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A Cost Allocation Problem Analyzed as a Partition Function Form Game

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Abstract

In the paper a cooperative game in partition function is proposed for the cost allocation problem. The game describes real situations in which payoff of any coalition does not only depend on the players in the coalition but also on the coalition structure of other players. Solution concepts are analyzed. It is shown that the core of the game in partition function form is equal to the core of an appropriately formulated game in characteristic function form. On the base of the theoretical results, the core, the stable sets and several nucleoli concepts can be derived and utilized for decision support.

1 Introduction

In the paper (Krus, Bronisz 2000) the cost allocation problem has been considered in the case of a development project giving on an output a vector of goods. Actors interested in obtaining the goods can create coalitions to implement the project jointly. The cost allocation problem deals with allocation of the cost of the project among the actors as well as among the goods. In the paper a family of cooperative games in characteristic function form, including pricing mechanism has been proposed and analyzed. Different solution concepts have been presented together with algorithms which allow derive the solutions. In the paper, similarly as in other papers dealing with the cost allocation problems (Young, Okada, Hashimoto 1980, Seo, Sakawa 1987, Legros 1986) utilizing cooperative games in characteristic function form it is assumed that the payoff of each coalition depends on the players who create it.

There are also practical situations where the payoff of any coalition depends not only on the players creating it but also on the coalition structure of the other players (more general on partition of the players). It is typical in the case of firms sharing a given market of goods or services. If several firms decide to create a coalition, its gain depends also on other firms, whether the other firms will act independently or create other coalition. This paper deals with cost allocation problems in such situations. It is assumed that the actors - players try to obtain some goods and are ready to cover required cost. To reach the goods they can act independently or create coalitions. The costs which have to be covered depend not only on the coalition itself but also on the coalition structure of other players. The problem deals with cost allocation among the players, but also among the goods. In the paper a cooperative game in partition function form is proposed to model the above decision situation. The theory of such games is developed as the base for decision support. In sections 2 and 3 the considered problem is formally defined. In section 4 the cooperative game in partition function form is given and considered as a model which can be used for analysis of the problem. Solution concepts to the game are proposed in section 5, based on the introduced domination relation. Properties of the solution concepts have been shown in five theorems.

2 General formulation of the problem

Let $N = \{1, \ldots, n\}$ be a finite set of players, and let $\mathcal{N}$ be the set of all nonempty subsets of N.

Let S denote a given coalition of players, $S \in \mathcal{N}$. Each player is interested in the same set of goods $M = \{1, \ldots, m\}$, which can be obtained by covering some costs.

For any $S \in \mathcal{N}$ let $Q = \{P_1, ..., P_r\}$ be a partition of S, i.e.

$$\bigcup_{i=1}^{r} P_i = S, \ \forall j \ P_j \neq 0, \ \forall k \ P_j \cap P_k = \emptyset \text{ if } k \neq j, \tag{1}$$

and let Π_S denote the set of all partitions of S. For simplicity we will denote by Π the set Π_N. Let P_I be the partition consisting of individual players, i.e. $P_I = \{\{1\}, \{2\}, \ldots, \{n\}\}$.

For each coalition $S \in \mathcal{N}$ and partition $P \in \Pi$, such that $S \in P$ there are given functions $f_{S,P} : \mathbb{R}^m \to \mathbb{R}$ describing the cost required to obtain the vector of goods $x \in \mathbb{R}^m$. We assume, that $x = (x_1, \ldots, x_m) \in \mathbb{R}^m_+$, what means, that the players are interested only in positive vectors of the goods. In other words, the functions $f_{S,P}$ depend on the total amount of the goods and do not depend on a division of them among players in S, but can depend on the partition of other players in P.

Let $z = (z_1, \ldots, z_n) \in I\!R_+^{n \times m} = I\!R^{NM}$, where:
$I\!R^{MN}$ is the space of goods of the grand coalition N,
$z_i = (z_{i1}, \ldots, z_{im}) \in I\!R_+^m$,
z_{ij} is the volume of the good j assigned to the player i, for $j \in M$, $i \in N$.

