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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Bifurcation of vortex dipole in the viscous incompressible flow around an airfoil

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THE VISCOUS incompressible flow around a NACA0012 airfoil at angle of attack equal to 5° and Reynolds numbers (Re) of 5 000, 10 000, and 30 000 is considerd. The Navier-Stokes equations written in terms of stream function and vorticity are solved by a finite-difference predictor-corrector algorithm. Distinctive feature of the algorithm is the strict fulfilment of pressure uniqueness condition in the stream. Special attention is paid to displaying the dispersion properties in the computed viscous flow. Development of the flow separation on the airfoil, surface vorticity waves (streaming waves) which intensify in the separation region and, especially, at the trailing edge (TE), leads to strong vortex-pressure (acoustic) transition in vicinity of TE, to generation of vortex spots and to bifurcation of vortex dipole at the leading (LE) and TE. The complex vortex disturbances are perturbed which partly are emitted from LE into upstreamflow, and partly merge into velocity-pressure disturbances and surface vorticity waves. Found are the physical grounds for inevitable development of dispersion in the vorticity fields at Re > 10 000. The vortex dipole bifurcation is clearly defined as the main reason for dispersion impulse formation.

1. Introduction

THE VISCOUS INCOMPRESSIBLE flow around an airfoil at low and moderate Reynolds numbers was studied by many authors using different numerical algorithms, see [1 – 6] for example. The Reynolds number (Re) of 10 000 is almost the limiting value for available solvers of complete Navier-Stokes equations. One must note that the essential differences in various authors' results obtained for Re = 10 000 $\alpha = 5°$ (the reliability of every one is assumed and does not rise any doubts) show that we really have a strong dependence of solution on some physical factors which manifest themselves in different properties of the computed viscous flows. For this reason the aim of the paper is not only to assess the accuracy of presented results but to depict essential physical processes which influence essentially the "boundary" problems for the flows under study (laminar-turbulent transition). Such approach allows us to exclude the deterioration of computational study of viscous incompressible flow at Reynolds numbers characterized by basic variation of flow properties which are revealed in practice.

The case with Re = 10 000 and angle of attack equal to 5° for NACA0012 and Joukowski airfoils is presented in the works of four authors: GHIA *et al.* [1, 2], WU *et al.* [4], CHOI (referred to in [5]) and ZAKHARENKOV [3, 7, 8]. These results present three different solutions.

i) Results of Ghia show two different forms of streamline behaviour in the region of trailing edge (TE) for Re = 1 000. Their difference is related to the angle of TE (for a finite angle and cusp). Time-history of the flow is not shown and flows are presented as two different steady flows. The values of Cx (drag) and Cy (lift) coefficients for Re = 1 000 do not coincide with unpublished results of Zakharenkov (however, the Ghia's pressure distribution can be reconstructed). The flow at Re = 10 000 is essentially unsteady, with strong separation being at the airfoil leeward side.

ii) Results of Zakharenkov for Re = 10 000 (and unpublished results for Re = 1 000) show that the flow around NACA0012 airfoil TE is unsteady. This flow is similar to Ghia's results for Re = 1 000 and, what is essential, reveal that both types of TE flow obtained by Ghia are really united in the unsteady streamline behavior. There is no strong dispersion in the flow at Re = 10 000 (in contrast with Ghia's results).

iii) Results of WU *et al.* for Re = 10 000 give the concidence with results of Zakharenkov for $Cy = 0.49$ (for circulation of velocity around an airfoil equal to -0.21). But WU do not show the dependence of lift upon the velocity circulation. This dependence presents the non-classic law as Zakharenkov has shown in [3].

iv) Results of Choi reveal the Kármán vortex street in the wake. It is obvious that such flow possesses essentially different acoustic properties considered in [5]. The vortex spots discussed in [9, 10] present a more sophisticated acoustic field.

v) GHIA in [2] finds the attractor in the solution; in [9], Zakharenkov investigates soliton-like properties of the flow; these are two different problems.

Such differences of results and goals of many authors make it difficult to compare the accuracy of the obtained results because the main properties of computational codes and meshes are quite different. New investigations are necessary to reveal the reasons for such differences in solutions. One of the problems is the dispersion of vorticity fields in CFD solution.

In GHIA *et al.* [1] the solution for Re= 10 000 and angle of attack $\alpha = 5°$ is presented. Much attention is paid to both specific properties of finite-difference approximation on the C type computational mesh and setting the vorticity boundary condition for this mesh. Nevertheless, the numerical solution (illustrated in [1] by equal vorticity lines) displays essential dispersion of vorticity which is often categorized as computational errors. Such type of dispersion and dissipation of vorticity is characteristic of vortical flows simulated by finite-difference schemes whose (first) differential approximation (FDA) essentially depends on the coefficients at third- and fourth-order derivatives of vorticity (the first two terms of approximation residual), see [11–13]. Nevertheless the main feature of the flow at Re= 10 000 consists in there being fundamental reasons for vorticity dispersion and energy dissipation; these reasons are connected with vortex dipole bifurcation. The vortex dipoles characterize the integral properties of vortex fields in the airfoil boundary flow for this value of Re (see [3, 7]), and are unstable under perturbation induced by trailing edge flow separation. The "dipole bifurcation" notion was introduced by SADOVSKY and TAGANOV in [14] for inviscid flow, but the coincidence of behavior of streamlines illustrated in [14] with streamlines obtained numerically in [3, 7, 15] allows us to adjoin this notion to the viscous flow. The physical phenomena in a viscous flow underline the sustained vortex dipole bifurcation.

The notion of "dipole bifurcation" corresponds to the phenomenon when the intensity of dipole in the flow (in a present case the dipole is related to a vorticity field in a boundary layer, see [9]) varies in time. As SADOVSKY and TAGANOV showed in [14], this variation leads to generation of vortex patterns. The period of vortex pattern generation depends on a number of factors but the process as a whole may be assessed in analogy with flow bifurcation known from the flow instability theory. Essentially, the dispersion properties of numerical solutions presented in this paper are mostly related to perturbations of vorticity field. The comparison with results of papers [12, 13] allows us to assess this perturbation with dispersion terms of F.D.A. for vorticity transport equation. For this reason we can say about the vorticity dispersion defined by specific terms of F.D.A. This selection is a major precondition for formulating more complex rheological laws characterizing this flow. These are the reasons defining the employment of the notions "vorticity dispersion", "dispersion impulse", and "vortex dipole bifurcation".

As we shall see later, the dipole bifurcation process in a viscous flow is associated with dispersion. For this reason, the solution presented in [1] can be regarded as a pioneering result which may be recommended for estimating the possibility to describe laminar-turbulent transition provoked by dispersion. This is a rather novel approach in comparison with the traditional direct simulation of transition due to wave – packet instability. It must be mentioned that GHIA *et al.* [1] consider their results as the laminar unsteady flow only.

The recent numeric results obtained in [16, 17] for Reynolds number from 1 000 to 100 000 show that many new physical phenomena are inherent in viscous incompressible flow around an airfoil. Examples include vortex lock-up of vorticity layers [16]; bursting of vortex layers, [16]; microdipole instability of vortex layer [7, 12], etc. Also, there exists the unique situation (in [16]): vortical boundary layer (BL) at Re = 10 000, does perform nonequilibrium transition to the laminar sublarger in BL at Re = 100 000. The generation and instability of critical layers considered in [18, 19] must also be mentioned as the major processes determining the effectiveness of numeric simulation of layered media. Heat transfer and dependence of viscosity on temperature are the physical factors which can play a crucial role in nonequilibrium transition of vortical layer which forms the boundary layer at Re = 10 000 into the laminar sublayer at Re = 100 000.

When vortex structure of BL transforms from a two-layer pattern (with windward and leeward vorticity layers) to a multi-layer pattern at Re > 100 000, we observe generation of vortex spots discussed in [9, 10] which are intermediate between vortex waves and the vortices themselves [10]. Ryzhov *et al.* showed in [20], that the moving vortex spots perturb the BL momentum thickness (momentum losses), and this perturbation participating in nonlinear process becomes a soliton. Thus a vortex spot reveals wave-specific properties in integral estimation of flow momentum itself (inherent in a vortex spot), see [20]. As a rule, the equations describing solitons and vortex spots contain dispersion and energy dissipation terms.

We have some examples [12, 13] that the dispersion and energy dissipation define the complex phenomena of large vortex destruction and explain the possible physical mechanism of vorticity imbalance in the temperature spots. For this reason it is natural to suppose that these processes (dispersion and energy dissipation) are the decisive factors in the formation, development and destruction of the spots posessing different physical/mechanical characteristics, i.e. temperature, momentum, turbulence spots.

The phenomena mentioned above demand a profound study before we can assess the critical values of dispersion/dissipation coefficients at FDA terms (which can be interpreted as new terms of Navier-Stokes equations employing a more complex rheological law than the Stokes law [13, 21]). Simultaneously the special property of computational model of viscous flow (to simulate the dispersion and energy dissipation properties) gives us the simple tool to study the dipole bifurcation by directly controlled computational simulation.

2. Problem statement and computational algorithm

The two-dimensional flow around an airfoil NACA0012 is studied by computational simulation based on the solution of Navier-Stokes equations written in the terms of stream function Ψ and vorticity Ω. These functions are defined by relations

$$u = \frac{\partial \Psi}{\partial y}, \qquad v = -\frac{\partial \Psi}{\partial x}, \qquad \Omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}, \tag{2.1}$$

where u and v are velocity components in the $\{x, y\}$ coordinates. Then the continuity condition is met throught introduction of a stream function, and relations (2.1) led to the Poisson equation

$$\Delta \Psi = \Omega, \tag{2.2}$$

where Δ is the Laplace operator. The momentum equations are transformed to the vorticity transport equation (i.e., equation of vortex dynamics)

$$\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = \frac{1}{\mathrm{Re}}\Delta\Omega, \tag{2.3}$$

where $\mathrm{Re} = U_\infty c/\nu$ is the Reynolds number, U_∞ the velocity of undisturbed flow, c is the airfoil chord, and ν the coefficient of kinematic viscosity.

The no-slip condition on the airfoil surface s, the conditions of uniform flow (the asymptotics of far – field velocity and vorticity) on the outer boundary s_∞ of the computational domain, the initial conditions of rest (for the fluid and airfoil) complete the problem statement. The pressure is calculated from Gromeka-Lamb form of Navier-Stokes equations where the vorticity and velocity are known from solution of (2.2)–(2.3). The pressure is calculated by integrating Gromeka-Lamb equations along the different paths [22, 23]. Pressure must not be dependent on the integration path.

The computational algorithm is detailed in [8, 9, 23]. The pseudo-spectral direct method is employed for solution of Eq. (2.2); the ADI method is used for (2.3). The two-parameter approximation is used for calculation of boundary vorticity Ω_s, which simultaneously serves as the boundary condition for Eq. (2.3) solved separately. The novel stage of computational algorithm deals with the correction of the vorticity field. This correction, in the first place, recovers the vorticity in the wake where the vorticity dissipation takes place on the coarse mesh, and, in the second place, introduces the pressure compatibility condition explicitly into the calculation of vorticity on the airfoil surface, and further into the boundary condition for Eq. (2.3).

A complete description of the method is given in [23] and the correction stage is derived in [24].

3. Results of computation

The "O-type" ortogonal curvilinear computational mesh is used with 128 mesh nodes in the η direction (along an airfoil surface), and with 80 mesh nodes in the ξ direction (from an airfoil surface to the outer boundary of computational region which is nearly the circle of radius equal to ten airfoil chord lengths). The additional compression in the ξ direction is employed [3, 23]. The minimum mesh sizes at the TE are $h_n = 3.529 \times 10^{-5}$, $h_\tau = 9.009 \times 10^{-4}$, and at the LE are $h_n = 71.343 \times 10^{-4}$, $h_\tau = 3.955 \times 10^{-3}$, where n and τ are the indices of normal and tangential directions.

The relative characteristics discussed below are obtained as the difference of these characteristics at the time instant t from those at the time $t + \Delta t$ where $\Delta t = 0.1$. Dependence of the discussed phenomenon (at least, its characteristics shown in the fields of relative velocity components and vorticity) on Δt may be rather high. This is the same problem as a dependence of turbulent flow characteristics on the interval taken for averaging. Nevertheless, the previous study of viscous flow around a NACA0012 airfoil ([3, 7]) shows that the integral characteristics (the lift and drag coefficients) oscillate with a period equal to $0.1 - 0.2$. These oscillations are related to generation of vortex spots near TE ([10]) (the momentum spots). These results had been obtained at different time steps in computational algorithms. The present results were computed with $dt = 0.00125 - 0.0025$. The time increment $\Delta t = 0.1$ used for relative fields shown in Figs. 2 through 7 is chosen because this is a basic period of TE spot generation (at least for $\mathrm{Re} = 10000$).

In Figs. 1a through 1f the flow around airfoil NACA0012 at $\mathrm{Re} = 1000$ and angle of attack of $5°$ is represented by the lines of equal values of relative V_ξ velocity component drawn for successive time moments with the interval $\Delta t = 0.1$. Disturbances in Fig. 1 can be characterized as the "dispersion impulse", which gives rise to the complex wave disturbances. The lines of equal values of relative V_η velocity components are in Fig. 1g, and in Fig. 1h the relative vorticity is presented. Disturbances of V_ξ and V_η are small, but the disturbances of vorticity are great and have the maximum on the airfoil surface where the velocity is zero. The outer boundary of the vortex layer is the second region of possible strong vorticity perturbation, see [17]. The vortex structures in the wake flow in Fig. 1h have the opposite signs.

At $\mathrm{Re} = 10\,000$ the disturbances in the boundary layer most clearly look like mainly the vortex disturbances. Figure 2 depicts the lines of equal values of the relative V_ξ velocity component; Fig. 3, the same ones for relative V_η velocity component; and Fig. 4, lines for relative vorticity. Disturbances of V_ξ and V_η are small again: in Figs. 2 and 3, zeroes of these relative quantities take place within the boundary layer, whereas in the wake, only the lines with values $k \times 0.0125$, $(k = 1, 2, ..., 5)$ of these quantities are revealed. We see that the vorticity disturbances look like the usual λ-structure of the boundary layer, see also [7, 9], and have the maximum on the airfoil surface. We can assert that the greatest disturbances are spread over the airfoil surface and form a vortex wave (note that the velocity at the surface is zero). Hence, it becomes clearly that the strong interaction of a surface vortex wave with the physical fields (such as pressure and temperature) around the flow separation point and in the vicinity of the trailing edge is the decisive phenomenon in the flow around an airfoil. In principle, we can say about analogy with streaming/shedding waves (see [25]) which here assume the form of shedding waves of vorticity (the surface tension waves). Again, the vortex structures in the wake flow in Fig. 4 have opposite signs.

The primary development of vorticity waves is clearly shown in the case of the flow at $\mathrm{Re} = 30000$ also. Figure 5 demonstrates the lines of equal values of relative V_ξ components; Fig. 6, relative V_η components; and Fig. 7, relative vorti-

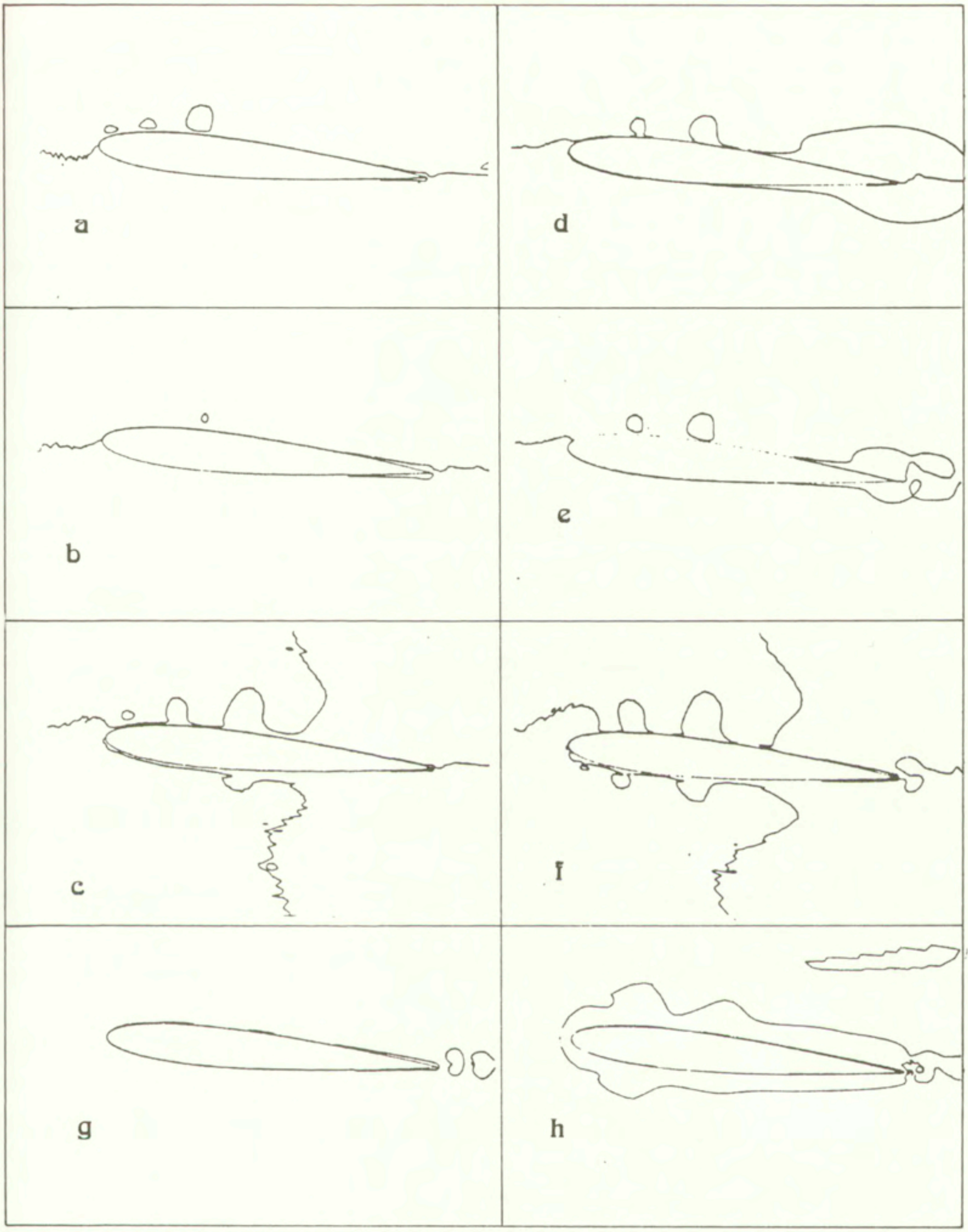

FIG. 1. The lines of equal values ($\overline{V}_\xi = (k-1) \times 0.0125 - 0.1375$, $k = 1, 2, \ldots 5$) of relative $\overline{V}_\xi$ component of velocity (a–f), relative $\overline{V}_\eta$ component of velocity (g) around NACA0012 airfoil at Re = 1000 and angle of attack of 5°. The lines of equal values of relative vorticity (h) for the values $\Omega_i = (i-1) \cdot 0.25$, $i = 1, \ldots, 11$; $\Omega_k = (k-1) \times 20 - 50$, $k = 1, 2, 3$; $\Omega_m = (m-1) \times 0.25$, $m = 1, \ldots, 11$; $\Omega_n = 20 \times n$, $n = 1, 2, 3$. Note that (g) and (h) correspond to the time instants of (f).

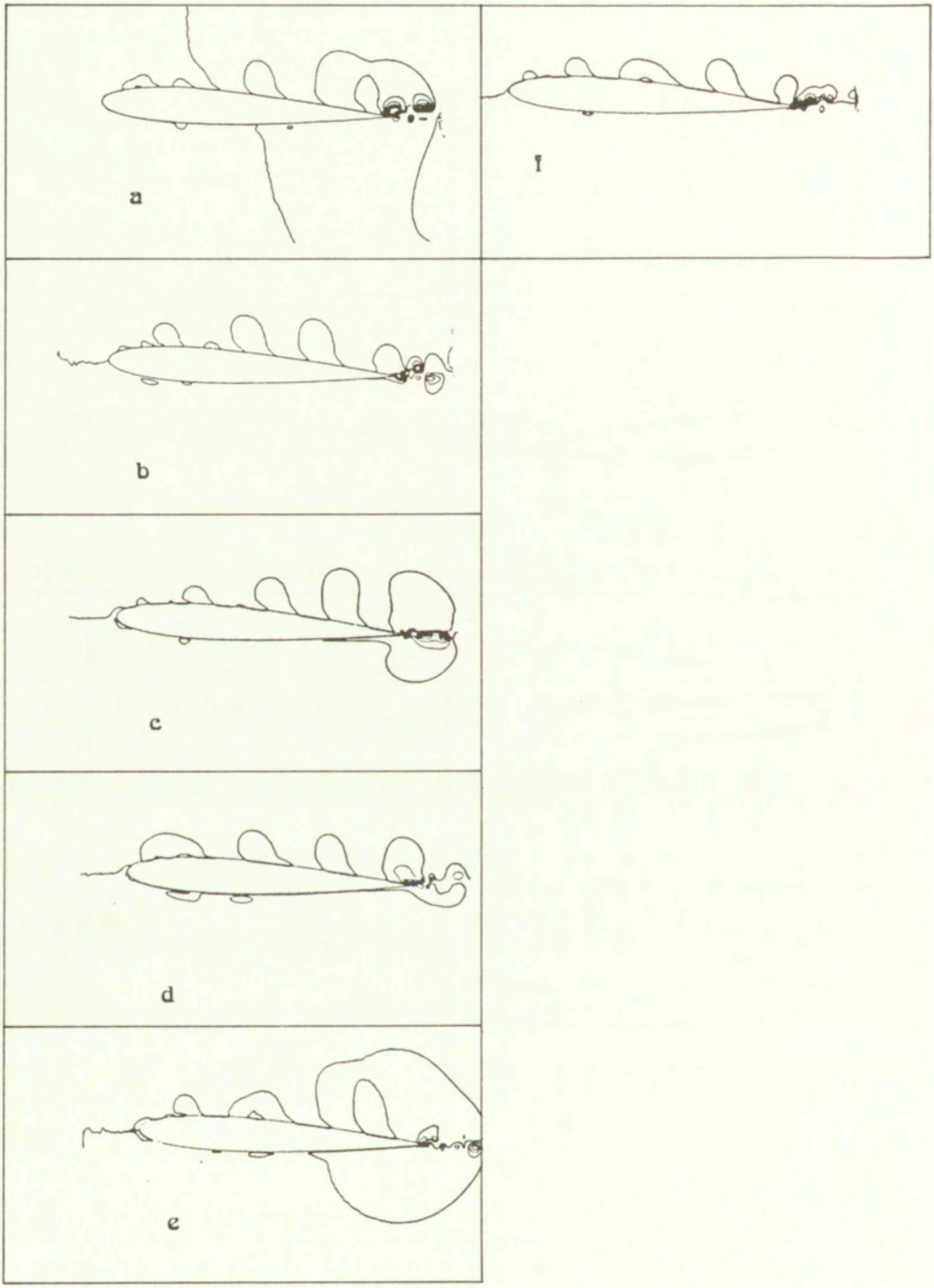

FIG. 2. The lines of equal values (which correspond to Fig. 1) of relative $\overline{V}_\xi$ component of velocity (a–f) around NACA0012 airfoil at Re = 10 000 and angle of attack of 5°.

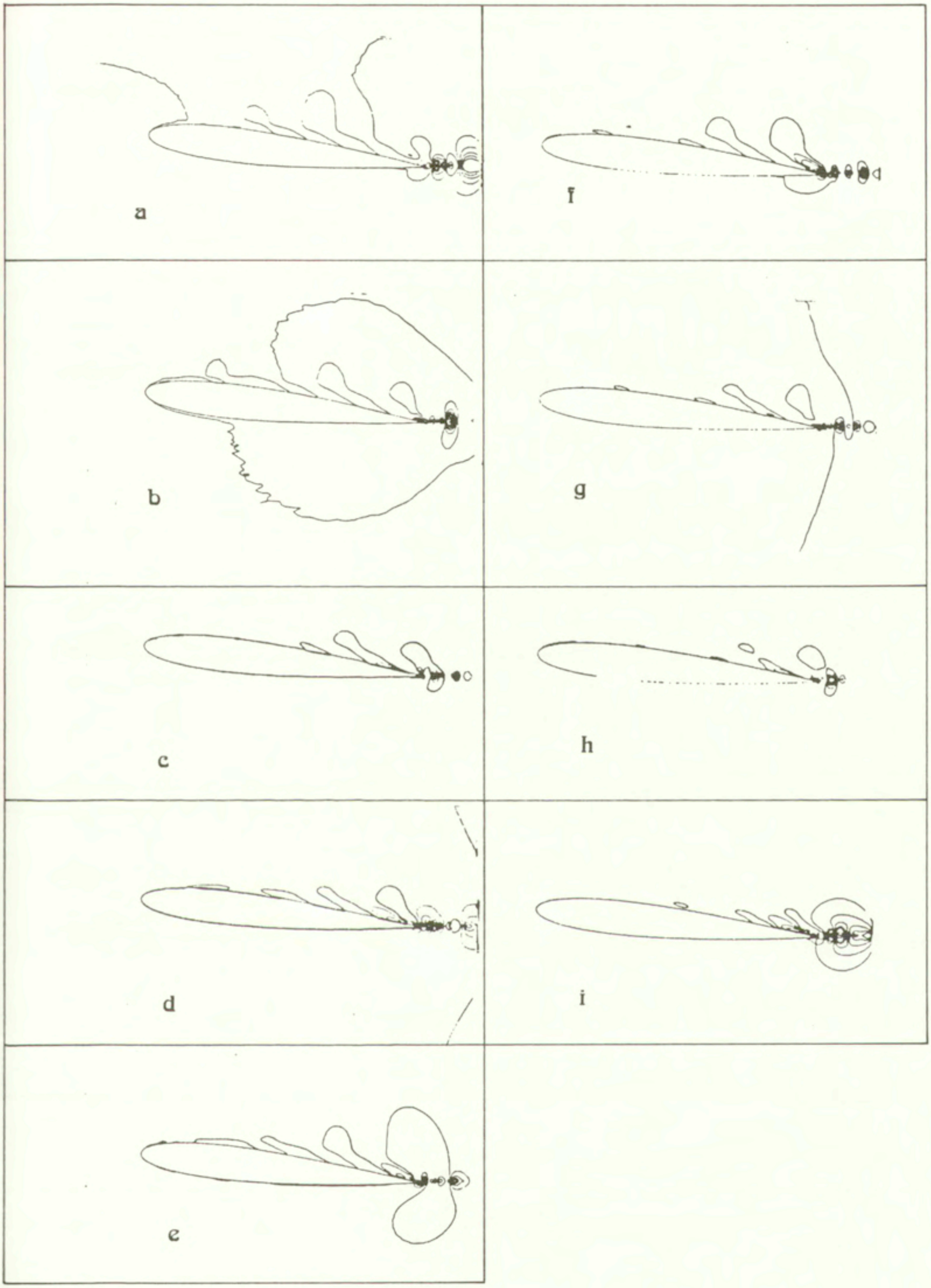

FIG. 3. The lines of equal values (which correspond to Fig. 1) of relative $\overline{V}_\eta$ component of velocity (a–i) around NACA0012 airfoil at Re = 10 000 and angle of attack of 5°.

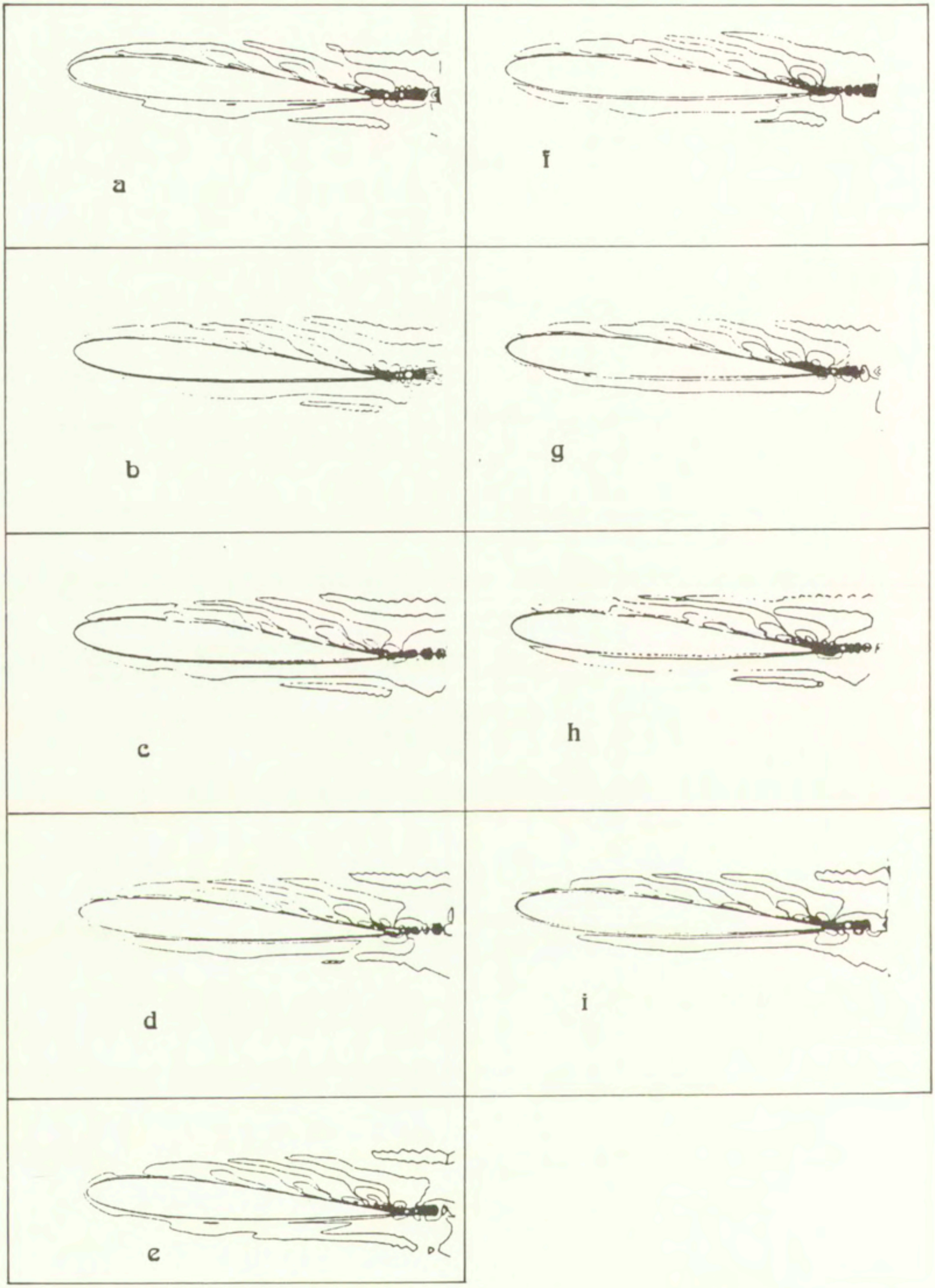

FIG. 4. The lines of equal values (which correspond to Fig. 1) of relative vorticity (a–i) around NACA0012 airfoil at Re = 10 000 and angle of attack equal to 5°.

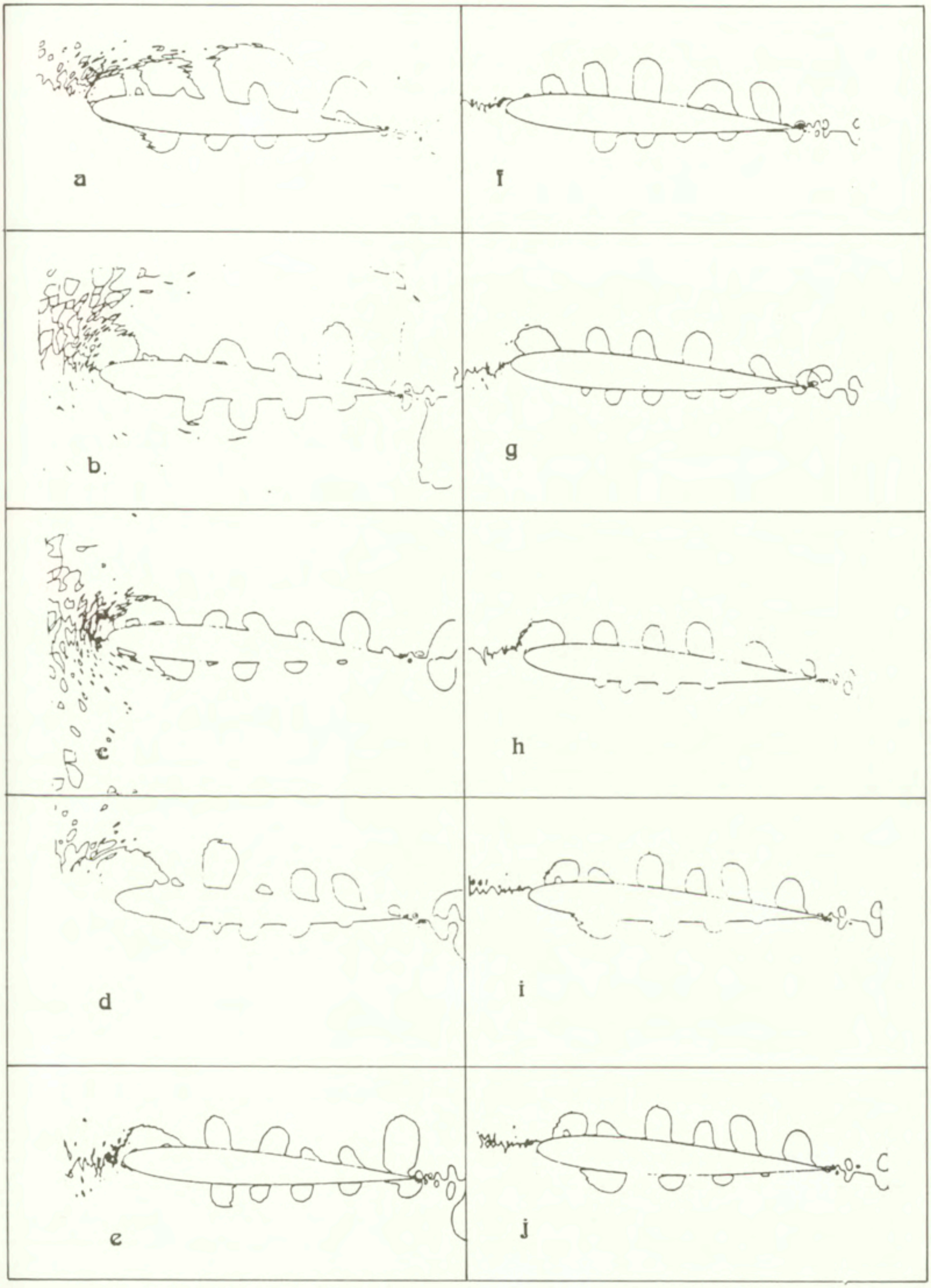

FIG. 5. The lines of equal values (which correspond to Fig. 1) of relative $\overline{V}_\xi$ component of velocity (a–k) around NACA0012 airfoil at Re 30 000 and angle of attack equal to 5°.

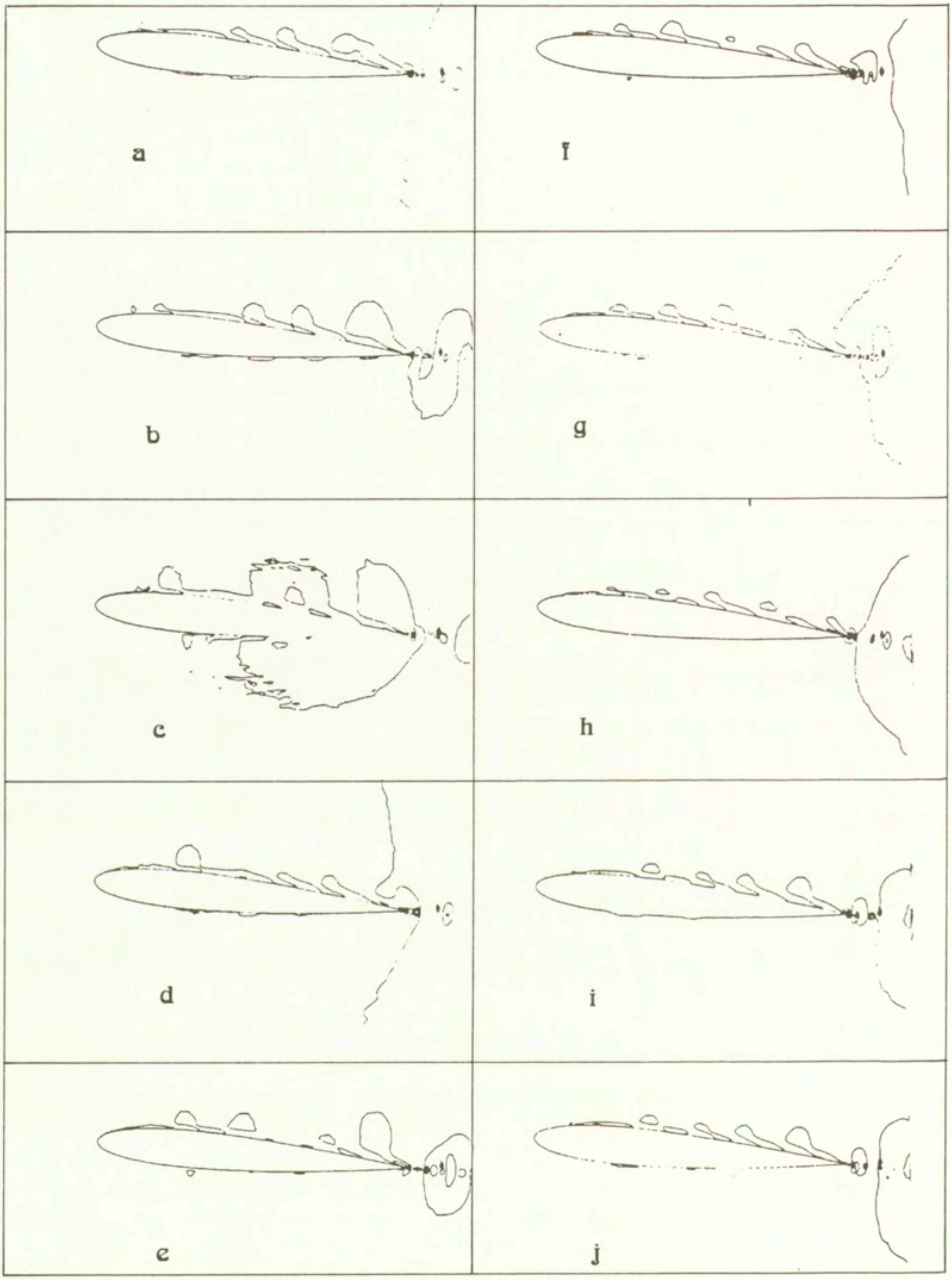

FIG. 6. The lines of equal values (which correspond to Fig. 1) of relative $\overline{V}_\eta$ component of velocity (a–k) around NACA0012 airfoil at Re= 30 000 and angle of attack equal to 5°.

