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L. Kruś

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Zakładu zgłaszający pracę:
dr inż. Lech Kruś

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A NUCLEOLUS EXPRESSING PREFERENCES OF PLAYERS IN A MULTICRITERIA COOPERATIVE GAME

Lech Kruś
Systems Research Institute, Polish Academy of Sciences
Newelska 6, 01 447 Warsaw, Poland
e-mail:krus@ibspan.waw.pl

Abstract

The paper deals with decision situations in which several decision makers consider realization of some projects. Each of them can realize a respective project independently. However they can create the grand coalition or smaller coalitions to realize one or several greater projects jointly and have some benefits due to the scale effects. Decision support problems in such situations are discussed.

The decision situations are described by multicriteria cooperative games without side payments in which a payoff of each player is measured by a vector of criteria. Each player has in general a different vector of the criteria and acts according to his own preferences. A nucleolus concept expressing preferences of all players is proposed as a solution to the game.

Key words: multicriteria decision support, multicriteria cooperative games without side payments, nucleolus.

1 Introduction

The paper deals with decision situations in which several decision makers consider realization of some projects. Each of them can realize a respective project independently. However they can create the grand coalition or smaller coalitions to realize one or several greater projects jointly and have some benefits due to the scale effects. Questions arise: - what can be coalition structure? - when the grand coalition will be created? - how to divide benefits from the cooperation among players? Decision support problems in such situations are discussed.

The decision situation are described by cooperative games without side payments in which a payoff of each player is measured by a vector of criteria. Each player has in general a different vector of the criteria and acts according to his own preferences. The criteria are conflicting in the case of each particular player as well as among the players.

The theory of cooperative games has been intensively investigated, especially in the case of games with side payments. The most interesting results can be found in Shapley (1953), Schmeidler (1969), Aumann, Maschler (1964), to mention only the leading papers. The theory of the games without side payments has not been so developed, however important results have been obtained, among others, by Aumann (1961), Peleg (1963), Stearns (1964), Kalai (1975). The theory of cooperative games has been developed under a general assumption that the players outcomes are measured by explicitly given utility function satisfying some assumptions. In practical problems the outcomes are usually measured by some number of criteria, and the utility function are not explicitly given. Papers dealing with cooperative games with multicriteria outcomes of players are relatively rare. We would like to mention the paper by Bergstressen, Yu (1977), where Yu' domination structures were utilized to define and analyze the cooperative games with side payments.

In this paper we make an attempt to develop the theory of cooperative games without side payments in the case of multicriteria payoffs of the players. We do not assume any a priory given utility function of a player. However it is assumed that every player acts according to has his own preferences among the criteria.

There is a wide literature dealing with multicriteria decision making, several different approaches have been developed, computer based decision support systems have been constructed and are implemented in practice. General idea is the following: the decision maker assumes a control parameter and the system calculates a Pareto solution related to the control parameter. With use of the decision support system, the decision maker can generate in this way some number of Pareto solutions in an interactive learning procedure so that he can select the most preferable one. Dealing with the cooperative games we formulate solution concepts related to control parameters describing the player preferences. Similarly as in the reference point method

(Wierzbicki 1982, 1986), we assume the control parameters to be reference points in the spaces of criteria of the players.

In this paper a nucleolus concept expressing preferences of all players is proposed as a solution to the game. Properties of the nucleolus are shown. The nucleolus can be base for construction of an interactive procedure supporting the players in decision analysis and agreeable solution selection. Ideas of such a nucleolus and proposals of decision support have been presented in papers (Kru, Bronisz 1995, Kru 2008). Extensions of the results are presented in this paper. Especially formal theorems and proofs are given justifying construction of the excess function used in the formulation of the nucleolus, and showing properties of the nucleolus.

