where $a_1, a_2 \in A$, $a_1 \neq a_2$, $\lambda > 0$ and $x = \lambda(a_1 - a) - \lambda(a_2 - a)$, i.e. $x \in \text{span}(A - a)$.

Now, take any nonzero element x of $\text{span}(A - a)$. It can be represented as $x = \sum_{i=1}^{k} \lambda_i[a_i - a]$, where $a_1, a_2, ..., a_k \in A$ and $\lambda_1, ..., \lambda_k \in \mathbb{R}$ and not all λ_i are zero's. If all λ_i are nonnegative, we immediately get $x \in \text{cone}(A - A)$. Now, suppose e.g. that $\lambda_1 < 0$. Then

$$x = \sum_{i=2}^{k} \lambda_i \left(\sum_{i=2}^{k} \frac{\lambda_i}{\sum_{i=2}^{k} \lambda_i} (a_i - a) \right) + (-\lambda_1)(a - a_1),$$

which proves that $x \in \text{cone}(A - A)$.

By (1), $\text{aff}(A) = x + \text{cone}(A - A)$ for every $x \in A$ and $\text{cone}(A - A)$ is a subspace parallel to $\text{aff}(A)$.

Lemma 2.2. *For any convex subset A a a linear space X the following are equivalent:*

(i) $a \in pri(A)$,

(ii) $cone(A - a) + a = aff(A)$.

Proof. Let $a \in \text{pri}(A)$. By (1), $\text{cone}(A-a)+a \subset \text{aff}(A)$. To see the inverse inclusion, take any $v \in \text{aff}(A)$. Hence,

$$v = \sum_{i=1}^{k} \lambda_i a_i, \quad a_i \in A, \quad i = 1, ..., k, \quad \sum_{i=1}^{k} \lambda_i = 1.$$

If all $\lambda_i \geq 0$, we have $v = a + \sum_{i=1}^{k} \lambda_i(a_i - a)$ and $v \in \text{cone}(A - a) + a$. If some $\lambda_i < 0$, then $(a - a_i) \in \text{cone}(A - a)$, and consequently $(-\lambda_i)(a - a_i) \in \text{cone}(A - a)$ and $v \in a + \text{cone}(A - a)$.

If $\text{cone}(A - a) + a = \text{aff}(A)$, then $\text{aff}(A - a)$ is a linear subspace, i.e. $\text{cone}(A - a)$ is a subspace and $a \in \text{pri}(A)$. $\square$

2.2 Topological concepts

Let X be a Hausdorff topological vector space (H.t.v.s., for short). By $\text{int}(A)$, $\text{rint}(A)$, $\overline{A}$, we denote the *interior*, the *relative interior*, i.e. the interior with respect to $\text{aff}(A)$, the closure of A, respectively.

Definition 2.3 (Ponstein [9], def 3.3.4). *The* relative interior *of a nonempty subset $A \subset X$, denoted by $ri(A)$ is the interior of A relative to its closed affine hull $\overline{aff(A)}$, i.e.*

$$ri(A) := \begin{cases} rint(A) \text{ if } aff(A) \text{ is closed} \\ \emptyset \quad \text{otherwise} \end{cases}$$

Clearly, $\text{ri}(A) \subset \text{icr}(A)$ and $\text{rint}(A) = \text{ri}(A)$ if $\text{aff}(A)$ is closed.

Lemma 2.4 (Borwein & Goebel [10], Lemma 2.5). *Let X be a Banach space and let A be a convex subset of X. If $aff(A)$ is closed, then $icr(A) = ri(A)$.*

Definition 2.5 (Borwein & Lewis [11]). *The* quasi relative interior *$qri(A)$ of a convex subset A of a H.t.v.s. X, is defined as*

$$qri(A) := \{a \in A \ : \ \overline{cone}(A - a) \text{ is a linear subspace}\}.$$

The quasi interior *$qi(A)$ of a convex subset A of a Banach space X, is defined as*

$$qi(A) := \{a \in A \ : \ \overline{cone}(A - a) = X\}.$$

Let us recall that the contingent cone $T_A(a)$ of a subset A of a normed space X at $a \in \overline{A}$ is given as

$$T_A(a) := \{v \in X \ : \ \liminf_{h \downarrow 0+} \frac{d(a+hv, A)}{h} = 0\},$$

where $d(x, C) := \inf\{\|x - c\| \ : \ c \in C\}$ for any set $C \subset X$. If $A \subset X$ is convex, then (see e.g. Frankowska & Aubin [12]. Proposition 4.2.1, p.138)

$$T_A(a) = \overline{\text{cone}}(A - a).$$

Hence,

$$\text{qri}\,(A) := \{a \in A \ : \ T_A(a) \text{ is a linear subspace}\}, \quad \text{qi}\,(A) := \{a \in A \ : \ T_A(a) = X\}.$$

Let X be a Banach space and X^* be its topological dual.