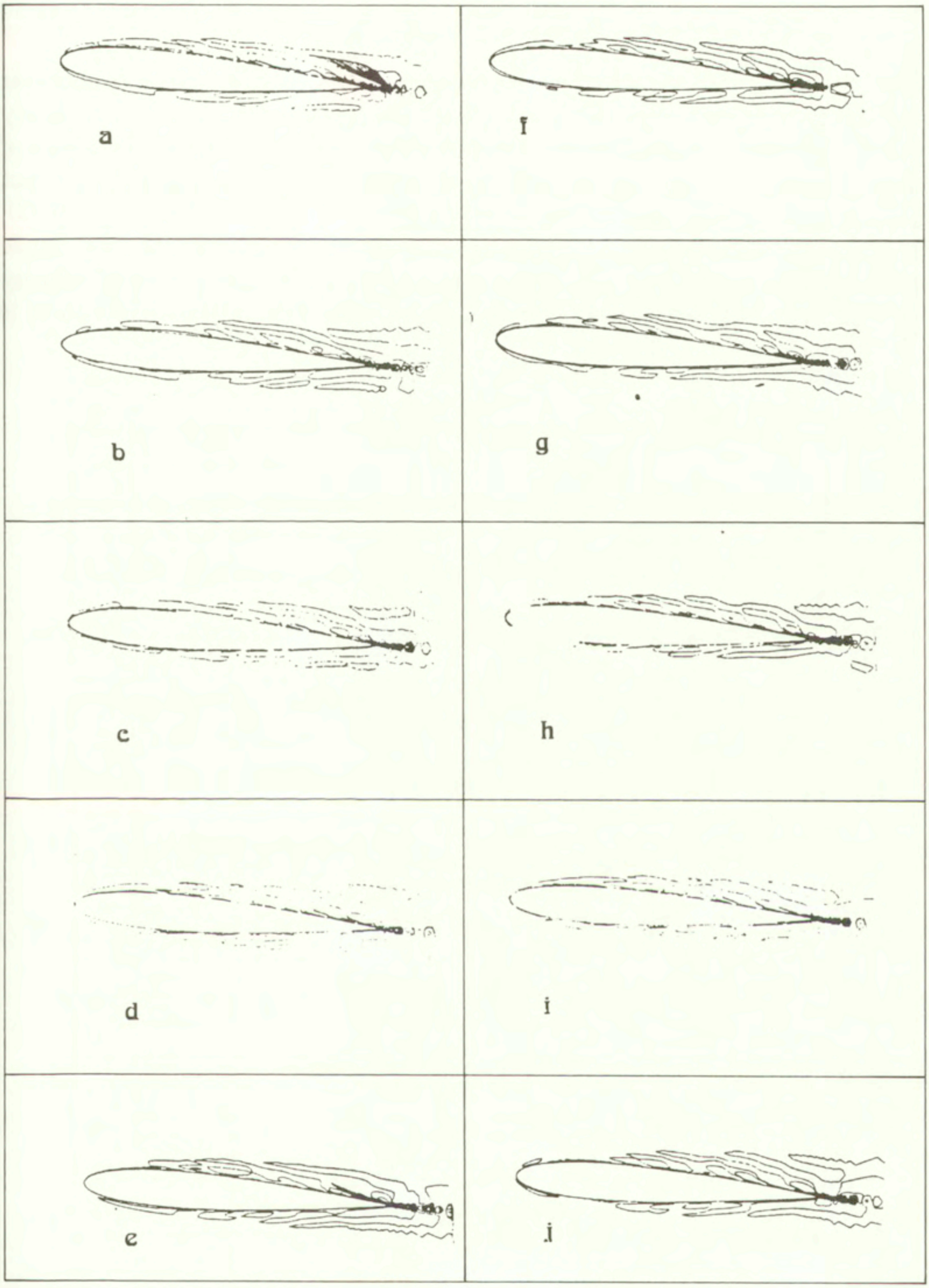

FIG. 7. The lines of equal values (which correspond to Fig. 1) of relative vorticity (a–k) around NACA0012 airfoil at Re = 30 000 and angle of attack of 5°.

city (the vortex structures in the wake have opposite signs). We can note also that the intensity of surface vorticity waves (i.e., the surface tension waves) is closely related to a corresponding pressure disturbance. Simultaneously, the existence of strong vortex-pressure transition in the vicinity of TE is confirmed by the fact that the pressure at the trailing edge in the Fig. 8 obtained for Re = 10 000, $\alpha = 5°$, $\Gamma = -0.21$ and $Dy = 4$ (where Dy is the intensity of the first dipole term with the axis directed along the vertical axis Oy in asymptotics of velocity at a far boundary, see [3], and Γ is intensity of circulation in the asymptotics) is

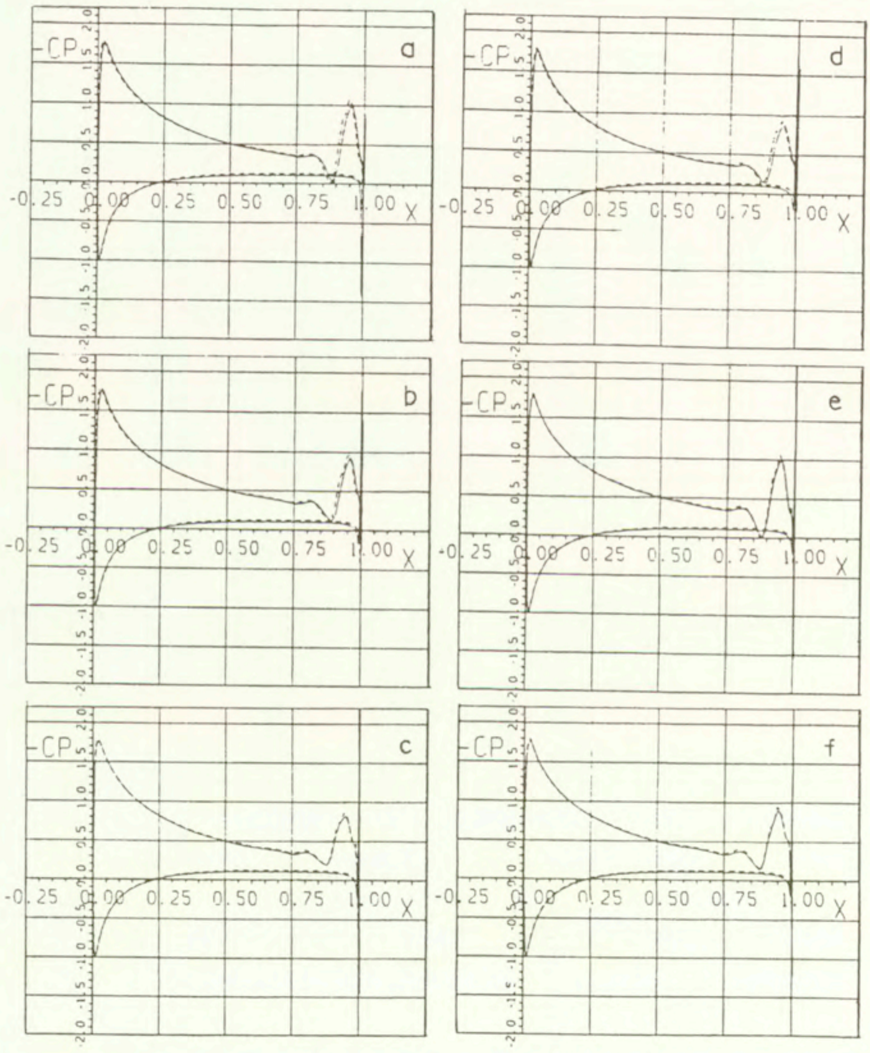

FIG. 8. The pressure coefficient distribution around NACA0012 airfoil at Re = 10000 and angle of attack equal to 5° (a–f) correspond to successive time instants with interval $\Delta t = 0.1$.

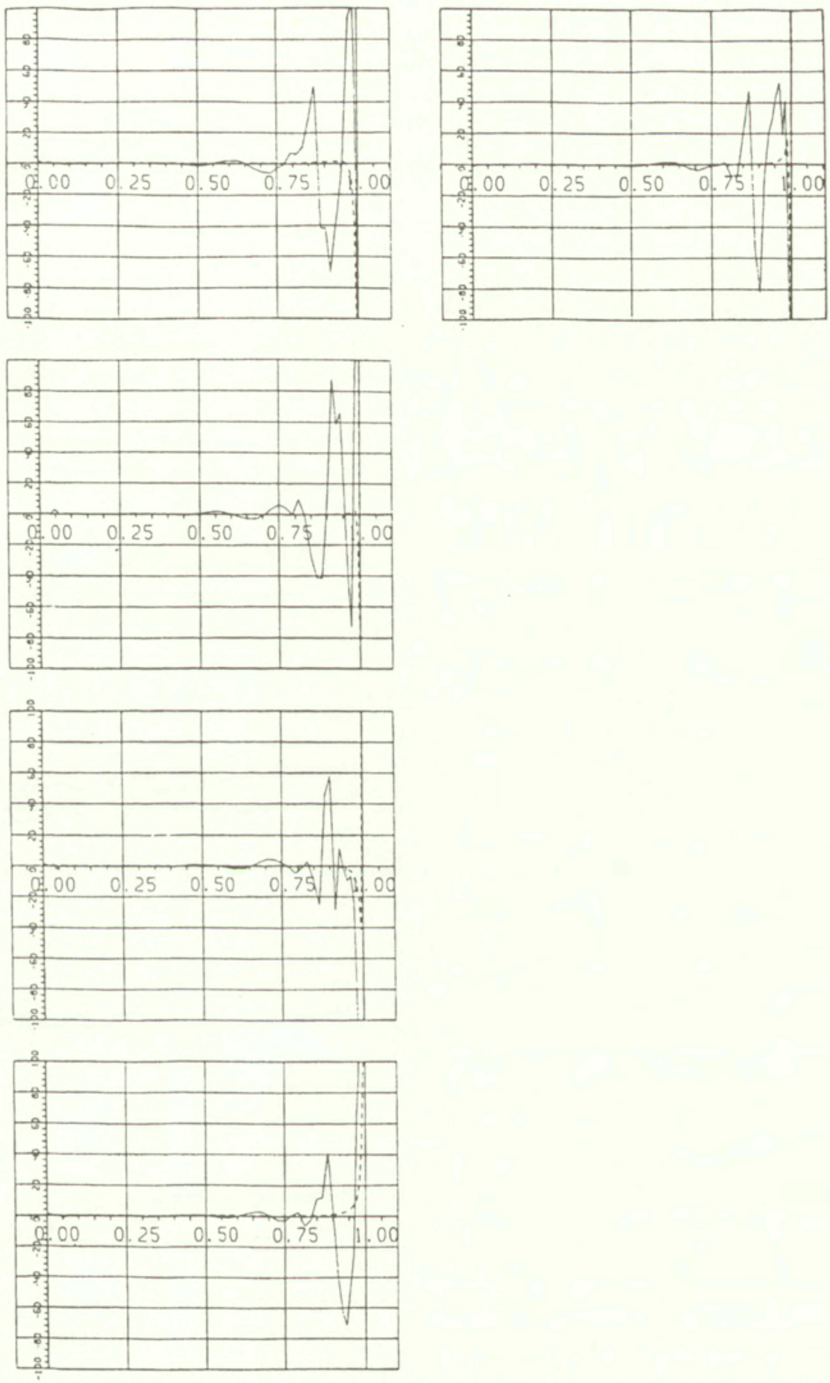

FIG. 9. The surface relative vorticity distribution along the airfoil NACA0012 at Re = 10 000 and angle of attack equal to 5°. The solid line corresponds to the upper part and dotted line to the lower part of an airfoil surface. Time increment equals 0.1.

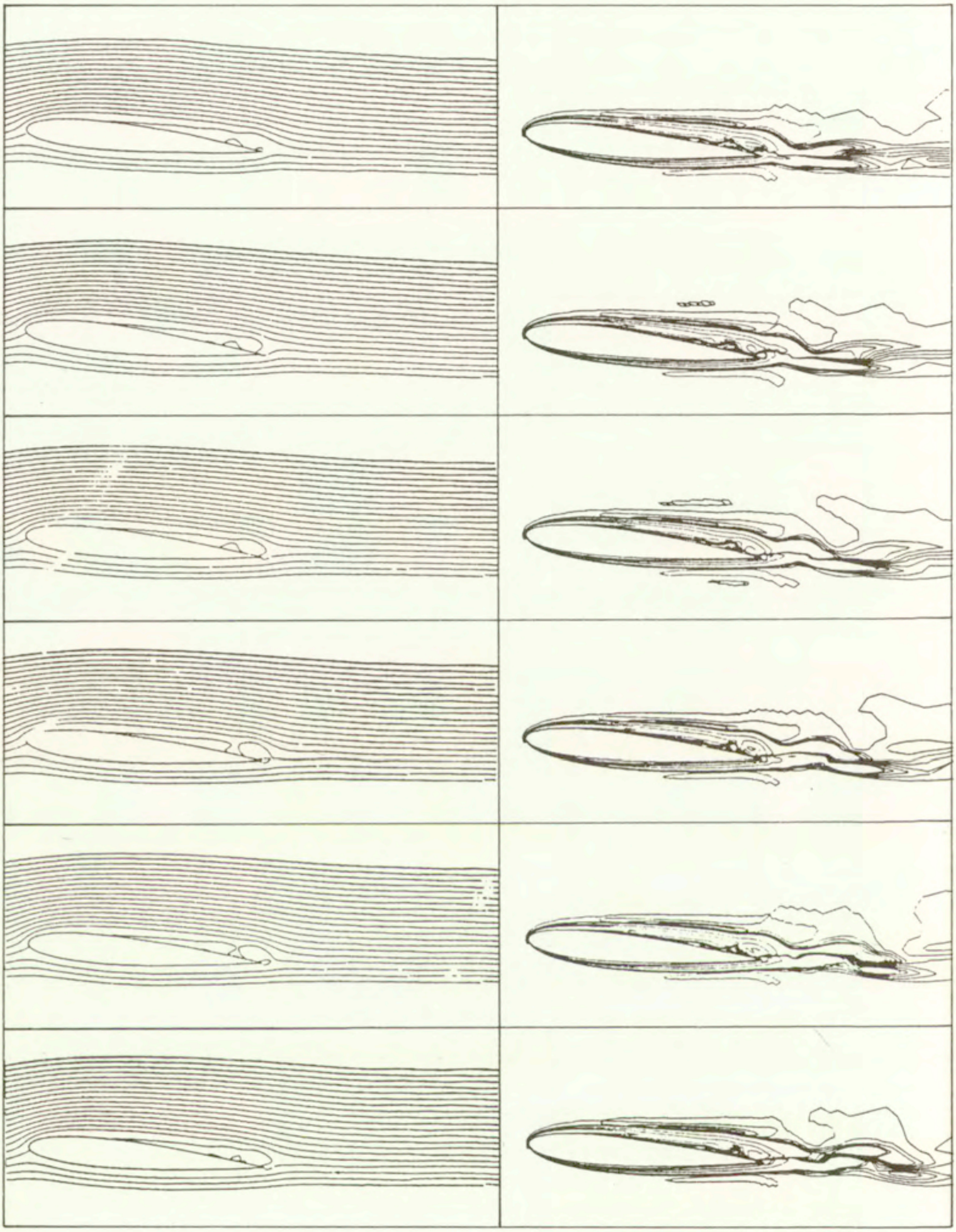

FIG. 10. Streamlines (a–f) and equal vorticity lines (g–m) around NACA0012 airfoil at Re $= 10\,000$ and angle of attack equal to $5°$. (a–f), (g–m) correspond to succesive time instants with interval $\Delta t = 0.1$ (they coincide with the time instants of Fig. 8). The values of ψ are $\psi_i = (i-1) \times 0.025 - 0.075$, $i = 1, \ldots, 22$; the values of vorticity: $\Omega_k = (k-1) \times 0.25 - 1.25$, $k = 1, \ldots, 5$; $\Omega_l = (l-1) \times 20 - 50$, $l = 1, 2, 3$; $\Omega_m = (m-1) \times 0.25$, $m = 1, \ldots, 5$; $\Omega_n = (n-1) \times 20 + 10$, $n = 1, 2, 3$.

greater than pressure at the stagnation point (the solid lines corresponds to pressure obtained by integration of Gromeka – Lamb equation from s_∞ to s, and dotted line corresponds to the pressure obtained by integration over the airfoil surface). This is the case of pressure impulse formation. The coincidence of pressure over an airfoil obtained by integration along two different paths confirms that the solution is accurate. The surface (relative) vorticity waves are presented in Fig. 9. Streamlines for this flow are depicted in Fig. 10. The results in Figs. 8 – 10 confirm that unsteady TE flow described in [7, 9, 10] is not related to pressure imbalance which had been observed in these earlier calculations.

Streamlines and equal vorticity lines for the viscous flow around NACA0012 airfoil at Re = 10 000 are presented in [3, 7–9, 12, 16, 17] and at Re = 100 000 in [16].

4. Discussion

We must underline that the problem of vortex dipole stability is a very complex problem of mathematical physics. Simultaneously, some phenomena which can only be studied by direct numerical simulation take the lead in this process. Flow separation on the airfoil and in vicinity of the trailing edge impose disturbing action on the vortex dipoles (at TE and LE). As the result, the surface vorticity waves running over the airfoil surface from LE to TE are derived, see Fig. 9. These waves are intensified at the flow separation point and, essentially, in the vicinity of TE. Two vortex waves run into each other at TE and the complex vortex-acoustic-heat interaction characterizes this collision. Accumulation of disturbances provokes generation of vortex spots [9, 10]. The shedding of vortex spots from TE imposed again disturbance of the vortex dipole at LE. This process has the known theoretical basis described below.

Generation of vortex spot in vicinity of TE reveals generation of vorticity of a certain sign in the boundary layer. An equal portion of vorticity of opposite sign must be generated in the flow due to the total vorticity conservation law. The airfoil surface (especially around LE) is the generator of vorticity (note that vorticity can also be generated in critical layers [23]). We repeat here that the vortex spot is intermediate between a vortex and a vortex wave. The corresponding vortex spot (with vorticity of opposite sign to the vortex spot at TE) near LE is difficult to appear as a consequence of the LE spot instability. For this reason, vortex spot formation at TE provokes the complex perturbations at LE. Some of these perturbations are realized as surface vorticity waves, and other ones – as a pressure perturbation and upstream perturbation of potential flow which thereafter run onto the airfoil. We cannot exclude the appearance of local anisotropic flow at LE because the transition of essential vorticity perturbation into disturbances of upstream potential flow (or generation of vortex spots in upcoming flow) is a quite complex phenomenon. This is the reason for generation of dispersion impulse in the flow. Thus, the perturbations of dipole reveal self-induction, and we have the analogy with self-induced flow separation ([26]).

The difficulties in theoretical description of dipole bifurcation consist in a special initial phase of this phenomenon which is characterized by emission of disturbances in a microwave part of the wave spectrum (this is revealed by perturbation of line $\{V_\xi = 0\}$ running on to the airfoil LE, see Figs. 1 trough 3). But the final stage of dipole bifurcation takes place in the short- and long-wave disturbances in the boundary layer. The transition from microwave disturbances to the short-wave ones is accompanied by dispersion. This is the reason why we choose such characteristic as "dispersion impulse" for the wave process in the Fig. 1. But the phenomenon as a whole, of course, is well defined by the notion of "dipole bifurcation". Note again that the last conclusion is based on a qualitative concidence of zero streamlines behavior in the vicinity of TE (this kind of behavior is detailed in [7, 9]) with the one presented in [14] theoretically describing dipole bifurcation. We must emphasize that there is a theoretical problem on the smallest scale for incompressible vicous flow ([27]). This problem is inherent in dipole bifurcation phenomenon.

The complexity of numerical simulation of this phenomenon by finite-difference methods is connected with the fact that the minimum length of the waves which are correctly described by computational models of continuum media is equal to the mesh step. The dipole bifurcation process features microwaves whose wavelengths are smaller than the mesh step. This may be the main reason of dispersion impulse in solution of N-S equations by finite-difference algorithms. But we do not have to reject the possibility of there being real physical foundations for the phenomenon. Undoubtedly, any complication of the rheology law at initial turbulence leads to inclusion of dispersion and energy dissipation terms in N-S equations ([21, 28]) which thereafter will describe the vortex erosion effects, bursting, dispersion impulse, soliton-like structures, see [12, 13]. This idea is close to consideration of some oscillating solutions (wiggles) proposed by Gresho *et al.* [29].

We must formulate conclusion for the numerical solution and can suppose for theoretical one that the vortex/vorticity disturbances and the surface vorticity waves are the main physical factor absorbing the dispersion impusle perturbation. In such a way, the dipole bifurcation near LE can give rise to complex perturbation which we can concider as a novel type of vortex-wave perturbations. The latter are different from the Tollmien–Schlichting waves, Rayleigh waves. But after interaction with LE, this vortex-wave perturbation transforms into the Tollmien–Schlichting waves, Rayleigh waves and vortex waves. Thereby, all types of disturbances are generated and each of these is realized in the region containing the most suitable conditions for this kind of waves. Note that the disturbances around LE facilitates the development of critical layers.

The results obtained by the conventional computational scheme described in [23] with these obtained by predictor-corrector scheme [24] were compared to show that the pressure impulse at TE and the dispersion impulse at LE are closely related. Fulfillment of the pressure compatibility condition in computations reduces the dispersion impulse at LE and increases the pressure impulse at the TE. But the dispersion impulse at LE is conserved and we can expect that its significance increases with taking into account the heat and compressibility effects.

If we consider such complex phenomenon as a strong dipole hysteresis [30] then we find the direct analogy with the dipole bifurcation perturbations which in this case are provoked by external dipole. The great pressure gradient is derived at the TE in the case of a strong dipole hysteresis. Then, vorticity waves on the upper surface cannot leave TE, and the corresponding pressure impulse increases the region of flow separation at initial stage of hysteresis. Thereafter the vortex disturbances in the separation region are growing and the spread separation develops as a consequence of vorticity accumulation.

Finally we must note that, in order to take into account the dependence of minimum wavelength on the mesh steps, it is necessary to investigate the frequency of dipole bifurcation (for example, at $\mathrm{Re} = 10\,000$ from the computational model parameters. The more comprehensive research requires parametric approximation of nonlinear terms in Eq. (2.3) as proposed (and implemented) in [12, 13] for the problem is under study.

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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Dimensional analysis and asymptotic expansions of the equilibrium equations in nonlinear elasticity Part I : The membrane model

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IN THIS PAPER, we develop a new asymptotic constructive approach in nonlinear plate theory. The dimensional analysis of the three-dimensional equilibrium equations naturally leads to dimensionless numbers which reflect the geometry of the structure and the magnitude of forces. These numbers also define the domain of validity of the two-dimensional models which will later be obtained by asymptotic expansions. For nonlinear plates, we prove that the two-dimensional models we obtain by asymptotic expansions are determined by the magnitude of the forces applied. In this first part, we consider a plate subjected to large loads. In this case, we prove that the nonlinear plate model we obtain by asymptotic expansions is a membrane model. In the second part of this article, we will consider a plate subjected to smaller applied forces.

1. Introduction

THE FIRST RIGOROUS JUSTIFICATION of the nonlinear plate model has been obtained at the end of the 70's by P.G. CIARLET and P. DESTUYNDER [1, 2]. In these works, the asymptotic expansion method is applied to a mixed variational formulation (in terms of stresses and displacements) of the three-dimensional elasticity problem. Afterwards, a variational approach was formulated only in terms of displacements [3, 16]. These works have also been extended to some linear [4 – 7], [8, 12], [17 – 20] and nonlinear [11, 13] shell models.

Nevertheless, to our knowledge, two problems still exist concerning the justification of the nonlinear plate model by asymptotic expansions:

1 – The choice of *a priori* scalings on the components of the displacements. These scalings determine the order of magnitude of the ratio between the normal and the tangential displacement.

2 – The difficult physical interpretation of the scalings on the applied loads. Indeed with these scalings, the applied body forces density depends on the thickness. However in usual elasticity problems, the plate is often subjected only to the gravity body force (ρg) which is independent of the thickness.

We propose to solve the two previous problems by extending to the nonlinear case, the new asymptotic approach we have developed in the linear case [14]. This new constructive approach, directly derived from the asymptotic expansion method, successfully used in fluid mechanics, can be considered as a continuation of A.L. GOLDENVEIZER'S works [10]. It is based on the dimensional analysis of the three-dimensional equilibrium equations and needs no *a priori* assumption on the ratio between the normal and the tangential displacements. The dimensional analysis of the three-dimensional equilibrium equations naturally leads to dimensionless numbers which reflect the geometry of the structure and the forces level. These numbers also define the domain of validity of the two-dimensional models which will later be obtained by asymptotic expansions.

In this paper, we prove that the reference scales of the normal and the tangential displacement and the corresponding two-dimensional model we obtain are determined by the forces level. Indeed, in order to allow for large displacements, the reference scales of the normal displacement u_{3r} and of the tangentiel displacement V_r are first assumed to be equal to L_0, diameter of the middle surface of the plate whose thickness is $2h_0$. For large applied forces, the asymptotic expansion leads to a membrane model which differs from the von Kármán model. Then going back to the variational formulation, we prove that this membrane model and the one obtained by D. FOX *et al.* [5] from a three-dimensional variational formulation are identical.

In the second part of this article we will consider a plate subjected to moderate forces. Then we prove that these moderate forces lead to new reference scales ($u_{3r} = h_0$ and $V_r = \varepsilon h_0$) for the normal and the tangential displacement. The new reference scales we obtain as a consequence of the forces magnitude are formally equivalent to the scaling assumptions generally made in the literature and naturally lead to the two-dimensional nonlinear von Kármán model.

2. The three-dimensional problem

In what follows, Greek indices take their values in $\{1,2\}$ and Latin indices take their values in $\{1,2,3\}$. We assume that an origin O and an orthonormal basis (e_1, e_2, e_3) have been chosen in the three-dimensional Euclidian space which will later be identified with $\mathbb{R}^3$. We denote by a superscribed asterisk $(^*)$ all the dimensional variables. On the other hand, within the framework of large displacements, the reference and the current configuration cannot be confused. Thus the reference configuration variables will be marked by $(_0)$.

Let ω_0^* be an open bounded connected set of $\mathbb{R}^2$ in the plane spanned by the vectors (e_α) with a "smooth enough" boundary γ_0^*. Let L_0 be the diameter of γ_0^*. Let us consider now a thin plate of thickness $2h_0$, whose middle surface is ω_0^*. The plate itself occupies the open set $\Omega_0^* = \omega_0^* \times]-h_0, h_0[$ of $\mathbb{R}^3$ in its reference configuration.

We denote by $X^* = (x^*, x_3^*)$ the generic point of Ω_0^* where $x^* \in \omega_0^*$. Let $\Gamma_{0\pm}^* = \overline{\omega}_0^* \times \{\pm h_0\}$ be the upper and the lower faces of the plate. In what follows, we consider only thin plates ($h_0 \ll L_0$) subjected to dead loads (independent of the configuration). Finally, in this paper we will use the following notations:

$\partial/\partial X^*$ and Div^* denote the gradient and the divergence in the three-dimensional space,

$\partial/\partial x^*$ and div^* denote the two-dimensional gradient and the two-dimensional divergence.

If U and V are two vectors of $\mathbb{R}^3$, we denote by $U\overline{V}$ the tensorial product of U and V (the overbar denotes the transposition operator).

We assume that the plate, subjected to the applied body forces $f^* = f_t^* + f_3^* e_3 : \overline{\Omega_0^*} \to \mathbb{R}^3$ and surface forces $g^{*\pm} = g_t^{*\pm} + g_3^{*\pm} e_3 : \Gamma_{0\pm}^* \to \mathbb{R}^3$, occupies the set Ω^* in its deformed configuration.

The unknown of the problem is then the displacement $U^* : \overline{\Omega_0^*} \to \mathbb{R}^3$ such that if $X^* \in \overline{\Omega_0^*}$ denotes the initial position of a material point, its position in the deformed configuration is $X^* + U^*(X^*)$. Moreover, as the plate is assumed to be clamped on its lateral surface $\Gamma_0^* = \gamma_0^* \times [-h_0, h_0]$, we have $U^* = 0 \quad \text{on } \Gamma_0^*$.

Within the framework of nonlinear elasticity, the displacement $U^* : \overline{\Omega_0^*} \longrightarrow \mathbb{R}^3$ and the second Piola-Kirchhoff tensor Σ^* solve the equilibrium equations

$$\begin{aligned} \mathrm{Div}^*(\Sigma^* \overline{F^*}) &= -f^* \quad \text{in } \Omega_0^*, \\ U^* &= 0 \qquad \text{on } \Gamma_0^*, \\ (F^*\Sigma^*).n^{\pm} &= g^{*\pm} \quad \text{on } \Gamma_{0\pm}^*, \end{aligned} \tag{2.1}$$

where

$$F^* = \frac{\partial \psi^*(X^*)}{\partial X^*} = I + \frac{\partial U^*}{\partial X^*}$$

denotes the linear map tangent to the mapping function $X^* \to \psi^*(X^*) = X^* + U^*(X^*)$, and $n^{\pm}$ is the external unit normal to the upper and the lower faces $\Gamma_{0\pm}^*$.

These equilibrium equations can be completed with the mass conservation law $\rho^* \det F^* = \rho_0^*$ where ρ_0^* and ρ^* denote, respectively, the voluminal mass of the material in the reference and the deformed configuration. In what follows, we assume ρ^* to be bounded, which can be written as:

$$\det F^* = \det\left(\frac{\partial \psi^*}{\partial X^*}\right) \geqslant a > 0 \qquad \text{in } \Omega_0^*, \tag{2.2}$$

where $a > 0$ is a constant. This condition will be used later.

Limiting our study to Saint-Venant-Kirchhoff materials, the constitutive relation takes the following form:

$$\begin{aligned} \Sigma^* &= \lambda \,\mathrm{Tr} E^* I + 2\mu E^*, \\ E^* &= \frac{1}{2}(\overline{F^*}F^* - I) = \frac{1}{2}\left(\overline{\frac{\partial U^*}{\partial X^*}} + \frac{\partial U^*}{\partial X^*}\right) + \frac{1}{2}\overline{\frac{\partial U^*}{\partial X^*}}\frac{\partial U^*}{\partial X^*} = e^* + \gamma^*, \end{aligned} \tag{2.3}$$

where e^* and γ^* denote respectively the linear and nonlinear part of the Green-Lagrange strain tensor E^* and I the identity of $\mathbb{R}^3$.

REMARK 1

Constitutive relation of Saint-Venant-Kirchhoff is obtained by linearizing more general Lagrangian constitutive relations with respect to E^*. Therefore our model is limited to small deformations even if large displacements are allowed. ■

Let us decompose the equilibrium equations so as to separate the linear from the nonlinear part. Writing E^* as $E^* = e^* + \gamma^*$, we get

$$\begin{aligned}\mathrm{Div}^*(\Sigma^*\overline{F}^*) &= \mathrm{Div}^*\Sigma^* + \mathrm{Div}^*(\Sigma^*\overline{\frac{\partial U^*}{\partial X^*}}) \\ &= \mathrm{Div}^*(\lambda \mathrm{Tr} e^* I + 2\mu e^*) + \mathrm{Div}^*\Gamma^* + \mathrm{Div}^* E_u^*\end{aligned}$$

which leads to

$$\mathrm{Div}^*(\Sigma^*\overline{F}^*) = (\lambda+\mu)\mathrm{Grad}^*(\ \mathrm{Div}^* U^*) + \mu\Delta_3^* U^* + \mathrm{Div}^*\Gamma^* + \mathrm{Div}^* E_u^*$$

where:

$$\begin{aligned}\Gamma^* &= \lambda \mathrm{Tr}\gamma^* I + 2\mu\gamma^*, \\ E_u^* &= \lambda \mathrm{Tr} E^* \overline{\frac{\partial U^*}{\partial X^*}} + 2\mu E^* \overline{\frac{\partial U^*}{\partial X^*}}.\end{aligned}$$

Now decomposing U^* into a tangential and a normal component :

$$U^* = V^* + u_3^* e_3 \ ,$$

we get:

$$\frac{\partial U^*}{\partial X^*} = \begin{pmatrix} \dfrac{\partial V^*}{\partial x^*} & \dfrac{\partial V^*}{\partial x_3^*} \\ \\ \overline{\mathrm{grad}\ u_3^*} & \dfrac{\partial u_3^*}{\partial x_3^*} \end{pmatrix} \qquad \text{and} \qquad \overline{\frac{\partial U^*}{\partial X^*}} = \begin{pmatrix} \overline{\dfrac{\partial V^*}{\partial x^*}} & \mathrm{grad}\ u_3^* \\ \\ \overline{\dfrac{\partial V^*}{\partial x_3^*}} & \dfrac{\partial u_3^*}{\partial x_3^*} \end{pmatrix}.$$

Thus we have

$$\gamma^* = \frac{1}{2}\overline{\frac{\partial U^*}{\partial x^*}}\frac{\partial U^*}{\partial x^*}$$

$$= \frac{1}{2}\begin{pmatrix} \overline{\dfrac{\partial V^*}{\partial x^*}}\dfrac{\partial V^*}{\partial x^*} + \mathrm{grad}\ u_3^*\ \overline{\mathrm{grad}\ u_3^*} & \overline{\dfrac{\partial V^*}{\partial x^*}}\dfrac{\partial V^*}{\partial x_3^*} + \dfrac{\partial u_3^*}{\partial x_3^*}\mathrm{grad}\ u_3^* \\ \\ \overline{\dfrac{\partial V^*}{\partial x_3^*}}\dfrac{\partial V^*}{\partial x^*} + \dfrac{\partial u_3^*}{\partial x_3^*}\ \overline{\mathrm{grad}\ u_3^*} & \overline{\dfrac{\partial V^*}{\partial x_3^*}}\dfrac{\partial V^*}{\partial x_3^*} + \left(\dfrac{\partial u_3^*}{\partial x_3^*}\right)^2 \end{pmatrix}$$

and

$$2\mathrm{Tr}\,\gamma^* = \mathrm{Tr}\left(\overline{\frac{\partial V^*}{\partial x_3^*}}\frac{\partial V^*}{\partial x^*}\right) + \|\mathrm{grad}\ u_3^*\|^2 + \left\|\frac{\partial V^*}{\partial x_3^*}\right\|^2 + \left(\frac{\partial u_3^*}{\partial x_3^*}\right)^2.$$

Therefore we can express Γ^* and E_u^* as a function of V^*, u_3^* and of their derivatives. We will use the following matrix notations:

$$\Gamma^* = \begin{pmatrix} \Gamma_t^* & \Gamma_s^* \\ \overline{\Gamma_s^*} & \Gamma_n^* \end{pmatrix} \qquad \text{and} \qquad E_u^* = \begin{pmatrix} E_{ut}^* & E_{us}^* \\ \overline{Q}^* & E_{un}^* \end{pmatrix}.$$

The explicit expressions of E_u^* and Γ^* are not written here to simplify our derivations. We will directly introduce their nondimensional expressions.

So the equilibrium equations can be written in $\Omega_0^* = \omega_0^* \times]-h_0, h_0[$ as:

$$\begin{aligned}(\lambda+\mu)\mathrm{grad}^*(\mathrm{div}^* V^*) + \mu\Delta^* V^* + (\lambda+\mu)\mathrm{grad}^*\ \frac{\partial u_3^*}{\partial x_3^*} + \mu\frac{\partial^2 V^*}{\partial x_3^{2*}}&\\ + \mathrm{div}^*[E_{ut}^* + \Gamma_t^*] + \frac{\partial}{\partial x_3^*}(\Gamma_s^* + Q^*) &= -f_t^*,\end{aligned}$$

(2.4)

$$\begin{aligned}(\lambda+\mu)\frac{\partial}{\partial x_3^*}\mathrm{div}^* V^* + \mu\Delta^* u_3^* + (\lambda+2\mu)\frac{\partial^2 u_3^*}{\partial x_3^{2*}} + \mathrm{div}^*[E_{us}^* + \Gamma_s^*]&\\ + \frac{\partial}{\partial x_3^*}(E_{un}^* + \Gamma_n^*) &= -f_3^*,\end{aligned}$$

with the boundary conditions on the upper and the lower faces:

$$\mu(\mathrm{grad}^* u_3^* + \frac{\partial V^*}{\partial x_3^*}) + Q^* + \Gamma_s^* = \pm g_t^{*\pm} \qquad \text{on } \overline{\omega}_0^* \times \{\pm h_0\},$$

(2.5)

$$(\lambda+2\mu)\frac{\partial u_3^*}{\partial x_3^*} + \lambda\mathrm{div}^*\ V^* + E_{un}^* + \Gamma_n^* = \pm g_3^{*\pm} \qquad \text{on } \overline{\omega}_0^* \times \{\pm h_0\}.$$

3. Dimensional analysis of the equilibrium equations

Like in the linear case [14], let us define the following dimensionless physical data and dimensionless unknowns of the problem:

$$V = \frac{V^*}{V_r}, \qquad u_3 = \frac{u_3^*}{u_{3r}}, \qquad x = \frac{x^*}{L_0}, \qquad x_3 = \frac{x_3^*}{h_0},$$

$$f_3 = \frac{f_3^*}{f_{3r}}, \qquad g_3^{\pm} = \frac{g_3^{*\pm}}{g_{3r}}, \qquad f_t = \frac{f_t^*}{f_{tr}}, \qquad g_t^{\pm} = \frac{g_t^{*\pm}}{g_{tr}},$$

where the variables with subscript r are the reference ones. The new variables which appear (without the asterisk) are dimensionless.

To avoid any assumption concerning the order of magnitude of the displacement components, the reference scales u_{3r} and V_r are firstly assumed to be equal to L_0. Thus we *a priori* allow large displacements. If necessary, it will always be possible to define new reference scales for the tangential and the normal displacement.