2 Decision making problem as a cooperative game

2.1 Decisions and payoffs

Let $N = 1, 2, ..., n$, and $\aleph$ be the set of all nonempty subsets of N. For any coalition $C \in \aleph$:
$E^C = \times_{i \epsilon C} E_i$ be a decision space of players in C, where E_i is Euclid decision space player i,
$Y^C = \times_{i \epsilon C} Y_i$ be a multicriteria space of payoffs of players in C, where Y_i is m_i dimensional space of criteria of player i,
We assume to be given: set of admissible decisions $X_{0i} \subset E_i$ for each player $i \in N$ and a vector function $W : E^C \rightarrow Y^C$ defining payoffs of the players.

For simplicity of notation, we assume that each player tries to maximize all his criteria. Each player can have different criteria and the number of criteria of each player can be different. We assume, that for any vector $y = (y_i)_{i \in N} \in Y^N$, vector $y^C = (y_i)_{i \in C} \in Y^C$ denotes payoffs of the players in coalition C, where $y_i = (y_{i1}, y_{i2}, \dots y_{im_i}) \epsilon Y_i \subseteq I\!R^{m_i}$.

A cooperation of players creating different coalitions is defined by a collection of sets ${V^C}_{C \epsilon \aleph}$, where $V^C \subset Y^C$ denotes the set of attainable multicriteria payoffs of players in C.

A convention has been assumed, that $z, y \in I\!R^m$, for any m :
$z \geq y$ denotes $z_i \geq y_i$ for $i = 1, 2, \dots, m$,
$z > y$ denotes , $z_i \geq y_i, z \neq y$ for $i = 1, 2 \dots, m$,
$z \gg y$ denotes $z_i > y_i$ for $i = 1, 2 \dots, m$.

Vector $z \in I\!R^m$ is **weekly Pareto optimal** in set $Y_0 \subset I\!R^m$ if $z \in Y_0$ and there is no $y \in Y_0$ such, that $y \gg z$,
Vector $z \in I\!R^m$ is **Pareto optymalny** in set $Y_0 \subset R^m$ jeli $z \in Y_0$ and there is no $y \in Y_0$ such, that $y \geq z$.

2.2 Multicriteria cooperative game without side payments

Definition 2.1

A multicriteria n-person cooperative game without side payments (n-person MCC game) is described by a collection
$V = \{V^C\}_{C \epsilon \aleph}$ of sets V^C satisfying the following conditions:

1. V^C is closed and nonempty subset of space Y^C,

2. V^C is upper bounded, i.e. there exists $y^C \epsilon Y^C$ such that $V^C \subset \{z^C \epsilon Y^C, z^C \leq y^C\}$,

3. for any $y^C \epsilon V^C$, $z^C \epsilon V^C$, if there is $z^C < y^C$, then $z^C \epsilon V^C$,

4. for each two coalitions $C, T \epsilon \aleph$, such that $C \cap T = \emptyset$, $V^C \times V^T \subset V^{C \cup T}$.

□

The formulation of MCC game is closely related to the formulation of cooperative game without side payments given by Aumann (1967) in the case of scalar payoffs of players.
The set V^C represents all payoffs that the coalition C can assure itself. The set has to be closed, nonempty and bounded (conditions 1 and 2).
Condition 3 (comprehensiveness) stipulates that if a coalition can assure a payoff y, it can also assure itself $z < y$, i.e. it can also assure itself anything coordinate-wise less.
The condition 4 assures superadditivity of the game, i.e. any payoff whose components can be obtained by each of two disjoint coalitions acting separately can be also obtained by them when acting together.

3 Solution concepts

Let Ω denote a class of games satisfying the above conditions.

A solution concept is a function F, which associates to each game $V \epsilon \Omega$ a set of payoffs $F(V) \subset V^N$.

Definition 3.1

A **core** of the game V is the set:

core$(V) = \{y \epsilon V^N$: for every coalition C there is no $z^C \epsilon V^C$ such that $z_i > y_i$ for every $i \epsilon C\}$.

A **week core** of the game V is the set:

wcore$(V) = \{y \epsilon V^N$: for every coalition C there is no $z^C \epsilon V^C$ such that $z_i \gg y_i$ for every $i \epsilon C\}$.