The general problem consists in the formulation of the cooperative games in partition function form and looking for solution concepts in the games. To formulate the game we apply a mechanism of cost allocation using prices for the vector of goods.

3 Cost allocation problem

Let $f : I\!R_+^m \to I\!R$ be a cost function, which has continuous first partial derivatives, $f(0) = 0$, and is defined for $x \geq 0$. According to the definition of the cost function, the value $f(x)$ means the cost required for obtaining the vector $x = (x_1, \ldots, x_m)$ of goods.

Definition 3.1
The pair (f, x) satisfying the above assumptions is called the **cost allocation problem** of the order m. □

The set of all cost allocation problems of the order m is denoted by P^m, and the set of all cost allocation problems by $P = \bigcup_{m \geq 1} P^m$

Definition 3.2
The function $c : P \to \bigcup_{m \geq 1} I\!R^m$ is called the **cost allocation procedure**, if

for a given cost allocation problem, it assigns a vector of prices, i.e.
$(f, x) \in P^m \rightarrow c(f, x) \in \mathbb{R}^m$,
where $c(f, x) = (c_1(f, x), \ldots, c_j(f, x), \ldots, c_m(f, x))$,
and $c_j(f, x)$ is the price of j-th good. □

4 Formulation of the cooperative game in partition function form

For a given vector of goods $z = (z_1, \ldots, z_n) \in \mathbb{R}^{NM}$, we can define the cooperative game in partition function form utilizing the cost allocation mechanism. In the following formulations we omit z in indexes to simplify the notation.

Definition 4.1

The multi-items cooperative game in partition function form is defined by a pair (N, F), where N is the set of players and F is a function which for each partition $P \in \Pi$, $P = \{P_1, P_2, ..., P_r\}$ assigns r-dimensional real vector $F_P = (F_P(P_1), ..., F_P(P_r))$. Components of the vector are given by:
$F_P(P_k) = \sum_{i \in P_k} f_{\{i\}P_I}(z) - f_{P_k,P}(z)$ for all $P_k \in P$. □

Applying the cost allocation procedure the components $F_P(P_k)$ can be rewritten as follows:

$$F_P(P_k) = \sum_{i \in P_k} \sum_{j=1}^{m} \left[c_j(f_{\{i\},P_I}, z_i) \times z_{ij}\right] - \sum_{j=1}^{m} \left[c_j(f_{P_k,P}, \sum_{i \in P_k} z_i) \times \sum_{i \in P_k} z_{ij}\right].$$

In the definition, the part:
$\sum_{j=1}^{m} \left[c_j(f_{\{i\},P_I}, z_i) \times z_{ij}\right]$ describes the cost of individual action of the i-th player, and the part:

$\sum_{j=1}^{m}\left[c_j(f_{P_k,P}, \sum_{i\in P_k} z_i) \times \sum_{i\in P_k} z_{ij}\right]$ describes the cost of joint action of the players in the coalition P_k.

$F_P(P_k)$ describes the benefit the players can obtain acting together in coalition P_k in comparison to their individual actions. Let us see that $F_P(P_k)$ depends on the partition P, that means it depends on possible coalition structure of other players.

For each nonempty coalition $S \in \mathcal{N}$ and each partition $Q \in \Pi_S$ we define the following functions:

$$v(S) = \min_{\{P\in\Pi: S\in P\}} F_P(S), \quad v(\emptyset) = 0, \tag{2}$$

$$u(Q) = \min_{\{P\in\Pi: Q\subset P\}} \sum_{T\in Q} F_P(T), \tag{3}$$

$$\overline{v}(S) = \max_{\{Q\in\Pi_S\}} u(Q), \quad \overline{v}(\emptyset) = 0. \tag{4}$$

Intuitively, $v(S)$ denotes the maximal worth of a coalition S independent on the behavior of the players, $u(Q)$ denotes the maximal amount which is guarantied for the players arranged in Q independent on the behavior of the others players, $\overline{v}(S)$ denotes maximal amount which is guaranteed for the players in S independent on the behavior of the other players.