Introducing the previously defined dimensionless variables into Eqs. (2.4) and (2.5), we obtain a new nondimensional problem posed on $\Omega_0 = \omega_0 \times]-1,1[$:

$$\varepsilon^2\Big[(1+\beta)\mathrm{grad}(\mathrm{div}V) + \Delta V + \mathrm{div}\,(E_{ut} + \Gamma_t)\Big] + \varepsilon\left[(1+\beta)\mathrm{grad}\,\frac{\partial u_3}{\partial x_3} + \frac{\partial}{\partial x_3}(Q+\Gamma_s)\right] + \frac{\partial^2 V}{\partial x_3^2} = -\varepsilon \mathcal{F}_t f_t,$$

(3.1)

$$\varepsilon^2\Big[\Delta u_3 + \mathrm{div}\,(E_{us} + \Gamma_s)\Big] + \varepsilon\left[(1+\beta)\frac{\partial}{\partial x_3}\mathrm{div}\ V + \frac{\partial}{\partial x_3}(E_{un} + \Gamma_n)\right] + (2+\beta)\frac{\partial^2 u_3}{\partial x_3^2} = -\varepsilon \mathcal{F}_3 f_3.$$

The boundary conditions on the upper and the lower faces become:

$$\varepsilon\Big[\mathrm{grad}\,u_3 + Q + \Gamma_s\Big] + \frac{\partial V}{\partial x_3} = \pm\varepsilon\ \mathcal{G}_t g_t^{\pm} \quad \text{for} \quad x_3 = \pm 1,$$

(3.2)

$$\varepsilon\Big[\beta\,\mathrm{div}\ V + E_{un} + \Gamma_n\Big] + (2+\beta)\frac{\partial u_3}{\partial x_3} = \pm\varepsilon\ \mathcal{G}_3 g_3^{\pm} \quad \text{for } x_3 = \pm 1,$$

with $\beta = \dfrac{\lambda}{\mu}$ and

$$\Gamma_t = \frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) I_2 + \frac{\beta}{2}\|\mathrm{grad}\ u_3\|^2 I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x} + \mathrm{grad}\ u_3\ \overline{\mathrm{grad}\ u_3} + \frac{\beta}{2\varepsilon^2}\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right] I_2,$$

$$\Gamma_s = \frac{1}{\varepsilon}\left\{\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\mathrm{grad}\ u_3\right\},$$

$$\Gamma_n = \frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) + \frac{\beta}{2}\|\mathrm{grad}\ u_3\|^2 + \frac{1}{\varepsilon^2}\left\{(1+\frac{\beta}{2})\Big[\Big\|\frac{\partial V}{\partial x_3}\Big\|^2 + \Big(\frac{\partial u_3}{\partial x_3}\Big)^2\Big]\right\},$$

$$E_{ut} = \left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial x} + \overline{\frac{\partial V}{\partial x}} + \beta\ \mathrm{div}V\ I_2 + \frac{\beta}{2}\|\mathrm{grad}\ u_3\|^2\ I_2\right.$$

$$\left. + \mathrm{grad}\ u_3\ \overline{\mathrm{grad}\ u_3}\right]\overline{\frac{\partial V}{\partial x}} + \frac{1}{\varepsilon}\left\{\mathrm{grad}\ u_3\ \overline{\frac{\partial V}{\partial x_3}} + \beta\frac{\partial u_3}{\partial x_3}\overline{\frac{\partial V}{\partial x}}\right\}$$

$$+\frac{1}{\varepsilon^2}\left\{\frac{\beta}{2}\left[\Big\|\frac{\partial V}{\partial x_3}\Big\|^2 + \Big(\frac{\partial u_3}{\partial x_3}\Big)^2\right]\overline{\frac{\partial V}{\partial x}} + \left[\left(I_2 + \overline{\frac{\partial V}{\partial x}}\right)\frac{\partial V}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\mathrm{grad}\ u_3\right]\overline{\frac{\partial V}{\partial x_3}}\right\},$$

$$E_{us} = \left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial x} + \overline{\frac{\partial V}{\partial x}} + \beta\,\mathrm{div}\ V\ I_2\right.$$

$$\left. +(1+\frac{\beta}{2})\|\mathrm{grad}\ u_3\|^2\ I_2\right]\mathrm{grad}\ u_3 + \frac{1}{\varepsilon}\left\{(1+\beta)\frac{\partial u_3}{\partial x_3}\ \mathrm{grad}\ u_3\right\}$$

$$+\frac{1}{\varepsilon^2}\left\{\left[\frac{\beta}{2}\Big\|\frac{\partial V}{\partial x_3}\Big\|^2 + (1+\frac{\beta}{2})\Big(\frac{\partial u_3}{\partial x_3}\Big)^2\right]\mathrm{grad}\ u_3 + \frac{\partial u_3}{\partial x_3}\left[I_2 + \overline{\frac{\partial V}{\partial x}}\right]\frac{\partial V}{\partial x_3}\right\},$$

$$Q = \frac{\partial V}{\partial x}\ \mathrm{grad}\ u_3 + \frac{1}{\varepsilon}\left\{\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) I_2 + \frac{\partial V}{\partial x}\overline{\frac{\partial V}{\partial x}} + \beta\mathrm{div}V\ I_2\right.\right.$$

$$\left.\left. +\frac{\beta}{2}\|\mathrm{grad}\ u_3\|^2\ I_2 + \frac{\partial V}{\partial x}\right]\frac{\partial V}{\partial x_3} + \frac{\partial V}{\partial x}\frac{\partial u_3}{\partial x_3}\mathrm{grad}\ u_3\right\} + \frac{1}{\varepsilon^2}(2+\beta)\frac{\partial u_3}{\partial x_3}\frac{\partial V}{\partial x_3}$$

$$+\frac{1}{\varepsilon^3}(1+\frac{\beta}{2})\left[\Big\|\frac{\partial V}{\partial x_3}\Big\|^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\frac{\partial V}{\partial x_3},$$

$$E_{un} = \|\mathrm{grad}\ u_3\|^2 + \frac{1}{\varepsilon}\left\{\left[\frac{\beta}{2}\mathrm{Tr}(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}) + \beta\mathrm{div}V\right.\right.$$

$$\left.\left. +(1+\frac{\beta}{2})\|\mathrm{grad}\ u_3\|^2\right]\frac{\partial u_3}{\partial x_3} + \overline{\frac{\partial V}{\partial x_3}}\left[I_2 + \frac{\partial V}{\partial x}\right]\mathrm{grad}\ u_3\right\}$$

$$+\frac{1}{\varepsilon^2}(2+\beta)\Big(\frac{\partial u_3}{\partial x_3}\Big)^2 + \frac{1}{\varepsilon^3}\left[\left(1+\frac{\beta}{2}\right)\left[\ \Big\|\frac{\partial V}{\partial x_3}\Big\|^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\frac{\partial u_3}{\partial x_3},\right.$$

where I_2 denotes the identity of $\mathbb{R}^2$.

Hence, the dimensional analysis of the equilibrium equations leads naturally to the same dimensionless numbers as in the linear case [14]:

$$\varepsilon = \frac{h_0}{L_0}, \qquad \mathcal{F}_3 = \frac{h_0 f_{3r}}{\mu}, \qquad \mathcal{F}_t = \frac{h_0 f_{tr}}{\mu}, \qquad \mathcal{G}_3 = \frac{g_{3r}}{\mu}, \qquad \mathcal{G}_t = \frac{g_{tr}}{\mu}.$$

On the other hand, writing the condition (2.2) in a nondimensional form, we obtain

$$\det\left(L_0 \frac{\partial \psi}{\partial X}\frac{\partial X}{\partial X^*}\right) = L_0^3 \det\left(\frac{\partial \psi}{\partial X}\right) \det\left(\frac{\partial X}{\partial X^*}\right) \geqslant a \qquad \text{in } \Omega_0$$

with $\psi = \dfrac{\psi^*}{L_0}$. Since we have

$$\det\left(\frac{\partial X}{\partial X^*}\right) = \frac{1}{h_0\ L_0^2} \quad ,$$

the condition (2.2) becomes:

$$\det F = \det\left(\frac{\partial \psi}{\partial X}\right) \geqslant \varepsilon a > 0 \qquad \forall \varepsilon > 0. \tag{3.3}$$

3.1. Interpretation of the dimensionless numbers

i) The shape ratio $\varepsilon = \dfrac{h_0}{L_0}$ of the initial thickness of the plate to the diameter of the middle surface ω_0^* is a known parameter of the problem. For thin plates, ε is a small parameter.

ii) The ratios of forces $\mathcal{F}_t = \dfrac{h_0 f_{tr}}{\mu}$, $\mathcal{F}_3 = \dfrac{h_0 f_{3r}}{\mu}$ and $\mathcal{G}_t = \dfrac{g_{tr}}{\mu}$, $\mathcal{G}_3 = \dfrac{g_{3r}}{\mu}$ represent respectively the ratios of the resultant body forces (acting across the thickness of the plate) or of the surface forces to μ considered as a reference stress. These numbers depend only on known physical quantities and are known data of the problem.

3.2. Reduction to a single-scale problem

In order to obtain a single-scale problem, ε is chosen as the reference parameter. Therefore the other dimensionless numbers must be linked to ε.

In usual elasticity problems, the body force f^* is often due to the gravity. As an example, let us consider a thin steel plate, of 1m diameter and 10^{-2}m thickness, whose Young's modulus, Poisson's ratio and voluminal mass are, respectively, $E = 2.1\ 10^{11}$ Pa, $\nu = 0.285$ and $\rho = 7800$ kg/m^3. If we assume that the plate is subjected only to the gravity force, we find $\mathcal{F}_3 = \dfrac{\rho g h}{\mu} = 10^{-8} = \varepsilon^4$. In this example the tangential component of the weight is equal to zero, so that we have $\mathcal{F}_t = 0$. Accordingly, in this paper we consider the body forces level to be of the order of magnitude of the weight which satisfies $\mathcal{F}_3 = \varepsilon^4$ and $\mathcal{F}_t = 0$.

Nevertheless we will distinguish two different magnitudes of surface forces which lead to two different two-dimensional models. In this first part of the article, we consider large surface forces such as $\mathcal{G}_3 = \mathcal{G}_t = \varepsilon$ to obtain large displacements. In the second part of this article, we will consider a plate subjected to the same moderate surface forces as in the linear case [14]: $\mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$.

However it would be possible to take into account other body forces (like centrifugal acceleration) whose tangential component is not equal to zero. In this case, $\mathcal{F}_t$ must be linked to ε.

4. The nonlinear two-dimensional membrane model

Let us consider in this section a thin plate subjected to body forces such as $\mathcal{F}_3 = \varepsilon^4$, $\mathcal{F}_t = 0$ and to the important surface forces of the magnitude $\mathcal{G}_3 = \mathcal{G}_t = \varepsilon$. Problem (3.1)–(3.2) being reduced to a single-scale problem with ε as a small parameter, the standard asymptotic expansion method leads to write the nondimensional solution (V, u_3) as a formal expansion with respect to ε:

$$\begin{aligned} V &= V^0 + \varepsilon V^1 + \varepsilon^2 V^2 + \cdots, \\ u_3 &= u_3^0 + \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \cdots. \end{aligned} \tag{4.1}$$

Therefore, we obtain the following result:

RESULT 1

For a plate subjected to applied forces such as $\mathcal{F}_3 = \varepsilon^4$ and $\mathcal{G}_t = \mathcal{G}_3 = \varepsilon$, the leading term (V^0, u_3^0) of the asymptotic expansion of (V, u_3) depends only on $x = (x_1, x_2)$ and solves the following membrane problem:

$$\begin{aligned} &\operatorname{div}\Big(N_t^0 \Big[I_2 + \overline{\frac{\partial V^0}{\partial x}} \Big] \Big) = -p_t, \\ &\operatorname{div}(N_t^0 \text{ grad } u_3^0) = -p_3, \\ &V^0 = 0 \quad \text{and} \quad u_3^0 = 0 \quad \text{on } \gamma_0 = \partial\omega_0, \end{aligned}$$

where:

$$N_t^0 = \frac{4\beta}{2+\beta} \operatorname{Tr} E_t^0 \; I_2 + 4E_t^0,$$

$$E_t^0 = \frac{1}{2}\left(\frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}} + \overline{\frac{\partial V^0}{\partial x}} \frac{\partial V^0}{\partial x} + \text{grad } u_3^0 \; \overline{\text{grad } u_3^0} \right),$$

$$p_3 = g_3^+ + g_3^- \qquad p_t = g_t^+ + g_t^- .$$

P r o o f. The proof of this result is divided into several steps from i) to iv).

i) V^0 and u_3^0 depends only on (x_1, x_2)

Replacing V and u_3 by their expansions (4.1) in the nondimensional equilibrium equations (3.1) and (3.2) and equating to zero the factors of successive powers of ε, we obtain the coupled problems $\mathcal{P}_{-2}, \mathcal{P}_{-1}, \mathcal{P}_0 \ldots$.
The problem $\mathcal{P}_{-2}$ can be written as

$$\frac{\partial}{\partial x_3}(Q^{-3} + \Gamma_s^{-3}) = 0 \qquad \text{in } \Omega_0, \tag{4.2}$$

$$\frac{\partial}{\partial x_3}(E_{un}^{-3} + \Gamma_n^{-3}) = 0 \qquad \text{in } \Omega_0, \tag{4.3}$$

$$Q^{-3} + \Gamma_s^{-3} = 0 \qquad \text{for } x_3 = \pm 1, \tag{4.4}$$

$$E_{un}^{-3} + \Gamma_n^{-3} = 0 \qquad \text{for } x_3 = \pm 1, \tag{4.5}$$

with:

$$Q^{-3} = \left(1 + \frac{\beta}{2}\right) \left[\left\| \frac{\partial V^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 \right] \frac{\partial V^0}{\partial x_3},$$

$$E_{un}^{-3} = \left(1 + \frac{\beta}{2}\right) \left[\left\| \frac{\partial V^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 \right] \frac{\partial u_3^0}{\partial x_3},$$

$$\Gamma_3^{-3} = 0 \qquad \Gamma_n^{-3} = 0.$$

Taking into account the boundary conditions (4.4) and (4.5), equations (4.2) and (4.3) imply the relations

$$\left[\left\| \frac{\partial V^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 \right] \frac{\partial V^0}{\partial x_3} = 0 \qquad \text{in } \Omega_0,$$

$$\left[\left\| \frac{\partial V^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 \right] \frac{\partial u_3^0}{\partial x_3} = 0 \qquad \text{in } \Omega_0,$$

which lead to $\dfrac{\partial u_3^0}{\partial x_3} = 0$ and $\dfrac{\partial V^0}{\partial x_3} = 0$ in Ω_0. Hence, we finally obtain:

$$
\begin{aligned}
V^0 &= V^0(x_1, x_2),\\
u_3^0 &= u_3^0(x_1, x_2).
\end{aligned}
\tag{4.6}
$$

ii) Determination of u_3^1 and V^1

Considering expression (4.6), we have $Q^{-2} = E_{un}^{-2} = \Gamma_n^{-2} = 0$ so that problems $\mathcal{P}_{-1}$ and $\mathcal{P}_0$ are evidently satisfied. Equating to zero the coefficient of ε^1, problem $\mathcal{P}_1$ can be written in the form

$$\frac{\partial}{\partial x_3}(Q^0 + \Gamma_s^0) + \frac{\partial^2 V^1}{\partial x_3^2} = 0 \qquad \text{in } \Omega_0, \tag{4.7}$$

$$\frac{\partial}{\partial x_3}(\Gamma_n^0 + E_{un}^0) + (2+\beta)\frac{\partial^2 u_3^1}{\partial x_3^2} = 0 \qquad \text{in } \Omega_0, \tag{4.8}$$

$$Q^0 + \Gamma_s^0 + \frac{\partial V^1}{\partial x_3} + \text{grad } u_3^0 = 0 \qquad \text{for } x_3 = \pm 1, \tag{4.9}$$

$$\Gamma_n^0 + E_{un}^0 + \beta \text{div } V^0 + (2+\beta)\frac{\partial u_3^1}{\partial x_3} = 0 \qquad \text{for } x_3 = \pm 1, \tag{4.10}$$

with

$$\Gamma_s^0 = \frac{\partial u_3^1}{\partial x_3} \text{ grad } u_3^0 + \overline{\frac{\partial V^0}{\partial x}} \frac{\partial V^1}{\partial x_3},$$

$$\Gamma_n^0 = \frac{\beta}{2}\text{Tr}\left(\frac{\partial V^0}{\partial x}\frac{\partial V^0}{\partial x}\right) + \frac{\beta}{2}\|\text{grad } u_3^0\|^2 + \left(1 + \frac{\beta}{2}\right)\left[\left\|\frac{\partial V^1}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3^1}{\partial x_3}\right)^2\right],$$

$$Q^0 = \left[\beta\,\text{Tr}E_t^0\; I_2 + \frac{1+\beta}{2}\left(\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 - 1\right) I_2 + \frac{\partial V^0}{\partial x}\overline{\frac{\partial V^0}{\partial x}} + \frac{\partial V^0}{\partial x}\right]\frac{\partial V^1}{\partial x_3}$$
$$+ \frac{\partial V^0}{\partial x}\text{grad } u_3^0\left(1 + \frac{\partial u_3^1}{\partial x_3}\right),$$

$$E_{un}^0 = \left[\beta\,\text{Tr}E_t^0 + \left(\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 - 1\right)\right]\frac{\partial u_3^1}{\partial x_3} + \overline{\frac{\partial V^1}{\partial x_3}}\left[I_2 + \frac{\partial V^0}{\partial x}\right]\text{grad } u_3^0$$
$$+ \left(1 + \frac{\partial u_3^1}{\partial x_3}\right)\left\|\text{grad } u_3^0\right\|^2,$$

and

$$E_t^0 = \frac{1}{2}\left(\frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}} + \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x} + \text{grad } u_3^0 \ \overline{\text{grad } u_3^0}\right).$$

Let us define

$$\psi^1 = V^1 + (x_3 + u_3^1)e_3 = \begin{pmatrix} V^1 \\ x_3 + u_3^1 \end{pmatrix}_{(e_i)}$$

so we have

$$\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 = 1 + \left\|\frac{\partial V^1}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3^1}{\partial x_3}\right)^2 + 2\frac{\partial u_3^1}{\partial x_3}.$$

The endomorphism E_t^0 represents the membrane strain due to the displacement (V^0, u_3^0). In particular, we obtain

$$\text{Tr } E_t^0 = \text{div } V^0 + \frac{1}{2}\text{Tr}\left(\overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x}\right) + \frac{1}{2}\|\text{grad } u_3^0\|^2.$$

REMARK 2

It is natural to introduce the functional mapping ψ^1 to simplify the equations. Indeed, introducing the functional mapping ψ^*, we have

$$\psi^* = X^* + U^* = x_\alpha^* \, e_\alpha + x_3^* e_3 + U^*. \quad \blacksquare$$

We then obtain the following nondimensional equation:

$$\psi = \frac{\psi^*}{L} = (x_\alpha e_\alpha + U) + \varepsilon \, x_3 e_3$$

where $U = V + u_3 e_3$. The expansion (4.1) of (V, u_3) with respect to ε implies then the following expansion of ψ:

$$\psi = \underbrace{x_\alpha \, e_\alpha + V^0 + u_3^0 e_3}_{\psi^0} + \varepsilon \left[\underbrace{(x_3 + u_3^1)e_3 + V^1}_{\psi^1}\right] + \cdots \tag{4.11}$$

The functional mapping ψ^0 represents the deformed middle surface ω_0 at the leading order of the expansion. With these notations, the membrane strain E_t^0 can also be written

$$E_t^0 = \frac{1}{2}\left(\overline{\frac{\partial \psi^0}{\partial x}}\frac{\partial \psi^0}{\partial x} - I_2\right) \qquad \text{with} \qquad \frac{\partial \psi^0}{\partial x} = \begin{pmatrix} I_2 + \dfrac{\partial V^0}{\partial x} \\ \\ \overline{\text{grad } u_3^0} \end{pmatrix}_{(e_i)}. \tag{4.12}$$

Considering the boundary conditions (4.9) and (4.10), Eqs. (4.7) and (4.8) lead to

$$Q^0 + \Gamma_s^0 + \frac{\partial V^1}{\partial x_3} + \text{grad } u_3^0 = 0 \qquad \text{in } \Omega_0, \tag{4.13}$$

$$\Gamma_n^0 + E_{un}^0 + \beta \text{div } V^0 + (2+\beta)\frac{\partial u_3^1}{\partial x_3} = 0 \qquad \text{in } \Omega_0. \tag{4.14}$$

Since we have

$$\Gamma_n^0 + \beta \, \text{div } V^0 + (2+\beta)\frac{\partial u_3^1}{\partial x_3} = \beta \, \text{Tr } E_t^0 + \left(1 + \frac{\beta}{2}\right)\left(\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 - 1\right),$$

equations (4.13) and (4.14) become

$$A \, \frac{\partial V^1}{\partial x_3} + \left(I_2 + \frac{\partial V^0}{\partial x}\right) B = 0,$$

$$A \left(1 + \frac{\partial u_3^1}{\partial x_3}\right) + \overline{\text{grad } u_3^0} \, B = 0,$$

with

$$A = \beta \, \text{Tr } E_t^0 + \left(1 + \frac{\beta}{2}\right)\left(\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 - 1\right),$$

$$B = \left(1 + \frac{\partial u_3^1}{\partial x_3}\right) \text{grad } u_3^0 + \overline{\left(I_2 + \frac{\partial V^0}{\partial x}\right)} \frac{\partial V^1}{\partial x}.$$

Using matrix notations, the last equation can be written as

$$A \begin{pmatrix} \dfrac{\partial V^1}{\partial x_3} \\ 1 + \dfrac{\partial u_3^1}{\partial x_3} \end{pmatrix} + \begin{pmatrix} I_2 + \dfrac{\partial V^0}{\partial x} \\ \overline{\text{grad } u_3^0} \end{pmatrix} B = 0,$$

and considering the expressions of ψ^0 and ψ^1, we get

$$A \, \frac{\partial \psi^1}{\partial x_3} + \frac{\partial \psi^0}{\partial x} \, B = 0. \tag{4.15}$$

Now we finally use the mass conservation law. As ψ^0 is independent of the variable x_3, the expansion of the condition (3.3) gives

$$\frac{\partial \psi^1}{\partial x_3} \cdot \left(\frac{\partial \psi^0}{\partial x_1} \wedge \frac{\partial \psi^0}{\partial x_2} \right) + O(\varepsilon) \geqslant a > 0 \qquad \forall \varepsilon > 0,$$

where $\wedge$ denotes the cross product in $\mathbb{R}^3$. Thus, we must have

$$\frac{\partial \psi^1}{\partial x_3} \cdot \left(\frac{\partial \psi^0}{\partial x_1} \wedge \frac{\partial \psi^0}{\partial x_2} \right) > 0 \qquad \text{in } \Omega_0.$$

This condition implies that the vectors $\left(\frac{\partial \psi^0}{\partial x_1}, \frac{\partial \psi^0}{\partial x_2}, \frac{\partial \psi^1}{\partial x_3} \right)$ are linearly independent, and form an $\mathbb{R}^3$ basis.

Going back to equation (4.15) where $\frac{\partial \psi^0}{\partial x} = \left(\frac{\partial \psi^0}{\partial x_1}, \frac{\partial \psi^0}{\partial x_2} \right)$ represents the local basis of the deformed configuration $\psi^0(\omega_0)$, we have

$$A = \beta \,\mathrm{Tr}\, E_t^0 + \left(1 + \frac{\beta}{2}\right) \left(\left\| \frac{\partial \psi^1}{\partial x_3} \right\|^2 - 1 \right) = 0 \tag{4.16}$$

and

$$B = \left(1 + \frac{\partial u_3^1}{\partial x_3}\right) \operatorname{grad} u_3^0 + \left(I_2 + \overline{\frac{\partial V^0}{\partial x}}\right) \frac{\partial V^1}{\partial x} = 0. \tag{4.17}$$

The second equation (4.17) can also be written as

$$\overline{\frac{\partial \psi^0}{\partial x}} \, \frac{\partial \psi^1}{\partial x_3} = 0$$

and implies that

$$\frac{\partial \psi^1}{\partial x_3} = \eta N,$$

where N denotes the unit normal to the surface $\omega = \psi^0(\omega_0)$.

Thanks to equation (4.16) it is now possible to determine the norm $\left\| \frac{\partial \psi^1}{\partial x_3} \right\|$. Indeed we have

$$\left\| \frac{\partial \psi^1}{\partial x_3} \right\|^2 = 1 + \frac{2\beta}{2+\beta} \mathrm{Tr}\, E_t^0 = 1 + \frac{2\lambda}{\lambda + 2\mu} \mathrm{Tr}\, E_t^0. \tag{4.18}$$

Finally we obtain the following expression of ψ^1:

$$\begin{aligned} \psi^1 &= \eta N + \tilde{u}^1(x_1, x_2), \\ \eta^2 &= 1 + \frac{2\lambda}{\lambda + 2\mu} \mathrm{Tr}\, E_t^0. \end{aligned} \tag{4.19}$$

iii) The first membrane equation

The cancellation of the coefficient of ε^2 leads to the problem $\mathcal{P}_2$ in Ω_0:

$$(4.20)\qquad (1+\beta)\mathrm{grad}(\mathrm{div}\ V^0) + \Delta V^0 + \mathrm{div}(E^0_{ut} + \Gamma^0_t) + (1+\beta)\,\mathrm{grad}\ \frac{\partial u^1_3}{\partial x_3} + \frac{\partial}{\partial x_3}(Q^1 + \Gamma^1_s) + \frac{\partial^2 V^2}{\partial x_3^2} = 0,$$

$$(4.21)\qquad \Delta u^0_3 + \mathrm{div}(E^0_{us} + \Gamma^0_s) + (1+\beta)\frac{\partial}{\partial x_3}\mathrm{div}\ V^1 + \frac{\partial}{\partial x_3}\Big(\Gamma^1_n + E^1_{un}\Big) + (2+\beta)\frac{\partial^2 u^2_3}{\partial x_3^2} = 0,$$

with the following boundary conditions on the upper and the lower faces:

$$(4.22)\qquad \mathrm{grad}\, u^1_3 + Q^1 + \Gamma^1_s + \frac{\partial V^2}{\partial x_3} = \pm g^{\pm}_t \qquad \text{for } x_3 = \pm 1,$$

$$(4.23)\qquad \beta\,\mathrm{div}\ V^1 + \Gamma^1_n + E^1_{un} + (2+\beta)\frac{\partial u^2_3}{\partial x_3} = \pm g^{\pm}_3 \qquad \text{for } x_3 = \pm 1.$$

Considering equations (4.17) and (4.18), the expression of E^0_{ut}, Γ^0_s, E^0_{us} and Γ^0_t can be written as:

$$E^0_{ut} = \left[\frac{2\beta}{2+\beta}\mathrm{Tr}E^0_t\ I_2 + \frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}} + \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x} + \mathrm{grad}\ u^0_3\ \overline{\mathrm{grad}\ u^0_3}\right]\overline{\frac{\partial V^0}{\partial x}},$$

$$\Gamma^0_s = \frac{\partial u^1_3}{\partial x_3}\ \mathrm{grad}\ u^0_3 + \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^1}{\partial x_3},$$

$$E^0_{us} = \left[\frac{2\beta}{2+\beta}\mathrm{Tr}E^0_t\ I_2 + \frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}} + \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x} + \|\mathrm{grad}\ u^0_3\|^2\ I_2\right]\mathrm{grad}\ u^0_3,$$

$$\Gamma^0_t = \frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x}\right)I_2 + \frac{\beta}{2}\|\mathrm{grad}\ u^0_3\|^2 I_2 + \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x} + \mathrm{grad}\ u^0_3\ \overline{\mathrm{grad}\ u^0_3} + \frac{\beta}{2}\left[\left\|\frac{\partial V^1}{\partial x_3}\right\|^2 + \left(\frac{\partial u^1_3}{\partial x_3}\right)^2\right] I_2.$$

Then using the fact that

$$\mathrm{grad}(\mathrm{div}\ V^0) = \mathrm{div}\Big(\mathrm{div}\ V^0\ I_2\Big) \qquad \text{and} \qquad \mathrm{grad}\ \frac{\partial u^1_3}{\partial x_3} = \mathrm{div}\left(\frac{\partial u^1_3}{\partial x_3}\ I_2\right)$$

and taking into account the boundary conditions (4.22), integration by parts of (4.20) leads to

$$(4.24)\qquad \int_{-1}^{1}\left[\operatorname{grad}(\operatorname{div} V^0)+\Delta V^0+\operatorname{div}\left(E^0_{ut}+\Gamma^0_t+\beta\frac{\partial u^1_3}{\partial x_3}\, I_2 + \beta\operatorname{div} V^0\, I_2\right)\right]dx_3=-p_t$$

with $p_t = g_t^+ + g_t^-$.

On the other hand, considering the expression of Γ^0_t, we get:

$$\Gamma^0_t+\beta\frac{\partial u^1_3}{\partial x_3}\, I_2+\beta\operatorname{div} V^0\, I_2=\frac{2\beta}{2+\beta}\operatorname{Tr}E^0_t\, I_2+\overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x}+\operatorname{grad}\, u^0_3\ \overline{\operatorname{grad}\, u^0_3}.$$

Using the relations

$$\operatorname{grad}(\operatorname{div} V^0)=\operatorname{div}(\operatorname{grad} V^0)=\operatorname{div}\Big(\frac{\partial V^0}{\partial x}\Big)\qquad\text{and}$$

$$\Delta V^0=\operatorname{div}(\overline{\operatorname{grad} V^0})=\operatorname{div}\left(\overline{\frac{\partial V^0}{\partial x}}\right),$$

equation (4.24) becomes

$$\operatorname{div}\left(N^0_t\Big[I_2+\overline{\frac{\partial V^0}{\partial x}}\Big]\right)=-p_t$$

with

$$N^0_t=\frac{4\beta}{2+\beta}TrE^0_t\, I_2+4E^0_t,$$

$$E^0_t=\frac{1}{2}\left(\frac{\partial V^0}{\partial x}+\overline{\frac{\partial V^0}{\partial x}}+\overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x}+\operatorname{grad}\, u^0_3\ \overline{\operatorname{grad}\, u^0_3}\right),$$

$$p_t=g_t^+ + g_t^-.$$

iv) The second membrane equation

Considering the boundary conditions (4.23), an integration from -1 to 1 of equation (4.21) leads to

$$(4.25)\qquad \int_{-1}^{1}\Big[\,\Delta u^0_3+\operatorname{div}\left(E^0_{us}+\Gamma^0_s+\frac{\partial V^1}{\partial x_3}\right)\Big]dx_3=-p_3$$

with $p_3 = g_3^+ - g_3^-$.
Using the property $\Delta u_3^0 = \mathrm{div}(\mathrm{grad} u_3^0)$, we obtain

$$\Delta u_3^0 + \mathrm{div}\left(\Gamma_s^0 + \frac{\partial V^1}{\partial x_3}\right) = \mathrm{div} B = 0$$

thanks to the orthogonality condition (4.17). Finally equation (4.25) becomes:

$$\mathrm{div}\left(N_t^0 \ \mathrm{grad}\ u_3^0\right) = -p_3$$

and the RESULT 1 is then proved.

4.1. Associated variational formulation

Let us define the space of admissible functional mappings

$$\mathcal{Q}(\omega_0) = \left\{\psi : \omega_0 \mapsto \mathbb{R}^3, \text{ “smooth”}, \quad \psi = I_2 \quad \text{on } \gamma_0\right\}$$

and the space of kinematically admissible displacements:

$$V(\omega_0) = \left\{v^0 : \omega_0 \mapsto \mathbb{R}^3, \text{ “smooth”}, \quad v = 0 \quad \text{on } \gamma_0\right\}.$$

If the displacements are assumed to be smooth enough, then the two-dimensional membrane equations of the RESULT 1 can be written in the following variational form which depends only on ψ^0 (instead of V^0 and u_3^0):

RESULT 2

$\psi^0 \in \mathcal{Q}(\omega_0)$ satisfies the following variational problem:

$$\text{(4.26)} \qquad \int\limits_{\omega_0} \mathrm{Tr}\left[N_t^0 \overline{\frac{\partial \psi^0}{\partial x}}\ \frac{\partial v^0}{\partial x}\right] d\omega_0 = \int\limits_{\omega_0} p.v^0 \ d\omega_0 \qquad \forall v^0 \in V(\omega_0)$$

where

$$N_t^0 = \frac{4\beta}{2+\beta}\ \mathrm{Tr} E_t^0(\psi^0) I_2 + 4E_t^0(\psi^0),$$

$$E_t^0(\psi^0) = \frac{1}{2}\left(\overline{\frac{\partial \psi^0}{\partial x}} \frac{\partial \psi^0}{\partial x} - I_2\right),$$

$$p = p_t + p_3 e_3.$$

Indeed, considering the expression (4.11) for $\psi^0 = x_\alpha e_\alpha + u_3^0 e_3 + V^0$, the membrane strain E_t^0 can be written as:

$$E_t^0 = \frac{1}{2}\left(\overline{\frac{\partial \psi^0}{\partial x}}\frac{\partial \psi^0}{\partial x} - I_2\right).$$

Now, if we consider a displacement field of $V(\omega_0)$ of the form $v^0 = v_t^0 + v_3^0 e_3$, the membrane equations of the RESULT 1 becomes:

$$\int_{\omega_0} \operatorname{div}\Big[N_t^0\Big(I_2 + \overline{\frac{\partial V^0}{\partial x}}\Big)\Big]\; v_t^0 \; d\omega_0 + \int_{\omega_0} \operatorname{div}\Big[N_t^0 \text{ grad } u_3^0\Big]\; v_3^0 \; d\omega_0$$

$$= -\int_{\omega_0} p_t . v_t^0 \; d\omega_0 - \int_{\omega_0} p_3 v_3^0 \; d\omega_0.$$

Setting $p = p_t + p_3 e_3$, we get

$$\int_{\omega_0} \operatorname{Tr}\Big[N_t^0\Big(I_2 + \overline{\frac{\partial V^0}{\partial x}}\Big)\;\frac{\partial v_t^0}{\partial x}\Big]\; d\omega_0 + \int_{\omega_0} Tr\Big[N_t^0 \text{ grad } u_3^0 \; \overline{\text{grad } v_3^0}\Big] d\omega_0$$

$$= \int_{\omega_0} p . v^0 d\omega_0.$$

Finally, using matrix notations, we have:

$$\frac{\partial \psi^0}{\partial x} = \begin{pmatrix} I_2 + \dfrac{\partial V^0}{\partial x} \\ \overline{\text{grad } u_3^0} \end{pmatrix} \qquad \text{and} \qquad \frac{\partial v^0}{\partial x} = \begin{pmatrix} \dfrac{\partial v_t^0}{\partial x} \\ \overline{\text{grad } v_3^0} \end{pmatrix}.$$

Thus we easily obtain the variational formulation of the RESULT 2.

This variational formulation of membrane equations is identical (more and less the coefficient μ) to the one obtained by D. FOX *et al.* in [9] from the three-dimensional variational formulation of the problem. Nevertheless, the asymptotic approach developed in this paper which leads to the membrane equations of the RESULT 1 presents some advantages:

• we naturally obtain a decomposition of the equilibrium equations into a tangential component (in the plane (O, e_1, e_2) of the initial middle surface ω_0) and a normal component.

• the naturally introduced dimensionless numbers define the domain of validity of the two-dimensional membrane model. Indeed the membrane model is valid for surface forces level such as $\mathcal{G}_t = \mathcal{G}_3 = \varepsilon$. These forces lead to displacements of order L_0, i.e to large displacements.

This remark concerning the domain of validity of the membrane model adds some importance to a limitation of Lagrangian approaches where the loads are assumed to be dead. Indeed the membrane model we have obtained is valid for large displacements, and in this case the dead loads hypothesis is not justified. However, we can notice that an Eulerian approach for plates with large displacements has been developed in [15]. With this Eulerian approach, the dead loads hypothesis can be dropped and the Eulerian membrane model we obtain takes into account the real physical forces.

4.2. Back to the dimensional variables

The return to the physical variables in the membrane equations of the RESULT 1 leads to the relations:

$$\begin{aligned} V^{*0} &= V_r V^0 = L_0 V^0, \\ u_3^{*0} &= u_{3r} u_3^0 = L_0 u_3^0. \end{aligned}$$

Therefore, we have the following result:

RESULT 3

For applied forces f^* and g^* such as $\mathcal{F}_3 = \varepsilon^4$ and $\mathcal{G}_t = \mathcal{G}_3 = \varepsilon$, the displacement (V^{*0}, u_3^0) depends only on $x^* = (x_1^*, x_2^*)$ and satisfies the following nonlinear membrane model:

$$h_0 \operatorname{div}^*\Big(N_t^{*0}\Big[I_2 + \overline{\frac{\partial V^{*0}}{\partial x^*}}\Big]\Big) = -p_t^*,$$

$$h_0 \operatorname{div}^*(N_t^{*0} \operatorname{grad}^* u_3^{*0}) = -p_3^*,$$

$$V^{*0} = 0 \quad \text{and} \quad u_3^{*0} = 0 \quad \text{on } \gamma_0^* = \partial\omega_0^*,$$

where:

$$\begin{aligned} N_t^{*0} &= \frac{4\lambda\mu}{\lambda + 2\mu} \operatorname{Tr} E_t^{*0}\, I_2 + 4\mu E_t^{*0}, \\ E_t^{*0} &= \frac{1}{2}\left(\frac{\partial V^{*0}}{\partial x^*} + \overline{\frac{\partial V^{*0}}{\partial x^*}} + \overline{\frac{\partial V^{*0}}{\partial x^*}}\frac{\partial V^{*0}}{\partial x^*} + \operatorname{grad}^* u_3^{*0}\, \overline{\operatorname{grad}^* u_3^{*0}} \right), \end{aligned}$$

$$p_3^* = g_3^{*+} + g_3^{*-}, \qquad p_t^* = g_t^{*+} + g_t^{*-}.$$

In the first part of this article we have proved, without any assumption, that the nonlinear membrane model is valid for large surface forces level. Precisely, the dimensional analysis of the three-dimensional equilibrium equations naturally leads to dimensionless numbers ($\mathcal{F}_3 = h_0 f_{3r}/\mu, \mathcal{G}_t = g_{tr}/\mu$ and $\mathcal{G}_3 = g_{3r}/\mu$) which reflect the forces level. Thus the membrane model we have obtained is valid for forces level such as $\mathcal{F}_3$ is of ε^4 order, $\mathcal{G}_t$ and $\mathcal{G}_3$ are of ε order.

In the second part of this article, we will consider a plate subjected to moderate forces level. In this case the dimensional analysis and the asymptotic expansions of the equilibrium equations lead to the nonlinear von Kármán model.

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Dimensional analysis and asymptotic expansions of the equilibrium equations in nonlinear elasticity Part II: The two-dimensional von Kármán model

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In the second part of this article, which is a continuation of [8] which dealt with a plate subjected to large loads, we consider a plate subjected to moderate applied forces level within the framework of nonlinear elasticity. We then apply the new constructive asymptotic approach developed in the first part which needs no *a priori* assumption. For these moderate forces, we prove that the two-dimensional model we obtain by asymptotic expansions is the von Kármán one. Finally the two-dimensional stress field in the plate is deduced from the three-dimensional constitutive equations without any *a priori* assumption.

1. Introduction

This paper is a continuation of [8] to which we will refer for the definitions and the notations not explained here.

In the first part we have proved that for a plate subjected to large applied forces such as $\mathcal{F}_3 = \varepsilon^4$ and $\mathcal{G}_t = \mathcal{G}_3 = \varepsilon$, the asymptotic expansion of the three-dimensional equilibrium equations leads to the nonlinear membrane model. In this part we assume the plate to be subjected to the same moderate applied forces as in the linear case [6]: $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$. We recall that in the linear case such forces lead to the two-dimensional linear Kirchhoff-Love model.

The aim of this second part is now to prove, as in the first part, that the reference scales of the displacement and the corresponding two-dimensional models we obtain by asymptotic expansions are determined by the magnitude of forces. Indeed, we prove that the force magnitude considered here ($\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$) leads to new reference scales ($u_{3r} = h_0$ and $V_r = \varepsilon h_0$) for the normal and tangential displacement. These new reference scales we obtain as a consequence of the forces applied, are formally equivalent to the scaling assumptions generally made in the literature [1–4, 9]. They naturally lead to the two-dimensional nonlinear "von Kármán model", even if the designation "von Kármán" is often reserved for particular boundary conditions. Finally, the two-dimensional stress field in the plate is deduced from the three-dimensional constitutive equations without any *a priori* assumption.

2. Determination of the reference scales

Let us consider now the same applied forces level as in the linear case [6], such as $\mathcal{G}_t = \varepsilon^3$, $\mathcal{F}_t = 0$ and $\mathcal{G}_3 = \mathcal{F}_3 = \varepsilon^4$. Then for these forces, the variational formulation of the membrane equations obtained in the first part of this article (result 2 of [8]) becomes:

$$\psi^0 \in \mathcal{Q}(\omega_0) = \Big\{\psi : \omega_0 \mapsto \mathbb{R}^3, \quad \text{“smooth”}, \quad \psi = I_2 \quad \text{on } \gamma_0\Big\},$$

$$\int\limits_{\omega_0} \mathrm{Tr}\left[N_t^0 \overline{\frac{\partial\psi^0}{\partial x}}\ \frac{\partial v^0}{\partial x}\right] d\omega_0 = 0\ , \qquad \forall v^0 \in V(\omega_0).$$

The solutions of this membrane problem without a right-hand term are the smooth enough inextensional, mapping functions defined on ω_0 ([5]). So we have $\psi^0 \in \mathcal{S}_0$ where $\mathcal{S}_0 = \Big\{\psi^0 : \omega_0 \to \mathbb{R}^3\ ;\ \overline{\frac{\partial\psi^0}{\partial x}}\frac{\partial\psi^0}{\partial x} = I_2\Big\}$ denotes the space of the inextensional mapping functions defined on ω_0. On the other hand, as the plate is assumed to be clamped on its lateral surface, the space of the inextensional mapping fuctions $\mathcal{S}_0$ reduces to the identity I_2 of $\mathbb{R}^2$. Hence we have $\psi^0 = I_2$ which implies that $V^0 = u_3^0 = 0$ (see relation (4.11) of [8]).