□

A payoff belongs to the core if for every coalition, there is no payoff improving at least one criterion of each member. A payoff belongs to a weak core if for every coalition, there is no payoff improving all criteria of each member.

Definition 3.2

A function $l_C : Y^N \times \Omega \rightarrow I\!R$ is called **excess function** for coalition C, if it satisfies the following conditions:

1. if $y, z \epsilon Y^N$ are such that $y_i = z_i$ for every $i \epsilon C$, then for every game V, $l_C(y, V) = l_C(z, V)$.
2. If $y, z \epsilon Y^N$ are such that $y_i > z_i$ for every $i \epsilon C$, then for every game V, $l_C(y, V) < l_C(z, V)$.
3. For any game V, if $y^C \epsilon bd(V^C) = \{v^C \epsilon V^C$: there is no $z^C \epsilon V^C$ such, that $y_i >> v_i$ for every $i \epsilon C\}$, then $l_C(y, V) = 0$, where $bd(V^C)$ denotes boundary of the set V^C.

4. $l_C(y, V)$ is continuous jointly with respect to y and V.

Function $l_C : Y^N \times \Omega \rightarrow I\!R$ is called **weak excess function** for coalition C, if it satisfies conditions 1,3,4 and condition
2a. If $y, z \epsilon Y^N$ are such that $y_i \gg z_i$ for every $i \epsilon C$, then for every game V, $l_C(y, V) < l_C(z, V)$. □

Kalai (1975) has proposed conditions defining the excess function for the classical cooperative games without side payment in the case of scalar payoffs of players. The above conditions given in Definition 3.2 extend his proposal on the case of multicriteria payoffs of players.

Definition 3.3
A payoff $y \epsilon V$ is called **individually rational** if it belongs to the set
$IR(V) = \{y \epsilon V^N :$ for every $i \in N$: there is no $z \epsilon V^{\{i\}}$ for which $z_i > y_i$ $\}$.
A payoff $y \epsilon V$ is called (**weakly individually rational**) if it belongs to the set
$IR(V) = \{y \epsilon V^N :$ for every $i \in N$: there is no $z \epsilon V^{\{i\}}$ for which $z_i \gg y_i\}$.
A payoff $y \epsilon V$ satisfies condition of **group rationality** if it belongs to the set $GR(V) = \{y \epsilon V^N$: : there is no $z \epsilon V^N$ for which $z_i > y_i$ for every $i \in N$ $\}$.
A payoff $y \epsilon V$ satisfies condition of (**weak group rationality**) if it belongs to the set $GR(V) = \{y \epsilon V^N$: there is no $z \epsilon V^N$ for which $z_i \gg y_i$ for every $i \in N$ $\}$.

□

Individual rationality means that no player agrees on a payoff "worse" than he can get acting individually. Group rationality says that each player tries to maximize his payoff and share all benefits resulting from cooperation.

Theorem 3.1

For any collection of weak excess functions $\{l_C\}_{C\epsilon\aleph}$*, for every game* $V\epsilon\Omega$

$\boldsymbol{wcore}(V) = \{y\epsilon wGR(V) : l_C(y, V) \leq 0,\ \textit{for every}\ C\epsilon\aleph\backslash\{N\}\}$.

For any collection of weak excess functions $\{l_C\}_{C\epsilon\aleph}$*, for every game* $V\epsilon\Omega$

$\boldsymbol{core}(V) = \{y\epsilon GR(V) : l_C(y, V) \leq 0,\ \textit{for every}\ C\epsilon\aleph\backslash\{N\}\}$.

■

Proof

Let V be a given MCC game and let $V\epsilon\Omega$, and $\{l_C\}_{C\epsilon\aleph}$ be a given collection of weak excess function (excess functions). The theorem 3.1 follows immediately from definition 3.1 of a weak core (core), from definition 3.3, of weak group rationality (group rationality), and from properties 2 and 3 (2a and 3) in the definition 3.2 of a weak excess function (an excess function). ◊

Let $\Theta(y)$ denotes vector in $I\!R^{|\aleph|-1}$ obtained by arranging values of the excess functions $l_C(y, V)$ of all coalitions C in $\aleph$, $C \neq N$ in the nonincreasing order. Let respectively $w\Theta(y)$ denotes such a vector in the space obtained by arranging values of the weak excess functions.