Example 1.

To illustrate the functions introduced above let us consider a game with four players. Let the the functions $F_P(P_k)$ be as follows:

for each different $i, j, k, m \in N$

(i) $F_P(\{i\}) = a$, for any partition P such that $\{i\} \in P$

(ii) $F_P(\{i,j\}) = b$, for $P = \{\{i,j\},\{k\},\{m\}\}$,

(iii) $F_P(\{i,j\}) = c$, for $P = \{\{i,j\},\{k,m\}\}$,

(iv) $F_P(\{i,j,k\}) = d$ for $P = \{\{i,j,k\},\{m\}\}$,

(v) $F_P(N) = e$, for $P = \{N\}$.

In such a case the function v(S) for any nonempty coalition $S \in N$ takes the values:

$$v(\{i\}) = a,$$
$$v(\{i,j\}) = min(b,c),$$
$$v(\{i,j,k\}) = d,$$
$$v(N) = e$$

and the values of the function $\overline{v}(S)$ are as follows:

$$\overline{v}(\{i\}) = a,$$
$$\overline{v}(\{i,j\}) = max[min(b,c), 2a],$$
$$\overline{v}(\{i,j,k\}) = max[d, b+a, 3a],$$
$$\overline{v}(N) = max[e, d+a, 2c, b+2a, 4a].$$

◇

It is easy to verify that for any multi-items cooperative game in partition function form (N, F) the following inequalities hold:

$$\overline{v}(\{i\}) = v(\{i\}) \quad \text{for each } i \in N, \tag{5}$$

$$v(S) \leq \overline{v}(S) \quad \text{for each } S \subset N, \tag{6}$$

$$u(Q_1) + u(Q_2) \leq u(Q_1 \cup Q_2) \quad \text{for each } Q_1 \in \Pi_S,\ Q_2 \in \Pi_T \text{ where } S,T \subset N \text{ such that } S \cap T = \emptyset. \tag{7}$$

$$\overline{v}(S) + \overline{v}(T) \leq \overline{v}(S \cup T) \quad \text{for each } S,T \subset N \text{ such that } S \cap T = \emptyset \tag{8}$$

5 Solution concepts

Definition 5.1

A vector $x = (x_1, \dots x_n)$ is called an **imputation** if

$$x_i \geq \overline{v}(\{i\}) \quad \text{for each } i \in N, \tag{9}$$

$$\sum_{i \in N} x_i = \sum_{S \in P} F_P(S) \quad \text{for some } P \in \Pi. \tag{10}$$

□

Conditions (9) and (10) are called individual rationality and realizability, respectively. The individual rationality means that nobody will agree to obtain payoff lower than his payoff when he acts independently. The realizability means the there exists a partition that can realize the payoffs. Let R denote the set of all imputations, and let R^P denote the set of all imputations realized by partition $P \in \Pi$.

Definition 5.2

Let S be a nonempty subset of N and let $x, y \in R$. Then x **dominates** y **via** S (denoted $x\ Dom_S\ y$) if

$$x_i > y_i \quad \text{for each } i \in S, \tag{11}$$

and there exists $Q \in \Pi_S$ such that

$$\sum_{i \in S} x_i \leq u(Q), \tag{12}$$

$$\sum_{i \in N} x_i = \sum_{T \in P} F_P(T) \text{ for some } P \in \Pi \text{ such that } Q \subset P. \tag{13}$$

□

Condition (11) says that each player in S prefers his payoff in x to that in y. Condition (12) states that the players in S can form such partition $Q \in \Pi_S$ that they can assure realization of payoffs x_i $i \in S$. Condition (13) states that the payoff x is realizable by some partition P.

We say that x dominates y (denoted by $x\,Dom\,y$) if $x\,Dom_S\,y$ for some $S \subset N$. It is easy to show that relation Dom is neither transitive nor antisymmetric.