Since we have proved that $V^0 = u_3^0 = 0$, we get

$$V \ = \ \frac{V^*}{V_r} = \frac{V^*}{L_0} = \varepsilon V^1 + \varepsilon^2 V^2 + \cdots\ ,$$

$$u_3 \ = \ \frac{u_3^*}{u_{3r}} = \frac{u_3^*}{L_0} = \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \cdots\ ,$$

which is equivalent to

$$\widetilde{V} = \frac{V^*}{\varepsilon V_r} = \frac{V^*}{h_0} = V^1 + \varepsilon V^2 + \cdots = \widetilde{V^0} + \varepsilon\widetilde{V^1} + \varepsilon^2\widetilde{V^2} + \cdots\ ,$$

$$\widetilde{u_3} = \frac{u_3^*}{\varepsilon u_{3r}} = \frac{u_3^*}{h_0} = u_3^1 + \varepsilon u_3^2 + \cdots = \widetilde{u_3^0} + \varepsilon\widetilde{u_3^1} + \varepsilon^2\widetilde{u_3^2} + \cdots\ .$$

Hence for these forces, the reference scales of the tangential displacement $V_r = L_0$ and of the normal displacement $u_{3r} = L_0$ are not properly chosen. $\widetilde{V}$ and $\widetilde{u_3}$ will be of the order of one unit provided the reference scales of the displacement satisfy the condition $V_r = u_{3r} = h_0$. Therefore the new reference scales of the tangential displacement V^* and of the normal displacement u_3^* we have to consider are $V_r = u_{3r} = h_0$.

Consequently, the dimensionless equilibrium equations are written again with $V_r = u_{3r} = h_0$ as reference scales. The dimensionless components of the displacement will still be denoted by V and u_3. Thus for the magnitude of forces such as $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$, $\mathcal{F}_t = 0$ and $\mathcal{G}_t = \varepsilon^3$, the dimensionless equilibrium Eqs. (2.4) and (2.5) of [8] become in $\Omega_0 = \omega_0 \times]-1,1[$:

$$(2.1)\quad \begin{aligned} &\varepsilon^2\Big[(1+\beta)\mathrm{grad}(\mathrm{div}V) + \Delta V\Big] + \varepsilon\left[(1+\beta)\mathrm{grad}\frac{\partial u_3}{\partial x_3} + \mathrm{div}\,(E_{ut}+\Gamma_t)\right] \\ &\qquad + \frac{\partial}{\partial x_3}(Q+\Gamma_s) + \frac{\partial^2 V}{\partial x_3^2} = 0\,, \\ &\varepsilon^2\Delta u_3 + \varepsilon\left[(1+\beta)\frac{\partial}{\partial x_3}\mathrm{div}\,V + \mathrm{div}\,(E_{us}+\Gamma_s)\right] + \frac{\partial}{\partial x_3}(E_{un}+\Gamma_n) \\ &\qquad + (2+\beta)\frac{\partial^2 u_3}{\partial x_3^2} = -\varepsilon^4 f_3\,. \end{aligned}$$

The boundary conditions on the upper and the lower faces become:

$$(2.2)\quad \begin{aligned} &\varepsilon\,\mathrm{grad}\,u_3 + Q + \Gamma_s + \frac{\partial V}{\partial x_3} = \pm\varepsilon^3 g_t^{\pm} \quad \text{for } x_3 = \pm 1\,, \\ &\varepsilon\,\beta\,\mathrm{div}\,V + E_{un} + \Gamma_n + (2+\beta)\frac{\partial u_3}{\partial x_3} = \pm\varepsilon^4 g_3^{\pm} \quad \text{for } x_3 = \pm 1\,, \end{aligned}$$

where $\beta = \lambda/\mu$ and

$$\Gamma_t = \varepsilon^2\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2 + \frac{\beta}{2}\|\mathrm{grad}\;u_3\|^2 I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x} + \mathrm{grad}\;u_3\;\overline{\mathrm{grad}\;u_3}\right] + \frac{\beta}{2}\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right]I_2\,,$$

$$\Gamma_s = \varepsilon\left[\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\mathrm{grad}\;u_3\right],$$

$$\Gamma_n = \varepsilon^2\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) + \frac{\beta}{2}\|\mathrm{grad}\;u_3\|^2\right] + \left(1+\frac{\beta}{2}\right)\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right],$$

$$E_{ut} = \varepsilon^3\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x} + \frac{\beta}{2}\|\mathrm{grad}\;u_3\|^2\;I_2 + \;\mathrm{grad}\;u_3\overline{\mathrm{grad}\;u_3}\right]\overline{\frac{\partial V}{\partial x}}$$

[cont.]

$$+\varepsilon^2\left[\beta\,\mathrm{div}V\ I_2+\frac{\partial V}{\partial x}+\overline{\frac{\partial V}{\partial x}}\right]\overline{\frac{\partial V}{\partial x}}+\varepsilon\left\{\left(1+\frac{\partial u_3}{\partial x_3}\right)\mathrm{grad}\ u_3\ \overline{\frac{\partial V}{\partial x_3}}\right.$$

$$\left.+\beta\frac{\partial u_3}{\partial x_3}\overline{\frac{\partial V}{\partial x}}+\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3}\overline{\frac{\partial V}{\partial x_3}}+\frac{\beta}{2}\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\overline{\frac{\partial V}{\partial x}}\right\}+\frac{\partial V}{\partial x_3}\overline{\frac{\partial V}{\partial x_3}}\,,$$

$$E_{us}=\varepsilon^3\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2+\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}+\left(1+\frac{\beta}{2}\right)\|\mathrm{grad}\ u_3\|^2\ I_2\right]\mathrm{grad}\ u_3$$

$$\varepsilon^2\left[\beta\,\mathrm{div}V\ I_2+\frac{\partial V}{\partial x}+\overline{\frac{\partial V}{\partial x}}\right]\mathrm{grad}\ u_3+\varepsilon\left\{\left[\frac{\beta}{2}\left\|\frac{\partial V}{\partial x_3}\right\|^2+\left(1+\frac{\beta}{2}\right)\right.\right.$$

$$\left.\left.\times\left(\frac{\partial u_3}{\partial x_3}\right)^2+\ (1+\beta)\frac{\partial u_3}{\partial x_3}\right]\mathrm{grad}\ u_3+\frac{\partial u_3}{\partial x_3}\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3}\right\}+\frac{\partial u_3}{\partial x_3}\frac{\partial V}{\partial x_3}\,,$$

$$Q=\varepsilon^2\left\{\left(1+\frac{\partial u_3}{\partial x_3}\right)\frac{\partial V}{\partial x}\ \mathrm{grad}\ u_3+\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2+\frac{\partial V}{\partial x}\overline{\frac{\partial V}{\partial x}}\right.\right.$$

$$\left.\left.+\frac{\beta}{2}\|\mathrm{grad}\ u_3\|^2\ I_2\right]\frac{\partial V}{\partial x_3}\right\}+\varepsilon\left[\beta\mathrm{div}V\ I_2+\frac{\partial V}{\partial x}\right]\frac{\partial V}{\partial x_3}$$

$$+\left(1+\frac{\beta}{2}\right)\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2+2\frac{\partial u_3}{\partial x_3}\right]\frac{\partial V}{\partial x_3}\,,$$

$$E_{un}=\varepsilon^2\left\{\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)+\frac{\beta}{2}\|\mathrm{grad}\ u_3\|^2\right]\frac{\partial u_3}{\partial x_3}+\left(1+\frac{\partial u_3}{\partial x_3}\right)\|\mathrm{grad}\ u_3\|^2\right.$$

$$\left.+\overline{\frac{\partial V}{\partial x_3}}\frac{\partial V}{\partial x}\mathrm{grad}\ u_3\right\}+\varepsilon\left\{\beta\,\mathrm{div}V\ \frac{\partial u_3}{\partial x_3}+\overline{\frac{\partial V}{\partial x_3}}\ \mathrm{grad}\ u_3\right\}$$

$$+\left(1+\frac{\beta}{2}\right)\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2+2\frac{\partial u_3}{\partial x_3}\right]\frac{\partial u_3}{\partial x_3}\,.$$

We assume again that there exists a formal expansion with respect to ε of the new dimensionless solution (V, u_3) of the dimensionless problem (2.1)–(2.2):

$$\begin{aligned} V &= V^0 + \varepsilon V^1 + \varepsilon^2 V^2 + \cdots \,, \\ u_3 &= u_3^0 + \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \cdots \,. \end{aligned} \tag{2.3}$$

We then obtain the following result:

RESULT 1

For applied forces magnitude such as $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$, we have $V^0 = 0$.

P r o o f. The proof of this result is divided into two steps.

i) u_3^0 and V^0 depend on (x_1, x_2) only.

Replacing V and u_3 by their expansions (2.3) in the dimensionless equilibrium Eqs. (2.1) and (2.2), problem $\mathcal{P}_0$ can be written in $\Omega_0 = \omega_0 \times]-h_0, h_0[$ in the form

$$\frac{\partial}{\partial x_3}(Q^0 + \Gamma_s^0) + \frac{\partial^2 V^0}{\partial x_3^2} = 0 \,, \tag{2.4}$$

$$\frac{\partial}{\partial x_3}(\Gamma_n^0 + E_{un}^0) + (2+\beta)\frac{\partial^2 u_3^0}{\partial x_3^2} = 0 \,, \tag{2.5}$$

with the boundary conditions on the upper and the lower faces

$$Q^0 + \Gamma_s^0 + \frac{\partial V^0}{\partial x_3} = 0 \quad \text{for } x_3 = \pm 1 \,, \tag{2.6}$$

$$\Gamma_n^0 + E_{un}^0 + (2+\beta)\frac{\partial u_3^0}{\partial x_3} = 0 \quad \text{for } x_3 = \pm 1 \,, \tag{2.7}$$

and where

$$Q^0 = \left(1 + \frac{\beta}{2}\right)\left[\left\|\frac{\partial V^0}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3}\right)^2 + 2\frac{\partial u_3^0}{\partial x_3}\right]\frac{\partial V^0}{\partial x_3} \,,$$

$$E_{un}^0 = \left(1 + \frac{\beta}{2}\right)\left[\left\|\frac{\partial V^0}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3}\right)^2 + 2\frac{\partial u_3^0}{\partial x_3}\right]\frac{\partial u_3^0}{\partial x_3} \,,$$

$$\Gamma_n^0 = \left(1 + \frac{\beta}{2}\right) \left[\left\| \frac{\partial V^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 \right],$$

$$\Gamma_s^0 = 0 .$$

It is then possible to introduce the mapping function

$$\psi^1 = V^0 + (x_3 + u_3^0) e_3 = \begin{pmatrix} V^0 \\ x_3 + u_3^0 \end{pmatrix}$$

to simplify the expressions of Q^0 and E_{un}^0. So we have:

$$Q^0 = \left(1 + \frac{\beta}{2}\right) \left(\left\| \frac{\partial \psi^1}{\partial x_3} \right\|^2 - 1 \right) \frac{\partial V^0}{\partial x_3} ,$$

$$E_{un}^0 = \left(1 + \frac{\beta}{2}\right) \left(\left\| \frac{\partial \psi^1}{\partial x_3} \right\|^2 - 1 \right) \frac{\partial u_3^0}{\partial x_3} .$$

REMARK 1

It is natural to introduce ψ^1 as a function of (V^0, u_3^0). Indeed, going back to the mapping function ψ^*, we get:

$$\psi^* = I + U^* = x_\alpha^* \, e_\alpha + V^* + (x_3^* + u_3^*) \, e_3.$$

Taking into account the expansion of (V, u_3) with respect to ε, the dimensionless form of the previous equation with $V_r = u_{3r} = h_0$ can be written as

$$\psi = \frac{\psi^*}{L} = \underbrace{x_\alpha \, e_\alpha}_{\psi^0 = I_2} + \varepsilon \left[\underbrace{(x_3 + u_3^0) e_3 + V^0}_{\psi^1} \right] + \cdots . \tag{2.8}$$

Thus we obtain $\psi^0 = I_2$ which complies with the new reference scales of the displacements. ■

According to the boundary conditions (2.7), Eqs. (2.5) leads to

$$\Gamma_n^0 + E_{un}^0 + (2 + \beta) \frac{\partial u_3^0}{\partial x_3} = 0 \qquad \text{in } \Omega_0 ,$$

which can also be written as

$$\left(1 + \frac{\partial u_3^0}{\partial x_3}\right) \left(\left\| \frac{\partial \psi^1}{\partial x_3} \right\|^2 - 1 \right) = 0 \qquad \text{in } \Omega_0 .$$

Consequently we obtain:

$$\frac{\partial u_3^0}{\partial x_3} = -1, \qquad \text{or}$$
(2.9)
$$\left\| \frac{\partial \psi^1}{\partial x_3} \right\| = 1.$$

We must use again the mass conservation law to conclude the analysis. Indeed, taking into account the new reference scales of the displacements, the expansion of the condition (3.3) of [8] gives

$$\frac{\partial \psi^1}{\partial x_3} . \left(\frac{\partial \psi^0}{\partial x_1} \wedge \frac{\partial \psi^0}{\partial x_2} \right) > 0 \, .$$

As $\psi^0 = I_2$, we get $\dfrac{\partial \psi^1}{\partial x_3} . e_3 > 0$ which becomes

$$\left(1 + \frac{\partial u_3^0}{\partial x_3}\right) > 0 \qquad \text{in } \Omega_0 \, , \tag{2.10}$$

according to the expression of ψ^1. Hence Eq. (2.9) gives

$$\left\| \frac{\partial \psi^1}{\partial x_3} \right\| = 1 \, . \tag{2.11}$$

Finally, according to the boundary conditions (2.6) and to Eq. (2.11), Eq. (2.4) becomes

$$\frac{\partial V^0}{\partial x_3} = 0 \qquad \text{in } \Omega_0$$

and yields

$$V^0 = V^0(x_1, x_2) \qquad \text{in } \Omega_0 \, . \tag{2.12}$$

Therefore equation (2.11) becomes

$$\left(\frac{\partial u_3^0}{\partial x_3} \right)^2 + 2 \, \frac{\partial u_3^0}{\partial x_3} = 0$$

and according to the condition (2.10) we have $\dfrac{\partial u_3^0}{\partial x_3} = 0$, or equivalently

$$u_3^0 = u_3^0(x_1, x_2) \qquad \text{in } \Omega_0 \, . \tag{2.13}$$

ii) Equation satisfied by V^0.

Taking into account Eqs. (2.12) and (2.13), we have $E^0_{us} = 0$. Thus the second equation and the second boundary condition of problem $\mathcal{P}_1$ can be written as:

$$\frac{\partial}{\partial x_3}(\Gamma^1_n + E^1_{un}) + (2+\beta)\frac{\partial^2 u^1_3}{\partial x_3^2} = 0 \qquad \text{in } \Omega_0, \tag{2.14}$$

$$\Gamma^1_n + E^1_{un} + (2+\beta)\frac{\partial u^1_3}{\partial x_3} + \beta \operatorname{div} V^0 = 0 \qquad \text{for } x_3 = \pm 1 . \tag{2.15}$$

These two equations lead to a relation which couples u^1_3 and V^0 :

$$\frac{\partial u^1_3}{\partial x_3} = -\frac{\beta}{2+\beta} \operatorname{div} V^0 . \tag{2.16}$$

The cancellation of the coefficient of ε^2 leads to the problem $\mathcal{P}_2$ whose first equation and first boundary condition can be written in the form

$$(1+\beta)\operatorname{grad}(\operatorname{div} V^0) + \Delta V^0 + (1+\beta)\operatorname{grad} \frac{\partial u^1_3}{\partial x_3} + \frac{\partial}{\partial x_3}(Q^2 + \Gamma^2_s) + \frac{\partial^2 V^2}{\partial x_3^2} = 0 , \tag{2.17}$$

$$\operatorname{grad} u^1_3 + Q^2 + \Gamma^2_s + \frac{\partial V^2}{\partial x_3} = 0 \qquad \text{for } x_3 = \pm 1 . \tag{2.18}$$

Using the boundary condition (2.18), the integration from -1 to 1 of Eq. (2.17) gives:

$$\int_{-1}^{1} \left[(1+\beta)\operatorname{grad}(\operatorname{div} V^0) + \Delta V^0 + \beta \operatorname{grad}\left(\frac{\partial u^1_3}{\partial x_3}\right)\right] dx_3 = 0 .$$

Then replacing $\partial u^1_3/\partial x_3$ by expression (2.16), the previous equation becomes

$$\operatorname{div}(n^0_t) = 0$$

with

$$n^0_t = \frac{4\beta}{2+\beta}\operatorname{Tr} e^0_t(V^0)\ I_2 + 4e^0_t(V^0) ,$$

$$e^0_t(V^0) = \frac{1}{2}\left(\frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}}\right) = \frac{1}{2}\Big(\operatorname{grad} V^0 + \overline{\operatorname{grad} V^0}\Big) .$$

As the plate is assumed to be clamped on its lateral surface, we obtain a linear boundary problem:

$$\begin{aligned} \operatorname{div}(n_t^0) &= 0 \qquad \text{in } \omega_0 , \\ V^0|_{\partial\omega_0} &= 0 , \end{aligned}$$

which has (if we assume that V^0 is smooth enough) a unique solution $V^0 = 0$ in $\left[H_0^1(\omega_0)\right]^2$.

Therefore for these forces level, the reference scale of the tangential displacement $V_r = h_0$ is still not properly chosen. We have to consider $V_r = \varepsilon h_0$. Thus the dimensionless equilibrium equations must be written again with $u_{3r} = h_0$ and $V_r = \varepsilon h_0$ as the reference scales. The dimensionless components of the displacement will still be denoted by V and u_3.

3. The two-dimensional von Kármán model

With the new reference scales of the displacement ($u_{3r} = h_0$ and $V_r = \varepsilon h_0$) which are obtained as a consequence of the order of the forces applied, the dimensionless equilibrium equations assume in $\Omega_0 = \omega_0 \times]-1,1[$ the form

$$(3.1) \qquad \begin{aligned} &\varepsilon^2[(1+\beta)\operatorname{grad}\operatorname{div}V + \Delta V] + (1+\beta)\operatorname{grad}\frac{\partial u_3}{\partial x_3} + \frac{\partial^2 V}{\partial x_3^2} \\ &\qquad + \operatorname{div}\,(E_{ut} + \Gamma_t) + \frac{1}{\varepsilon}\frac{\partial}{\partial x_3}(Q + \Gamma_s) = 0 , \\ &\varepsilon^2\left[(1+\beta)\frac{\partial}{\partial x_3}\operatorname{div}V + \Delta u_3\right] + \varepsilon\,\operatorname{div}\,(E_{us} + \Gamma_s) + (2+\beta)\frac{\partial^2 u_3}{\partial x_3^2} \\ &\qquad + \frac{\partial}{\partial x_3}(E_{un} + \Gamma_n) = -\varepsilon^4 f_3 . \end{aligned}$$

The boundary conditions on the upper and the lower faces $\Gamma_{0\pm}$ become:

$$(3.2) \qquad \begin{aligned} \operatorname{grad} u_3 + \frac{\partial V}{\partial x_3} + \frac{1}{\varepsilon}(Q + \Gamma_s) &= \pm\varepsilon^2 g_t^{\pm} \qquad \text{for } x_3 = \pm 1 , \\ (2+\beta)\frac{\partial u_3}{\partial x_3} + \beta\varepsilon^2\operatorname{div}V + E_{un} + \Gamma_n &= \pm\varepsilon^4 g_3^{\pm} \qquad \text{for } x_3 = \pm 1 , \end{aligned}$$

where

$$\Gamma_t = \varepsilon^4\left\{\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right\} + \varepsilon^2\left\{\frac{\beta}{2}\left(\|\mathrm{grad}\ u_3\|^2 + \left\|\frac{\partial V}{\partial x_2}\right\|^2\right)I_2 + \mathrm{grad}\ u_3\ \overline{\mathrm{grad}\ u_3}\right\} + \frac{\beta}{2}\left(\frac{\partial u_3}{\partial x_3}\right)^2 I_2\ ,$$

$$\Gamma_s = \varepsilon^3\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3} + \varepsilon\frac{\partial u_3}{\partial x_3}\mathrm{grad}\ u_3\ ,$$

$$\Gamma_n = \varepsilon^4\ \frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) + \varepsilon^2\left\{\frac{\beta}{2}\|\mathrm{grad}\ u_3\|^2 + \left(1+\frac{\beta}{2}\right)\left\|\frac{\partial V}{\partial x_3}\right\|^2\right\} + \left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3}{\partial x_3}\right)^2,$$

$$E_{ut} = \varepsilon^6\left\{\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)\overline{\frac{\partial V}{\partial x}} + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\overline{\frac{\partial V}{\partial x}}\right\} + \varepsilon^4\left\{\left[\beta\ \mathrm{div}V\ I_2 + \frac{\beta}{2}\left(\left\|\frac{\partial V}{\partial x_3}\right\|^2 + \|\mathrm{grad}\ u_3\|^2\ \right)I_2 + \left(\frac{\partial V}{\partial x} + \overline{\frac{\partial V}{\partial x}}\right) + \mathrm{grad}\ u_3\ \overline{\mathrm{grad}\ u_3}\right]\overline{\frac{\partial V}{\partial x}} + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3}\overline{\frac{\partial V}{\partial x_3}}\right\}$$
$$+\ \varepsilon^2\left\{\left[\frac{\partial V}{\partial x_3} + \mathrm{grad}\ u_3 + \frac{\partial u_3}{\partial x_3}\mathrm{grad}\ u_3\right]\overline{\frac{\partial V}{\partial x_3}} + \left[\beta\frac{\partial u_3}{\partial x_3} + \frac{\beta}{2}\left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\overline{\frac{\partial V}{\partial x}}\right\},$$

$$E_{us} = \varepsilon^5\left\{\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right\}\mathrm{grad}\ u_3 + \varepsilon^3\left\{\left[\beta\mathrm{div}\ V\ I_2 + \frac{\beta}{2}\left\|\frac{\partial V}{\partial x_3}\right\|^2 I_2 + \left(1+\frac{\beta}{2}\right)\|\mathrm{grad}\ u_3\|^2\ I_2 + \left(\frac{\partial V}{\partial x} + \overline{\frac{\partial V}{\partial x}}\right)\right]\mathrm{grad}\ u_3 + \frac{\partial u_3}{\partial x_3}\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3}\right\}$$
$$+\ \varepsilon\left\{\left[\left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3}{\partial x_3}\right)^2 + (1+\beta)\frac{\partial u_3}{\partial x_3}\right]\mathrm{grad}\ u_3 + \frac{\partial u_3}{\partial x_3}\frac{\partial V}{\partial x_3}\right\},$$

$$Q = \varepsilon^5 \left\{ \frac{\beta}{2} \mathrm{Tr} \left(\overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x} \right) \frac{\partial V}{\partial x_3} + \frac{\partial V}{\partial x} \overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x_3} \right\} + \varepsilon^3 \left\{ \left[\left[\beta \operatorname{div} V \ I_2 \right.\right.\right.$$

$$\left.+ \frac{\beta}{2} \|\mathrm{grad}\ u_3\|^2 dis + \left(1 + \frac{\beta}{2}\right) \left\| \frac{\partial V}{\partial x_3} \right\|^2 \right] \frac{\partial V}{\partial x_3} + \frac{\partial V}{\partial x} \left[\frac{\partial V}{\partial x_3} + \mathrm{grad}\ u_3 \right.$$

$$\left.\left. + \frac{\partial u_3}{\partial x_3} \mathrm{grad}\ u_3 \right] \right\} + \varepsilon \left[\left(1 + \frac{\beta}{2}\right) \left(\frac{\partial u_3}{\partial x_3} \right)^2 + (2 + \beta) \frac{\partial u_3}{\partial x_3} \right] \frac{\partial V}{\partial x_3} ,$$

$$E_{un} = \varepsilon^4 \left\{ \frac{\beta}{2} \mathrm{Tr} \left(\overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x} \right) \frac{\partial u_3}{\partial x_3} + \overline{\frac{\partial V}{\partial x_3}} \frac{\partial V}{\partial x} \mathrm{grad}\ u_3 \right\}$$

$$+ \varepsilon^2 \left\{ \left[\beta \operatorname{div} V + \left(1 + \frac{\beta}{2}\right) \|\mathrm{grad}\ u_3\|^2 + \left(1 + \frac{\beta}{2}\right) \left\| \frac{\partial V}{\partial x_3} \right\|^2 \right] \frac{\partial u_3}{\partial x_3} \right.$$

$$\left. + \overline{\frac{\partial V}{\partial x_3}} \mathrm{grad}\, u_3 + \|\mathrm{grad}\ u_3\|^2 \right\} + \left[(2 + \beta) \frac{\partial u_3}{\partial x_3} + \left(1 + \frac{\beta}{2}\right) \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right] \frac{\partial u_3}{\partial x_3} .$$

The asymptotic expansion method enables us to write again the new dimensionless solution (V, u_3) of the problem (3.1) – (3.2) as a formal expansion with respect to ε :

$$\begin{aligned} V &= V^0 + \varepsilon V^1 + \varepsilon^2 V^2 + \cdots , \\ u_3 &= u_3^0 + \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \cdots . \end{aligned} \tag{3.3}$$

Then replacing V and u_3 by their expansions in equations (3.1) – (3.2) and equating to zero the coefficients of the powers of ε, we obtain again a chain of coupled problems.

RESULT 2

For the applied given forces of the magnitude such as $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$, the leading term (V^0, u_3^0) of the expansion of (V, u_3) is a Kirchhoff-Love displacement which satisfies the following conditions

i) $u_3^0 = \zeta_3^0(x_1, x_2)$, $\qquad V^0 = \zeta_t^0(x_1, x_2) - x_3 \ \mathrm{grad}\ \zeta_3^0$;

ii) $\zeta^0 = (\zeta_t^0, \zeta_3^0)$ is a solution of the following problem:

$$\operatorname{div} n_t^0 = -p_t \quad \text{in } \omega_0 ,$$

$$\operatorname{div}(\operatorname{div} m_t^0) + \operatorname{div}(n_t^0 \operatorname{grad} u_3^0) = -p_3 - \operatorname{div} M_t \quad \text{in} \quad \omega_0 ,$$

$$\zeta_3^0 = \frac{\partial \zeta_3^0}{\partial \nu} = 0 \quad \text{and} \quad \zeta_t^0 = 0 \quad \text{on } \gamma_0 ,$$

where ν denotes the external unit normal to γ_0, and where

$$n_t^0 = \frac{4\beta}{2+\beta} \operatorname{Tr} E_t^0(\zeta^0) I_2 + 4E_t^0(\zeta^0) ,$$

$$E_t^0(\zeta^0) = \frac{1}{2}\Big(\operatorname{grad} \zeta_t^0 + \overline{\operatorname{grad} \zeta_t^0} + \operatorname{grad} \zeta_3^0 \, \overline{\operatorname{grad} \zeta_3^0}\Big) ,$$

$$m_t^0 = -\Big\{ \frac{4\beta}{3(2+\beta)} \Delta u_3^0 I_2 + \frac{4}{3} \operatorname{grad} (\operatorname{grad} u_3^0) \Big\} ,$$

$$p_3 = \int_{-1}^{1} f_3 dx_3 + g_3^+ + g_3^- , \qquad p_t = g_t^+ + g_t^- , \qquad M_t = g_t^+ - g_t^- .$$

P r o o f. The proof of this result is split into several steps from i) to iii).

i) u_3^0 et V^0 is a Kirchhoff-Love displacement.

The cancellation of the factor of ε^0 leads to the problem $\mathcal{P}_0$:

(3.4) $$(1+\beta) \operatorname{grad} \frac{\partial u_3^0}{\partial x_3} + \frac{\partial^2 V^0}{\partial x_3^2} + \frac{\partial}{\partial x_3}(Q^1 + \Gamma_s^1) + \operatorname{div} (\Gamma_t^0) = 0 \quad \text{in } \Omega_0 ,$$

(3.5) $$(2+\beta) \frac{\partial^2 u_3^0}{\partial x_3^2} + \frac{\partial}{\partial x_3}(E_{un}^0 + \Gamma_n^0) = 0 \quad \text{in } \Omega_0 ,$$

(3.6) $$\frac{\partial V^0}{\partial x_3} + \operatorname{grad} u_3^0 + Q^1 + \Gamma_s^1 = 0 \quad \text{for } x_3 = \pm 1 ,$$

(3.7) $$(2+\beta) \frac{\partial u_3^0}{\partial x_3} + E_{un}^0 + \Gamma_n^0 = 0 \quad \text{for } x_3 = \pm 1 ,$$

with

$$Q^1 = \left[\left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3^0}{\partial x_3}\right)^2 + (2+\beta)\frac{\partial u_3^0}{\partial x_3}\right]\overline{\frac{\partial V^0}{\partial x_3}}\,, \quad \Gamma_s^1 = \frac{\partial u_3^0}{\partial x_3}\,\mathrm{grad}\, u_3^0,$$

$$E_{un}^0 = \left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3^0}{\partial x_3}\right)^3 + (2+\beta)\left(\frac{\partial u_3^0}{\partial x_3}\right)^2\,, \quad \Gamma_t^0 = \frac{\beta}{2}\left(\frac{\partial u_3^0}{\partial x_3}\right) I_2,$$

$$\Gamma_n^0 = (1+\frac{\beta}{2})\left(\frac{\partial u_3^0}{\partial x_3}\right)^2.$$

Equations (3.5) and (3.7) lead to:

$$(2+\beta)\frac{\partial u_3^0}{\partial x_3} + E_{un}^0 + \Gamma_n^0 = 0 \qquad \text{in } \Omega_0\,.$$

Equivalently we get

$$(2+\beta)\frac{\partial u_3^0}{\partial x_3}\left[1+\frac{3}{2}\frac{\partial u_3^0}{\partial x_3} + \frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_3}\right)^2\right] = 0 \qquad \text{in } \Omega_0$$

which gives

$$\frac{\partial u_3^0}{\partial x_3} = 0 \quad \text{or} \quad \frac{\partial u_3^0}{\partial x_3} = -1 \quad \text{or} \quad \frac{\partial u_3^0}{\partial x_3} = -2 \qquad \text{in } \Omega_0\,.$$

Then using the mass conservation law condition (2.10):

$$\left(1+\frac{\partial u_3^0}{\partial x_3}\right) > 0 \qquad \text{in } \Omega_0\,,$$

which is still valid with the new reference scales, we have $\dfrac{\partial u_3^0}{\partial x_3} = 0$ which implies

(3.8) $$u_3^0 = \zeta_3^0(x_1, x_2) \qquad \text{in } \Omega_0\,.$$

On the other hand, since the condition (2.10) can be extended to the upper and the lower faces $\Gamma_\pm$, equation (3.7) leads to

(3.9) $$\frac{\partial u_3^0}{\partial x_3} = 0 \qquad \text{on } \Gamma_\pm\,.$$

REMARK 2

In [2] and [3] appears a similar indeterminacy in the evaluation of $\partial u_3^0/\partial x_3$, even if a variational formulation of the problem is used. To remove this problem, P.G. Ciarlet and P. Destuynder assume that $\dfrac{\partial u_3^0}{\partial x_3} \in C^0(\overline{\Omega}_0)$. With the new approach presented in this paper, the condition (2.10), which is obtained as a consequence of the mass conservation law, enables us to drop this additional assumption concerning the regulariy of $\partial u_3^0/\partial x_3$. ∎

Then according to (3.8) and (3.9), equations (3.4) and (3.6) become

$$\frac{\partial^2 V^0}{\partial x_3^2} = 0 \qquad \text{in } \Omega_0 ,$$

$$\frac{\partial V^0}{\partial x_3} + \text{grad } u_3^0 = 0 \qquad \text{for } x_3 = \pm 1 ,$$

and lead to

(3.10) $$V^0 = \zeta_t^0(x_1, x_2) - x_3 \text{grad } \zeta_3^0 ,$$

where $\zeta^0 = (\zeta_t^0, \zeta_3^0)$ represents the displacement of the middle surface ω_0. Hence step i) of the RESULT 2 is proved.

Since the problems are two-two coupled in the linear case, problem $\mathcal{P}_1$ leads to a similar result with u_3^1 and V^1. Even if this result is not necessary to prove the steps ii) and iii), it will be used for the stress calculus. So it is explained here.

Taking into account (3.8), (3.9) and (3.10), problem $\mathcal{P}_1$ can be written as

(3.11) $$(1+\beta)\text{grad}\frac{\partial u_3^1}{\partial x_3} + \frac{\partial^2 V^1}{\partial x_3^2} + \frac{\partial}{\partial x_3}(Q^2 + \Gamma_s^2) = 0 \qquad \text{in } \Omega_0 ,$$

(3.12) $$(2+\beta)\frac{\partial^2 u_3^1}{\partial x_3^2} = 0 \qquad \text{in } \Omega_0 ,$$

(3.13) $$\frac{\partial V^1}{\partial x_3} + \text{grad } u_3^1 + Q^2 + \Gamma_s^2 = 0 \qquad \text{for } x_3 = \pm 1 ,$$

(3.14) $$(2+\beta)\frac{\partial u_3^1}{\partial x_3} = 0 \qquad \text{for } x_3 = \pm 1 ,$$

with

$$Q^2 + \Gamma_s^2 = (2+\beta)\frac{\partial u_3^1}{\partial x_3}\frac{\partial V^0}{\partial x_3} + \frac{\partial u_3^1}{\partial x_3}\text{grad } u_3^0 .$$

Then equations (3.12) and (3.14) immediately lead to

(3.15) $$u_3^1 = \zeta_3^1(x_1, x_2) .$$

Hence equations (3.11) and (3.13) give

$$\frac{\partial V^1}{\partial x_3} + \text{grad } u_3^1 = 0 \qquad \text{in } \Omega_0 ,$$

or equivalently

(3.16) $$V^1 = \zeta_t^1(x_1, x_2) - x_3 \text{ grad } \zeta_3^1 .$$

Therefore the second term (V^1, u_3^1) of the expansion of (V, u_3) is a Kirchhoff-Love displacement as well.

ii) Extension-compression equation

According to the results previously obtained, the cancellation of the coefficient of ε^2 leads to the problem $\mathcal{P}_2$ in Ω_0 :

(3.17) $$(1+\beta)\text{grad div}V^0 + \Delta V^0 + \beta\text{grad } \frac{\partial u_3^2}{\partial x_3} + \frac{\partial}{\partial x_3}(\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3}) + \text{div } \Gamma_t^2 + \frac{\partial}{\partial x_3}(Q^3 + \Gamma_s^3) = 0 ,$$

(3.18) $$(1+\beta)\frac{\partial}{\partial x_3}\text{div}V^0 + \Delta u_3^0 + (2+\beta)\frac{\partial^2 u_3^2}{\partial x_3^2} + \frac{\partial}{\partial x_3}\Gamma_n^2 = 0 ,$$

(3.19) $$\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3} + Q^3 + \Gamma_s^3 = \pm g_t^{\pm} \qquad \text{for } x_3 = \pm 1 ,$$

(3.20) $$(2+\beta)\frac{\partial u_3^2}{\partial x_3} + \beta\,\text{div}V^0 + \Gamma_n^2 = 0 \qquad \text{for } x_3 = \pm 1 ,$$

with

$$Q^3 = \left[\beta\,\text{div } V^0 + \frac{\beta}{2}\|\text{grad } u_3^0\|^2 + \left(1+\frac{\beta}{2}\right)\left\|\frac{\partial V^0}{\partial x_3}\right\|^2\right]\frac{\partial V^0}{\partial x_3} + (2+\beta)\frac{\partial u_3^2}{\partial x_3}\frac{\partial V^0}{\partial x_3}$$

$$\Gamma_t^2 = \frac{\beta}{2}\left(\|\text{grad } u_3^0\|^2 + \left\|\frac{\partial V^0}{\partial x_3}\right\|^2 \right) I_2 + \text{grad } u_3^0\ \overline{\text{grad } u_3^0} ,$$

$$\Gamma_n^2 = \frac{\beta}{2}\|\text{grad } u_3^0\|^2 + \left(1+\frac{\beta}{2}\right)\left\|\frac{\partial V^0}{\partial x_3}\right\|^2 , \qquad \Gamma_s^3 = \frac{\partial u_3^2}{\partial x_3}\text{grad } u_3^0 + \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x_3}$$

Then replacing the expression (3.10) of V^0 in (3.17) and integrating the equation so obtained from -1 to 1, we have

$$2(1+\beta)\ \text{grad}\ (\text{div}\ \zeta_t^0) + 2\Delta\ \zeta_t^0 + \int_{-1}^{1} \beta\ \text{grad}\ \frac{\partial u_3^2}{\partial x_3} dx_3 + \int_{-1}^{1} \text{div}\ (\Gamma_t^2) dx_3 = -(g_t^+ + g_t^-) \tag{3.21}$$

thanks to the boundary condition (3.19).