Definition 2.6 (Peressini [17]). *A point $a \in A$ is called a* nonsupport point *of a convex subset A of X if every closed supporting hyperplane to A at a contains A. Equivalently, for any $x^* \in X^*$, if $\langle A - a, x^* \rangle \geq 0$, then $\langle A - a, x^* \rangle = 0$.*

Lemma 2.7 (Borwein & Goebel [10], Lemma 2.7, Borwein & Lewis [11], Proposition 2.16). *Let A be a convex subset of a Banach space X. A point $a \in A$ is a nonsupport point if and only if $a \in qri(A)$.*

Proof. Let $a \in A$ be a nonsupport point. If $C = \overline{\text{cone}}(A - a)$ is not a subspace, then $y \in C$ and $-y \notin C$ for some $y \in X$. By separation argument, for some $x^* \in X^*$ it is $\langle x^*, C \rangle > \langle x^*, -y \rangle$. Since C is a cone, $\langle x^*, C \rangle \geq 0$. In particular, $\langle x^*, A - a \rangle \geq 0$. On the other hand, since $\langle x^*, -y \rangle < 0$ and $y \in C$ there exists $x \in A$ and $\lambda \geq 0$ such that $\langle x^*, \lambda(x - a) \rangle > 0$. This contradicts the assumption.

Now assume that $a \in \text{qri}\,(A)$. If, for some $x^* \in X^*$ we have $\langle A - a, x^* \rangle \geq 0$, then also $\langle \overline{\text{cone}}(A - a), x^* \rangle \geq 0$ and by the linearity of $\overline{\text{cone}}(A - a)$ we get $\langle \overline{\text{cone}}(A - a), x^* \rangle = 0$. In particular $\langle A - a, x^* \rangle = 0$, i.e., a is a nonsupport point. $\square$

Let $N_A(a) \subset X^*$ be the normal cone to A at a,

$$N_A(a) := \{\phi \in X^* \ : \ \phi(x - a) \leq 0 \ \ \forall_{x \in A}\}.$$

Proposition 2.8 (Borwein & Lewis [11], Proposition 2.8). *Let X be locally convex and let $A \subset X$ be a convex subset of X, $a \in A$. Then $a \in qri(A)$ if and only if $N_A(a)$ is a subspace of X^*.*

Proof. For any subset $C \subset X$, the polar of C is given by

$$C^\circ = \{\phi \in X^* \mid \phi(x) \leq 1 \ \forall x \in C\} = \{\phi \in X^* \mid \phi(x) \leq 0 \ \forall x \in C\},$$

if C is a cone. Similarily, for a cone $L \in X^*$

$${}^\circ L = \{x \in X \mid \phi(x) \leq 0 \ \forall \phi \in L\}$$

Clearly, if C is a subspace, then C° is a subspace and if L is a subspace, then ${}^\circ L$ is a subspace.

Now, for $\phi \in X^*$, $\phi(x - a) \leq 0$ for all $x \in A$ if and only if $\phi(u) \leq 0$ for all $u \in \overline{\text{cone}}(A - a)$ by the continuity of ϕ. Thus, $N_A(a) = (\overline{\text{cone}}(A - a))^\circ$.

on the other hand, by Theorem 12C of Holmes [7]

$${}^\circ N_A(a) = {}^\circ((\overline{\text{cone}}(A - a))^\circ) = \overline{\text{cone}}(\{0\} \cup \overline{\text{cone}}(A - a)) = \overline{\text{cone}}(A - a)$$

which proves the assertion. $\square$

A subset $C \subset X$ is *CS-closed* (see Borwein and Lewis [11]) if for any $\lambda_n \geq 0$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and any $x_n \in C$, $n \in \mathbb{N}$ for which $\sum_{n=1}^{N} \lambda_n x_n \to \bar{x}$ we have $\bar{x} \in C$. Clearly, every CS-closed set is convex. In Banach spaces, all convex sets which are closed, open, finite-dimensional or G_δ are CS - closed (see Borwein [2]).