Let X be subset of R. Then

$$\begin{aligned} Dom_S\,X &= \{y \in R: \; x\,Dom_S\,y \quad \text{for some } x \in X\}, \\ Dom\,X &= \{y \in R: \; x\,Dom\,y \quad \text{for some } x \in X\}. \end{aligned}$$

Definition 5.3

A set of imputations K is a **stable set** if

$$K \cap Dom\,K = \emptyset, \tag{14}$$

$$K \cup Dom\,K = R. \tag{15}$$

□

Definition 5.4

A set of imputations C is a **core** if

$$C = R \setminus Dom\,R. \tag{16}$$

□

Condition (14) says that if x and y are in K then neither dominates the other, condition (15) states that if z is not in K then there exists x in K which dominates z. The above formal definition is based on the idea that instead of one imputation which every coalition is satisfied with, there is a set of imputations, so that if we take any imputation outside the set, there is an imputation inside the set, which is more beneficial for some coalition and the coalition has an incentive to obtain it. Not everyone might be satisfied with this new imputation, and some subset of players might force a change to another imputation outside the set. But the new imputation is again dominated by an imputation inside the set. Thus the bargaining process resolves around the set. Therefore, whole the set can be considered as a possible solution. All the imputations in the set are as important as one another. So there is no domination among the imputations in the set. The relation (14) is called as the internal stability condition, and the relation (15) as the external stability condition.

The core is the set of nondominated imputations in R, i.e. for any partition P there is no coalition $S \in P$ that gives its members payoffs better than payoffs in the core. Clearly, the core is contained in every stable set.

The definitions of the stable set and the core are described in a similar way to those proposed in (Thrall and Lucas 1963) and in (Lucas 1965) but they are based on the weaker domination relation. Thrall and Lucas assumed that given coalition $S \in P$ can not be subdivided. In our approach we assume that if subdividing coalition S gives better result for the coalition then it is possible to realize it.

For each partition $P \in \Pi$ in a game (N, F) let

$$\|P\| = \sum_{S \in P} F_P(S). \tag{17}$$

Definition 5.5

Any imputation x has the property of **group rationality** if

$$\sum_{i \in N} x_i = \max_{\{P \in \Pi\}} \|P\|. \tag{18}$$

□

Let $R^{\max}$ denote the set of all imputations satisfying the property of group rationality in the game (N, F).

If the players choose an imputation $x \in R^{\max}$ as the payoff at the end of the game it means that they divide maximal possible gain in the game. It will be shown that imputations belonging to the concepts presented above fulfill this property.

Theorem 5.1

The core C of a game (N, F) is a subset of the core proposed in (Thrall and Lucas 1963). Moreover, each imputation $x \in C$ has the property of group rationality. ■

Proof. Let $x \in C$ and $x \notin R^{\max}$. Then $\sum_{i \in N} x_i = \max_{\{P \in \Pi\}} \|P\| - M$, where $M > 0$. If $y \in R^{\max}$ is defined by $y_i = x_i + M/n$ for each $i \in N$ then $y \, Dom_N \, x$. Contradiction.

If $x \, dom_S \, y$ for some $S \subset N$ in the sense proposed in (Thrall and Lucas 1963) then $x \, Dom_S \, y$. Therefore $C = R \setminus Dom\, R \subset R \setminus dom\, R$. ◇

It can happen that the core C is empty though the core proposed in (Thrall and Lucas 1963) is nonempty.

On the base of the definition of function $\overline{v}$ (see equation 4) i.e. by the condition $\overline{v}(\emptyset) = 0$ and superadditivity condition (8) we have that the pair

$(N, \overline{v})$ is a well defined cooperative game in characteristic function form with side payments. The following theorem shows relation between cores defined for games in partition function form and games in characteristic function form.