On the other hand, taking into account (3.10), Eq. (3.18) can be written as

$$\frac{\partial}{\partial x_3}[(2+\beta)\frac{\partial u_3^2}{\partial x_3} + \beta\ \text{div} V^0 + \Gamma_n^2] = 0\ ,$$

and becomes, in view of (3.20),

$$\frac{\partial u_3^2}{\partial x_3} = -\frac{\beta}{2+\beta}\text{div} V^0 - \frac{1}{2+\beta}\Gamma_n^2\ ,$$

or equivalently

$$\frac{\partial u_3^2}{\partial x_3} = -\frac{\beta}{2+\beta}\ \text{div}\ V^0 - \frac{\beta}{2(2+\beta)}\left\|\text{grad}\ u_3^0\right\|^2 - \frac{1}{2}\|\frac{\partial V^0}{\partial x_3}\|^2. \tag{3.22}$$

Then using (3.22) and the relation

$$\text{div}(\alpha I_2) = \text{grad}\ \alpha \quad \text{for all scalar fields}\ \alpha\ ,$$

equation (3.21) finally becomes

$$\frac{4+6\beta}{2+\beta}\ \text{grad div}\ \zeta_t^0 + 2\Delta\zeta_t^0 + \text{div}\ \Big(\frac{2\beta}{2+\beta}\|\text{grad}\ u_3^0\|^2 I_2 + 2\text{grad}\ u_3^0\ \overline{\text{grad}\ u_3^0}\ \Big) = -(g_t^+ + g_t^-)\ ,$$

or equivalently

$$\text{div}\ n_t^0 = -p_t\ ,$$

with

$$n_t^0 = \frac{4\beta}{2+\beta}\text{Tr}\ E_t^0(\zeta^0) I_2 + 4E_t^0(\zeta^0)\ ,$$

$$E_t^0(\zeta^0) = \frac{1}{2}(\text{grad}\ \zeta_t^0 + \overline{\text{grad}\ \zeta_t^0} + \text{grad}\ u_3^0\ \overline{\text{grad}\ u_3^0})\ ,$$

$$p_t = g_t^+ + g_t^-\ .$$

Step ii) of the result 2 is then proved.

iii) Equation of bending

Now replacing the expression (3.10) of V^0 in (3.17), we obtain

$$-x_3(2+\beta)\text{grad } \Delta\, u_3^0 + \beta \text{ grad } \frac{\partial u_3^2}{\partial x_3} + \frac{\partial}{\partial x_3}\left(\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3}\right)$$
$$+[(1+\beta)\text{grad } (\text{div } \zeta_t^0) + \Delta\, \zeta_t^0] + \text{div } \Gamma_t^2 + \frac{\partial}{\partial x_3}(Q^3 + \Gamma_s^3) = 0\,.$$

Then after multiplying the last equation by x_3 and taking its divergence, an integration between -1 and 1 leads to

$$(3.23)\qquad -\frac{2}{3}(2+\beta)\Delta^2 u_3^0 + \beta \int\limits_{-1}^{1} x_3 \frac{\partial}{\partial x_3}\Delta u_3^2 dx_3 + \int\limits_{-1}^{1} x_3 \frac{\partial}{\partial x_3}(\Delta u_3^2$$
$$+\frac{\partial}{\partial x_3}\text{div} V^2)dx_3 + \int\limits_{-1}^{1} x_3 \frac{\partial}{\partial x_3}\text{div } (Q^3 + \Gamma_s^3) = 0\,.$$

Now using (3.22) it is possible to express the second term of (3.23) in terms of u_3^0. Indeed we have

$$(3.24)\qquad \int\limits_{-1}^{1} x_3 \frac{\partial}{\partial x_3}\Delta u_3^2 dx_3 = \frac{2\beta}{3(2+\beta)}\Delta^2 u_3^0\,.$$

On the other hand, integration by parts of the third and fourth terms of (3.23) leads to

$$\int\limits_{-1}^{1} x_3 \frac{\partial}{\partial x_3}\Big[\Delta u_3^2 + \frac{\partial}{\partial x_3}\text{div} V^2 + \text{div } (Q^3 + \Gamma_s^3)\Big] dx_3 = \Big[x_3 \text{ div } (\text{grad } u_3^2$$
$$+\frac{\partial V^2}{\partial x_3} + Q^3 + \Gamma_s^3)\Big]_{-1}^{1} - \int\limits_{-1}^{1}\Big[\Delta\, u_3^2 + \frac{\partial}{\partial x_3}\text{div} V^2 + \text{div}(Q^3 + \Gamma_s^3)\Big] dx_3\,,$$

and with the boundary condition (3.19) we obtain

$$\Big[x_3 \text{ div}(\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3} + Q^3 + \Gamma_s^3)\Big]_{-1}^{1} = \text{div}(g_t^+ - g_t^-)\,.$$

Finally equation (3.23) becomes

$$(3.25)\qquad \frac{8}{3}\frac{\beta+1}{\beta+2}\Delta^2 u_3^0 + \int\limits_{-1}^{1}\Big\{\Delta\, u_3^2 + \frac{\partial}{\partial x_3}\mathrm{div}V^2 + \mathrm{div}(Q^3 + \Gamma_s^3)\Big\}dx_3 = \mathrm{div}M_t$$

where

$$M_t = g_t^+ - g_t^- \ .$$

Now in order to eliminate the unknowns u_3^2 and V^2 we have to analyse the problem $\mathcal{P}_4$. The second equation and the second boundary condition of the problem $\mathcal{P}_4$ are given by

$$(3.26)\qquad (1+\beta)\frac{\partial}{\partial x_3}\mathrm{div}V^2 + \Delta u_3^2 + (2+\beta)\frac{\partial^2 u_3^4}{\partial x_3^2} + \mathrm{div}(E_{us}^3 + \Gamma_s^3) + \frac{\partial}{\partial x_3}(E_{un}^4 + \Gamma_n^4) = -f_3 \ \text{ on } \ \Omega_0 \ ,$$

$$(3.27)\qquad (2+\beta)\frac{\partial u_3^4}{\partial x_3} + \beta\,\mathrm{div}V^2 + E_{un}^4 + \Gamma_n^4 = \pm g_3^\pm \quad \text{for} \quad x_3 = \pm 1 \ ,$$

with

$$(3.28)\qquad E_{us}^3 = \Big[\beta\,\mathrm{div}\ V^0 + \Big(1+\frac{\beta}{2}\Big)\|\mathrm{grad}\ u_3^0\|^2 + \frac{\beta}{2}\Big\|\frac{\partial V^0}{\partial x_3}\Big\|^2\Big]\mathrm{grad}\ u_3^0 + \left(\frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}}\right)\mathrm{grad}\ u_3^0 + (1+\beta)\frac{\partial u_3^2}{\partial x_3}\mathrm{grad}\ u_3^0 + \frac{\partial u_3^2}{\partial x_3}\frac{\partial V^0}{\partial x_3} \ ,$$

$$\Gamma_s^3 = \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x_3} + \frac{\partial u_3^2}{\partial x_3}\mathrm{grad}\ u_3^0 \ .$$

Then integrating (3.26) between -1 and 1, and using the boundary condition (3.27), we get

$$\int\limits_{-1}^{1}\Big[\frac{\partial}{\partial x_3}\mathrm{div}V^2 + \Delta u_3^2\Big]dx_3 + \int\limits_{-1}^{1}\mathrm{div}(E_{us}^3 + \Gamma_s^3)dx_3 = -p_3$$

where

$$p_3 = \int\limits_{-1}^{1} f_3 dx_3 + g_3^+ + g_3^- .$$

Hence we have:

$$\int_{-1}^{1}\Big(\Delta u_3^2+\frac{\partial}{\partial x_3}\mathrm{div}V^2+\mathrm{div}(Q^3+\Gamma_s^3\Big)dx_3=-p_3+\int_{-1}^{1}\mathrm{div}(Q^3-E_{us}^3)dx_3\ .$$

Replacing then Q^3 and E_{us}^3 by their respective expressions and using the coupling relation (3.22) we obtain

$$Q^3-E_{us}^3=-\left[\frac{2\beta}{2+\beta}\,\mathrm{div}\,V^0+\frac{2+2\beta}{2+\beta}\|\mathrm{grad}\ u_3^0\|^2\right]\ \mathrm{grad}\ u_3^0$$

$$-\left(\overline{\frac{\partial V^0}{\partial x}}+\frac{\partial V^0}{\partial x}\right)\ \mathrm{grad}\ u_3^0\ .$$

Finally the bending equation (3.25) becomes

$$\frac{8}{3}\frac{\beta+1}{\beta+2}\Delta^2u_3^0-\mathrm{div}\ \left[\left(\frac{4\beta}{2+\beta}\mathrm{div}\ \zeta_t^0\ +\frac{4+4\beta}{2+\beta}\|\mathrm{grad}\ u_3^0\|^2\right)\ \mathrm{grad}\ u_3^0\right]$$

$$-2\mathrm{div}\ \left[\left(\overline{\frac{\partial V^0}{\partial x}}+\frac{\partial V^0}{\partial x}\right)\ \mathrm{grad}\ u_3^0\right]=p_3+\mathrm{div}M_t\ ,$$

or equivalently

$$\frac{8}{3}\frac{\beta+1}{\beta+2}\Delta^2u_3^0-\mathrm{div}(n_t^0\ \mathrm{grad}\ u_3^0)=p_3+\mathrm{div}M_t\ ,$$

with

$$n_t^0\ =\ \frac{4\beta}{2+\beta}\mathrm{Tr}E_t^0(\zeta^0)I_2+4E_t(\zeta^0)\ ,$$

$$E_t^0(\zeta^0)\ =\ \frac{1}{2}(\mathrm{grad}\ \zeta_t^0+\overline{\mathrm{grad}\ \zeta_t^0}+\mathrm{grad}\ u_3^0\ \overline{\mathrm{grad}\ u_3^0})\ ,$$

$$p_3\ =\ \int_{-1}^{1}f_3dx_3+g_3^++g_3^-\ ,\qquad M_t=g_t^+-g_t^-\ .$$

It is also possible to write the last equation in a more classical form. To this end, let us define the dimensionless tensor of bending moments:

$$m_t^0=-\left\{\frac{4\beta}{3(2+\beta)}\Delta u_3^0I_2+\frac{4}{3}\mathrm{grad}(\mathrm{grad}\ u_3^0)\right\}\ .$$

Therefore we have

$$\operatorname{div}(\operatorname{div} m_t^0) + \operatorname{div}(n_t^0 \operatorname{grad} u_3^0) = -p_3 - \operatorname{div} M_t \ ,$$

which concludes the proof of the RESULT 2.

4. Stress analysis

Now the components of the stress tensor can be deduced from the three-dimensional constitutive equations and from the previous results. We recall that the constitutive equation of Saint-Venant-Kirchhoff materials is given by

$$\Sigma^* = \lambda \operatorname{Tr} E^* I + 2\mu E^* \ ,$$

$$E^* = \frac{1}{2}(\overline{F^*}F^* - I) = \frac{1}{2}\left(\overline{\frac{\partial U^*}{\partial x^*}} + \frac{\partial U^*}{\partial x^*}\right) + \frac{1}{2}\overline{\frac{\partial U^*}{\partial x^*}}\frac{\partial U^*}{\partial x^*} = e^* + \gamma^* \ ,$$

where e^* and γ^* denote, respectively, the linear and the nonlinear parts of the Green-Lagrange strain tensor E^*. So it is possible to write Σ^* in the following form:

$$\Sigma^* = \sigma^* + \Gamma^* \ ,$$

where

$$\sigma^* = \lambda \operatorname{Tr} e^* I + 2\mu\, e^* \ ,$$

$$\Gamma^* = \lambda \operatorname{Tr} \gamma^* I + 2\mu \gamma^* \ .$$

Then decomposing U^* as $U^* = V^* + u_3^* e_3$ and writing the components of Σ^* in a dimensionless form with $u_{3r} = h_0$ and $V_r = \varepsilon u_{3r}$, we obtain

$$\frac{\Sigma_t^*}{\mu} = \beta \frac{\partial u_3}{\partial x_3} I_2 + \varepsilon^2 \left\{ 2e_t(V) + \frac{\beta}{2}\left(2 \operatorname{div} V + \|\operatorname{grad} u_3\|^2 + \|\frac{\partial V}{\partial x_3}\|^2\right) I_2 \right.$$
$$\left. + \operatorname{grad} u_3\, \overline{\operatorname{grad} u_3} \right\} + O(\varepsilon^3) \ ,$$

$$\frac{\Sigma_s^*}{\mu} = \varepsilon \left\{ \operatorname{grad} u_3 + \frac{\partial V}{\partial x_3} + \frac{\partial u_3}{\partial x_3} \operatorname{grad} u_3 \right\} + \varepsilon^3 \overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x_3} + O(\varepsilon^4) \ ,$$

$$\frac{\Sigma_n^*}{\mu} = (2+\beta)\frac{\partial u_3}{\partial x_3} + \left(1 + \frac{\beta}{2}\right)\left(\frac{\partial u_3}{\partial x_3}\right)^2 + \varepsilon^2 \left\{ \beta \operatorname{div} V + \frac{\beta}{2}\|\operatorname{grad} u_3\|^2 \right.$$
$$\left. + \left(1 + \frac{\beta}{2}\right)\left\|\frac{\partial V}{\partial x_3}\right\|^2 \right\} + \varepsilon^4 \left\{ \frac{\beta}{2} \operatorname{Tr}\left(\overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x} \right) \right\} + O(\varepsilon^5) \ .$$

Now let us replace the dimensionless components of the displacement (V, u_3) by their expansions (3.3) and let us use some of the previous results to simplify the formulas. On the one hand, the first and the second terms of the expansion of (V, u_3) satisfy the kinematic Kirchhoff-Love hypothesis :

$$\frac{\partial u_3^0}{\partial x_3} = 0 , \qquad \frac{\partial V^0}{\partial x_3} + \text{grad } u_3^0 = 0 \quad \text{and} \quad \frac{\partial u_3^1}{\partial x_3} = 0 , \qquad \frac{\partial V^1}{\partial x_3} + \text{grad } u_3^1 = 0 .$$

On the other hand, we easily remark that the problem $\mathcal{P}_3$ leads to a coupling equation similar to (3.22):

$$\frac{\partial u_3^3}{\partial x_3} = -\frac{\beta}{2+\beta} \text{ div } V^1 - \frac{\beta}{2(2+\beta)}\Big[\|\text{grad } u_3^1\|^2 + 2\overline{\text{grad } u_3^0} \text{ grad } u_3^2\Big] - \frac{1}{2}\left[\left\|\frac{\partial V^1}{\partial x_3}\right\|^2 + 2\frac{\overline{\partial V^0}}{\partial x_3}\frac{\partial V^2}{\partial x_3}\right]. \tag{4.1}$$

Then using the previous relations, the formulas for the components of the stress tensor assume the form :

$$\frac{\Sigma_t^*}{\varepsilon^2\mu} = \frac{1}{2}n_t^0 + \frac{3}{2}x_3 m_t^0 + O(\varepsilon) ,$$

$$\frac{\Sigma_s^*}{\varepsilon^2\mu} = \varepsilon\left\{\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3} + \frac{\partial u_3^2}{\partial x_3}\text{grad } u_3^0 + \frac{\overline{\partial V^0}}{\partial x}\frac{\partial V^0}{\partial x_3}\right\} + O(\varepsilon^2) ,$$

$$\begin{aligned}\frac{\Sigma_n^*}{\varepsilon^2\mu} = \varepsilon^2\Bigg\{&(2+\beta)\frac{\partial u_3^4}{\partial x_3} + (1+\frac{\beta}{2})\Big(\frac{\partial u_3^2}{\partial x_3}\Big)^2 + \beta \text{ div} V^2 + \frac{\beta}{2}\Big[\|\text{grad } u_3^1\|^2 \\ &+ 2\overline{\text{grad } u_3^0} \text{ grad } u_3^2\Big] + \Big(1+\frac{\beta}{2}\Big)\left[\left\|\frac{\partial V^1}{\partial x_3}\right\|^2 + 2\frac{\overline{\partial V^0}}{\partial x_3}\frac{\partial V^2}{\partial x_3}\right] \\ &+ \frac{\beta}{2}\text{Tr}\Big(\frac{\overline{\partial V^0}}{\partial x}\frac{\partial V^0}{\partial x}\Big)\Bigg\} + O(\varepsilon^3) .\end{aligned}$$

The reference scale $\Sigma_r = \varepsilon^2\mu$ of the stresses naturally appears so that at least one component of $\Sigma = \dfrac{\Sigma^*}{\Sigma_r}$ is of order 1.

The existence of an expansion of the solution (V, u_3) into a power series of ε then implies the existence of an expansion of Σ with respect to ε :

$$\Sigma = \Sigma^0 + \varepsilon\Sigma^1 + \varepsilon^2\Sigma^2 + \cdots , \tag{4.2}$$

where

$$
(4.3)\qquad
\begin{aligned}
\Sigma_t^0 &= \frac{1}{2}n_t^0 + \frac{3}{2}x_3 m_t^0 \,,\\
\Sigma_s^1 &= \text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3} + \frac{\partial u_3^2}{\partial x_3}\text{grad } u_3^0 + \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x_3} \,,\\
\Sigma_n^2 &= (2+\beta)\frac{\partial u_3^4}{\partial x_3} + \left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3^2}{\partial x_3}\right)^2 + \beta\,\text{div } V^2 + \frac{\beta}{2}\Big[\|\text{grad } u_3^1\|^2\\
&\quad + 2\overline{\text{grad } u_3^0}\text{ grad } u_3^2\Big] + (1+\frac{\beta}{2})\Big[\|\frac{\partial V^1}{\partial x_3}\|^2 + 2\overline{\frac{\partial V^0}{\partial x_3}}\frac{\partial V^2}{\partial x_3}\Big]\\
&\quad + \frac{\beta}{2}\text{Tr}\left(\overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x}\right) \,,
\end{aligned}
$$

$$
\Sigma_s^0 = 0, \qquad \Sigma_n^i = 0 \quad i = 0,1 \,.
$$

The explicit expressions of Σ_s^1 and Σ_n^2 are obtained by integration across the thickness. This process is similar to the one developed by P. G. CIARLET and P. DESTUYNDER in [2].

4.1. Evaluation of Σ_s^1

Taking the gradient of (3.22) and inserting the obtained expression in (3.17), we get, according to (4.3),

$$
\begin{aligned}
\frac{\partial}{\partial x_3}\Sigma_s^1 &= -\text{div } \Sigma_t^0 \quad &\text{on } \Omega_0 \,,\\
\Sigma_s^1 &= \pm g_t^{\pm} \quad &\text{on } \Gamma_{0\pm} \,.
\end{aligned}
$$

The integration of Σ_s^1 leads then to the classical expression:

$$
\Sigma_s^1 = \frac{3}{4}\left(1 - x_3^2\right)\text{div}(m_t^0) + \frac{g_t^+ - g_t^-}{2} + \frac{x_3}{2}\left(g_t^+ + g_t^-\right) \,.
$$

4.2. Evaluation of Σ_n^2

The equation (3.26) can be written as

$$(4.4) \qquad \frac{\partial}{\partial x_3}\Big[(2+\beta)\frac{\partial^2 u_3^4}{\partial x_3^2} + \beta\,\mathrm{div}V^2\Big] + \mathrm{div}\Big[\frac{\partial V^2}{\partial x_3} + \mathrm{grad}\ u_3^2\Big] + \mathrm{div}(E_{us}^3 + \Gamma_s^3) + \frac{\partial}{\partial x_3}(E_{un}^4 + \Gamma_n^4) = -f_3\,,$$

with

$$E_{us}^3 = \left[\beta\ \mathrm{div}V^0 + \left(1+\frac{\beta}{2}\right)\|\mathrm{grad}\ u_3^0\|^2 + \frac{\beta}{2}\left\|\frac{\partial V^0}{\partial x_3}\right\|^2 + \beta\frac{\partial u_3^2}{\partial x_3}\right]\mathrm{grad}\ u_3^0 + \left(\frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}}\right)\mathrm{grad}\ u_3^0\,,$$

$$E_{un}^4 = \left[\overline{\frac{\partial V^0}{\partial x_3}}\frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^2}{\partial x_3}} + \overline{\mathrm{grad}\ u_3^2}\right]\mathrm{grad}\ u_3^0 + \frac{\partial u_3^2}{\partial x_3}\|\mathrm{grad}\ u_3^0\|^2\,,$$

$$\Gamma_s^3 = \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x_3} + \frac{\partial u_3^2}{\partial x_3}\mathrm{grad}\ u_3^0\,,$$

$$\Gamma_n^4 = \left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3^2}{\partial x_3}\right)^2 + \frac{\beta}{2}\left[\left\|\mathrm{grad}\ u_3^1\right\|^2 + 2\overline{\mathrm{grad}\ u_3^0}\ \mathrm{grad}\ u_3^2\right] + \left(1+\frac{\beta}{2}\right)\left[\left\|\frac{\partial V^1}{\partial x_3}\right\|^2 + 2\overline{\frac{\partial V^0}{\partial x_3}}\frac{\partial V^2}{\partial x_3}\right] + \frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x}\right).$$

The intermediate steps of the previous calculation are not difficult, so they will not be discussed in details. In these steps, use is made of relations (3.8), (3.10), (3.15), (3.16) and (3.22).

On the other hand, a simple analysis leads to

$$\frac{\partial V^2}{\partial x_3} + \mathrm{grad}\ u_3^2 + \Gamma_s^3 = \Sigma_s^1\,,$$

$$E_{us}^3 = \Sigma_t^0\ \mathrm{grad}\ u_3^0\,,$$

$$(2+\beta)\frac{\partial u_3^4}{\partial x_3} + \beta\,\mathrm{div}\ V^2 + \Gamma_n^4 = \Sigma_n^2\,,$$

$$E_{un}^4 = \overline{\Sigma_s^1}\,\mathrm{grad}\ u_3^0\,.$$

Hence Eq. (4.4) and the boundary conditions on the upper and the lower faces give us

$$\frac{\partial}{\partial x_3}\,\Sigma_n^2 = -\mathrm{div}(\Sigma_t^0 \mathrm{grad}\ u_3^0) \quad - \mathrm{div}(\Sigma_s^1) - \frac{\partial}{\partial x_3}(\overline{\Sigma}_s^1 \mathrm{grad}\ u_3^0) - f_3 \ \text{ on } \ \Omega_0 \ ,$$

$$\Sigma_n^2 = \pm g_3^{\pm} \ \text{ on } \ \Gamma_{0\pm}.$$

Finally, the integration of Σ_n^2 leads to the classical result:

$$\Sigma_n^2 = -\frac{x_3}{4}(1 - x_3^2)\mathrm{div}(\mathrm{div}\ m_t^0) + \frac{3}{4}(1 - x_3^2)\mathrm{Tr}\left[m_t^0 \ \mathrm{grad\ grad}\ u_3^0\right]$$

$$+\frac{1 + x_3}{2}\int_{-1}^{1} f_3 dx_3 - \int_{-1}^{x_3} f_3(z)dz + \frac{1}{2}(g_3^+ - g_3^-) + \frac{x_3}{2}(g_3^+ + g_3^-)$$

$$+\frac{1}{4}(1 - x_3^2)\mathrm{div}\ (g_t^+ + g_t^-) - \mathrm{grad}\ \zeta_3^0.\left\{\frac{1}{2}(g_t^+ - g_t^-) + \frac{x_3}{2}(g_t^+ + g_t^-)\right\} .$$

5. Passage to the initial variables

Now let us go back to the initial domain Ω_0^* and to the physical variables V^*, u_3^*, f^* and g^*. To do this, let us define

$$V^{*0} = V_r V^0 = \varepsilon h_0 V^0 \ ,$$

$$u_3^{*0} = u_{3r} u_3^0 = h_0 u_3^0 \ .$$

Thus we have the following result :

RESULT 3

For applied forces of order f^* and g^* such that $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$, (V^{*0}, u_3^{*0}) is a Kirchhoff-Love displacement which satisfies

$$u_3^{*0} = \zeta_3^{*0}(x_1^*, x_2^*) \ , \qquad V^{*0} = \zeta_t^{*0}(x_1^*, x_2^*) - x_3^* \ \mathrm{grad}^* \zeta_3^{*0} \ ;$$

where $\zeta^{*0} = (\zeta_t^{*0}, \zeta_3^{*0})$ is the solution of the following nonlinear problem :

$$h_0^3 \ \mathrm{div}^*(\mathrm{div}^* m_t^{0*}) + h_0 \ \mathrm{div}^*\left(n_t^{0*} \mathrm{grad}^* \zeta_3^{*0}\right) = -p_3^* - \mathrm{div} M_t^* \ ,$$

$$h_0 \ \mathrm{div}^*(n_t^{*0}) = -p_t^* \ ,$$

$$\zeta_3^{*0} = \frac{\partial \zeta_3^{*0}}{\partial \nu} = 0 \qquad \text{and} \qquad \zeta_t^{*0} = 0 \quad \text{on } \gamma_0^* ,$$

with

$$n_t^{*0} = \frac{4\lambda\mu}{\lambda + 2\mu} \mathrm{Tr} E_t^{*0}(\zeta^{*0}) I_2 + 4\mu E_t^{*0}(\zeta^{*0}) ,$$

$$m_t^{*0} = -\Big\{ \frac{4\lambda\mu}{3(\lambda + 2\mu)} \Delta^* \zeta_3^{*0} I_2 + \frac{4\mu}{3} \mathrm{grad}^*(\mathrm{grad}^* \zeta_3^{*0}) \Big\} ,$$

$$E_t^{*0}(\zeta^{*0}) = \frac{1}{2}\Big(\mathrm{grad}^* \zeta_t^{*0} + \overline{\mathrm{grad}^* \zeta_t^{*0}} + \mathrm{grad}\ \zeta_3^{*0}\ \overline{\mathrm{grad}\ \zeta_3^{*0}} \Big) ,$$

$$p_3^* = \int_{-h_0}^{h_0} f_3^* dx_3^* + g_3^{*+} + g_3^{*-} , \qquad p_t^* = g_t^{*+} + g_t^{*-} , \qquad M_t^* = h_0(g_t^{*+} - g_t^{*-}) .$$

In the same way, let us define

$$\Sigma_t^{*0} = \varepsilon^2 \mu \Sigma_t^0 , \qquad \Sigma_s^{*0} = \varepsilon^3 \mu \Sigma_s^1 , \qquad \Sigma_n^{*0} = \varepsilon^4 \mu \Sigma_n^2 .$$

Then the expression of the physical stresses are as follows :

RESULT 4

$$\Sigma_t^{*0} = n_t^{*0} + \frac{3x_3^*}{2} m_t^{*0} ,$$

$$\Sigma_s^{*0} = -\frac{3}{4}(h_0^2 - (x_3^*)^2) \mathrm{div}^*(m_t^{*0}) + \frac{1}{2}(g_t^{*+} - g_t^{*-}) + \frac{x_3^*}{2h_0}(g_t^{*+} + g_t^{*-}) ,$$

$$\begin{aligned}
\Sigma_n^{*0} = & -\frac{x_3^*}{4}(h_0^2 - (x_3^*)^2) \mathrm{div}^*\ (\mathrm{div}^*\ m_t^{*0}) + \frac{3}{4}(h_0^2 - (x_3^*)^2) \mathrm{Tr}\Big[m_t^{*0} \mathrm{grad}^*(\mathrm{grad}^*\ u_3^{*0}) \Big] \\
& + \frac{1}{2}(1 + \frac{x_3^*}{h_0}) \int_{-h_0}^{h_0} f_3^* dx_3^* - \int_{-h_0}^{x_3^*} f_3^*(z^*) dz^* + \frac{1}{2}(g_3^{*+} - g_3^{*-}) \\
& + \frac{x_3^*}{2h_0}(g_3^{*+} + g_3^{*-}) + \frac{1}{4}\left(1 - \left(\frac{x_3^*}{h_0}\right)^2\right) \mathrm{div}^* \Big[h_0(g_t^{*+} + g_t^{*-}) \Big] \\
& - \frac{1}{2} \mathrm{grad}\ \zeta_3^{*0} . \Big\{ g_t^{*+} - g_t^{*-} + \frac{x_3^*}{h_0}(g_t^{*+} + g_t^{*-}) \Big\} .
\end{aligned}$$

The return to the initial variables does not create any difficulty. The procedures are similar to those exposed in [6] and [7].

6. Conclusion

The results obtained in the first part [8] and in the present article prove that the reference scales of the displacement and the corresponding two-dimensional model we obtain are determined by the magnitude of forces, mainly by the surface forces. Thus the scalings of the displacements generally used in the nonlinear case cannot be considered anymore as a simple change of functions. Indeed, as it has been proved in the first part, there exists another change of functions (corresponding to $V_r = u_{3r} = L_0$) which leads to the nonlinear membrane model for large forces and not to the von Kármán's one.

On the other hand, we have proved in this second part that for moderate applied forces, the standard asymptotic expansion method leads to the nonlinear two-dimensional von Kármán model. Its domain of validity is then specified thanks to the dimensionless numbers naturally introduced. Indeed the von Kármán model is valid for applied forces magnitude so that $\mathcal{F}_3 = h_0 f_{3r}/\mu$ and $\mathcal{G}_3 = g_{3r}/\mu$ are of order $\left(\dfrac{h_0}{L}\right)^4$, $\mathcal{G}_t = g_{3r}/\mu$ of order $\left(\dfrac{h_0}{L}\right)^3$ (where $\mathcal{F}_3$, $\mathcal{G}_3$ and $\mathcal{G}_t$ are known data of the problem). These forces lead to deflections of the order of thickness.

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Power evaluation of the influence of roughness on the value of contact stress for interaction of rough cylinders

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THE PAPER deals with mathematical modelling of contact interaction of rough cylindrical bodies, what significantly reduces the complexity of investigation of contact stress in practice. The contact problem for a rough elastic disk and isotropic plate with a rough cylindrical hole has been considered. The explicit approximate solution of the integral equation is presented in this paper. It allows us to determine the influence of the rough layer characteristics on the distribution of contact stresses.

NOTATIONS

U	energy of elastic deformation of a rough layer
P	load intensity
R and r	radii of cylindrical bodies
E_i, $(i = 1, 2)$	Young's moduli
ν_i, $(i = 1, 2)$	Poisson's ratios
a	half-length of contact
$P(t)$	normal contact pressure
$u_m, v_m (m = \overline{1, 2})$	components of displacements for the plate with a hole ($m = 1$), for an elastic disk ($m = 2$)
δ	displacement of disk center
L	area of contact in non-Hertz theory of interior interaction of cylinders
$\sigma_r(\theta)$	normal contact stress in non-Hertz theory of interior interaction of cylinders
$\mu_m, (m = \overline{1, 2})$	Lamé's coefficient
$\varphi_m(W), \psi_m(W)$	Kolosov-Mushelishvili complex potentials
$z = x + iy$	complex variable in frame X0Y connected with the center of hole
$s = x' + iy'$	complex variable in a frame X'0Y' connected with a center of disk and belonging to its exterior area
$\Phi_m(W) = \varphi'_m(W)$	
$\Psi_m(W) = \psi'_m(W)$	
$\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$	coefficients of the integral equation
$\tau = t = Re^{i\theta}$	
$\zeta = h = re^{i\theta}$	
α_0	contact half-angle
$p_{\max}$	maximum contact pressure

For the state of plane deformation:

$$G_{1m} = (1 - \nu_m^2),\ G_{2m} = (1 + \nu_m);\ \kappa_m = (3 - 4\nu_m)$$

For the state of plane stress:

$$G_{1m} = G_{2m} = 1;\ \kappa_m = (3 - \nu_m)/(1 + \nu_m).$$

1. Introduction

THE CONTACT STRESS is one of the major factors leading to the wear of working surfaces and determining limit loads of machine parts [1]. However, roughness of real bodies has an essential influence on the distribution of the contact stress [1 – 6]. It considerably reduces the pressure which becomes much smaller than for smooth bodies in the case of small loads. In addition, the area of contact considerably increases [4 – 6]. Therefore the principal attention is directed to the determination of elastic interaction of rough bodies. Many authors studied this problem, which is connected with solving both the applied and fundamental problems [2, 3].

It is necessary to point out that the mechanics of rough surfaces contains several peculiarities. They are due to the fact that the roughness, formed after technological processing, has various heights distribution [3, 6]. The irregularity of roughness leads to the necessity of application of the probability methods for the determination of rigidity of the element of a rough surface [3, 6, 7] where the pressure is constant. Just the similarity of dimensions of the element and contact a ea explains methodical complexities of solving contact problems for real bodies.

The posed problem was solved in several investigations under the assumption, that distribution of pressure becomes parabolic, while the deformations of elastic half-space in comparison with the deformations of a rough layer can be neglected [6]. This approach leads to essential increasing of the complexity of integral equations and enables us to perform only the numerical analysis of the influence of roughness on the stress [4]. But the curvature of loaded surfaces is transformed and decreases at the expense of deformations of microirregularities. It essentially enables us to simplify the derivation of analytical solutions in the case of interaction on the elliptic area of the contact of elastic bodies. It is possible to carry out this research by estimating the influence of energy of elastic deformation of a rough layer on the displacement in the area of contact. This approach is explained in this paper for the case of contact of two rough cylinders.

2. Power evaluation of the influence of roughness on the deformed surface of interaction of two semi-infinite bodies

Let us consider an analog of the Hertz problem for a case of contact interaction of two elastic plane semi-infinite bodies S_1 and S_2 (Fig. 1). The frame XOY is selected so, that the line of contact L is a segment [-a, a] of the real axis OX. Average Steklov's values of deformation and stress [8] are

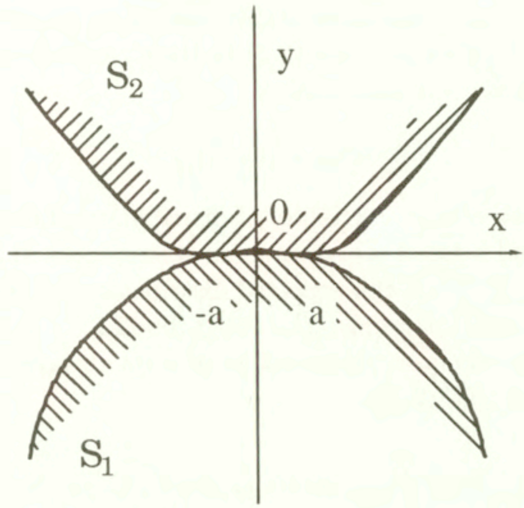

FIG. 1. Contact interaction of two elastic plane semi-infinite bodies.

$$\varepsilon_b^*(t) = \begin{cases} \frac{1}{2b}\int\limits_{a-2b}^{a} \varepsilon^*(x)dx, & t \in [a-b, a], \\ \frac{1}{2b}\int\limits_{t-b}^{t+b} \varepsilon^*(x)dx, & t \in [-a+b, a-b], \\ \frac{1}{2b}\int\limits_{-a}^{-a+2b} \varepsilon^*(x)dx, & t \in [-a, -a+b], \end{cases} \tag{2.1}$$

$$\sigma_b^*(t) = \begin{cases} \frac{1}{2b}\int\limits_{a-2b}^{a} \sigma^*(x)dx, & t \in [a-b, a], \\ \frac{1}{2b}\int\limits_{t-b}^{t+b} \sigma^*(x)dx, & t \in [-a+b, a-b], \\ \frac{1}{2b}\int\limits_{-a}^{-a+2b} \sigma^*(x)dx, & t \in [-a, -a+b], \end{cases} \tag{2.2}$$

where $2b(a \geqslant b)$ is the base length of a measurement of characteristics of microdeviations, $[-a, a]$ is the segment of contact, σ^*, ε^* – microstress and microdeformation for a rough layer (σ^*, ε^* – integrable functions), and [3]

$$\varepsilon_b^*(t) = (C_0 \sigma_b^*(t))^{\chi}, \tag{2.3}$$

where C_0, χ are real and rational constants defined by parameters of microdeviations. Some authors [3] did not use (2.3) in their research work but applied the relation between stress and deformation

$$\varepsilon(t) = (C_0\sigma(t))^\chi.$$

It is incorrect in view of the probability of distribution of the material in a rough layer and since a and b are of the same order of smallness. It is important to note the functions (2.1), (2.2) are the expected distributions of the contact stress and deformation in the contact problem for rough bodies.

Then the energy of elastic deformation of a rough layer U will be expressed in the following way ([9], (2.3)):

$$(2.4) \qquad U = \frac{\chi}{(1+\chi)C_0} 2h \int\limits_0^a \varepsilon_b^*(s)^{\chi+1/\chi} ds = \frac{\chi C_0^\chi}{(1+\chi)} 2h \int\limits_0^a \sigma_b^*(s)^{\chi+1} ds$$

$$\approx \frac{\chi C_0^\chi}{(1+\chi)} V(\tilde{\sigma}_b^*)^{\chi+1} \left[1 + O\left(\frac{(\chi+1)\chi}{2a} \int\limits_0^a \left(\frac{\sigma_b^*(s)}{\tilde{\sigma}_b^*} - 1\right)^2 ds\right)\right],$$

where $V = 2ah$, h is the thickness of the rough layer, and $\tilde{\sigma}_b^* = \dfrac{1}{a}\displaystyle\int\limits_0^a \sigma_b^*(s)ds$.

On the other hand, from (1.3) we obtain

$$(2.5) \qquad \tilde{\varepsilon}_b^* \approx (C_0\tilde{\sigma}_b^*)^\chi \left(1 + O\left(\frac{(\chi-1)\chi}{2a} \int\limits_0^a \left(\frac{\sigma_b^*(s)}{\tilde{\sigma}_b^*} - 1\right)^2 ds\right)\right);$$

here $\tilde{\varepsilon}_b^* = \dfrac{1}{a}\displaystyle\int\limits_0^a \varepsilon_b^*(s)ds$.

Since the bodies are in the elastic equilibrium, then

$$\varepsilon_b^*(t) = \varepsilon_b(t), \qquad \sigma_b^*(t) = \sigma_b(t),$$

where $\varepsilon_b(t)$, $\sigma_b(t)$ are the average Steklov values of deformation $\varepsilon(t)$ and stress $\sigma(t)$ within the limits of the base length for smooth bodies. Therefore equalities (2.4) and (2.5) hold for these functions too.

Functions are continuous on $[-a, a]$ and differentiable on $]-a, a[$. Then taking into account (2.4) – (2.5), we can obtain the following approximate equality:

$$(2.6) \qquad U \approx \frac{\chi C_0^\chi}{(1+\chi)} V(\tilde{\sigma})^{\chi+1}[1 + O(H_1)],$$

$$\tilde{\varepsilon} \approx (C_0\tilde{\sigma})^{\chi}(1 + O(H_2)), \tag{2.7}$$

where

$$H_1 = \frac{(\chi+1)\chi}{2a}\int\limits_0^a \left(\frac{\sigma_b(s)}{\tilde{\sigma}_b} - 1\right)^2 ds + \frac{(\chi+1)b^2}{2a}\left|\frac{\sigma'(a-b)}{\tilde{\sigma}}\right|,$$

$$H_2 = \frac{(1-\chi)\chi}{2a}\int\limits_0^a \left(\frac{\sigma_b(s)}{\tilde{\sigma}_b} - 1\right)^2 ds + \frac{\chi b^2}{2a}\left|\frac{\sigma'(a-b)}{\tilde{\sigma}}\right| + \frac{b^2}{2a}\left|\frac{\varepsilon'(a-b)}{\tilde{\varepsilon}}\right|,$$

$$\tilde{\sigma} = \frac{1}{a}\int\limits_0^a \sigma(s)ds, \qquad \tilde{\varepsilon} = \frac{1}{a}\int\limits_0^a \varepsilon(s)ds.$$

Thus the energy U of deformation of a rough layer in the case of contact interaction of rough bodies within H_1 and $\tilde{\varepsilon}$ within H_2 are fixed if the value of $\tilde{\sigma}$ is fixed.

3. Generalization of the Hertz theory to the case of contact of rough cylinders

Let us assume that there is no friction in the area of contact L of bodies S_1 and S_2 (Fig. 1). The stresses and rotations are equal to zero at infinity [10].

The equations of the boundaries of bodies before the deformation are [10]

$$y_1 = -\frac{t^2}{2R_1}, \qquad y_2 = \frac{t^2}{2R_2},$$

and after the deformation we have

$$v_1 - v_2 = \left(\frac{t^2}{2R_1} + \frac{t^2}{2R_2}\right) + v^*, \qquad t \in L$$

$$v^* = \frac{\Delta}{2}(a^2 - t^2),$$

where v^* is the displacement of the rough layer, Δ is a constant determined by the parameters of the rough layer. Then from (2.7) we obtain

$$\Delta \approx \frac{3}{a^2}h\left(C_0\frac{P}{2a}\right)^{\chi}.$$

Taking into account all the conditions we obtain, after some transformations, the following solution in case when the contact pressure $p(t)$ $(p(t) = -\sigma(t))$ is equal to zero at points $-a$ and a :

$$p(t) = \left(\frac{R + r - \Delta Rr}{Rr}\right) \frac{1}{K} \sqrt{(a^2 - t^2)},$$

$$K = \frac{\kappa_1 + 1}{4\mu_1} + \frac{\kappa_2 + 1}{4\mu_2},$$

where $\kappa_m = (3 - 4\nu_m)$ for the state of plane deformation; $\kappa_m = (3 - \nu_m)/(1 + \nu_m)$ for the state of plane stress and a is the solution of nonlinear equation

$$a^2 - 3\frac{Rrh}{(r + R)}\left(C_0 \frac{P}{2a}\right)^{\chi} = \frac{2PrRK}{\pi(r + R)}; \tag{3.1}$$

here P is the intensity of the load; R, r are radii of the curvature of the interacting bodies.

The results of numerical analysis of (3.1) show that the roughness has an essential role only for small operating loads, and with their growth the influence of roughness decreases (Fig. 2). It is necessary to point out that relative errors H_1 and H_2 from (2.6), (2.7) do not exceed 9% and 15% of the examined values.