Definition 3.4

Nucleolus is defined as a set:

$\mathfrak{N}(V) = \{y\epsilon IR(v) : \Theta(y) \leq_{lex} \Theta(z)$ for any $z\epsilon IR(V)\}$.

Weak nucleolus is defined as a set :

$w\mathfrak{N}(V) = \{y\epsilon IR(v) : w\Theta(y) \leq_{lex} w\Theta(z)$ for any $z\epsilon IR(V)\}$. □

For any vectors $y, z\epsilon I\!R^M$, $y \leq_{lex} z$ means, that $z = y$, or that there is an integer $k, 1 \leq k \leq m$, such that $y_i = z_i$ dla $1 \leq i < k$ and $y_k < z_k$.

We exclude $l_C(y, N)$ because $l_C(y, N) = 0$ for all $y\epsilon IR(V)$

Theorem 3.2

For any collection of weak excess functions $\{l_C\}_{C\epsilon\aleph}$, for every game V, nucleolus $\mathfrak{N}(V)$ is nonempty and if the core is nonempty then
$\mathfrak{N}(V) \subset \boldsymbol{core}(V)$.

For any collection of weak excess functions $\{l_C\}_{C\epsilon\aleph}$, for every game V, weak nucleolus $w\mathfrak{N}(V)$ is nonempty and if the weak core is nonempty then
$w\mathfrak{N}(V) \subset \boldsymbol{wcore}(V)$. ■

Proof

Let V be multicriteroia game without side payments $V\epsilon\Omega$, and $\{l_C\}_{C\epsilon\aleph}$ be a collection of excess functions (weak excess functions). Let $m = R^{|\aleph|-1}$, and $\Theta(y) = (\Theta_1(y), \Theta_2(y), ..., \Theta_m(y))$ be defined according to Definition 3.2. Due to property 4 of Definition 3.2, the funcions $l_C(y, V)$ are continuous with respect to variable y, so the functions $\Theta, i = 1, 2, ..., m$ are also continuous, defined respectively as defined as minimum and maximum of a finite number of continuous functions $l_C(y, V)$. Let

$A_1 = \{y \in IR(V) : C_1(y) \leq \Theta_1(z)$ for every $z \in IR(V)$, and

$A_{i+1} = \{y \in A_1 : \Theta_{i+1}(y) \leq \Theta_{i+1}(z)$ for every $z \in A_1$

for $i = 1, 2, ..., m - 1$.

Because Θ_i are continuous and the set $IR(V)$ is compact, then the sets $A_i i = 1, 2, ..., m$ are compact and nonempty. But we have $\mathfrak{N}(V) = A_m$, so the nucleolus is nonempty..

Let the game have a nonempty core, let $y \in \mathbf{core}(V)$, and $z \in \mathfrak{N}(V)$. Then $y \in GR(V)$, and maximal components of the vector Θ satisfy $\Theta_i(z) \leq \Theta_i(y)$. From the Theorem 3.1 we have $\Theta_i(z) \leq \Theta_i(y) \leq 0$, so $z \in \mathbf{core}(V)$.
In the similar way we can prove the second part of the theorem. ◊

4 Nucleolus expressing preferences of players

In this section an excess function and a nucleolus are proposed, which can be used in a system supporting decision analysis. Each player compares attainable payoffs in the space of his criteria and looks for the payoff according to his preferences. We assume that preferences of each player can be defined on the basis of two points in his criteria space: reference and reservation points. Reservation point can be defined as a payoff preferred by the player acting independently. The reference point can be defined by the aspiration levels assumed by the player in his criteria space, when he cooperates with other players. When the players cooperate, the payoffs of all players are considered in the space which is a cartesian product of the criteria spaces of particular players. The reference point expressing preferences of all the players can be defined by:

$\underline{y} = (\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n) \in wIR(V)$, $\underline{y}_i \in V^{\{i\}}$ i

and respectively the reservation point of all the players:

$\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \in Y^N$, $\bar{y} \gg \underline{y}$,

where $\underline{y}$ denotes preferred payoffs of players acting independently, and $\bar{y}$ - preferred payoffs of the players assumed according to their aspirations.