Theorem 5.2

The core of the game (N, F) is equal to the core of the cooperative game in characteristic function form $(N, \overline{v})$, i.e. it satisfies the following conditions:

$$\sum_{i \in S} x_i \geq \overline{v}(S) \quad \text{for each } S \subset N, \tag{19}$$

$$\sum_{i \in N} x_i = \overline{v}(N). \tag{20}$$

■

Proof. Let CR denote the core of the game $(N, \overline{v})$. Let $x \in C$ and $x \notin CR$. From (5) it follows that $x_i \geq \overline{v}(\{i\})$, theorem 5.1 states that $\sum_{i \in N} x_i = \overline{v}(N)$, so x is an imputation in the game $(N, \overline{v})$. Because $x \notin CR$ then there exists an imputation y and a coalition S in the game $(N, \overline{v})$, such that $y_i > x_i$ for each $i \in S$ and $\sum_{i \in S} y_i \leq \overline{v}(S)$. Let Q be a partition of S such that $\overline{v}(S) = u(Q)$. Consider a partition of N such that $P = Q \cup \{\{i_1\}, \{i_2\}, \ldots \{i_{n-s}\}\}$ where s denotes the number of players in S, $i_j \in N \setminus S$, $j = 1, 2, \ldots, n - s$ and an imputation z of the game (N, F) defined by

$$z_i = \begin{cases} y_i & \text{for each } i \in S, \\ F_P(\{i\}) + M/(n - s) & \text{for each } i \in N \setminus S, \end{cases}$$

where $M = \sum_{T \in Q} F_P(T) - \sum_{i \in S} y_i \geq 0$.
Because $z \, Dom_S \, x$ then $x \notin C$.

Contradiction.

Let $x \in CR$ and $x \notin C$. x is an imputation in the game (N, F). Because $x \notin C$ then there exists a coalition $S \subset N$, a partition $Q \in \Pi_S$ and an imputation y in the game (N, F) such that $y_i > x_i$ for each $i \in S$, $\sum_{i \in S} y_i \leq u(Q)$ and $\sum_{i \in N} y_i = \sum_{T \in P} F_P(T)$ for some $P \in \Pi$, $Q \subset P$.

Consider an imputation z in the game $\overline{v}$ defined by

$$z_i = \begin{cases} y_i & \text{for each } i \in S, \\ y_i + M/(n-s) & \text{for each } i \in N \setminus S \end{cases}$$

where $M = \overline{v}(N) - \sum_{T \in P} F_P(T) \geq 0$.

It follows that z dominates x via S in the game $(N, \overline{v})$, so $x \notin CR$.

Contradiction. This proves the theorem. ◇

The following theorem shows that stable sets are defined in the rational form.

Theorem 5.3

If K is any stable set of the game (N, F) then each imputation $x \in K$ has the property of group rationality. ■

Proof. Let $x \in R \setminus R^{\max}$ and y be an imputation defined as in the proof of theorem 5.1. We have that $y\, Dom_N\, x$. If $y \in K$ then $x \in Dom\, K$. If $y \notin K$ then $y \in Dom_S\, K$ for some $S \subset N$, so there exists $z \in K$ such that $z\, Dom_S\, y$. But it is easy to verify that if $z\, Dom_S\, y$ and $y\, Dom_S\, y$ and $y\, Dom_N\, x$ then $z\, Dom_S\, x$. Therefore $x \in Dom\, K$. This proves the theorem. ◇

From theorem 5.1 and 5.3 it follows that when discussing the core and the stable sets, without loosing of generality we can restrict our considerations to the imputations satisfying the property of group rationality. It can happen that for a game (N, F) there is no stable set, there is one stable set or there are many stable sets. We can prove the following result:

Theorem 5.4
For an n-person game with $\tilde{P} = \{N\}$ such that $\|\tilde{P}\| > \|P\|$ for each $P \in \Pi$, $P \neq \tilde{P}$, there exists unique stable set $K = R^{\{N\}} = R^{\max}$. ■