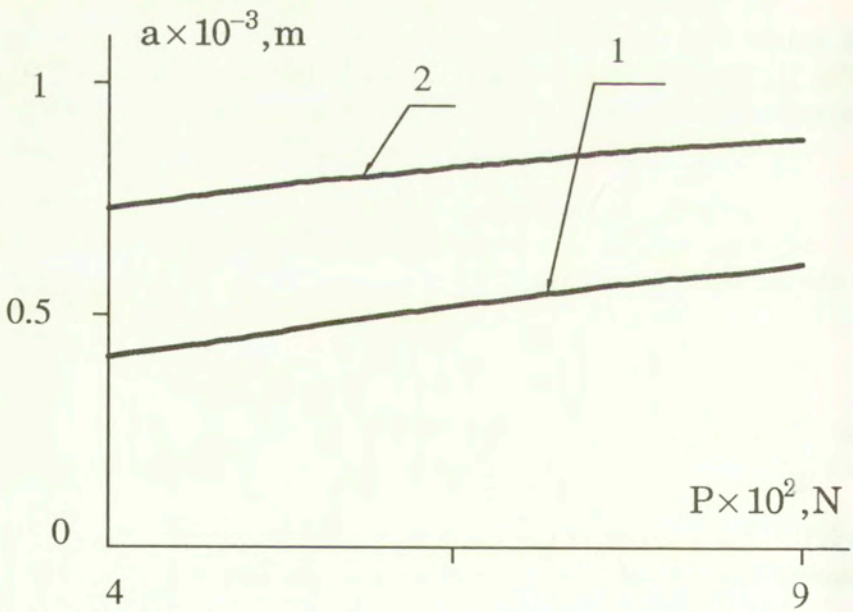

FIG. 2. Relation between a and P: 1 – for smooth cylinders; 2 – for rough cylinders ($C_0 = 3.812 \cdot 10^{-11}$ m^2/N, $K = 7.289 \cdot 10^{-11}$ m^2/N, $R/r = 10$, $\chi = 2/9$).

4. Interior contact of rough elastic disk and plate with cylindrical hole

Consider an elastic isotropic plate with a cylindrical hole of radius R. An elastic isotropic disk of radius r is inserted into the hole. It will be assumed that ε^2, ε/R ($\varepsilon = R - r > 0$) are small values. Force P acts along the y – axis (Fig. 3). Due to the fact that displacements in the area of contact L are negligible in comparison with the dimensions of the bodies, one obtains:

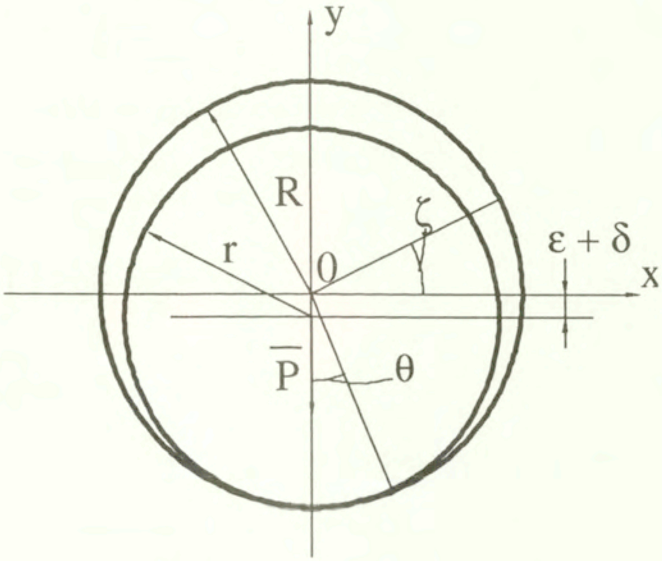

FIG. 3. Scheme of bodies location

$$(4.1) \qquad (x_1 + u_1^{**})^2 + (y_1 + v_1^{**})^2 = (x_2 + u_2^{**})^2 + (y_2 + v_2^{**} - \delta)^2,$$

where

$$x_1 = R\cos(\zeta), \qquad y_1 = R\sin(\zeta),$$

$$x_2 = r\cos(\zeta), \qquad y_2 = r\sin(\zeta) - \varepsilon,$$

$u_m^{**}, v_m^{**} (m = \overline{1,2})$ are components of the displacements of the plate with the hole ($m = 1$), for the elastic disk ($m = 2$); δ is the displacement of the disk center. It is easy to see that equation (4.1) reduces to

$$\varepsilon + u_1^{**}\cos(\zeta) + v_1^{**}\sin(\zeta) = u_2^{**}\cos(\zeta) + (v_2^{**} - \delta - \varepsilon)\sin(\zeta).$$

Let

$$u_m^{**} = u_m^* + u_m, \qquad v_m^{**} = v_m^* + v_m,$$

where u_m, v_m, u_m^*, v_m^* are displacements of the basic material and the surface rough layer, respectively.

We shall assume, that the elastic radial displacement in the area of contact being determined by the deformation of a micro-irregularity, is given by the following expression [11]:

$$v_r^* = u_1^* \cos(\zeta) + v_1 \sin(\zeta) - u_2^* \cos(\zeta) - v_2^* \sin(\zeta) = \Delta(\cos(\zeta) - \cos(\alpha_0)).$$

We obtain, similarly to Sec. 1, that

$$\Delta \approx \frac{\alpha_0}{\sin(\alpha_0) - \alpha_0 \cos(\alpha_0)} h \left(C_0 \frac{P}{2R\sin(\alpha_0)} \right)^{\chi},$$

where α_0 is the contact half-angle. After transformation we obtain

$$\begin{aligned}(4.2)\qquad &(\varepsilon - \Delta\cos(\alpha_0)) - 2\frac{\partial u_1}{\partial \zeta}\sin(\zeta) + 2\frac{\partial v_1}{\partial \zeta}\cos(\zeta) + \left(\frac{\partial^2 u_1}{\partial \zeta^2}\cos(\zeta)\right.\\ &\left. + \frac{\partial^2 v_1}{\partial \zeta^2}\sin(\zeta)\right) = -2\frac{\partial u_2}{\partial \zeta}\sin(\zeta) + 2\frac{\partial v_2}{\partial \zeta}\cos(\zeta) + \left(\frac{\partial^2 u_2}{\partial \zeta^2}\cos(\zeta)\right.\\ &\left. + \frac{\partial^2 v_2}{\partial \zeta^2}\sin(\zeta)\right).\end{aligned}$$

But on the contour of the hole we have [12]

$$(4.3)\qquad \frac{1}{R_m}\left(\frac{\partial v_{\zeta m}}{\partial \theta} + v_m\right) = \frac{1}{E_m}(G_{1m}\sigma_{\zeta m} - \nu_m G_{2m}\sigma_r),$$

where $R_m = R(m = 1)$ and $R_m = r(m = 2)$; $\nu_m, (m = \overline{1,2})$ is Poisson's ratio; $E_m, (m = \overline{1,2})$ is Young's modulus; $G_{1m} = (1 - \nu_m^2)$, $G_{2m} = (1 + \nu_m)$ for the state of plane deformation; $G_{1m} = G_{2m} = 1$ for the state of plane stress; $\sigma_{\zeta m}$, σ_r are normal components of stress. Then, using (4.2), (4.3) we obtain:

$$\begin{aligned}(4.4)\qquad &\varepsilon - \Delta\cos(\alpha_0) + \frac{R}{E_1}(G_{11}\sigma_{\zeta 1} - \nu_1 G_{21}\sigma_r) + \frac{\partial}{\partial \zeta}\left(\frac{\partial u_1}{\partial \zeta}\cos(\zeta) + \frac{\partial v_1}{\partial \zeta}\sin(\zeta)\right)\\ &= \frac{r}{E_2}(G_{12}\sigma_{\zeta 2} - \nu_2 G_{22}\sigma_r) + \frac{\partial}{\partial \zeta}\left(\frac{\partial u_2}{\partial \zeta}\cos(\zeta) + \frac{\partial v_2}{\partial \zeta}\sin(\zeta)\right).\end{aligned}$$

It is known that [10]

$$\begin{aligned}&\sigma_{\zeta m} + \sigma_r = 2\left[\Phi_m(W) + \overline{\Phi_m(W)}\right],\\ (4.5)\qquad &\sigma_{\zeta m} - \sigma_r + 2i\tau_{r\zeta m} = 2e^{2i\zeta}\left[\overline{W}\Phi'(W) + \Psi(W)\right],\\ &2\mu_m(u_m + iv_m) = \kappa_m\varphi_m(W) - W\overline{\Phi_m(W)} - \overline{\psi_m(W)}.\end{aligned}$$

Here $w = z(m = 1)$, $w = s(m = 2)$; $i = \sqrt{-1}$; $\varphi_m(W)$, $\psi_m(W)$ are Kolosov-Mushelishvili complex potentials; $\mu_m, (m = \overline{1,2})$ is Lamé's coefficient; $\varphi'_m(W) = \Phi_m(W), \psi'_m(W) = \Psi_m(W)$.

From (4.4), (4.5) we obtain

$$(4.6)\qquad (\varepsilon - \Delta\cos(\alpha_0)) + \frac{R}{E_1}(2G_{11}[\Phi_1(t) + \overline{\Phi_1(t)}] - (G_{11} + \nu_1 G_{21})\sigma_r) + R\frac{\partial}{\partial\zeta}\left(\frac{(\kappa_1+1)}{4\mu_1} i[\Phi_1(t) - \overline{\Phi_1(t)}]\right) = \frac{r}{E_2}(2G_{12}[\Phi_2(h) + \overline{\Phi_2(h)}] - (G_{12} + \nu_2 G_{22})\sigma_r) + r\frac{\partial}{\partial\zeta}\left(\frac{(\kappa_2+1)}{4\mu_2} i[\Phi_2(h) - \overline{\Phi_2(h)}]\right),$$

$$1/h = r/(rt), \qquad t = Rh/r - i\varepsilon.$$

Then using the Eqs. (4.6), (4.7) [10, 13]

$$(4.7)\qquad \begin{aligned} \Phi_1(z) &= \frac{\kappa_1}{2\pi(1+\kappa_1)}\frac{iP}{z} - \frac{1}{2\pi i}\int_L \frac{\sigma_r(\tau)d\tau}{\tau - z}, \\ \Phi_2(s) &= \frac{-iP}{2\pi(1+\kappa_2)}\frac{1}{s} - \frac{iP}{2\pi(1+\kappa_2)}\frac{s}{r^2} + \frac{1}{2\pi i}\int_L \frac{\sigma_r(\xi)d\xi}{\xi - s} - \frac{1}{4\pi i}\int_L \frac{\sigma_r(\xi)d\xi}{\xi}, \end{aligned}$$

we arrive at the integral equation:

$$(4.8)\qquad \frac{t}{\pi i}\int_L \frac{\sigma'_r(\tau)d\tau}{\tau - t} = \gamma_1\sigma_r(\tau) - \frac{iP}{\pi}\gamma_2\left(\frac{1}{t} - \frac{t}{R^2}\right) - \gamma_3\frac{P}{\pi} - \gamma_4 b - \gamma_5(\varepsilon - \Delta\cos(\alpha_0)),$$

where

$$\gamma_1 = \frac{(G_{12} - \nu_2 G_{22})E_1 Rr - (G_{11} - \nu_1 G_{21})E_2 R^2}{2(R^2 E_2 G_{11} + r^2 E_1 G_{12})}$$

$$\gamma_2 = \frac{(1+\nu_2)E_1 Rr + \kappa_1(1+\nu_1)E_2 R^2}{4(R^2 E_2 G_{11} + r^2 E_1 G_{12})},$$

$$\gamma_3 = \frac{G_{12}\varepsilon R E_1}{2r(1+\kappa_2)(R^2 E_2 G_{11} + r^2 E_1 G_{12})},$$

$$\gamma_4 = \frac{G_{11}E_2}{(R^2E_2G_{11} + r^2E_1G_{12})},$$

$$\gamma_5 = \frac{E_1E_2}{2(R^2E_2G_{11} + r^2E_1G_{12})},$$

$$\frac{b}{R^2} = -\frac{1}{2\pi i}\int\limits_L \frac{\sigma_r}{\tau}d\tau, \qquad t = Re^{i\zeta}.$$

The results of the investigations [10, 14] show that the approximate solution of (4.8) can be reduced to the following form:

$$(4.9) \qquad \sigma_r(\theta) = -P\frac{\sqrt{2}}{R}\left[\gamma_2\frac{2}{\pi} + \frac{\gamma_1}{\alpha_0 - \cos(\alpha_0)\sin(\alpha_0)}\right]\sqrt{\cos(\theta) - \cos(\alpha_0)}$$

$$\cos(\theta/2) + 2\left[P\left(\frac{\gamma_3}{\pi} + \frac{\gamma_1\cos(\alpha_0)}{R(\alpha_0 - \cos(\alpha_0)\sin(\alpha_0))}\right) + \gamma_4 b\right.$$

$$\left. + \gamma_5(\varepsilon - \Delta\cos(\alpha_0))\right] \times \ln\left[\frac{\sqrt{1+\cos(\theta)} - \sqrt{\cos(\theta) - \cos(\alpha_0)}}{\sqrt{1+\cos(\alpha_0)}}\right];$$

here

$$P = -2R\int\limits_0^{\alpha_0}\sigma_r(\theta)\cos(\theta)d\theta, \qquad b = -\frac{R^2}{\pi}\int\limits_0^{\alpha_0}\sigma_r(\theta)d\theta.$$

It is necessary to emphasize that for elastic constants of isotropic materials, which are widely used in machines, the error of approximation (4.9) of the solution of the equation (4.8) with respect to $\sigma_r^{\max}$ is less than 4%.

It has been established that the obtained dependence of the half-angle of contact on the non-dimensional parameter, introduced by I.Y. Staerman, is analogous to the dependences established by M. I. TEPLY for the state of plane deformation [13]. This confirms a high accuracy of the approximate solution (4.9).

The obtained solutions for smooth bodies can be used as a zero approximation in the analysis of the influence of roughness on the distribution of normal radial stresses. The obtained results show that in the case of interaction of rough bodies, the contact half-angle increases in comparison with the contact half-angle for smooth machine parts, and the greatest contact stress decreases (Figs. 4, 5). It is necessary to note that in this case, the relative errors H_1 and H_2 from (2.6), (2.7) do not exceed 10% and 11% of the investigated values.

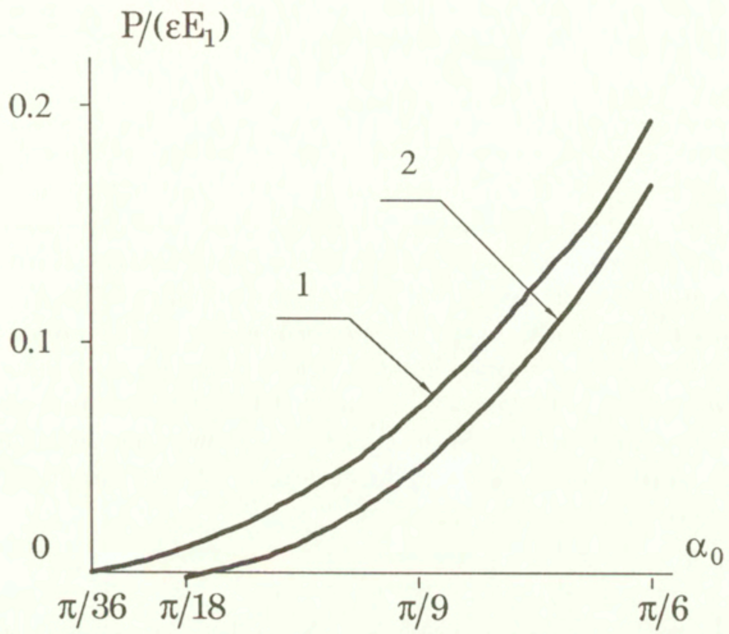

FIG. 4. The relation between α_0 and parameter $P/\varepsilon E$: 1 – for a smooth hole; 2 – for a rough hole.

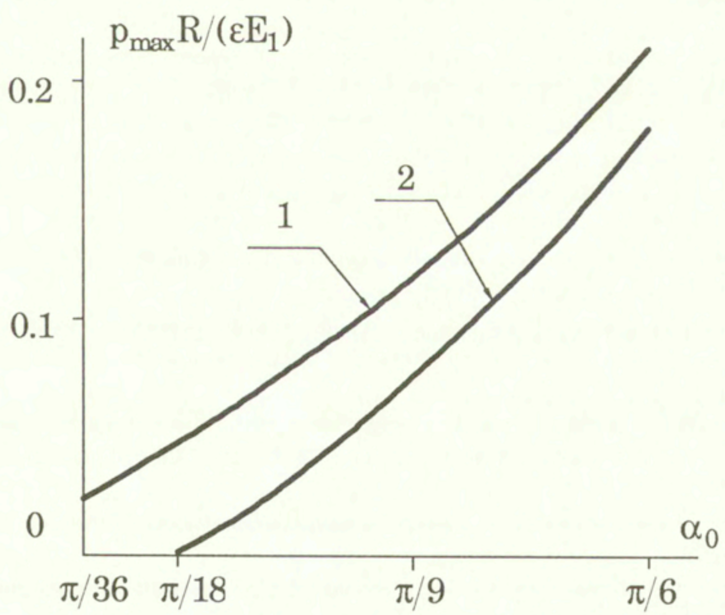

FIG. 5. The relation between α_0 and parameter $p_{\max}R/(\varepsilon E_1)(p_{\max} = -\sigma_r^{\max})$: 1 – for a smooth hole; 2 – for a rough hole.

5. Conclusions

The problems of elastic contact interaction of rough cylindrical bodies is solved by taking into account not only their geometry and relative location, but also the geometric characteristics of their surfaces. It allows us to determine the influence of basic parameters of the problem on the contact stress. It considerably reduces the complexity of investigating contact stresses in practice.

The comparison of the data of the stress analysis in the area of contact for various combinations of elastic characteristics of interacting bodies with the results of paper [13] confirms high effectiveness of the approach proposed here.

The suppositions made here and the conclusions drawn from the experimental research enable us to take into account the geometric peculiarities of the surfaces of interacting bodies.

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Objective corotational rates and unified work-conjugacy relation between Eulerian and Lagrangean strain and stress measures

H. XIAO[1], O. T. BRUHNS and A. MEYERS (BOCHUM)

BY VIRTUE OF OBJECTIVE corotational rates and related corotating frames, a unified work-conjugacy relation between Eulerian and Lagrangean strain and stress measures is established, which is a natural extension of Hill's work-conjugacy relation between Lagrangean strain and stress measures. It turns out that the latter is the particular case of the former one when a corotating frame with the well-known spin $\mathbf{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^T$ is concerned, where R is the rotation tensor defined by the polar decomposition of the deformation gradient. The work-conjugate stress measure of an arbitrary Hill's strain measure (either Eulerian or Lagrangean) with regard to any kind of objective corotational rate is determined in the sense of the introduced unified work-conjugacy relation. The result is presented both in the principal component form and explicit basis-free form valid for all cases of the principal stretches. In particular, the intrinsic, unique relationship between Hencky's logarithmic strain measures $\ln \mathbf{V}$ and $\ln \mathbf{U}$ and the fundamental mechanical quantities, i.e. the Eulerian and Lagrangean stretching tensors D and $\mathbf{R}^T\mathbf{D}\mathbf{R}$ and Eulerian and Lagrangean Kirchhoff stress measures $\boldsymbol{\sigma}$ and $\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$, are disclosed. It is shown that there are objective corotational rates of $\ln \mathbf{V}$ and $\ln \mathbf{U}$ that are identical with the Eulerian and Lagrangean stretching tensors D and $\mathbf{R}^T\mathbf{D}\mathbf{R}$ respectively, and further that only $\ln \mathbf{V}$ and $\ln \mathbf{U}$ enjoy the just-stated favourable properties. As a result, the two pairs of strain and stress measures, $(\ln \mathbf{V}, \boldsymbol{\sigma})$ and $(\ln \mathbf{U}, \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R})$, form a work-conjugate Eulerian strain-stress pair and a work-conjugate Lagrangean strain-stress pair, respectively, in the sense of the introduced work-conjugacy relation. Finally, application of the unified work-conjugacy notion in formulating the rate – type constitutive relations is indicated.

1. Introduction

IN SOLID MECHANICS and other related fields, there is a variety of strain and stress measures (actually infinitely many). It is well-known that strain measures and stress measures can be associated with each other via the stress power per unit volume, in a manner independent of any material behaviours. According to HILL [21 – 23] (see also WANG and TRUESDELL [53], OGDEN [37–38], *et al.*), a Lagrangean strain-stress pair $(\mathbf{E}, \mathbf{T})$ forms a work-conjugate pair if the inner product of the Lagrangean stress measure T and the material time rate $\dot{\mathbf{E}}$ of the Lagrangean strain measure E furnishes the stress power $\dot{w}$ (cf. the formulas (2.13) – (2.14) given later):

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$$\dot{w} = \mathbf{T} : \dot{\mathbf{E}} = T_{ij}\dot{E}_{ij}. \tag{1.1}$$

The just-stated Hill's work-conjugacy notion for Lagrangean strain and stress measures has found applications in constitutive modeling and proved to be fruitful (e.g., see HILL [21 – 23], RICE [42], HUTCHINSON and NEALE [27], NEMAT-NASSER [36], PALGEN and DRUCKER [39], OGDEN [38], *et al.*). However, such notion in general does not apply to Eulerian stress and strain measures, as proved by OGDEN [37 – 38] and HOGER [26], *et al.* Moreover, even within the scope of Lagrangean strain and stress measures, the aforementioned work-conjugacy notion excludes the possibility of associating certain significant stress measures with strain measures. Indeed, it is known that the *rotated Kirchhoff stress measure* $\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$, which is a Lagrangean stress measure useful in formulation of elastic and elastoplastic constitutive relations (e.g., see GREEN and NAGHDI [9], RICE [42], SIMO and MARSDEN [49]), can not be related to any strain measure via (1.1), as noted by HILL [21 – 23], RICE [42], PALGEN and DRUCKER [39], and OGDEN [38], *et al.*

The main objective of this article is to investigate the work-conjugacy notion in both the Lagrangean and Eulerian strain and stress measures in a broader sense and in a unified manner. It is shown that by virtue of the objective corotational rates and the related corotating frames, a unified work-conjugacy relation can be established for both the Eulerian and Lagrangean strain and stress measures. This unified work-conjugacy relation may be visualized as a natural extension of Hill's work-conjugacy notion in the sense that there exists a certain class of corotating frames relative to each of which the stress power $\dot{w}$ can be expressed as the inner product of a stress measure and the time rate of a strain measure. It turns out that Hill's work-conjugacy relation is the particular case when a corotating frame with the well-known spin $\boldsymbol{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^T$ is concerned, where $\mathbf{R}$ is the rotation tensor defined by the polar decomposition of the deformation gradient (see (2.1) given later). By applying a general expression for the spin tensors defining objective corotational rates derived in XIAO, BRUHNS and MEYERS [60], the conjugate stress measure of an arbitrary Hill's generalized strain measure (either Eulerian or Lagrangean) with regard to any kind of objective corotational rate is determined in the sense of the introduced unified work-conjugacy relation. The results are presented in both the principal component form and the explicit basis-free form. In particular, the intrinsic, unique relationship between Hencky's logarithmic strain measures $\ln\mathbf{V}$ and $\ln\mathbf{U}$ and the fundamental mechanical quantities, i.e. the Eulerian and Lagrangean stretching tensors $\mathbf{D}$ and $\mathbf{R}^T\mathbf{D}\mathbf{R}$ and Eulerian and Lagrangean Kirchhoff stress measures $\boldsymbol{\sigma}$ and $\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$, are disclosed. It is shown that there exist objective corotational rates of $\ln\mathbf{V}$ and $\ln\mathbf{U}$ that are identical with the Eulerian and Lagrangean stretching tensors $\mathbf{D}$ and $\mathbf{R}^T\mathbf{D}\mathbf{R}$, respectively, and furthermore that only $\ln\mathbf{V}$ and $\ln\mathbf{U}$ enjoy the just-stated favourable properties. As a result, the two pairs $(\ln\mathbf{V}, \boldsymbol{\sigma})$ and $(\ln\mathbf{U}, \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R})$ form a work-conjugate Eulerian strain-stress pair and a work-conjugate Lagrangean strain-stress pair respectively, where $\boldsymbol{\sigma}$ is the Kirchhoff stress measure. The fact concerning the Eulerian logarithmic strain measure $\ln \mathbf{V}$ has been established recently by various authors independently of their different points of view (see XIAO, BRUHNS and MEYERS [57 – 60]; see also LEHMANN, GUO and LIANG [29], REINHARDT and DUBEY [40 – 41], DUBEY and REINHARDT [7]). However, the fact concerning the Lagrangean logarithmic strain measure $\ln \mathbf{U}$ has been unknown until very recently (see XIAO, BRUHNS and MEYERS [61]), since for a long time it has been believed that the rotated stretching tensor $\mathbf{R}^T\mathbf{D}\mathbf{R}$ is not a direct flux of a strain measure (see HILL [21 – 23], RICE [42], PALGEN and DRUCKER [39], OGDEN [38], *et al.*). Finally, application of the unified work-conjugacy notion in formulating rate-type constitutive relations is indicated.

It should be pointed out that other extended work-conjugacy notions are possible and useful. For example, we refer to ZIEGLER and MACVEAN [64] and MACVEAN [32] for a discussion of work-conjugacy relation from a general point of view, and to HAUPT and TSAKMAKIS [18 – 19] and SVENDSEN and TSAKMAKIS [51] for a comprehensive account of associating strain and stress measures via the concept of dual variables, etc. In this article, we shall confine ourselves to the objective mentioned before.

2. Preliminaries in kinematics

To facilitate the succeeding account, in this section we recapitulate some related facts and results in kinematics of finite deformations of continua. For details, refer to TRUESDELL and TOUPIN [52], WANG and TRUESDELL [53], GURTIN [16], MARSDEN and HUGHES [34], and OGDEN [37 – 38], *et al.*

In this article, vector and tensor mean vectors and tensors over a three-dimensional Euclidean space.

2.1. Some fundamental kinematical quantities

Consider a material body experiencing finite deformation over a time interval $I \subset R$. A typical particle of this body is identified with a position vector $\mathbf{X}$ relative to a fixed reference state. The motion of the body is described by the current position vector $\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X}, t)$, $t \in I$. The velocity vector of the particle $\mathbf{X}$ is given by $\mathbf{v} = \dot{\mathbf{x}}$.

The state of the local rotation and deformation in a neighbourhood of a particle $\mathbf{X}$ at any instant $t \in I$ is characterized by the *deformation gradient*

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \tag{2.1}$$

while the rate-of-change of state of the rotation and deformation in a neighbourhood of a particle $\mathbf{X}$ at any instant $t \in I$ is described by the *velocity gradient*

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{2.2}$$

For the former, the following unique left and right polar decompositions hold:

$$\begin{aligned} \mathbf{F} &= \mathbf{VR} = \mathbf{RU}\,, \\ \mathbf{V}^2 &= \mathbf{B} = \mathbf{FF}^T, \\ \mathbf{U}^2 &= \mathbf{C} = \mathbf{F}^T\mathbf{F}\,, \\ \mathbf{RR}^T &= \mathbf{I}, \qquad \mathbf{R}^T\mathbf{R} = \hat{\mathbf{I}}, \end{aligned} \tag{2.3}$$

where the two tensors V and B and the two tensors B and C are known as the *left* and *right stretch tensors* and the *left* and *right Cauchy-Green tensors* respectively, each of which is symmetric, positive definite; the proper orthogonal tensor $\mathbf{R}$ is the *rotation tensor*. Throughout, I and $\hat{\mathrm{I}}$ are used to represent the *metric tensors* in the current configuration and the fixed reference configuration, respectively. On the other hand, the following unique additive decomposition holds:

$$\begin{aligned} \mathbf{L} &= \mathbf{D} + \mathbf{W}\,, \\ \mathbf{D} &= \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)\,, \\ \mathbf{W} &= \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)\,, \end{aligned} \tag{2.4}$$

where the tensor $\mathbf{D}$, the symmetric part of the velocity gradient $\mathbf{L}$, is known as the *stretching tensor*, and the tensor $\mathbf{W}$, the antisymmetric part of the velocity gradient $\mathbf{L}$, as the *vorticity tensor*.

In addition to those given above, there are other basic tensor quantities, some of which will be given in the next subsection.

2.2. Eulerian and Lagrangean tensors and their rotated correspondence

In a deforming material body, several types of tensor quantities are involved due to the different ways by which a fixed reference configuration and a current configuration are related, refer to, e.g., OGDEN [37 – 38] for details. There are three types of second order tensors: *Eulerian, Lagrangean and mixed-type*, for which the current configuration only, the reference configuration only, and both the reference and current configurations are related, respectively (see OGDEN [37 – 38]). In the tensor quantities mentioned before, the deformation gradient $\mathbf{F}$, the rotation tensor $\mathbf{R}$ and its transpose $\mathbf{R}^T$ are mixed-type, the right stretch tensor $\mathbf{U}$ and the right Cauchy–Green tensor $\mathbf{C}$ are Lagrangean, and the others, including the left stretch tensor $\mathbf{V}$, the left Cauchy-Green tensor $\mathbf{B}$, the velocity gradient $\mathbf{L}$, the vorticity tensor $\mathbf{W}$ and the stretching tensor $\mathbf{D}$ etc., are Eulerian. In this article, we are mainly concerned with Eulerian and Lagrangean second order tensors. Henceforth, *tensor* means *second order tensor*, if not otherwise indicated.

It is known that the transformation between the reference configuration and the current configuration can be effectuated by virtue of the rotation tensor $\mathbf{R}$. As a result, a natural correspondence between Eulerian and Lagrangean tensors can be established via the rotation tensor $\mathbf{R}$. Let $\mathbf{G}$ be an Eulerian tensor. Then the *Lagrangean counterpart* of $\mathbf{G}$, denoted by $\hat{\mathbf{G}}$, is defined as

$$\hat{\mathbf{G}} = \mathbf{R}^T\mathbf{G}\mathbf{R}\,. \tag{2.5}$$

Conversely, we call $\mathbf{G}$ the *Eulerian counterpart of the Lagrangean tensor* $\hat{\mathbf{G}}$ and we have

$$\mathbf{G} = \mathbf{R}\hat{\mathbf{G}}\mathbf{R}^T\,. \tag{2.6}$$

The above correspondence is called the *rotated correspondence* between Eulerian and Lagrangean tensors. It is evident that any two Eulerian and Lagrangean tensors $\mathbf{G}$ and $\hat{\mathbf{G}}$ associated by the rotated correspondence represent the same tensor quantity.

From (2.3) it follows

$$\mathbf{U} = \hat{\mathbf{V}} = \mathbf{R}^T\mathbf{V}\mathbf{R}\,, \tag{2.7}$$

$$\mathbf{C} = \hat{\mathbf{B}} = \mathbf{R}^T\mathbf{B}\mathbf{R}\,, \tag{2.8}$$

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T\,, \tag{2.9}$$

$$\mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^T\,. \tag{2.10}$$

Moreover, the Lagrangean counterparts of the stretching tensor $\mathbf{D}$ and the Kirchhoff stress tensor $\boldsymbol{\sigma}$, called the *Lagrangean stretching tensor* and the *Lagrangean Kirchhoff stress* (or the rotated Kirchhoff stress tensor according to SIMO and MARSDEN [49]) respectively, are given by

$$\hat{\mathbf{D}} = \mathbf{R}^T\mathbf{D}\mathbf{R}\,, \tag{2.11}$$

$$\hat{\boldsymbol{\sigma}} = \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}\,. \tag{2.12}$$

The following Eqs. represent two standard formulas for the stress power $\dot{w}$ *per unit reference state volume*

$$\dot{w} = \boldsymbol{\sigma} : \mathbf{D}\,, \tag{2.13}$$

$$\dot{w} = \hat{\boldsymbol{\sigma}} : \hat{\mathbf{D}}\,. \tag{2.14}$$

2.3. Rotating frames and objectivity

Let $\mathbf{Q}^*$ be an Eulerian time-dependent proper orthogonal tensor. Then a rotating frame $*$ relative to a fixed background frame is defined as follows[(2)]:

$$\mathbf{x}^*(\mathbf{X},t) = \mathbf{Q}^*\tilde{\mathbf{x}}(\mathbf{X},t) + \mathbf{x}_0(t)\,. \tag{2.15}$$

It is evident that the rotating frame $*$ is defined by the proper orthogonal tensor $\mathbf{Q}^*$. On the other hand, given the spin $\boldsymbol{\Omega}^*$ of the frame $*$, the latter is in turn determined by the first order tensorial differential equation

$$\dot{\mathbf{Q}}^{*T}\mathbf{Q}^* = -\mathbf{Q}^{*T}\dot{\mathbf{Q}}^* = \boldsymbol{\Omega}^* \tag{2.16}$$

up to a constant initial rotation. Thus, a rotating frame $*$ can also be defined by its spin. The latter definition serves our purpose and will be adopted. Let $\boldsymbol{\Omega}^*$ be a spin, i.e. an Eulerian time-dependent skewsymmetric tensor. Henceforth, by an $\boldsymbol{\Omega}^*$-frame we mean a rotating frame defined by (2.15) – (2.16).

Let $\mathbf{G}$ and $\hat{\mathbf{H}}$ be, respectively, an Eulerian and a Lagrangean tensors defined in a deforming material body. Following OGDEN [37 – 38], we say that the Eulerian tensor $\mathbf{G}$ and the Lagrangean tensor $\hat{\mathbf{H}}$ are *objective*, respectively, if they obey the following transformation rules with respect to any change of frame indicated by (2.15):

$$\mathbf{G}^* = \mathbf{Q}^*\mathbf{G}\mathbf{Q}^{*T}\,, \tag{2.17}$$

Eulerian e.g. V

$$\hat{\mathbf{H}}^* = \hat{\mathbf{H}}\,, \tag{2.18}$$

Lagrangean e.g. U

$U^2 = F^T F;\ (U^*)^2 = (F^*)^T F^*;\ F^* = Q^* F;\ (F^*)^T = F^T (Q^*)^T \Leftrightarrow (U^*)^2 = F^T F = U^2$ - objective

where the superscript $*$ indicates the association with a rotating frame defined by any continuously time-dependent rotation tensor $\mathbf{Q}^* = \mathbf{Q}^*(t)$, i.e. an $\boldsymbol{\Omega}^*$-frame, where $\boldsymbol{\Omega}^* = \dot{\mathbf{Q}}^{*T}\mathbf{Q}^*$.

The left stretch tensor $\mathbf{V}$, the left Cauchy–Green tensor $\mathbf{B}$ and the stretching tensor $\mathbf{D}$ are objective Eulerian tensors. The right stretch tensor $\mathbf{U}$ and the right Cauchy–Green tensor $\mathbf{C}$ are objective Lagrangean tensors. The velocity gradient $\mathbf{L}$ and the vorticity tensor $\mathbf{W}$ and their Lagrangean counterparts $\hat{\mathbf{L}}$ and $\hat{\mathbf{W}}$ provide, respectively, two examples of Eulerian and Lagrangean tensors which are not objective. For detail, refer to OGDEN [37 – 38].

2.4. Hill's generalized strain measures and their alternative expressions

A general class of Eulerian and Lagrangean strain measures, called Hill's *generalized strain measures*, was introduced by HILL [21 – 23] (see also WANG

(2) Generally, there is a time difference between the two frames, i.e. $t^* = t + t_0$, which is irrelevant to our purpose. Here we assume $t_0 = 0$ for the sake of simplicity.

and TRUESDELL [53] and OGDEN [38]). Let λ_1, λ_2 and λ_3 be the three principal stretches, i.e. the three eigenvalues (possibly repeated) of the stretch tensor $\mathbf{V}$ or $\mathbf{U}$. A set of three orthonormal eigenvectors of $\mathbf{V}$ (resp. $\mathbf{U}$) is called an *Eulerian triad* (resp. a *Lagrangean triad*), denoted by $\{\mathbf{n}_i\}$ (resp. $\{\mathbf{N}_i\}$). Hill's generalized strain measures are of the forms

$$\mathbf{e} = \mathbf{f}(\mathbf{V}) = \sum_{i=1}^{3} f(\lambda_i)\mathbf{n}_i \otimes \mathbf{n}_i\,, \tag{2.19}$$

$$\mathbf{E} = \mathbf{f}(\mathbf{U}) = \sum_{i=1}^{3} f(\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i\,. \tag{2.20}$$

In the above, the function $f:\ R^+ \to R$ is a smooth monotonic increasing function with the property $f(1) = f'(1) - 1 = 0$, which defines the strain measures $\mathbf{e}$ and $\mathbf{E}$ and is called the *scale function*. Since the tensor functions $\mathbf{f}(\mathbf{V})$ and $\mathbf{f}(\mathbf{U})$ are isotropic and the left and right stretch tensors $\mathbf{V}$ and $\mathbf{U}$ are objective, both the strain measures $\mathbf{e}$ and $\mathbf{E}$ defined through the scale function $f(\lambda)$, which are Eulerian and Lagrangean respectively, are objective. Moreover, they can be related to each other via the rotated correspondence indicated by (2.5) and (2.6), i.e.

$$\mathbf{e} = \mathbf{R}\mathbf{E}\mathbf{R}^T\,, \tag{2.21}$$

$$\mathbf{E} = \hat{\mathbf{e}} = \mathbf{R}^T\mathbf{e}\mathbf{R}\,. \tag{2.22}$$

It is known (see DOYLE and ERICKSEN [5], TRUESDELL and TOUPIN [52], SETH [48], HILL [21 – 23], OGDEN [38], *et al.*) that by choosing the scale function $f(\lambda)$ in the particular form

$$f(\lambda) = \frac{1}{m}(\lambda^m - 1) \tag{2.23}$$

and assigning several integers to the number m, Hill's generalized strain measures, i.e.

$$\mathbf{e}^{(m)} = \frac{1}{m}(\mathbf{V}^m - \mathbf{I})\,, \tag{2.24}$$

$$\mathbf{E}^{(m)} = \frac{1}{m}(\mathbf{U}^m - \hat{\mathbf{I}})\,, \tag{2.25}$$

supply all commonly-known objective Eulerian and Lagrangean strain measures. In particular, the limiting process $m \to 0$ or the logarithmic scale function $f(\lambda) = \ln\lambda$ results in Hencky's *logarithmic strain measures* (see HENCKY [20])

$$\ln\mathbf{V} = \sum_{i=1}^{3}(\ln\lambda_i)\mathbf{n}_i \otimes \mathbf{n}_i\,, \tag{2.26}$$

$$\ln \mathbf{U} = \sum_{i=1}^{3} (\ln \lambda_i) \mathbf{N}_i \otimes \mathbf{N}_i , \tag{2.27}$$

which have received much attention (e.g., see TRUESDELL and TOUPIN [52], HILL [21 – 23], RICE [42], FITZGERALD [8], GURTIN and SPEAR [17], HOGER [25 – 26], and LEHMANN and LIANG [30], *et al.*) and will be discussed in Sec. 6 of this article.

Since each principal stretch λ_i is always positive, we can give an alternative definition of Hill's strain measures **e** and **E** in terms of the left and right Cauchy–Green tensors **B** and **C**. Let $g: R^+ \to R$ be a new scale function defined by

$$g(\chi) = f(\sqrt{\chi}) \quad (\forall \chi > 0) . \tag{2.28}$$

Moreover, let $\chi_1, \cdots, \chi_r$ be the distinct eigenvalues of the left Cauchy–Green tensor **B** or, equivalently, the right Cauchy–Green tensor **C**, and $\mathbf{B}_\sigma$ and $\mathbf{C}_\sigma$ be the corresponding subordinate eigenprojections of **B** and **C** respectively, where $\sigma = 1, \cdots, r$. Then we have

$$\mathbf{e} = \mathbf{g}(\mathbf{B}) = \sum_{\sigma=1}^{r} g(\chi_\sigma) \mathbf{B}_\sigma , \tag{2.29}$$

$$\mathbf{E} = \mathbf{g}(\mathbf{C}) = \sum_{\sigma=1}^{r} g(\chi_\sigma) \mathbf{C}_\sigma . \tag{2.30}$$

Henceforth we shall adopt the latter definition. In so doing, the main results that will be derived in Sec. 5 can be expressed in terms of the Cauchy–Green tensors **B** and **C** instead of the stretch tensors **V** and **U**. Here we would mention the fact that once the deformation gradient **F** is known, it is much easier to calculate the Cauchy–Green tensors $\mathbf{B} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{C} = \mathbf{F}\mathbf{F}^T$ than to calculate the stretch tensors **V** and **U**, the latter being the complicated square roots $\sqrt{\mathbf{F}^T\mathbf{F}}$ and $\sqrt{\mathbf{F}\mathbf{F}^T}$, respectively. Moreover, the use of eigenprojections instead of eigenvectors will prove to be crucial. The use of eigenprojections has certain advantages over the use of eigenvectors. Here we mention the following three aspects only.