Theorem 2.9 (Borwein & Lewis [11], Theorem 2.19). *Suppose that (X, τ) is a topological vector space with either*

(a) (X, τ) is a separable Fréchet space, or

(b) $X = Y^*$ with Y a separable normed space and $\tau - \sigma(Y^*, Y)$.

If $A \subset X$ is CS-closed, then $qri(A) \neq \emptyset$.

The following theorem follows from Theorem 2.9.

Theorem 2.10 (Borwein & Goebel [10], Theorem 2.8,). *Every nonempty convex subset A of a separable Banach space X has nonempty quasi relative interior.*

2.3 Examples

(i) $X = L^p(\Omega)$, $p \in [1, +\infty)$

$$\text{qri}\,(K_{L^p}) = \{f \in L^p(\Omega) \mid f(x) > 0 \text{ a.e}\},$$

(ii) $X = \ell^p$, $p \in [1, +\infty)$

$$\text{qri}\,(K_{\ell^p}) = \{x \in \ell^p \mid \xi_n > 0 \ \forall n \in \mathbb{N}\},$$

3 Cone convex operators and their directional derivatives

Let Y be a topological vector space and let $K \subset Y$ be a closed convex pointed cone in Y. The cone K generates the ordering relation in Y as follows

$$x \leq_K y \ \Leftrightarrow \ y - x \in K \quad \text{for} \quad x, y \in Y.$$

A mapping $f : X \to Y$ is *K-convex in a subset* $\Omega \subset X$ if for any $x, y \in \Omega$ and $0 \leq \lambda \leq 1$ we have

$$\lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y) \in K.$$

Proposition 3.1. *Let $f : X \to Y$ be convex on X. Then, for any $x_0 \in X$, $d \in X$, $t \in \mathbb{R}_+$ and $t_1 \in (0, t)$ we have*

$$\frac{f(x_0 + t_1 d) - f(x_0)}{t_1} \leq_K \frac{f(x_0 + td) - f(x_0)}{t}.$$

Proof. Let $t_0, t_2 \in \mathbb{R}$, $t_0 < t_2$, and $t_0 < t_1 < t_2$. Then

$$0 \leq \lambda = \frac{t_1 - t_0}{t_2 - t_0} \leq 1 \quad \text{and} \quad \lambda t_2 + (1 - \lambda) t_0 = t_1.$$

Let $\phi : \mathbb{R} \to Y$,

$$\phi(t) := f(x_0 + td).$$

It is immediate that ϕ is convex on $\mathbb{R}$. Then

$$\begin{aligned} \frac{t_1 - t_0}{t_2 - t_0}\phi(t_2) + (1 - \frac{t_1 - t_0}{t_2 - t_0})\phi(t_0) - \phi(t_1) &= \tfrac{t_1 - t_0}{t_2 - t_0}(\phi(t_2) - \phi(t_0)) - (\phi(t_1) - \phi(t_0)) \\ &= (t_1 - t_0)\left(\tfrac{\phi(t_2) - \phi(t_0)}{t_2 - t_0} - \tfrac{\phi(t_1) - \phi(t_0)}{t_1 - t_0}\right) \in K. \end{aligned}$$

Hence, for any $t_1 \in (t_0, t_2)$ we have

$$\frac{\phi(t_2) - \phi(t_0)}{t_2 - t_0} - \frac{\phi(t_1) - \phi(t_0)}{t_1 - t_0} \in K. \tag{2}$$

Choose $t_2 = t > 0$, $t_1 \in (0, t)$, $t_0 = 0$. By (2),

$$\frac{\phi(t) - \phi(0)}{t} - \frac{\phi(t_1) - \phi(0)}{t_1} \in K \quad \text{for any} \quad t > 0.$$

□

The following result proved by Topchishvili, Maisuradze and Ehrgott [22] and Valadier [23] establishes the existence of directional derivatives of cone convex vector-valued mappings.

Theorem 3.2 ([22, 23]). *Let X be a linear space and let Y be a weakly complete Banach space which is partially ordered by a normal closed convex cone $K \subset Y$. Let $f : X \to Y$ be a cone convex mapping. Then for an arbitrary element $(x_0, d) \in X \times X$ the direction derivative $f'(x_0; d)$, i.e. the limit*

$$f'(x_0; d) := \lim_{t \downarrow 0} \frac{f(x_0 + td) - f(x_0)}{t}$$

exists in the strong topology of Y.