Proof. Let $x, y \in R^{\max}$ and let $x \, Dom \, y$. Domination may be realized only by the partition $\{N\}$, so we have $x_i > y_i$ for each $i \in N$. It follows that $\sum_{i \in N} x_i > \sum_{i \in N} y_i$. Contradiction.
Let $x \in R \setminus R^{\max}$ and let y be an imputation defined as in the proof of theorem 5.1. We have that $y \, Dom_N \, x$, so $x \in Dom \, R^{\max}$. This proves that $R^{\max}$ is a solution. It is unique solution by theorem 5.3. ◇

For n-person games in which the outcome to the partition $\{N\}$ is greater than the sum of the outcomes for any other partition, we have no trouble in finding solution. Moreover, the unique solution is the same as that in (Thrall and Lucas 1963).

The following theorem shows relation between stable sets defined for games in partition function form and games in characteristic function form.

Theorem 5.5
If $\hat{P} = \{\hat{P}_1, \ldots, \hat{P}_r\}$ is a partition of N such that $\|\hat{P}\| > \|P\|$ for each

$P \in \Pi$, $P \neq \hat{P}$ then the game (N, F) has the same stable sets as the game in characteristic function form $(N, \hat{v})$ defined by

$$\hat{v}(S) = \overline{v}(T) + \sum_{i \in S \setminus T} \overline{v}(\{i\}) \quad \text{for each } S \subset N, \quad \hat{v}(\emptyset) = 0, \tag{21}$$

where:

$$T \subset S, \qquad T = \bigcup_{i=1}^{r} \{\hat{P}_i : \ \hat{P}_i \subset S\}.$$

■

Proof. From (8) it is easy to verify that the game $(N, \hat{v})$ is well defined. x is an imputation in the game $(N, \hat{v})$ if $x_i \geq \hat{v}(\{i\}) = v(\{i\})$ and $\sum_{i \in N} x_i = \hat{v}(N) = \|\hat{P}\|$. In such a case the set of imputations in the game $(N, \hat{v})$ is equal to the set $R^{\max}$. Moreover, from theorem 5.3, imputations which are not in $R^{\max}$ play no role in the game (N, F) so we can only consider the set $R^{\max}$.

Let $x, y \in R^{\max}$ and let y dominates x in the game $(N, \hat{v})$. Then there exists a coalition $S \subset N$ such that $y_i > x_i$ for each $i \in S$ and $\sum_{i \in S} y_i \leq \hat{v}(S)$. It follows that $y_i > x_i$ for each $i \in T \subset S$ and
$\sum_{i \in T} y_i \leq \overline{v}(T) + \sum_{i \in S \setminus T}(v(\{i\}) - y_i) \leq \overline{v}(T)$ so $y \, Dom \, x$.

Let $x, y \in R^{\max}$ and let $y \, Dom \, x$. Then there exists a coalition $S = T$ such that $y_i > x_i$ for each $i \in S$ and $\sum_{i \in S} y_i \leq \hat{v}(S)$ so y dominates x in the game $(N, \hat{v})$. It proves the theorem. ◇

6 Final remarks

In the paper a cooperative game in partition function form has been proposed for the cost allocation problem. The game describes real situations in which payoff of any coalition does not only depend on the players in the coalition but also on the coalition structure of the other players. Theory of such games has been developed. In particular solution concepts like core and stable sets have been proposed on the base of introduced domination relations. Properties of the concepts have been analyzed. The concepts are similar to those presented by Thrall and Lucas (1963) but they have been formulated for weaker domination relation which seems to be more relevant.

It has been shown that the core of the game in partition function form is equal to the core of an appropriately formulated game in characteristic function form. This theoretical result is very important for construction of decision support systems. On the base of the result, the core can be derived and proposed to the players as the set describing frames of their negotiations. Different nucleoli can be calculated by solving a sequence of linear programming problems and presented to the players as mediation proposals. Further research on the decision support problems are planned. Another interesting problems for further research are indicated by the papers (Koczy 2007, 2008) dealing with coalitional games with externalities, in which pessimistic, optimistic cores and sequential coalition formation are analyzed.

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