1. The eigenprojections $\mathbf{B}_\sigma$ and $\mathbf{C}_\sigma$ are unique and this applies to all the cases for eigenvalues of **B** and **C**. In fact, the following Sylvester's formulas hold

$$\mathbf{C}_\sigma = \delta_{1r} \hat{\mathbf{I}} + \prod_{\tau \neq \sigma}^{r} \frac{\mathbf{C} - \chi_\tau \hat{\mathbf{I}}}{\chi_\sigma - \chi_\tau} , \tag{2.31}$$

$$\mathbf{B}_\sigma = \delta_{1r} \mathbf{I} + \prod_{\tau \neq \sigma}^{r} \frac{\mathbf{B} - \chi_\tau \mathbf{I}}{\chi_\sigma - \chi_\tau} ; \tag{2.32}$$

2. The explicit basis-free form of the main results for strain rates and conjugate stresses can easily be derived with the aid of Sylvester's formulas (2.31) – (2.32); and

3. All the procedure can be fulfilled merely by means of the following simple manipulation properties concerning the eigenprojections $\mathbf{B}_\sigma$ and $\mathbf{C}_\sigma$

$$\mathbf{C}_\sigma \mathbf{C}_\tau = \delta_{\sigma\tau}\mathbf{C}_\tau\,, \tag{2.33}$$

$$\sum_{\sigma=1}^{r} \mathbf{C}_\sigma = \hat{\mathbf{I}}\,, \tag{2.34}$$

and

$$\mathbf{B}_\sigma \mathbf{B}_\tau = \delta_{\sigma\tau}\mathbf{B}_\tau\,, \tag{2.35}$$

$$\sum_{\sigma=1}^{r} \mathbf{B}_\sigma = \mathbf{I}\,, \tag{2.36}$$

with no summation over repeated indices. Here, $\delta_{\sigma\tau}$ is used to denote the Kronecker delta.

The advantage of using the Cauchy–Green tensors and the eigenprojections was known by HOGER and CARLSON [24], CARLSON and HOGER [1], SCHEIDLER [46] and XIAO [56] *et al.*, and was further exploited by these authors (see XIAO, BRUHNS and MEYERS [57 – 63], esp. [65]).

Finally, it should be pointed out that once the deformation gradient $\mathbf{F}$ is given in a coordinate system other than the particular ones formed by the principal bases, usually it is not easy to calculate the generalized strain measures $\mathbf{e}$ and $\mathbf{E}$ for any nonpolynomial scale function $g(\chi)$, especially for any transcendental scale function $g(\chi)$ such as the logarithmic function $g(\chi) = \frac{1}{2}\ln\chi$ etc. This difficulty may be circumvented by using the formula (5.24) – (5.25) given later and by the explicit basis-free formulas derived from (2.29) – (2.32).

3. Material spins, corotating material frames and objective corotational rates

3.1. Corotational rates of Eulerian and Lagrangean tensors

Let $\mathbf{\Omega}^*$ be an Eulerian spin relative to a fixed background frame, i.e. a continuous time-dependent skewsymmetric Eulerian tensor, and let $\mathbf{G}$ be an objective Eulerian tensor. The Eulerian tensor defined by

$$\overset{\circ}{\mathbf{G}}{}^{*} = \dot{\mathbf{G}} + \mathbf{G}\mathbf{\Omega}^* - \mathbf{\Omega}^*\mathbf{G} \tag{3.1}$$

is called the *corotational rate of the tensor* G *defined by the Eulerian spin* $\boldsymbol{\Omega}^*$. To see what the term *corotational rate* means, let us consider an $\boldsymbol{\Omega}^*$-frame $*$ (see Sec. 2.3) and let $\mathbf{Q}^*$ be a continuously differentiable time-dependent proper orthogonal tensor determined by the spin $\boldsymbol{\Omega}^*$ (cf. (2.16)). In such an $\boldsymbol{\Omega}^*$-frame $*$, the Eulerian tensor G is of the form $\mathbf{Q}^*\mathbf{G}\mathbf{Q}^{*}$T. We have

$$
\begin{aligned}
\overline{(\mathbf{Q}^*\mathbf{G}\mathbf{Q}^{*T})}^{\cdot} &= \mathbf{Q}^*\dot{\mathbf{G}}\mathbf{Q}^{*T} + \dot{\mathbf{Q}}^*\mathbf{G}\mathbf{Q}^{*T} + \mathbf{Q}^*\mathbf{G}\dot{\mathbf{Q}}^{*T} \\
&= \mathbf{Q}^* \overset{\circ}{\mathbf{G}}{}^* \mathbf{Q}^{*T}.
\end{aligned} \tag{3.2}
$$

Note that the latter is the counterpart of $\overset{\circ}{\mathbf{G}}{}^*$ in the $\boldsymbol{\Omega}^*$-frame. Thus, *the corotational rate of an objective Eulerian tensor defined by an Eulerian spin* $\boldsymbol{\Omega}^*$ *(cf. (3.1)) is a material time derivative in an* $\boldsymbol{\Omega}^*$*-frame.* It should be noted that this interpretation can be made only when the tensor G is an *objective Eulerian tensor.*

The Lagrangean counterpart of the corotational rate $\overset{\circ}{\mathbf{G}}{}^*$ provided by the tensor $\mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^* \mathbf{R}$. From (2.5) we derive

$$
\dot{\mathbf{G}} = \overline{(\mathbf{R}\hat{\mathbf{G}}\mathbf{R}^T)}^{\cdot} = \mathbf{R}\dot{\hat{\mathbf{G}}}\mathbf{R}^T + \dot{\mathbf{R}}\hat{\mathbf{G}}\mathbf{R}^T + \mathbf{R}\hat{\mathbf{G}}\dot{\mathbf{R}}^T = \mathbf{R}(\dot{\hat{\mathbf{G}}} - \hat{\mathbf{G}}\hat{\boldsymbol{\Omega}}^R + \hat{\boldsymbol{\Omega}}^R\hat{\mathbf{G}})\mathbf{R}^T,
$$

where $\hat{\mathbf{G}}$ and $\hat{\boldsymbol{\Omega}}^R = \mathbf{R}^T\dot{\mathbf{R}}$ are the Lagrangean counterparts of the Eulerian tensors G and $\boldsymbol{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^T$ respectively. Applying the result just derived and the identity

$$
\mathbf{Q}(\mathbf{H}_1\mathbf{H}_2)\mathbf{Q}^T = (\mathbf{Q}\mathbf{H}_1\mathbf{Q}^T)(\mathbf{Q}\mathbf{H}_2\mathbf{Q}^T)
$$

for any two tensors $\mathbf{H}_1$ and $\mathbf{H}_2$ and for any orthogonal tensor Q, we arrive at the following formula

$$
\mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^* \mathbf{R} = \dot{\hat{\mathbf{G}}} + \hat{\mathbf{G}}(\hat{\boldsymbol{\Omega}}^* - \hat{\boldsymbol{\Omega}}^R) - (\hat{\boldsymbol{\Omega}}^* - \hat{\boldsymbol{\Omega}}^R)\hat{\mathbf{G}}, \tag{3.3}
$$

where

$$
\begin{aligned}
\hat{\mathbf{G}} &= \mathbf{R}^T\mathbf{G}\mathbf{R}, \\
\hat{\boldsymbol{\Omega}}^* &= \mathbf{R}^T\boldsymbol{\Omega}^*\mathbf{R}, \\
\hat{\boldsymbol{\Omega}}^R &= \mathbf{R}^T\boldsymbol{\Omega}^R\mathbf{R} = \mathbf{R}^T\dot{\mathbf{R}} = -\dot{\mathbf{R}}^T\mathbf{R},
\end{aligned} \tag{3.4}
$$

are the Lagrangean counterparts of the Eulerian tensors G and $\boldsymbol{\Omega}^*$ and $\boldsymbol{\Omega}^R$ respectively.

The Lagrangean counterpart of each corotational rate $\overset{\circ}{\mathbf{G}}{}^*$ of an Eulerian tensor G provides a rate measure of the Lagrangean counterpart $\hat{\mathbf{G}}$ of this tensor. Since the right-hand side of the formula (3.3) is of the same structure as that of

(3.1), we call the rate measure $\mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^* \mathbf{R}$, i.e. the right-hand side of (3.3), the *corotational rate of the Lagrangean tensor* $\hat{\mathbf{G}}$ *defined by the spin* $\hat{\boldsymbol{\Omega}}^*$, *denoted by* $\overset{o}{\hat{\mathbf{G}}}{}^*$. Thus, we have

$$(3.5) \qquad \overset{\circ}{\hat{\mathbf{G}}}{}^* = \mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^* \mathbf{R} = \dot{\hat{\mathbf{G}}} + \hat{\mathbf{G}}(\hat{\boldsymbol{\Omega}}^* - \hat{\boldsymbol{\Omega}}^R) - (\hat{\boldsymbol{\Omega}}^* - \hat{\boldsymbol{\Omega}}^R)\hat{\mathbf{G}}\,.$$

Note the difference between the two definitions (3.1) and (3.5): the latter is using the additional spin $\hat{\boldsymbol{\Omega}}^R = \mathbf{R}^T\dot{\mathbf{R}}$, while the former is not concerned with any other spin except the spin $\boldsymbol{\Omega}^*$. This difference arises from the fact that there is a relative rotation between the Eulerian triad and the Lagrangean triad, which is just given by the rotation tensor $\mathbf{R}$.

Let $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}^R$, i.e. $\hat{\boldsymbol{\Omega}}^* = \hat{\boldsymbol{\Omega}}^R$ and introduce the *polar rate* of a symmetric Eulerian tensor $\mathbf{G}$ (see GREEN and NAGHDI [9], DIENES [3 – 4] and SCHEIDLER [45], *et al.*)

$$(3.6) \qquad \overset{\circ}{\mathbf{G}}{}^R = \dot{\mathbf{G}} + \mathbf{G}\boldsymbol{\Omega}^R - \boldsymbol{\Omega}^R\mathbf{G}\,.$$

Then the formula (3.5) yields

$$(3.7) \qquad \dot{\hat{\mathbf{G}}} = \mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^R \mathbf{R}, \qquad \overset{\circ}{\mathbf{G}}{}^R = \mathbf{R}\dot{\hat{\mathbf{G}}}\mathbf{R}^T\,.$$

It turns out that *the Lagrangean counterpart of the polar rate of an objective Eulerian tensor is just the material time rate of the Lagrangean counterpart of this tensor* and *vice versa*, i.e. *the Eulerian counterpart of the material time rate of an objective Lagrangean tensor is the polar rate of the Eulerian counterpart of this tensor*. This fact indicates that the material time rate of an objective Lagrangean tensor is merely a particular kind of corotational rate of this tensor, which is defined by the spin $\hat{\boldsymbol{\Omega}}^R = \mathbf{R}^T\dot{\mathbf{R}}$.

We would emphasize that the formula (3.5) and hence the above fact apply to objective Eulerian and Lagrangean tensors only.

3.2. Material spins and objective corotational rates

More essentially, it is required that corotational rates of objective Eulerian and Lagrangean tensors be *objective rate measures* so that any superimposed rigid rotation motion has no effect on it. Moreover, to establish the extended work-conjugacy relation, this requirement is just what is needed, as will be shown in the next section. It can be readily proved that if an Eulerian tensor is objective, then its Lagrangean counterpart via the rotated correspondence (2.5) is also objective. The opposite is true, i.e. if a Lagrangean tensor is objective, then its Eulerian counterpart via the rotated correspondence (2.6) is also objective. In view of this fact and the rotated correspondence relationship between the Eulerian and

Lagrangean corotational rates as indicated in the last subsection, it suffices to consider objective corotational rates of objective Eulerian tensors.

Let $\mathbf{G}$ be an objective Eulerian tensor. Not every corotational rate $\overset{\circ}{\mathbf{G}}{}^{*}$ is objective. For instance, let $\boldsymbol{\Omega}^{*} = c\mathbf{W}$ with c – any given constant and $\mathbf{W}$ being the vorticity tensor (cf. $(2.4)_3$). Then, (3.1) defines infinitely many corotational rates of the Eulerian tensor $\mathbf{G}$, when the constant c runs over all the reals. However, only the one with $c = 1$, i.e. the Zaremba–Jaumann rate $\dot{\mathbf{G}} + \mathbf{GW} - \mathbf{WG}$, is objective. Generally, whether the corotational rate $\overset{\circ}{\mathbf{G}}{}^{*}$ is objective or not depends on its defining spin $\boldsymbol{\Omega}^{*}$. To arrive at objective corotational rates, the defining spin $\boldsymbol{\Omega}^{*}$ should be associated with the deformation and motion of the deforming material body under consideration in an appropriate manner, as has been shown for several well-known examples: $\boldsymbol{\Omega}^{*} = \mathbf{W}$ (Zaremba–Jaumann rate), $\boldsymbol{\Omega}^{*} = \dot{\mathbf{R}}\mathbf{R}^{T}$ (the polar rate or Green–Naghdi–Dienes rate), $\boldsymbol{\Omega}^{*} = \boldsymbol{\Omega}^{E}$, etc. Here the latter is the twirl tensor of the Eulerian triad $\{\mathbf{n}_i\}$, i.e. $\dot{\mathbf{n}}_i = \boldsymbol{\Omega}^{E}\mathbf{n}_i$.

Since the deformation gradient $\mathbf{F}$ and the velocity gradient $\mathbf{L}$ characterize the local deformation state and the rate-of-change of the local deformation state at a generic material particle, the most general form of the spin tensor $\boldsymbol{\Omega}^{*}$ associated with the deformation and rotation of a deforming body may be assumed as

$$\boldsymbol{\Omega}^{*} = \boldsymbol{\Upsilon}(\mathbf{F}, \mathbf{L}) \tag{3.8}$$

with $\boldsymbol{\Upsilon}(\mathbf{F}, \mathbf{L})$ being an antisymmetric tensor-valued function of the deformation gradient $\mathbf{F}$ and the stretching $\mathbf{D}$. Of course, such a general form is of little use. To make the corotational rate defined by the above spin a reasonable objective rate measure, the spin $\boldsymbol{\Omega}^{*}$ must fulfill certain necessary requirements. The latter place restrictions on the form of the tensor function $\boldsymbol{\Upsilon}(\mathbf{F}, \mathbf{L})$. Recently, these authors (see Xiao, Bruhns and Meyers [60]) have introduced the following necessary requirements for $\boldsymbol{\Omega}^{*}$:

(i) any superimposed constant rigid rotation has no effect on $\boldsymbol{\Omega}^{*}$, and, moreover, any superimposed constant uniform dilational deformation has also no effect on $\boldsymbol{\Omega}^{*}$;

(ii) the corotational rate of an Eulerian tensor defined by the spin $\boldsymbol{\Omega}^{*}$ depends linearly on the change of time scale,

(iii) the corotational rate of each time-differentiable objective Eulerian tensor field defined by the spin $\boldsymbol{\Omega}^{*}$ is objective, and

(iv) the tensor function $\boldsymbol{\Upsilon}(\mathbf{F}, \mathbf{L})$ is continuously differentiable at $\mathbf{L} = \mathbf{O}$.

From these requirements, a general form of spin $\boldsymbol{\Omega}^{*}$ has been derived (see Xiao, Bruhns and Meyers [60]):

$$\boldsymbol{\Omega}^{*} = \mathbf{W} + \sum_{\sigma,\tau=1}^{r} h\left(\frac{\chi_\sigma}{I}, \frac{\chi_\tau}{I}\right)\mathbf{B}_\sigma \mathbf{D} \mathbf{B}_\tau\,, \tag{3.9}$$

where

$$I = \mathrm{tr}\mathbf{B} = \mathrm{tr}\mathbf{C} = \mathrm{tr}\mathbf{F}\mathbf{F}^{\mathrm{T}}. \tag{3.10}$$

In the above, the function $h(x, y)$: $R^+ \times R^+ \to R$, which defines the spin tensor $\mathbf{\Omega}^*$ and is hence called the *spin function*, is antisymmetric, i.e.

$$h(x, y) = -h(y, x).$$

Each spin given by (3.9), which is the same kind of kinematical quantity as the vorticity tensor $\mathbf{W}$, is called a *material strain* in XIAO, BRUHNS and MEYERS [60]. In particular, a subclass of the above material spins is as follows

$$\begin{aligned} \mathbf{\Omega}^* &= \mathbf{W} + \sum_{\sigma,\tau=1}^{r} \tilde{h}\left(\frac{\chi_\sigma}{\chi_\tau}\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau\,, \\ \tilde{h}(z^{-1}) &= -\tilde{h}(z) \quad (\forall z > 0)\,. \end{aligned} \tag{3.11}$$

It has been shown (see XIAO, BRUHNS and MEYERS [60]) that all commonly-known spins, including the vorticity tensor $\mathbf{\Omega}^J = \mathbf{W}$, the *polar spin* $\mathbf{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^T$, the twirl tensors $\mathbf{\Omega}^E$ and $\mathbf{\Omega}^L$ of the Eulerian and Lagrangean triads (for these spins, refer to, e.g., HILL [23], OGDEN [38] and MEHRABADI and NEMAT-NASSER [35], for detail) and the newly discovered *logarithmic spin* $\mathbf{\Omega}^{\log}$ (see XIAO, BRUHNS and MEYERS [57 – 60]; see also LEHMANN, GUO and LIANG [29] and REINHARDT and DUBEY [40 – 41]), are incorporated into the above subclass by taking the simplified spin function $\tilde{h}(z)$ in the particular forms

$$\tilde{h}(z) = \tilde{h}^J(z) = 0\,, \tag{3.12}$$

$$\tilde{h}(z) = \tilde{h}^R(z) = \frac{1-\sqrt{z}}{1+\sqrt{z}}\,, \tag{3.13}$$

$$\tilde{h}(z) = \tilde{h}^E(z) = \frac{1+z}{1-z}\,, \tag{3.14}$$

$$\tilde{h}(z) = \tilde{h}^L(z) = \frac{2\sqrt{z}}{1-z}\,, \tag{3.15}$$

$$\tilde{h}(z) = \tilde{h}^{\log}(z) = \frac{1+z}{1-z} + \frac{2}{\ln z}\,, \tag{3.16}$$

respectively.

Henceforth, the objective corotational rates of the objective Eulerian tensor $\mathbf{G}$ and the objective Lagrangean tensor $\hat{\mathbf{G}}$ defined by the above five material spins and their Lagrangean counterparts, respectively, are denoted by

$$\overset{\circ}{\mathbf{G}}{}^M = \dot{\mathbf{G}} + \mathbf{G}\mathbf{\Omega}^M - \mathbf{\Omega}^M\mathbf{G}, \qquad M \in \{J, R, E, L, \log\}\,, \tag{3.17}$$

(3.18) $$\overset{\circ}{\hat{\mathbf{G}}}{}^{M} = \mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^{M}\mathbf{R} = \dot{\hat{\mathbf{G}}} + \hat{\mathbf{G}}(\hat{\mathbf{\Omega}}^M - \hat{\Omega}^R) - (\hat{\mathbf{\Omega}}^M - \hat{\Omega}^R)\hat{\mathbf{G}},$$
$$M \in \{J, R, E, L, \log\}.$$

Note that for any material spin $\mathbf{\Omega}^*$ (cf. (3.9)), the Lagrangean spin $\hat{\mathbf{\Omega}}^* - \hat{\mathbf{\Omega}}^R$ appearing in (3.5) and in particular in (3.18), i.e.

$$\hat{\mathbf{\Omega}}^* - \hat{\mathbf{\Omega}}^R = \sum_{\sigma,\tau=1} (h(\frac{\chi_\sigma}{I}, \frac{\chi_\tau}{I}) - \tilde{h}^R(\frac{\chi_\sigma}{\chi_\tau}))\mathbf{C}_\sigma\hat{\mathbf{D}}\mathbf{C}_\tau,$$

is independent of the vorticity tensor $\mathbf{W}$. In the above, the spin function $\tilde{h}^R(z)$ is given by $(3.13)_2$.

In particular, from (3.7) it follows

(3.19) $$\overset{\circ}{\hat{\mathbf{G}}}{}^{R} = \dot{\hat{\mathbf{G}}}$$

for any objective Lagrangean tensor $\hat{\mathbf{G}}$.

Rates of various strain measures (mainly the material time rate) and the material spins $\mathbf{W}$, $\mathbf{\Omega}^R$, $\mathbf{\Omega}^E$, $\mathbf{\Omega}^L$ and $\mathbf{\Omega}^{\log}$ have been studied by many authors and many results are available, refer to, e.g., HILL [21 – 23], DIENES [3 – 4], FITZGERALD [8], GURTIN and SPEAR [17], OGDEN [38], GUO and DUBEY [12], GUO [11], HOGER and CARLSON [24], CARLSON and HOGER [1], HOGER [25], MEHRABADI and NEMAT-NASSER [35], DUBEY [6], STICKFORTH and WEGENER [50], WHEELER [55], GUO, LEHMANN and LIANG [13], SCHEIDLER [44 – 47], WANG and DUAN [54], GUO, LEHMANN, LIANG and MAN [14], MACMILLAN [31], CHEN and WHEELER [2], MAN and GUO [33], XIAO [56], REINHARDT and DUBEY [40], and XIAO, BRUHNS and MEYERS [57 – 58, 60 – 61, 65], *et al.*

A unified study of time derivatives of tensor fields via Lie derivatives, which incorporates corotational rates as particular cases, was given earlier by GUO [10] and later by MARSDEN and HUGHES [34].

3.3. Material corotating frames

Each material spin $\mathbf{\Omega}^*$ of the form (3.9), which defines a kind of objective corotational rates of objective tensors, is associated with the rotation and deformation of a deforming material body in a suitable manner. Thus, an $\mathbf{\Omega}^*$-frame is a rotating frame that is embedded in a deforming material body in a suitable manner, and hence it traces the rotation and deformation of the deforming material body in an intrinsic way. In view of this, we call a rotating frame defined by a material spin through (2.15) – (2.16) a *corotating material frame*. Evidently, the Eulerian and Lagrangean triads are two corotating material frames when they are well-defined. Another two important examples are provided by the rotating frames defined by the vorticity tensor $\mathbf{\Omega}^J = \mathbf{W}$ and the polar spin $\mathbf{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$, respectively.

A significant fact for corotating material frames is as follows:
The time rate of an objective Eulerian tensor in a corotating material frame is an objective corotational rate and vice versa, i.e. an objective corotational rate of an objective Eulerian tensor is the time rate of this tensor in a corotating material frame.

This fact is essential for the subsequent considerations, as will be seen in the next section.

4. Unified work-conjugacy relation between Eulerian and Lagrangean strain and stress measures

Let $(\mathbf{e}, \mathbf{t})$ be a pair of objective Eulerian strain and stress measures, both being symmetric, and let $\mathbf{\Omega}^*$ be an Eulerian spin. In an $\mathbf{\Omega}^*$-frame (cf. (2.15) – (2.16)) relative to a fixed background frame, this pair becomes $(\mathbf{Q}^*\mathbf{e}\mathbf{Q}^{*T}, \mathbf{Q}^*\mathbf{t}\mathbf{Q}^{*T})$. Then an observer in the just-mentioned $\mathbf{\Omega}^*$-frame forms the inner product

$$(\mathbf{Q}^*\mathbf{t}\mathbf{Q}^{*T}) : \dot{\overline{(\mathbf{Q}^*\mathbf{e}\mathbf{Q}^{*T})}},$$

just as an observer in a fixed background frame does for a Lagrangean strain-stress pair. Following the same argument as that used in Hill's work-conjugacy notion (cf. (1.1)), which is concerned with a fixed background frame, the observer in the $\mathbf{\Omega}^*$-frame judges that the pair $(\mathbf{t}, \mathbf{e})$ is an *$\mathbf{\Omega}^*$-work-conjugate pair* if the just-mentioned inner product furnishes the stress power $\dot{w}$, i.e.

$$\dot{w} = (\mathbf{Q}^*\mathbf{t}\,\mathbf{Q}^{*T}) : \dot{\overline{(\mathbf{Q}^*\mathbf{e}\,\mathbf{Q}^{*T})}}\,, \tag{4.1}$$

or, equivalently,

$$\dot{w} = \mathbf{t} : \overset{\circ}{\mathbf{e}}{}^*\,, \tag{4.2}$$

where $\overset{\circ}{\mathbf{e}}{}^*$ is the corotational rate of the strain measure $\mathbf{e}$ defined by the spin $\mathbf{\Omega}^*$, i.e.

$$\overset{\circ}{\mathbf{e}}{}^* = \dot{\mathbf{e}} + \mathbf{e}\mathbf{\Omega}^* - \mathbf{\Omega}^*\mathbf{e}\,. \tag{4.3}$$

As has been shown, the above work-conjugacy relation is defined in a rotating frame. However, the definition itself does not mean that such relation is well-defined for every rotating frame, i.e. for every kind of corotational rates. Now we are in a position to find out in what rotating frames the aforementioned work-conjugacy relation can be defined. Since both $\dot{w}$ and $\mathbf{t}$ are objective and both $\mathbf{t}$ and $\mathbf{e}$ are symmetric, from (4.2) we conclude that *the relation (4.1), i.e. (4.2), may be defined only if the corotational rate $\overset{\circ}{\mathbf{e}}{}^*$ of the strain measure $\mathbf{e}$ is objective.*

The above fact justifies the introduction of objective corotational rates in Sec. 3.2. Furthermore, let the spin $\boldsymbol{\Omega}^*$ be associated with the rotation and deformation of the deforming material body at issue in a manner indicated by (3.8). Then, by applying the general result proved in XIAO, BRUHNS and MEYERS [60] we infer that the spin $\boldsymbol{\Omega}^*$ must be of the form given by (3.9), i.e., the spin $\boldsymbol{\Omega}^*$ must be a material spin. Accordingly, *the work-conjugacy relation (4.1), i.e. (4.2), may be defined only in a corotating material frame.*
However, we still can not say that the work-conjugacy relation (4.1), i.e. (4.2), can be defined in all possible corotating material frames. In fact, each objective corotational strain rate $\overset{\circ}{\mathbf{e}}{}^*$ defined by a material spin $\boldsymbol{\Omega}^*$ is of the form (see (5.1) given in Sec. 5)

$$\overset{\circ}{\mathbf{e}}{}^* = \mathcal{L}[\mathbf{D}],$$

where $\mathcal{L} = \tilde{\mathcal{L}}(\mathbf{B})$ is a fourth order tensor depending on the left Cauchy–Green tensor $\mathbf{B}$, i.e. a linear transformation between second order tensors, with the index symmetry properties

$$\mathcal{L}_{ijkl} = \mathcal{L}_{jikl} = \mathcal{L}_{ijlk} = \mathcal{L}_{klij}.$$

Hence, (4.2) may be recast in

$$\dot{w} = \mathbf{t} : (\mathcal{L}[\mathbf{D}]).$$

From the latter and the formula (2.13) we deduce that the equality

$$\mathbf{t} : (\mathcal{L}[\mathbf{D}]) = \boldsymbol{\sigma} : \mathbf{D}$$

must hold for each $\mathbf{D}$ and each $\boldsymbol{\sigma}$. Since $\mathcal{L}$ is a symmetric linear transformation, we derive

$$\mathcal{L}[\mathbf{t}] = \boldsymbol{\sigma}\,. \tag{4.4}$$

Thus, if the stress $\boldsymbol{\sigma}$ is not allowed to be restricted in any manner, as it should be, the fourth order tensor $\mathcal{L} = \tilde{\mathcal{L}}(\mathbf{B})$ must be a nonsingular linear transformation between symmetric second order tensors, and therefore the objective corotational strain rate $\overset{\circ}{\mathbf{e}}{}^*$ must be a *complete strain rate measure*. By the latter we mean that the strain rate $\overset{\circ}{\mathbf{e}}{}^*$ and the stretching tensor $\mathbf{D}$ constitutes a one-to-one correspondence for any given left Cauchy–Green tensor $\mathbf{B}$. By applying the expression for the strain rate $\overset{\circ}{\mathbf{e}}{}^*$ in terms of $\mathbf{B}$ and $\mathbf{D}$ (cf. (5.1)), from (4.4) we can derive the $\boldsymbol{\Omega}^*$-work-conjugate stress measure $\mathbf{t}$ of the strain measure $\mathbf{e}$ in terms of $\mathbf{B}$ and $\mathbf{D}$. We postpone the further discussion in this aspect to the next section.

Now we consider objective Lagrangean strain and stress measures. By virtue of the rotated correspondence (2.5), we can convert (4.2) to

$$\dot{w} = \hat{\mathbf{t}} : (\mathbf{R}^T \overset{\circ}{\mathbf{e}}{}^* \mathbf{R})\,, \tag{4.5}$$

where the identity

$$\mathbf{H}_1 : \mathbf{H}_2 = (\mathbf{Q}\mathbf{H}_1\mathbf{Q}^{\mathrm{T}}) : (\mathbf{Q}\mathbf{H}_2\mathbf{Q}^{\mathrm{T}})$$

for any two tensors $\mathbf{H}_1$ and $\mathbf{H}_2$ and for any orthogonal tensor $\mathbf{Q}$, is used. Applying the formula $(3.5)_1$, we further arrive at

$$\dot{w} = \mathbf{T} : \overset{\circ}{\mathbf{E}}{}^{*}, \tag{4.6}$$

where $\mathbf{T} = \hat{\mathbf{t}}$ and $\mathbf{E} = \hat{\mathbf{e}}$ are the Lagrangean counterparts of the Eulerian stress and strain measures $\mathbf{t}$ and $\mathbf{e}$ through the rotated correspondence (2.5), and $\overset{\circ}{\mathbf{E}}{}^{*}$ is the objective corotational rate of the objective Lagrangean strain measure $\mathbf{E}$ defined by the Lagrangean spin $\hat{\boldsymbol{\Omega}}^{*} = \mathbf{R}^{T}\boldsymbol{\Omega}^{*}\mathbf{R}$ (cf. (3.5)), i.e.

$$\overset{\circ}{\mathbf{E}}{}^{*} = \dot{\mathbf{E}} + \mathbf{E}(\hat{\boldsymbol{\Omega}}^{*} - \hat{\boldsymbol{\Omega}}^{R}) - (\hat{\boldsymbol{\Omega}}^{*} - \hat{\boldsymbol{\Omega}}^{R})\mathbf{E}. \tag{4.7}$$

Observing that (4.6) has the same structure as that of (4.2), we say that a pair of Lagrangean strain and stress measures, $(\mathbf{T}, \mathbf{E})$, is an $\hat{\boldsymbol{\Omega}}^{*}$*-work-conjugate pair* if (4.6) holds.

Let $(\boldsymbol{\Omega}^{*}, \hat{\boldsymbol{\Omega}}^{*})$, $(\mathbf{t}, \hat{\mathbf{t}})$ and $(\mathbf{e}, \hat{\mathbf{e}})$ be, respectively, the Eulerian-Lagrangean spin pair, stress pair and strain pair related by the rotated correspondence (2.5) and (2.6). Then, it is evident that

$(\mathbf{e}, \mathbf{t})$ is an $\boldsymbol{\Omega}^{*}$-work-conjugate pair $\Longleftrightarrow (\hat{\mathbf{e}}, \hat{\mathbf{t}})$ is an $\hat{\boldsymbol{\Omega}}^{*}$-work-conjugate pair.

Thus, via (4.2) and (4.6) we have established a unified work-conjugacy relation between Eulerian and Lagrangean strain and stress measures. The just-stated fact indicates that in the sense of this unified work-conjugacy relation, the rotated correspondence relationship via the rotation tensor $\mathbf{R}$ remains true for work-conjugate Eulerian strain-stress pairs and work-conjugate Lagrangean strain-stress pairs.

The introduced unified work-conjugacy relation is much broader than the Hill's work-conjugacy relation, even within the scope of objective Lagrangean strain and stress measures. In fact, via various kinds of corotating material frames, or, equivalently, via various kinds of objective corotational rates, a given strain measure may be related to different stress measures in the sense of the introduced work-conjugacy relation. It turns out that the introduced unified work-conjugacy relation incorporates the Hill's work-conjugacy relation into a particular case when an $\boldsymbol{\Omega}^{R}$-frame is concerned. In fact, by utilizing (3.19) we have

$$\dot{w} = \mathbf{T} : \dot{\mathbf{E}} = \mathbf{T} : \overset{\circ}{\mathbf{E}}{}^{R} = \mathbf{t} : \overset{\circ}{\mathbf{e}}{}^{R}$$

where $\mathbf{t} = \mathbf{R}\mathbf{T}\mathbf{R}^{T}$ and $\mathbf{e} = \mathbf{R}\mathbf{E}\mathbf{R}^{T}$ are the Eulerian counterparts of the objective Lagrangean stress and strain measures $\mathbf{T}$ and $\mathbf{E}$, and, moreover, $\overset{\circ}{\mathbf{E}}{}^{R}$ and $\overset{\circ}{\mathbf{e}}{}^{R}$

are the polar rates of E and e (cf. (3.17) and (3.18) with $M = R$), respectively. Thus,

$$(4.8)\quad \begin{cases} (\mathbf{E},\mathbf{T}) \text{ is a work-conjugate Lagrangean strain-stress pair} \\ \hfill \text{in Hill's sense}, \\ \Longleftrightarrow (\mathbf{e},\mathbf{t}) \text{ is an } \mathbf{\Omega}^R-\text{work-conjugate Eulerian strain-stress pair}. \end{cases}$$

In the above, the two strain-stress pairs are related to each other by the rotated correspondence (2.5) and (2.6).

ZIEGLER and MACVEAN [64] introduced a more general work-conjugacy notion. MACVEAN [32] studied work-conjugacy relation between certain commonly-known Eulerian and Lagrangean strain and stress measures. The general work-conjugacy notion was also adopted in HAUPT and TSAKMAKIS [19]. These studies allow for general objective rate measures including objective corotational rates. Moreover, the definition (4.2) was used by LEHMANN [28] in a thermodynamical setting and later used in some particular cases by LEHMANN, GUO and LIANG [29] and LEHMANN and LIANG [30]. It seems that the interpretation (cf. (4.1)) of this notion in terms of corotating material frames, and hence the fact that the introduced work-conjugacy relation is a natural extension of Hill's work-conjugacy relation, are disclosed first by these authors in XIAO, BRUHNS and MEYERS [57 – 58]. Moreover, it seems that the unified work-conjugacy relation for both Eulerian and Lagrangean strain and stress measures in a general sense is established here for the first time.

5. Work-conjugate stresses of generalized Eulerian and Lagrangean strain measures

Let $\mathbf{e} = \mathbf{g}(\mathbf{B})$ be any given Hill's Eulerian strain measure defined by the scale function $g(\chi)$ (cf. (2.28)–(2.29)) and moreover, let $\mathbf{\Omega}^*$ be any given Eulerian material spin characterized by the spin function $h(x,y)$ (cf. (3.9)). According to the formulas (31a) and (30) given in XIAO [56], we have

$$\dot{\mathbf{e}} = \sum_{\sigma,\tau=1}^{r} g_{\sigma\tau}\mathbf{B}_\sigma\dot{\mathbf{B}}\mathbf{B}_\tau,$$

where

$$g_{\sigma\tau} = \frac{g(\chi_\sigma)-g(\chi_\tau)}{\chi_\sigma-\chi_\tau}$$

with the limiting process $\lim_{\sigma\to\tau} g_{\sigma\tau} = g'(\chi_\sigma)$ understood when $\sigma = \tau$ in the summation. On the other hand, by using $(2.3)_2$ and (2.2) and $(2.4)_1$ we infer

$$\dot{\mathbf{B}} = \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T = (\mathbf{DB}+\mathbf{BD}) + (\mathbf{WB}-\mathbf{BW}).$$

Then, utilizing the above results and the equalities

$$\mathbf{B}\mathbf{B}_\theta = \mathbf{B}_\theta\mathbf{B} = \chi_\theta\mathbf{B}_\theta; \quad \mathbf{e}\mathbf{B}_\theta = \mathbf{B}_\theta\mathbf{e} = \mathbf{g}(\chi_\theta)\mathbf{B}_\theta,$$

for the material spin $\mathbf{\Omega}^*$ given by (3.9) we derive

$$\begin{aligned}\overset{\circ}{\mathbf{e}}{}^* = &\left(\sum_{\sigma,\tau=1}^{r} g_{\sigma\tau}\mathbf{B}_\sigma\dot{\mathbf{B}}\mathbf{B}_\tau\right) + \mathbf{e}\left(\sum_{\sigma,\tau=1}^{r} \mathbf{h}\left(\frac{\chi_\sigma}{I},\frac{\chi_\tau}{I}\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau\right) \\ &- \left(\sum_{\sigma,\tau=1}^{r} h\left(\frac{\chi_\sigma}{I},\frac{\chi_\tau}{I}\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau\right)\mathbf{e} \\ = &\sum_{\sigma,\tau=1}^{r}\left((\chi_\sigma+\chi_\tau)g_{\sigma\tau} + (g(\chi_\sigma)-g(\chi_\tau))h\left(\frac{\chi_\sigma}{I},\frac{\chi_\tau}{I}\right)\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau\,.\end{aligned}$$

Hence, the objective corotational strain rate $\overset{\circ}{\mathbf{e}}{}^*$ defined by the material spin $\mathbf{\Omega}^*$ is given by

$$\overset{\circ}{\mathbf{e}}{}^* = \tilde{\mathcal{L}}(\mathbf{B})[\mathbf{D}] = \sum_{\sigma,\tau=1}^{\mathbf{r}} \rho(\chi_\sigma,\chi_\tau)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau\,, \tag{5.1}$$

where

$$\rho(x,y) = \left((x+y) + (x-y)h\left(\frac{x}{I},\frac{y}{I}\right)\right)\frac{g(x)-g(y)}{x-y} \tag{5.2}$$

for any $x, y > 0$, where the invariant I is given by (3.10). In (5.1), $\chi_1,\cdots,\chi_r$ are the distinct eigenvalues of $\mathbf{B}$ and $\mathbf{B}_1,\cdots,\mathbf{B}_r$ are the corresponding subordinate eigenprojections of $\mathbf{B}$, and the fourth order tensor $\tilde{\mathcal{L}}(\mathbf{B})$ is defined by $(5.1)_2$. In order to obtain the conjugate stress $\mathbf{t}$ from (4.4), it is required to judge whether or not the strain rate $\overset{\circ}{\mathbf{e}}{}^*$ is a complete one, i.e. whether or not the fourth order tensor $\tilde{\mathcal{L}}(\mathbf{B})$ as a linear transformation between second order tensors is nonsingular for all $\mathbf{B}$, and, moreover, to work out the inverse of $\tilde{\mathcal{L}}(\mathbf{B})$. At first sight, it seems not easy to solve the just-mentioned two problems, since we have to deal with a fourth order tensor depending on a second order tensor. Fortunately, utilizing (2.35) – (2.36), from (5.1) we can derive a *spectral representation* of the fourth order tensor $\tilde{\mathcal{L}}(\mathbf{B})$ and hence the aforementioned tough problems become tractable.