4 Criticality for convex operators

Definition 4.1. *A point $x_0 \in X$ is a* relative critical point *of f if*

$$f'(x_0; d) \notin -ri(K) \quad \textit{for any} \quad d \in X.$$

Definition 4.2. *A point $x_0 \in X$ is a* local Pareto point *of f if there exists a neighbourhood V of x_0 such that*

$$f(X \cap V) \cap (f(x_0) - K) = \{f(x_0)\}.$$

Theorem 4.3. *Let X be a normed space and let Y be a Banach space ordered by a closed convex pointed cone $K \subset Y$, $ri K \neq \emptyset$. Let $f : X \to Y$ be a cone convex directionally differentiable mapping. If $x_0 \in X$ is a local Pareto point of f, then $x_0 \in X$ is a critical point of f.*

Proof. If x_0 is not a critical point of f, there exists $d \in X$ such that

$$f(x_0; d) \in -\text{ri}(K).$$

Let $\phi(t) := \frac{f(x_0+td)-f(x_0)}{t}$. We prove that for any decreasing sequence $t_n \downarrow 0$

$$\phi(t_n) = \frac{f(x_0 + t_n d) - f(x_0)}{t_n} \in \text{aff } (K) \text{ for all } n. \tag{3}$$

Otherwise there exists n_0 such that $\phi(t_{n_0}) \notin \text{aff } (K)$. Since $\text{aff}(K)$ is closed, there exists $x^* \in X^*$ such that

$$\langle x^*, \phi(t_{n_0}) \rangle < c < \langle x^*, \text{aff}(K) \rangle.$$

Since K is a cone it must be $\langle x^*, \text{aff}(K)\rangle = 0$. By $K-$convexity of f, ϕ is a non-decreasing function of t, i.e.

$$\phi(t) - \phi(t_1) \in K \quad \text{for} \quad 0 < t_1 < t.$$

Hence,

$$\langle x^*, \phi(t_n)\rangle = \langle x^*, \phi(t_{n_0})\rangle < c < 0 \quad \text{for all } n \geq n_0.$$

Consequently, $\langle x^*, f'(x_0; d)\rangle \leq c < 0$ contradictory to the fact that $f'(x_0, d) \in -\text{ri}\ (K)$. This proves (3). Clearly, there exists a neighbourhood V of $f'(x_0; d)$ and $N > 0$ such that

$$V \cap \text{aff}(K) \subset -K \quad \text{and} \quad \phi(t_n) \in V \cap \text{aff}(K) \quad \text{for} \quad n \geq N$$

which gives $\phi(t_n) \in -K$ for all $n \geq N$. This proves that x_0 is not a local Pareto point. $\square$

Definition 4.4. *A point $x_0 \in X$ is a* quasi relative critical point *of f if*

$$f'(x_0; d) \notin qri(-K) \quad \textit{for every} \quad d \in X.$$

Theorem 4.5. *Let X be a normed space and let Y be a Banach space ordered by a closed convex pointed cone $K \subset Y$, $qri(K) \neq \emptyset$. Let $f : X \to Y$ be a cone convex directionally differentiable mapping. If $x_0 \in X$ is a local Pareto point of f, then x_0 is a quasi relative critical point of f.*

Proof. If x_0 is not a quasi relative critical point of f, there exists $0 \neq d \in X$ such that

$$f'(x_0; d) \in \text{qri}\,(-K).$$

Let $\phi(t) := \frac{f(x_0+td)-f(x_0)}{t}$ and let $t_n \downarrow 0$ be a decreasing sequence of positive real numbers. Then

$$\phi(t_n) = \frac{f(x_0 + t_n d) - f(x_0)}{t_n} \in (-K) \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

To see this, suppose on the contrary that $\phi(t_{n_0}) \notin (-K)$ for some $n_0 \in \mathbb{N}$. Since K is closed and convex there exists $x^* \in X^*$ such that

$$\langle x^*, -K\rangle \geq c > \langle x^*, \phi(t_{n_0})\rangle = c_1, \quad c, c_1 \in \mathbb{R}.$$

Since K is a cone, it must be $\langle x^*, -K\rangle \geq 0$, i.e. $c = 0$, $c_1 < 0$ and $\langle x^*, f'(x_0; d)\rangle = c_2 \geq 0$. Hence, $\langle x^*, -K - f'(x_0; d)\rangle \geq c_2 \geq 0$ which means that $-x^* \in N_K(f'(x_0; d))$, where

$$N_K(\hat{x}) = \{x* \in X^* \ : \ x^*(k - \hat{x}) \leq 0 \ \ \forall_{k\in K}\}.$$