Let $w(\underline{y}, \bar{y}) \in Y^N$ denote preferred normalized improvement direction generated according to the aspirations of all players in the grand coalition, and $w^C(\underline{y}, \bar{y}) \in Y^C$ - preferred improvement direction of the players in a coalition C.

According to the reference point method (Wierzbicki 1982, 1986), applied in multicriteria decision support, we assume that points $\underline{y}$ i $\bar{y}$ indicated by players, define required improvement direction. The direction denoted by $w(\underline{y}, \bar{y}) \in Y^N$, can defined by:

$$w(\underline{y}, \bar{y}) = w_1(\underline{y}, \bar{y}), w_2(\underline{y}, \bar{y}), ..., w_n(\underline{y}, \bar{y})),$$

where

$$w_i(\underline{y}, \bar{y}) \in Y_i = I\!R_i^m, w_i(\underline{y}, \bar{y}) = w_{i1}(\underline{y}, \bar{y}), .., w_{ik}(\underline{y}, \bar{y}))$$

for $i \in N$, and

$$w_{ij}(\underline{y}, \overline{y}) = \frac{\overline{y}_{ij} - \underline{y}_{ij}}{\sum_{j=1}^{m_i} \left(\overline{y}_{ij} - \underline{y}_{ij}\right)}.$$

We can check that the formulation satisfies the normalization condition, such that for any player $i \in N$,

$$\sum_{i=1}^{m_i} w(\underline{y}, \bar{y}) = 1.$$

In the following we look for solution concept to the MCC game, which will depend on the players preferences. We assume that the preferences are expressed by the points $\underline{y}$ and $\bar{y}$.

Let us denote by $w^C(\underline{y}, \bar{y})$ a composition of improvement directions of the players creating a coalition C, i.e. $w^C(\underline{y}, \bar{y}) = (w_i(\underline{y}, \bar{y})_{i \in C} \in Y^C$.

The following measuring of the excess of the coalition C, is proposed for given points $\underline{y}$ i $\bar{y}$:

$$h_C(y, V, \underline{y}, \bar{y}) = sup\{t \in I\!R : y^C + t \cdot t^C(\underline{y}, \bar{y}) \circ w^C(\underline{y}, \bar{y})/|C| \in V^C\},$$

where

$$t^C(\underline{y}, \bar{y}) = (t_i(\underline{y}, \bar{y}))_{i \in C},$$

$$t_i(\underline{y}, \bar{y}) = sup\{t \in I\!R : (\underline{y}_i + t \cdot w_i(\underline{y}, \bar{y})) \in P^{\{i\}}(V^N)\},$$

$P^C : Y^N \longrightarrow Y^C$ is the projection Y^N on Y^C, i.e

$$P^C(V^N) = \{y^C : y \in V^N\},$$

$$t(\underline{y}, \bar{y}) \circ w(\underline{y}, \bar{y}) = (t_1 \cdot w_1(\underline{y}, \bar{y}), ..., t_n \cdot w_n(\underline{y}, \bar{y})) \in Y^N,$$

$$t^C(\underline{y}, \bar{y}) \circ w^C(\underline{y}, \bar{y}) = (t_i \cdot w_i(\underline{y}, \bar{y}))_{i \in C} \in Y^C,$$

$|C|$ denotes number of players in the coalition C.

Lemma 4.1

For given points $\underline{y}$ i $\bar{y}$, the function $h_C(y, V, \underline{y}, \bar{y})$ is a weak excess function of the MCC game.

■

The proof derives directly from the definition of the function $h_C(y, V, \underline{y}, \bar{y})$.

Definition 4.1

A payoff $u(\underline{y}, \bar{y})$ is called an **utopia payoff relative to points** $\underline{y}$ i $\bar{y}$ if for each $i \in N$:

$u(\underline{y}, \bar{y}) = sup\{y_i \in P^{\{i\}}(V^N) : y_i = \underline{y}_i + t \cdot (\bar{y}_i - \underline{y}_i) \text{ for some } t \in I\!R\}$. □

A payoff $u(\underline{y}, \bar{y})$ for a player i in grand coalition N is such that

$$(\underline{y}_1, \underline{y}_2, ...\underline{y}_{i-1}, u_i(\underline{y}, \bar{y}), \underline{y}_{i+1}, ..., \underline{y}_n) \in wGR(V).$$

That means, the payoff u_i defines nondominated values of criteria of the player i, he could obtain in the grand coalition assuming that the payoffs of other players $j \in N$, $j \neq i$ are on the levels $\underline{y}_j$. The payoff has been called "utopia" because other players in the coalition do not agree that all benefits from the cooperation would be given to one player.

Lemma 4.2

The excess function $h_C(y, V, \underline{y}, \bar{y})$ depends only on $\underline{y}$ and on direction w generated by $\bar{y}$, but does not depend on of $|\bar{y} - \underline{y}|$. ∎

Proof

It is easy to notice that the following equations are satisfied:

$$u(\underline{y}, \bar{y}) = \underline{y} + t(\underline{y}, \bar{y}) \circ w(\underline{y}, \bar{y}),$$

$$h_C(y, V, \underline{y}, \bar{y}) = h_C(y, V, \underline{y}, u(\underline{y}, \bar{y})),$$

and for any $z \in Y^N$ such that $z = \underline{y} + t \cdot w(\underline{y}, \bar{y})$ for some $t \in I\!R$, $t > 0$,

$$h_C(y, V, \underline{y}, \bar{y}) = h_C(y, V, \underline{y}, z).$$

◊

Position of the players are weighted in proportion to the distance $u_i - \underline{y}_i$. We can use the distance for particular player to define his bargaining power in the game.

Lemma 4.3

For any C the weak excess function $h_C(y, V, \underline{y}, \bar{y})$ is invariant on positive affine transformation of criteria, i.e. for any coalition $C \in \mathcal{N}$ and an arbitrary affine transformation $T = (T_1, T_2, ..., T_n) : Y^C \to Y_C$, such that $T_{ij}y = (a_{ij} \cdot y_{ij} + b_{ij})$, where numbers $a_{ij}, b_{ij} > 0$ for $i \in N, j = 1, 2, ..., m_i$,

$$h_C(Ty, TV, \underline{y}, \bar{y}) = Th_C(y, V, \underline{y}, \bar{y})$$

for given $\underline{y}, \bar{y}$. ∎

Proof

Let T be affine positive transformation of criteria $T^C = (T_i)_{i \in C} : Y^C \to Y^C$. We can notice, that for any $i \in N$ and j such that $1 \leq j \leq k_i$ we have

$$w_{ij}(T\underline{y}, T\bar{y}) = \frac{a_{ij}(\bar{y}_{ij} - \underline{y}_{ij})}{\sum_{j=1}^{m_i} a_{ij}(\bar{y}_{ij} - \underline{y}_{ij})}$$

and

$$\frac{t_i(\underline{y}, \bar{y})}{\sum_{j=1}^{m_i} a_{ij}(\bar{y}_{ij} - \underline{y}_{ij})} = \frac{t_i(T\underline{y}, T\bar{y})}{\sum_{j=1}^{m_i} a_{ij}(\bar{y}_{ij} - \underline{y}_{ij})}.$$