Since $f'(x_0; d) \in \text{qri}\,(-K)$, by Proposition 2.8, the normal cone $N_{-K}(f'(x_0; d))$ is a subspace, i.e., $x^* \in N_{-K}(f'(x_0; d))$ and consequently

$$0 \leq \langle -x^*, -K - f'(x_0; d)\rangle \leq c_2 \leq 0$$

which proves that it must be $\langle x^*, f'(x_0; d)\rangle = c_2 = 0$.

Since $f(x_0; d) \in \text{qri}\,(-K)$, by Proposition 2.7, $f(x_0; d)$ is a nonsupport point of $-K$, i.e. since $\langle x^*, -K - f'(x_0; d)\rangle \geq 0$, it must be $\langle x^*, -K\rangle = 0$.

On the other hand, by $K-$convexity of f, ϕ is a non-decreasing function of t, i.e.

$$\phi(t) - \phi(t_1) \in K \quad \text{for} \quad t_1 < t.$$

Hence,

$$\langle x^*, \phi(t_n)\rangle = \langle x^*, \phi(t_{n_0})\rangle < c_1 < 0 \quad \text{for } n \geq n_0.$$

Consequently, $\langle x^*, f'(x_0; d)\rangle \leq c_1 < 0$ contradictory to the fact that $\langle x^*, f'(x_0; d)\rangle = 0$. This proves (4) which implies that x_0 is not a local Pareto point. □

As a corollary of Theorem 4.5 and Theorem 3.2 we obtain the following result.

Theorem 4.6. *Let X be a normed space and let Y be a weakly complete Banach space ordered by a closed convex normal cone $K \subset Y$, $qri(K) \neq \emptyset$. Let $f : X \to Y$ be a cone convex mapping. If $x_0 \in X$ is a local Pareto point of f, then x_0 is a quasi relative critical point of f.*

Proof. It is enough to observe that by Theorem 3.2, the mapping f is directionally differntiable at any $x_0 \in X$ and any $d \in X$. The conclusion follows from Theorem 4.5 □

5 Generating cones

A cone $K \subset Y$ is *generating* if $Y = K - K$.

Theorem 5.1. *Let X be a normed space and let Y be a Banach space ordered by a closed convex pointed cone K. Assume that K is a generating cone in Y. Let $f : X \to Y$ be a directionally differentiable mapping. If $x_0 \in X$ is a local Pareto point, then*

$$f'(x_0; d) \notin qri(-K) \quad \text{for any} \quad d \in X.$$

Proof. On the contrary, suppose that there exists a direction $0 \neq d \in X$ such that

$$f'(x_0; d) \in \text{qri}\,(-K).$$

Take a sequence of real positive numbers (t_n), $t_n \downarrow 0$ and put

$$\phi(t) = f(x_0 + td) \quad \text{for} \quad t \in \mathbb{R}.$$

For any $n \in \mathbb{N}$ we have

$$\frac{\phi(t_{n+1}) - \phi(0)}{t_{n+1}} = \frac{\phi(t_n) - \phi(0)}{t_n} + \alpha_n,$$

where

$$\alpha_n := \frac{t_n\phi(t_{n+1}) - t_{n+1}\phi(t_n + (t_{n+1} - t_n)\phi(0)}{t_{n+1}t_n}.$$

Since K is generating, there exist α_n^1, $\alpha_n^2 \in K$ such that

$$\alpha_n = \alpha_n^1 - \alpha_n^2 \quad \text{for} \quad n \in \mathbb{N}.$$

Hence, for $n \in \mathbb{N}$

$$\frac{\phi(t_{n+1}) - \phi(0)}{t_{n+1}} = \frac{\phi(t_n) - \phi(0)}{t_n} + \alpha_n^1 - \alpha_n^2. \tag{5}$$

We show that

$$\phi(t_n) = \frac{f(x_0 + t_n d) - f(x_0)}{t_n} \in (-K) \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

To see this, suppose on the contrary that $\phi(t_{n_0}) \notin (-K)$ for some $n_0 \in \mathbb{N}$. Since K is closed and convex there exists $x^* \in X^*$ such that

$$\langle x^*, -K\rangle \geq c > \langle x^*, \phi(t_{n_0})\rangle = c_1, \quad c, c_1 \in \mathbb{R}.$$

Since K is a cone, it must be $\langle x^*, -K\rangle \geq 0$, i.e. $c = 0$, $c_1 < 0$ and $\langle x^*, f'(x_0; d)\rangle = c_2 \geq 0$. Hence, $\langle x^*, -K - f'(x_0; d)\rangle \geq c_2 \geq 0$ which means that $-x^* \in N_K(f'(x_0; d))$, where

$$N_K(\hat{x}) = \{x* \in X^* \ : \ x^*(k - \hat{x}) \leq 0 \ \ \forall_{k \in K}\}.$$

Since $f'(x_0; d) \in \operatorname{qri}(-K)$, by Proposition 2.8, the normal cone $N_{-K}(f'(x_0; d))$ is a subspace, i.e., $x^* \in N_{-K}(f'(x_0; d))$ and consequently

$$0 \leq \langle -x^*, -K - f'(x_0; d)\rangle \leq c_2 \leq 0$$

which proves that it must be $\langle x^*, f'(x_0; d)\rangle = c_2 = 0$.

Since $f(x_0; d) \in \operatorname{qri}(-K)$, by Proposition 2.7, $f(x_0; d)$ is a nonsupport point of $-K$, i.e. since $\langle x^*, -K - f'(x_0; d)\rangle \geq 0$, it must be $\langle x^*, -K\rangle = 0$.

On the other hand, by (5),

$$\langle x^*, \frac{\phi(t_{n+1}) - \phi(0)}{t_{n+1}}\rangle = \langle x^*, \frac{\phi(t_n) - \phi(0)}{t_n}\rangle = c_1 < 0 \quad \text{for} \quad n \geq n_0.$$

Consequently, $\langle x^*, f'(x_0; d)\rangle = c_1 < 0$ contradictory to the fact that $\langle x^*, f'(x_0; d)\rangle = 0$. This proves (6) which immediately gives that x_0 is not a local Pareto point.

□

6 Other classes of functions

Definition 6.1. *A function $f : X \to Y$ is* K-lower semicontinuous *at $\bar{x}$ (see [6, 15, 16])if for each sequence (x_n) converging to $\bar{x}$ there exists a sequence $(b_n) \subset Y$ converging to $f(\bar{x})$ such that*

$$b_n \leq_K f(x_n).$$

Below we strenghten Definition 6.1 by introducing monotonically K-lower semicontinuous functions

Definition 6.2. *A function $f : X \to Y$ is* directionally monotone K-lower semicontinuous *at $\bar{x} \in X$ if for every direction $d \in X$ there exist a decreasing sequence of real positive numbers $t_n \downarrow 0$ and a sequence $(b_n) \subset Y$ converging to $f(\bar{x})$ such that*

$$b_n \leq_K f(\bar{x} + t_n d) \quad \textit{and} \quad f(\bar{x} + t_{n+1} d) \geq_K f(\bar{x} + t_n d).$$

Definition 6.3. *A function $f : X \to Y$ is* directionally decreasing *at $\bar{x} \in X$ if for every direction $d \in X$ there exist a decreasing sequence of real positive numbers $t_n \downarrow 0$ such that*

$$t_n \left(f(\bar{x} + t_{n+1} d) - t_{n+1} f(\bar{x} + t_n d)\right) \geq_K (t_n - t_{n+1}) f(\bar{x}).$$

Proposition 6.4. *For a function $f : X \to Y$ which is directionally monotone K lower semicontinuous at $\bar{x} \in X$, for any direction $d \in X$, the respective difference quotients are nondecreasing, i.e.,*

$$\phi(t_{n+1}) \leq \phi(t_n) \quad \textit{for a decreasing sequence of positive numbers} t_n \downarrow 0.$$

Proof. By the directional monotonicity of f at $\bar{x} \in X$, for every direction $d \in X$ there exist a decreasing sequence of real positive numbers $t_n \downarrow 0$

$$f(\bar{x} + t_{n+1}) \leq_K f(\bar{x} + t_n d).$$

Hence, $t_{n+1} < t_n$ for $n \in \mathbb{N}$ and

$$\frac{f(\bar{x} + t_{n+1}d) - f(\bar{x})}{t_{n+1}} \leq_K \frac{f(\bar{x} + t_n d) - f(\bar{x})}{t_n} \frac{t_{n+1}}{t_n} \leq_K$$

□

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