We obtain

$$T_{ij}y_{ij} + t \cdot \frac{t_i(T\underline{y}, T\bar{y})}{s} \cdot w_{ij}(T\underline{y}, T\bar{y}) =$$

$$a_{ij} + b_{ij} + t \cdot \frac{t_i(\underline{y}, \bar{y}) \cdot \sum_{j=1}^{m_i} a_{ij}(\bar{y}_{ij} - \underline{y}_{ij})}{\sum_{j=1}^{m_i} s \cdot (\bar{y}_{ij} - \underline{y}_{ij})} \cdot \frac{a_{ij}(\bar{y}_{ij} - \underline{y}_{ij})}{\sum_{j=1}^{m_i} a_{ij}(\bar{y}_{ij} - \underline{y}_{ij})} =$$

$$a_{ij}(y_{ij} + t \cdot \frac{t_i(\underline{y}, \bar{y})}{s} \cdot \frac{(\bar{y}_{ij} - \underline{y}_{ij})}{\sum_{j=1}^{m_i} a_{ij}(\bar{y}_{ij} - \underline{y}_{ij})}) + b_{ij} =$$

$$T_{ij}(y_{ij} + t \cdot \frac{t_i(\underline{y}, \bar{y})}{s} \cdot w_{ij}(\underline{y}, \bar{y}).$$

That means, for any coalition C, any $y \in Y^N$ and $t > 0$

$$T^C y^C + t \cdot \frac{t^C(T\underline{y}, T\bar{y})}{s} \circ w^C(T\underline{y}, T\bar{y}) =$$

$$T^C(y^C + t \cdot \frac{t^C(\underline{y}, \bar{y})}{s} \circ w^C(\underline{y}, \bar{y}).$$

◊

Theorem 4.1

For given points $\underline{y}$ and $\bar{y}$, weak nucleolus $w\mathfrak{N}(V,\underline{y},\bar{y})$ generated by the function h_C is invariant on positive affine transformation of criteria, i.e.

$$w\mathfrak{N}(TV,T\underline{y},T\bar{y}) = T[w\mathfrak{N}(V,\underline{y},\bar{y})],$$

where T is a positive affine transformation. ■

The proof follows directly from Lemma 4.3 and the definition of the nucleolus.

It can be verified that the nucleolus generated by $h_C(y,V,\underline{y},\bar{y})$ function is a generalization of the nucleolus defined by Schmeidler (1969)for the cooperative games with side payments on the case of the mulicriteria cooperative games.

Let the game $V = \{V^C\}_{C\epsilon\aleph}$ be such that all subcoalitions containing more then one player and less then n players are trivial, i.e. if $|C| \neq 1$, $|C| \neq N$ then $V^C = \times_{i\in C}V^{\{i\}}$. Let (S,d) be n-person multicritaria bargaining problem defined as in (Krus, Bronisz 1993). If the generalized Raiffa-Kalai-Smrodinsky solution concept proposed in the paper is Pareto optimal in the set V^N, and $d = \underline{y}$, then it can verified that

$$w\mathfrak{N}(V,\underline{y},\bar{y}) = f^R(V^N,\underline{y},u(\underline{y},\bar{y})).$$

The solution is also generalization of the Raiffa-Kalai-Smorodinsky solution originally defined for the two person bargaining problem with scalar payoffs of players (i.e. if $m_i = 1$ for each player $i \in N$) by Raiffa (1953) and considered later by (Kalai, Smorodinsky 1975), (Thomson 1980). In the case of scalar payoffs of players and if the subcoalitions are trivial, the weak nucleolus proposed here coincides with the original Imai solution to the classical bargaining problem (Imai 1983). The Imai solution lexicographically improves the Raiffa-Kalai-Smorodinsky solution if the last one is not Pareto optimal.

5 Final remarks

In the paper a model of cooperative game without side payments is presented to describe cooperation of players having multicriteria payoffs. A decision making problem is formulated a the space being the cartesian product of multicriteria spaces of players payoffs. Theoretical results are presented including a formulation of general solution concepts like core and weak core, nucleolus and weak nucleolus, as well as some relations among the concepts. A nucleolus expressing players preferences is formulated. It is assumed that players express their preferences indicating aspiration and reservation points according to the ideas of reference point method applied in multicriteria decision support. Properties of the nucleolus are proved. The nucleolus expressing players preferences can be considered as a mediation proposal in negotiations regarding sharing of the cooperation benefits.

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