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two given second order tensors. We introduce the *Kronecker product* $\mathbf{H}_1 * \mathbf{H}_2$ of the tensors $\mathbf{H}_1$ and $\mathbf{H}_2$ by

$$(\mathbf{H}_1 * \mathbf{H}_2)[\mathbf{X}] = \mathbf{H}_1\mathbf{X}\mathbf{H}_2 \tag{5.3}$$

for any second order tensor $\mathbf{X}$. It is evident that the Kronecker product $\mathbf{H}_1 * \mathbf{H}_2$ defined above is a linear transformation between second order tensors, i.e. a fourth order tensor. With the help of the Kronecker product introduced, from (5.1) we derive

$$\tilde{\mathcal{L}}(\mathrm{B}) = \sum_{\sigma,\tau=1}^{r} \rho(\chi_\sigma, \chi_\tau)\mathrm{B}_\sigma * \mathrm{B}_\tau \,. \tag{5.4}$$

The crucial point is that *the above expression is exactly a spectral representation of the fourth order tensor $\tilde{\mathcal{L}}(\mathrm{B})$ as a linear transformation between second order tensors, in which each $\rho(\chi_\sigma, \chi_\tau)$ is an eigenvalue.* In fact, let $\mathcal{L}_1 \circ \mathcal{L}_2$ designate the composition of the two fourth order tensors $\mathcal{L}_1$ and $\mathcal{L}_2$ as two tranformations. Then, by utilizing (2.35) and (2.36) and the definition (5.3), we deduce

$$\begin{aligned}((\mathrm{B}_\alpha * \mathrm{B}_\beta) \circ (\mathrm{B}_\sigma * \mathrm{B}_\tau))[\mathbf{X}] &= (\mathrm{B}_\alpha * \mathrm{B}_\beta)[(\mathrm{B}_\sigma * \mathrm{B}_\tau)[\mathbf{X}]] \\ &= (\mathrm{B}_\alpha * \mathrm{B}_\beta)[\mathrm{B}_\sigma \mathbf{X} \mathrm{B}_\tau] \\ &= \mathrm{B}_\alpha \mathrm{B}_\sigma \mathbf{X} \mathrm{B}_\tau \mathrm{B}_\beta \\ &= \begin{cases} (\mathrm{B}_\sigma * \mathrm{B}_\tau)[\mathbf{X}] & \text{if} \quad \alpha = \sigma, \beta = \tau\,, \\ \mathbf{O} \quad \text{otherwise}\,, \end{cases}\end{aligned}$$

for any $1 \leqslant \alpha, \beta, \sigma, \tau \leqslant r$ and for any second order tensor $\mathbf{X}$, and, moreover, we have

$$\sum_{\sigma,\tau=1}^{r} \mathrm{B}_\sigma * \mathrm{B}_\tau = \mathbf{I} * \mathbf{I}.$$

The former yields

$$(\mathrm{B}_\alpha * \mathrm{B}_\beta) \circ (\mathrm{B}_\sigma * \mathrm{B}_\tau) = \begin{cases} \mathrm{B}_\sigma * \mathrm{B}_\tau & \text{if} \quad \alpha = \sigma, \beta = \tau\,, \\ \mathbf{O} \otimes \mathbf{O} & \text{otherwise}\,, \end{cases}$$

for any $1 \leqslant \alpha, \beta, \sigma, \tau \leqslant r$, where $\mathbf{O} \otimes \mathbf{O}$ is the fourth order null tensor. Moreover, the tensor $\mathbf{I} * \mathbf{I}$ gives the identity transformation between second order tensors, since

$$(\mathbf{I} * \mathbf{I})[\mathbf{X}] = \mathbf{X}$$

for any second order tensor $\mathbf{X}$. Thus, from the above facts and a well-known fact from the linear transformations we conclude that (5.4) is a spectral representation of $\tilde{\mathcal{L}}(\mathrm{B})$ and hence each $\rho(\chi_\sigma, \chi_\tau)$ is an eigenvalue.

From the fact just proved, we can derive the desired results immediately. First, we assert that $\tilde{\mathcal{L}}(\mathrm{B})$ is nonsingular if and only if each eigenvalue of it is nonzero, i.e. $\rho(\chi_\sigma, \chi_\tau) \neq 0$. The latter produces

$$(x + y) + (x - y)h(x, y) \neq 0, \qquad \forall x, y > 0, \qquad x \neq y\,. \tag{5.5}$$

In deriving the above, the condition $(g(x) - g(y))/(x - y) \neq 0$ is used. For the latter, we would mention that the scale function $g(z)$ is a monotonic increasing

function. Then, combining the condition (5.5) and the related result derived in Sec. 4, we conclude that
the $\mathbf{\Omega}^*$-work-conjugacy relation (4.1), i.e. (4.2) can be defined if and only if the spin $\mathbf{\Omega}^*$ is a material spin (cf. (3.9)) fulfilling the condition (5.5).
Next, for any given material spin $\mathbf{\Omega}^*$ (cf. (3.9)) fulfilling the condition (5.5), from (4.4) we derive the $\mathbf{\Omega}^*$-work-conjugate stress measure $\mathbf{t}$ of an arbitrary Hill's Eulerian strain measure $\mathbf{e}$ as follows

$$\mathbf{t} = (\tilde{\mathcal{L}}(\mathbf{B}))^{-1}[\sigma] = \sum_{\sigma,\tau=1}^{\mathbf{r}} \rho(\chi_\sigma, \chi_\tau)^{-1}\mathbf{B}_\sigma \boldsymbol{\sigma} \mathbf{B}_\tau\,, \tag{5.6}$$

where the symmetric function $\rho(x,y)$ is given by (5.4).

The formula (5.6), which provides the $\mathbf{\Omega}^*$-work-conjugate stress of the Eulerian strain measure $\mathbf{e}$ in terms of the related basic quantities, i.e. the left Cauchy–Green tensor B and the Kirchhoff stress measure $\boldsymbol{\sigma}$, is valid for any given Eulerian material spin $\mathbf{\Omega}^*$ (cf. (3.9)) fulfilling the condition (5.5) and for any given Eulerian strain measure $\mathbf{e}$ (cf. (2.19) or (2.29)). Moreover, by means of the rotated correspondence relationship as indicated by (4.8), we obtain the $\hat{\mathbf{\Omega}}^*$-work-conjugate stress measure of any given Lagrangean strain measure $\mathbf{E}$ (cf. (2.20) or (2.30)) as follows:

$$\mathbf{T} = \sum_{\sigma,\tau=1}^{\mathbf{r}} \rho(\chi_\sigma, \chi_\tau)^{-1}\mathbf{C}_\sigma \hat{\boldsymbol{\sigma}} \mathbf{C}_\tau\,. \tag{5.7}$$

We would point out that in the formulas (5.6) – (5.7), the limiting process $\lim\limits_{\chi_\sigma\to\chi_\tau} \dfrac{g(\chi_\sigma)-g(\chi_\tau)}{\chi_\sigma-\chi_\tau} = g'(\chi_\sigma)$ is meant when $\sigma=\tau$.

Substituting the four spin functions $h(x,y) = \tilde{h}^M\left(\dfrac{x}{y}\right)$ for $M \in \{J, R, L, \log\}$[(3)] given by (3.12) – (3.13) and (3.15) – (3.16) into (5.6) and (5.7) respectively, one can obtain the work-conjugate stress measures of any given Eulerian and Lagrangean strain measures $\mathbf{e}$ and $\mathbf{E}$ with regard to the material spins $\mathbf{\Omega}^M$ and $\hat{\mathbf{\Omega}}^M$ for $M \in \{J, R, L, \log\}$, respectively. In particular, substituting the spin function $h(x,y) = \tilde{h}^R\left(\dfrac{x}{y}\right)$ (cf. (3.13)) defining the spin $\mathbf{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^T$ into the formula (5.7), we derive the work-conjugate stress measure of any given Lagrangean strain measure E (cf. (2.20) or (2.30)) in Hill's work-conjugacy sense (cf. (1.1)), i.e. the $\hat{\mathbf{\Omega}}^R$-work-conjugate stress measure of the Lagrangean strain measure $\mathbf{E}$, as follows:

$$\mathbf{T}^R = \sum_{\sigma,\tau=1}^{r} (2\sqrt{\chi_\sigma\chi_\tau})^{-1}\frac{\chi_\sigma-\chi_\tau}{g(\chi_\sigma)-g(\chi_\tau)}\mathbf{C}_\sigma \hat{\boldsymbol{\sigma}} \mathbf{C}_\tau\,. \tag{5.8}$$

(3) The spin function given by (3.14) is excluded, since it fails to meet the condition (5.5), i.e., the strain rate $\overset{\circ}{\mathbf{e}}{}^{E}$ for any $\mathbf{e}$ is not a complete strain rate measure.

The above result is presented in terms of the right Cauchy–Green tensor $\mathbf{C}$ and the Lagrangean Kirchhoff stress measure or the *rotated Kirchhoff stress*, $\hat{\boldsymbol{\sigma}} = \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$. Other forms of expressions for $\mathbf{T}^R$ can be found in HILL [21 – 23], WANG and TRUESDELL [53], OGDEN [38], WANG and DUAN [54], and XIAO [56], *et al.*; see also HOGER [26], LEHMANN and LIANG [30], GUO and MAN [15], *et al.*, for some particular cases.

The formulas (5.6) – (5.7) are expressed in terms of the eigenprojections of the Cauchy–Green tensors B and C, respectively. Applying Sylvester's formulas (2.31) and (2.32), from (5.6) and (5.7) one can derive explicit basis-free expressions for the conjugate stresses t and T. In fact, we have

$$\mathbf{t} = \sum_{i,j=0}^{r-1} \varrho_{ij}\mathbf{B}^i\boldsymbol{\sigma}\mathbf{B}^j\,, \tag{5.9}$$

$$\mathbf{T} = \sum_{i,j=0}^{r-1} \varrho_{ij}\mathbf{C}^i\hat{\boldsymbol{\sigma}}\mathbf{C}^j\,, \tag{5.10}$$

where each coefficient $\varrho_{ij} = \varrho_{ji}$ is a symmetric function of the distinct eigenvalues $\chi_1, \cdots, \chi_r$ of $\mathbf{B}$ or $\mathbf{C}$, i.e. an invariant of the latter. The expressions for the coefficients ϱ_{ij} are given as follows.

(i) $r = 1:\ \chi_1 = \chi_2 = \chi_3 = \chi$.

$$\varrho_{00} = (2\chi g'(\chi))^{-1}\,. \tag{5.11}$$

(ii) $r = 2:\ \chi_1 \neq \chi_2 = \chi_3$.

The eigenprojections are of the forms

$$\begin{aligned} \mathbf{B}_1 &= \mathbf{R}\mathbf{C}_1\mathbf{R}^T = (\chi_1 - \chi_2)^{-1}(\mathbf{B} - \chi_2\mathbf{I})\,, \\ \mathbf{B}_2 &= \mathbf{R}\mathbf{C}_1\mathbf{R}^T = (\chi_2 - \chi_1)^{-1}(\mathbf{B} - \chi_1\mathbf{I})\,. \end{aligned} \tag{5.12}$$

The coefficients ϱ_{ij}, $i, j = 0, 1$, are given by

$$\varrho_{00} = \frac{1}{2}(\chi_2^2(\chi_1 g_1')^{-1} + \chi_1^2(\chi_2 g_2')^{-1} - 4\chi_1\chi_2\rho_{12}^{-1})(\chi_1 - \chi_2)^{-2}\,, \tag{5.13}$$

$$\varrho_{01} = \varrho_{10} = -\frac{1}{2}(\chi_2(\chi_1 g_1')^{-1} + \chi_1(\chi_2 g_2')^{-1} g' \\ -2(\chi_1 + \chi_2)\rho_{12}^{-1})(\chi_1 - \chi_2)^{-2}\,, \tag{5.14}$$

$$\varrho_{11} = \frac{1}{2}((\chi_1 g_1')^{-1} + (\chi_2 g_2')^{-1} - 4\rho_{12}^{-1})(\chi_1 - \chi_2)^{-2}\,. \tag{5.15}$$

(iii) $r = 3$: $\chi_1 \neq \chi_2 \neq \chi_3 \neq \chi_1$.

The eigenprojections are of the forms

$$(5.16)\qquad \begin{aligned} \mathbf{B}_1 &= \mathbf{R}\mathbf{C}_1\mathbf{R}^T = \frac{\chi_2-\chi_3}{\Delta}(\mathbf{B}-\chi_2\mathbf{I})(\mathbf{B}-\chi_3\mathbf{I}),\\ \mathbf{B}_2 &= \mathbf{R}\mathbf{C}_2\mathbf{R}^T = \frac{\chi_3-\chi_1}{\Delta}(\mathbf{B}-\chi_3\mathbf{I})(\mathbf{B}-\chi_1\mathbf{I}),\\ \mathbf{B}_3 &= \mathbf{R}\mathbf{C}_3\mathbf{R}^T = \frac{\chi_1-\chi_2}{\Delta}(\mathbf{B}-\chi_1\mathbf{I})(\mathbf{B}-\chi_2\mathbf{I}),\\ \Delta &= (\chi_3-\chi_2)(\chi_2-\chi_1)(\chi_1-\chi_3). \end{aligned}$$

The coefficients ϱ_{ij}, $i,j=0,1,2$, are given by

$$(5.17)\qquad \varrho_{00} = \frac{1}{2}\Delta^{-2}\sum_{(ijk)}\Big(({\chi_i}^2\chi_j-\chi_i{\chi_j}^2)^2(\chi_k g'_k)^{-1} \\ -4III(\chi_i-\chi_k)(\chi_j-\chi_k)\chi_k\rho_{ij}^{-1}\Big),$$

$$(5.18)\qquad \varrho_{11} = \frac{1}{2}\Delta^{-2}\sum_{(ijk)}\Big(({\chi_i}^2-{\chi_j}^2)^2(\chi_k g'_k)^{-1}-4({\chi_i}^2-{\chi_k}^2)({\chi_j}^2-{\chi_k}^2)\rho_{ij}^{-1}\Big),$$

$$(5.19)\qquad \varrho_{22} = \frac{1}{2}\Delta^{-2}\sum_{(ijk)}\Big((\chi_i-\chi_j)^2(\chi_k g'_k)^{-1}-4(\chi_i-\chi_k)(\chi_j-\chi_k)\rho_{ij}^{-1}\Big),$$

$$(5.20)\qquad \varrho_{01} = -\frac{1}{2}\Delta^{-2}\sum_{(ijk)}\Big((\chi_i-\chi_j)^2({\chi_i}^2\chi_j+\chi_i{\chi_j}^2)(\chi_k g'_k)^{-1} \\ -2(III+II\chi_k)(\chi_i-\chi_k)(\chi_j-\chi_k)\rho_{ij}^{-1}\Big),$$

$$(5.21)\qquad \varrho_{02} = \frac{1}{2}\Delta^{-2}\sum_{(ijk)}\Big(\chi_i\chi_j(\chi_i-\chi_j)^2(\chi_k g'_k)^{-1} \\ -2(\chi_i+\chi_j)(\chi_i-\chi_k)(\chi_j-\chi_k)\chi_k\rho_{ij}^{-1}\Big),$$

$$(5.22)\qquad \varrho_{12} = -\frac{1}{2}\Delta^{-2}\sum_{(ijk)}\Big((\chi_i+\chi_j)(\chi_i-\chi_j)^2(\chi_k g'_k)^{-1} \\ -2(I+\chi_k)(\chi_i-\chi_k)(\chi_j-\chi_k)\rho_{ij}^{-1}\Big),$$

where the notation $\sum_{(ijk)}$ means the summation for $(ijk)=(123),(231),(312)$. In the above, we denote

$$(5.23)\qquad g'_k = g'(\chi_k),\qquad \rho_{ij}=\rho(\chi_i,\chi_j).$$

Moreover, I, II and III are the three principal invariants of $\mathbf{B}$ or $\mathbf{C}$, i.e. for $\mathbf{G} = \mathbf{B},\ \mathbf{C}$,

$$
\begin{aligned}
I &= \chi_1 + \chi_2 + \chi_3 = \mathrm{tr}\mathbf{G}\,,\\
II &= \chi_1\chi_2 + \chi_2\chi_3 + \chi_3\chi_1 = \frac{1}{2}(\mathrm{tr}\mathbf{G})^2 - \frac{1}{2}\mathrm{tr}\mathbf{G}^2\,,\\
III &= \det\mathbf{G} = \chi_1\chi_2\chi_3 = \frac{1}{6}(\mathrm{tr}\mathbf{G})^3 - \frac{1}{2}(\mathrm{tr}\mathbf{G})(\mathrm{tr}\mathbf{G})^2 + \frac{1}{3}\mathrm{tr}\mathbf{G}^3\,.
\end{aligned}
\tag{5.24}
$$

The three eigenvalues (possibly repeated) of $\mathbf{B}$ or $\mathbf{C}$ are determined by the formula (see SAWYERS [43])

$$
\begin{aligned}
\chi_i &= \frac{1}{3}(I + 2\sqrt{I^2 - 3II}\cos\frac{1}{3}(\theta - 2\pi i)), \qquad i = 1,2,3\,,\\
\cos\theta &= \frac{2I^3 - 9I\cdot II + 27III}{2(I^2 - 3II)^{3/2}}, \qquad 0 \leqslant \theta < \pi\,.
\end{aligned}
\tag{5.25}
$$

In the above results, the formulas (5.6) and (5.7) in a principal component form are simple, but to apply them actually, one needs to calculate the eigenvalues and eigenvectors of $\mathbf{B}$ at each material particle. However, in a deforming material body, the Eulerian and Lagrangean triads may vary with the time and the spacial position of material particle during the course of deformation in a rather complicated manner. Thus, the just-mentioned eigenvalue/eigenvector calculation may become cumbersome, or even intractable. The explicit basis-free formulas, which have rather complicated forms, are just what are needed to avoid this unsatisfactory situation. They are valid for all coordinate system and hence independent of any particular coordinate system. Once the deformation gradient $\mathbf{F}$ is known under any coordinate system, by means of the explicit basis-free formulas given, one can directly calculate the desired results for conjugate stresses without recourse to the complicated eigenvalue/eigenvector calculation.

6. On Hencky's logarithmic strain measures

The logarithmic strain measures $\ln\mathbf{V}$ and $\ln\mathbf{U}$ (cf. (2.26) and (2.27)), introduced by HENCKY [20], have long been popular and enjoyed favoured treatment in solid mechanics, metallurgy and materials science, etc. In constitutive modeling, these measures and their rates are often chosen as basic strain measures and basic strain rate measures and have been shown to possess certain intrinsic advantages (e.g., see HILL [21 – 23], RICE [42], HUTCHINSON and NEALE [27], and OGDEN [38], *et al.*). Earlier, the usefulness of them was confined to some particular cases due to their complicated transcendental form, as pointed out by TRUESDELL and TOUPIN [52]. This situation was later improved by FITZGERALD [8], GURTIN and SPEAR [17], HOGER [25 – 26], and SCHEIDLER [44 – 46], *et al.*

The main objective of this section is to disclose intrinsic, unique relationship between the logarithmic strain measures and the fundamental mechanical quantities, i.e., the stretching tensors $\mathbf{D}$ and $\hat{\mathbf{D}}$ and the Kirchhoff stress measures $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$. (2.11) (2.12)

It has been known for a long time that neither of the Eulerian and Lagrangean stretching tensors $\mathbf{D}$ and $\hat{\mathbf{D}}$ can be written in a direct flux of a strain measure (cf. HILL [23]), although they are frequently referred to as the rate of deformation tensor, the Eulerian strain rate, etc. As a result, neither of the Eulerian and Lagrangean Kirchhoff stress tensors $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ is conjugate to any strain measure. Recently, these authors have proved (see XIAO, BRUHNS and MEYERS [57 – 60]) that an objective corotational rate of the Hencky's Eulerian logarithmic strain measure $\ln \mathbf{V}$ can be identical with the Eulerian stretching tensor $\mathbf{D}$, i.e., in a corotating material frame the latter is a *true* time rate of the former, and furthmore that in all strain measures only $\ln \mathbf{V}$ enjoys this property. Specifically, we have

$$(6.1) \qquad \overset{\circ}{\mathbf{e}}{}^{*} = \dot{\mathbf{e}} + \mathbf{e}\boldsymbol{\Omega}^{*} - \boldsymbol{\Omega}^{*}\mathbf{e} = \mathbf{D} \iff \mathbf{e} = \ln \mathbf{V} \;\&\; \boldsymbol{\Omega}^{*} = \boldsymbol{\Omega}^{\log},$$

where $\boldsymbol{\Omega}^{\log}$ is the logarithmic spin (cf. (3.11) and (3.16)). As a result, from (2.13) and (4.2) and

$$(6.2) \qquad \mathbf{D} = (\ln \overset{\circ}{\mathbf{V}})^{\log}$$

it follows that the pair $(\ln \mathbf{V}, \boldsymbol{\sigma})$ is an $\boldsymbol{\Omega}^{\log}$-work-conjugate Eulerian strain-stress pair. A particular case of the above fact was known first by LEHMANN, GUO and LIANG [29] and later by REINHARDT and DUBEY [40 – 41] and DUBEY and REINHARDT [7], where only $\mathbf{e} = \ln \mathbf{V}$, i.e.

$$(6.3) \qquad \mathbf{e} = \ln \mathbf{V} \;\&\; \boldsymbol{\Omega}^{*} = \boldsymbol{\Omega}^{\log} \Longrightarrow \overset{\circ}{\mathbf{e}}{}^{*} = \dot{\mathbf{e}} + \mathbf{e}\boldsymbol{\Omega}^{*} - \boldsymbol{\Omega}^{*}\mathbf{e} = \mathbf{D}$$

was considered, and hence the aforementioned unique property of $\ln \mathbf{V}$ was not realized. The aforementioned intrinsic, unique property of the Hencky's Eulerian strain measure $\ln \mathbf{V}$ has proved to be far-reaching and found applications in constitutive modeling (see REINHARDT and DUBEY [41], DUBEY and REINHARDT [7], BRUHNS, XIAO and MEYERS [62], XIAO, BRUHNS and MEYERS [58 – 59, 63]).

On the other hand, the corresponding question concerning the Lagrangean stretching tensor $\hat{\mathbf{D}}$ and the Lagrangean Kirchhoff stress measure $\hat{\boldsymbol{\sigma}}$ have been discussed recently by these authors (see XIAO, BRUHNS and MEYERS [61]). Here we supply a short alternative proof for the main results in the latter.

In fact, from the notion of objective corotational rate and the formula (3.5), the following fact follows immediately:

$$\overset{\circ}{\mathbf{E}}{}^{*} = \dot{\mathbf{E}} + \mathbf{E}(\hat{\mathbf{\Omega}}^{*} - \hat{\mathbf{\Omega}}^{\mathrm{R}}) - (\hat{\mathbf{\Omega}}^{*} - \hat{\mathbf{\Omega}}^{\mathrm{R}})\mathbf{E} = \hat{\mathbf{D}} \Longleftrightarrow \mathbf{E} = \ln \mathbf{U} \ \& \ \hat{\mathbf{\Omega}}^{*} = \hat{\mathbf{\Omega}}^{\log} . \tag{6.4}$$

Namely, an objective corotational rate of the Hencky's Lagrangean logarithmic strain measure $\ln \mathbf{U}$ is identical with the Lagrangean stretching tensor $\hat{\mathbf{D}}$, and furthermore that in all strain measures only $\ln \mathbf{U}$ enjoys this property. Thus, from (2.14) and (4.6) and

$$\hat{\mathbf{D}} = (\ln \overset{\circ}{\mathbf{U}})^{\log} \tag{6.5}$$

it follows that the pair $(\ln \mathbf{U}, \hat{\boldsymbol{\sigma}})$ is an $\hat{\mathbf{\Omega}}^{\log}$-work-conjugate Lagrangean strain-stress pair.

7. On Eulerian and Lagrangean formulations of rate-type constitutive relations

Let $\mathbf{E}$ be any given Lagrangean strain measure, defined by a scale function $g(\chi)$ (see (2.30)). The conjugate stress $\mathbf{T}^R$ of $\mathbf{E}$ in Hill's work-conjugacy sense (see (1.1) and (2.14)) is given by (5.8). Suppose the response of a material to incremental loading is rate-independent, i.e., either linear or piecewise linear. Following HILL [23], relative to a reference configuration we write down the Lagrangean rate-type constitutive relation

$$\dot{\mathbf{T}}^R = \hat{\mathcal{M}}^R[\dot{\mathbf{E}}] . \tag{7.1}$$

The fourth-order tensor of moduli, $\hat{\mathcal{M}}^R$, may depend on the stress and the deformation state, but not on $\dot{\mathbf{E}}$. In particular, $\hat{\mathcal{M}}^R$ depends on the choice of reference configuration and of measure, i.e., scale function $g(\chi)$. In HILL [23], certain significant properties of rate-type constitutive relation (7.1) are exploited by means of the class of generalized strain measures characterized by the scale function $g(\chi)$ as well as the work-conjugacy notion (1.1), such as the dependence of the moduli $\hat{\mathcal{M}}^R$ on the change of scale function $g(\chi)$, constitutive inequalities in terms of the scale function $g(\chi)$, the measure invariance, etc.

The unified work-conjugacy notion introduced enables us to broaden the scope of the foregoing study. Indeed, let $\mathbf{\Omega}^* \neq \mathbf{\Omega}^L$ be a material spin of the form (3.9), and $\mathbf{e}$ – any given Eulerian strain measure with the scale function $g(\chi)$. Then, the $\mathbf{\Omega}^*$-work-conjugate stress $\mathbf{t}$ of $\mathbf{e}$ is given by $(5.6)_2$ and (5.2). We propose the following Eulerian rate-type constitutive relation for a rate-independent elastoplastic material:

$$\overset{\circ}{\mathbf{t}}{}^{*} = \mathcal{M}^*[\overset{\circ}{\mathbf{e}}{}^{*}] , \tag{7.2}$$

where $\overset{\circ}{\mathbf{t}}{}^{*}$ and $\overset{\circ}{\mathbf{e}}{}^{*}$ are the objective corotational stress and strain rates given by (3.1) with $\mathbf{G} = \mathbf{e},\ \mathbf{t}$ and (3.11). The corresponding Lagrangean formulation of the above relation can be obtained by using the rotated correspondence relation (2.5)–(2.6). We have

$$\overset{\circ}{\mathbf{T}}{}^{*} = \hat{\mathcal{M}}^{*}[\overset{\circ}{\mathbf{E}}{}^{*}], \tag{7.3}$$

where the Lagrangean strain-stress pair $(\mathbf{E}, \mathbf{T})$ is the counterpart of the Eulerian strain-stress pair $(\mathbf{e}, \mathbf{t})$ via the rotated correspondence relation (see (2.5)–(2.6)); the objective corotational rates $\overset{\circ}{\mathbf{T}}{}^{*}$ and $\overset{\circ}{\mathbf{E}}{}^{*}$ are the Lagrangean counterparts of $\overset{\circ}{\mathbf{t}}{}^{*}$ and $\overset{\circ}{\mathbf{e}}{}^{*}$ (see (3.5)); and finally

$$\hat{\mathcal{M}}^{*} = \mathbf{R} \star \mathcal{M}^{*}; \quad (\mathbf{R} \star \mathcal{M}^{*})_{ijkl} = \mathbf{R}_{ip}\mathbf{R}_{jq}\mathbf{R}_{kr}\mathbf{R}_{ls}\mathcal{M}^{*}_{pqrs}\,. \tag{7.4}$$

The Eulerian and Lagrangean fourth order tensors of moduli, $\mathcal{M}^{*}$ and $\hat{\mathcal{M}}^{*}$, may depend on the stress and the deformation state, as well as certain internal variables (4) characterizing the internal state of material, etc. In particular, either of them depends on both the choice of strain measure and the choice of spin tensor, i.e. both the choice of scale function $g(\chi)$ and the choice of spin function $h(x, y)$. It should be pointed out that the just-mentioned double choices are arbitrary and independent of each other. Thus, the proposed Eulerian and Lagrangean formulations of rate-type constitutive models broaden the usual Lagrangean formulation with the material time rate and Eulerian formulations with several known objective corotational rates. The former allow for the double choices of scale function $g(\chi)$ and spin function $h(x, y)$, while the latter are concerned merely with the choice of scale function $g(\chi)$. In fact, the usual Lagrangean formulation (7.1) is incorporated into a particular case of the proposed, more general Lagrangean formulation (7.3) when $h(x, y) = \tilde{h}^{R}(y/x)$ (see (3.14)), i.e. $\mathbf{\Omega}^{*} = \mathbf{\Omega}^{R}$.

Since the stretching $\mathbf{D}$ is a simple, natural measure for the rate-of-change of deformation state, it is hopeful that the strain rate measure $\overset{\circ}{\mathbf{e}}{}^{*}$ is replaced by $\mathbf{D}$, as is most often done. This results in the noticeable fact (see XIAO, BRUHNS and MEYERS [58] or last section): the strain measure $\mathbf{e}$ must be the logarithmic strain measure $\ln \mathbf{V}$, the stress $\mathbf{t}$ must be the Kirchhoff stress $\boldsymbol{\sigma}$ and the spin $\mathbf{\Omega}^{*}$ must be the logarithmic spin $\mathbf{\Omega}^{\log}$. Such uniqueness yields the following formulations based on the logarithmic rate:

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{\log} = \mathcal{M}^{\log}[\mathbf{D}], \tag{7.5}$$

$$\overset{\circ}{\hat{\boldsymbol{\sigma}}}{}^{\log} = \hat{\mathcal{M}}^{\log}[\hat{\mathbf{D}}]. \tag{7.6}$$

For hypoelasticity and finit deformation elastoplasticity etc., further study shows (see XIAO, BRUHNS and MEYERS [58 – 59, 63] and BRUHNS, XIAO and MEYERS [62]) that the above formulations possess certain unique, far-reaching properties.

(4) For the sake of simplicity, the evolution equations of internal variables are not discussed here.

In general, by virtue of objective corotational stress rates and the unified work-conjugacy notion introduced, the structure and property of rate-type constitutive relations may be further exploited, such as the dependence of the moduli on both the scale function $g(\chi)$ and the spin function $h(x,y)$, constitutive inequalities in terms of both the scale function $g(\chi)$ and the spin function $h(x,y)$, the broader invariance relative to both strain measure and strain rate measure, etc. This line of investigation will be pursued elsewhere.

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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

On distortion of waves in a nonlinear magnetoelastic conductor

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WAVE DISTORTION and formation of shocks due to the elastic nonlinearity of the medium in the presence of a magnetic field are studied using the multiple scales technique for a one-dimensional travelling longitudinal wave. Condition for formation of shocks has been obtained for a sinusoidal signal.

1. Introduction

TRAVELLING WAVES in a nonlinear elastic medium are studied for the purpose of understanding the phenomena of distortion and formation of shocks. Problems of propagation of one-dimensional longitudinal and transverse waves in nonlinear elasticity have been studied by NAYFEH [1], LARDNER [2, 3]. Growth of amplitude and shock formation were investigated by them using the perturbation and multiple scales technique. The effect of a magnetic field on elastic waves was discussed by MAUGIN in [4], where problems of propagation of harmonic waves in hyperelastic non-linear magnetic dielectrics and shocks and simple waves in a perfectly conducting nonlinear elastic conductor have been considered. HEFNI *et al.* [5] have studied general one-dimensional bulk waves in a non-linear magnetoelastic conductor. They discussed both linear and nonlinear waves, starting from the general formulation of constitutive equations. However, the interesting phenomena of distortion as well as shock formation have not been treated there.

2. Basic equations

We consider a non-linear one-dimensional wave propagating in a perfectly conducting elastic medium in the presence of a uniform magnetic field $\mathbf{H}^0$ transverse to the direction of wave propagation. Maxwell's equations of the electromagnetic field are:

$$
\begin{aligned}
\operatorname{div}\mathbf{B} &= 0,\\
\operatorname{curl}\mathbf{H} &= \mathbf{J},\\
\operatorname{div}\mathbf{D} &= 0,\\
\operatorname{curl}\mathbf{E} &= -\mathbf{B}_t
\end{aligned}
\tag{2.1}
$$

where the displacement current has been neglected.

The constitutive equations are

$$\mathbf{B} = \mu\mathbf{H},$$
(2.2)
$$\mathbf{D} = \varepsilon_1\mathbf{E}.$$

Ohm's law gives

$$(2.2)_3 \qquad \mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}_t \times \mathbf{B}),$$

σ being the conductivity and ε_1 the electric permittivity.

Following BLAND [6], the equations of motion in a conducting medium with a magnetic field $\mathbf{B}$ are:

$$(2.3) \qquad \mathbf{L}_{ij\cdot j} + (\mathbf{J} \times \mathbf{B})_i = \rho u_{i_{tt}},$$

where

$$(2.4) \qquad \mathbf{L}_{ij} = \frac{\partial W}{\partial u_{i,j}}$$

is the Piola-Kirchhoff stress tensor, W being the strain-energy of the material per unit volume.

For a hyperelastic material, W may be taken (correct up to the third power of strain):

$$(2.5) \qquad W = \frac{1}{2}\lambda I_1^2 + GI_2 + \alpha\ I_1^3 + \beta\ I_1 I_2 + \gamma\ I_3.$$

I_1, I_2, I_3 are the strain invariants given by

$$(2.6) \qquad I_1 = e_{ii}, \quad I_2 = e_{ij}e_{ij}\ , \quad I_3 = e_{ij}e_{jk}e_{ki}.$$

The strain components in terms of displacement u are given by

$$(2.7) \qquad e_{ij} = \left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)/2.$$

$J \times B$ is the Lorentz force per unit volume due to the magnetic field $\mathbf{B}$ and the current density J.

3. Formulation

Referred to rectangular axes of coordinates (x, y, z), we consider a wave with displacement

$$(3.1) \qquad \mathbf{u} = (u(x,t),\ 0,\ 0)$$

propagating in the x-direction in a conducting medium with an initially uniform magnetic field

$$(3.2) \qquad \mathbf{H}^0 = (0,\ 0,\ H^0).$$

The perturbations of the electromagnetic field are

$$\begin{aligned} \mathbf{H} &= \mathbf{H}^0 + \mathbf{h}, \\ \mathbf{E} &= 0 + \mathbf{e}, \\ \mathbf{J} &= 0 + \mathbf{j}. \end{aligned} \tag{3.3}$$

For a perfectly conducting medium we have from $(2.2)_3$

$$\mathbf{e} + \mathbf{u}_t \times \mathbf{B} = 0. \tag{3.4}$$

By equations (2.1), (3.3) we get

$$\text{curl } \mathbf{e} = -\mu h_t. \tag{3.5}$$

Using (3.4), (3.5) gives

$$\mathbf{h}_t = \text{curl } (\mathbf{u}_t \times \mathbf{H}). \tag{3.6}$$

Equation (3.6) together with (3.1), (3.3) yields

$$\begin{aligned} h_{1_t} &= 0, \\ h_{2_t} &= -(h_2 u_t)_x, \\ h_{3_t} &= -H^0 u_{xt} - (h_3 u_t)_x, \end{aligned} \tag{3.7}$$

where $\mathbf{h} = (h_1, h_2, h_3)$.

We now use a scaling parameter ε and consider the displacement u to be of order ε. The magnetic field $\mathbf{h}$ being dependent on u is also of order ε.

From $(3.7)_1$ it follows that h_1 is a function of x only. We take $h_1 = 0$ in this wave problem. For the determination of h_2, h_3 and u, we use the perturbation and multiple scales. We introduce only one scale, namely $\xi = \varepsilon x$. The perturbation expansions of h_2, h_3 and u are taken in the form:

$$\begin{aligned} h_2(x,t) &= \varepsilon h_{20}(x,\xi,t) + \varepsilon^2 h_{21}(x,\xi,t) + 0(\varepsilon^3), \\ h_3(x,t) &= \varepsilon h_{30}(x,\xi,t) + \varepsilon^2 h_{31}(x,\xi,t) + 0(\varepsilon^3), \\ u(x,t) &= \varepsilon U_0(x,\xi,t) + \varepsilon^2 U_1(x,\xi,t) + 0(\varepsilon^3). \end{aligned} \tag{3.8}$$

Substituting (3.8) in $(3.7)_2$, $(3.7)_3$ and equating the coefficients of ε, ε^2, we get

$$\begin{aligned} h_{20_t} &= 0, \\ h_{21_t} &= -h_{20_x} U_{0_t} - h_{20} U_{0_{tx}}, \\ h_{30_t} + H^0 U_{0_{xt}} &= 0, \\ h_{31_t} + H^0 U_{1_{xt}} + (h_{30} U_{0_t})_x + H^0 U_{0_{\xi t}} &= 0. \end{aligned} \tag{3.9}$$

Equation $(3.9)_1$ shows h_{20} to be independent of t. We therefore take $h_{20} = 0$ for wave solution. Putting $h_{20} = 0$ in $(3.9)_2$, we get $h_{21_t} = 0$ which implies $h_{21} = 0$. h_2 is therefore zero to within the accuracy of $o(\varepsilon^2)$. Also the Lorentz force components are

$$J \times B = \mu[\text{curl } (0.0.h_3) \times (0,0,H^0 + h_3)] = [-\mu(H^0 + h_3)h_{3_x}.0.0]. \tag{3.10}$$

Using (3.10) in Eq. (2.3), the only equation of motion not identically satisfied is

$$c_1^2 u_{xx} + 2c_3^2 u_x u_{xx} - \frac{\mu}{\rho}(H^0 + h_3)h_{3_x} = u_{tt}, \tag{3.11}$$

$$c_1^2 = \frac{\lambda + 2G}{\rho}, \qquad c_3^2 = \frac{(3/2)\lambda + 3\gamma + 3\beta + 3G + 3\alpha}{\rho}.$$

Substituting (3.8) in (3.11) and equating the terms of order ε, ε^2 separately to zero, we get

$$c_1^2 U_{0_{xx}} - \frac{\mu H^0}{\rho} h_{30_x} - U_{0_{tt}} = 0, \tag{3.12}$$

$$c_1^2 U_{1_{xx}} - U_{1_{tt}} + 2c_1^2 U_{0_{x\xi}} + 2c_3^2 U_{0_x} U_{0_{xx}} - \frac{\mu H^0}{\rho}\left(h_{31_x} + h_{30_\xi}\right) - \frac{\mu h_{30}}{\rho} h_{30_x} = 0. \tag{3.13}$$

Integrating $(3.9)_3$ and $(3.9)_4$ partially with respect to time and neglecting the time-independent term, we obtain

$$h_{30} = -H^0 U_{0_x}, \tag{3.14}$$

$$h_{31} = -H^0 U_{1x} - H^0 U_{0\xi} + H^0 \int (U_{0x} U_{0t})_x \, dt. \tag{3.15}$$

From Eqs. (3.12), (3.14) we get

$$c^2 U_{0_{xx}} = U_{0_{tt}}, \tag{3.16}$$

where

$$c^2 = c_1^2 + c_H^2, \tag{3.17}$$

$$c_H^2 = \mu H^{0^2}/\rho, \tag{3.18}$$

c_H being the Alfvén wave velocity, and c_1 the P-wave velocity in linear elastic solids.

4. Travelling wave solution

Solution of (3.16) suitable for a wave progressing in the positive x-direction is

(4.1) $$U_0 = F(\theta, \xi),$$

where

(4.2) $$\theta = t - \frac{x}{c}.$$

From equations (3.15) and (4.1),

(4.3) $$h_{31_x} = H^0 U_{1_{xx}} + \frac{H^0}{c} F_{\xi\theta} - \frac{2H^0}{c^3} F_\theta F_{\theta\theta}.$$

Substituting from (3.14), (3.15). (4.1), and (4.3) in equation (3.13), one obtains

(4.4) $$c^2 U_{1_{xx}} - U_{1_{tt}} = 2cF_{\theta\xi} + \frac{2c_3^2 - 3c_H^2}{c^3} F_\theta F_{\theta\theta}.$$

On using the transformation $\theta = t - \dfrac{x}{c}$, $\phi = t + \dfrac{x}{c}$ in Eq. (4.4), it takes the form

(4.5) $$4U_{1_{\theta\phi}} = -2cF_{\theta\xi} - \frac{2c_3^2 - 3c_H^2}{c^3} F_\theta F_{\theta\theta}.$$

Hence from (4.5), it follows

(4.6) $$4U_1 = -\left(2cF_\xi + \frac{2c_3^2 - 3c_H^2}{2c^3} F_\theta^2\right)\phi + \text{complementary function.}$$

For U_1 to be finite for large t, the coefficient of ϕ must be zero. Thus

(4.7) $$2cF_\xi + \frac{2c_3^2 - 3c_H^2}{2c^3} F_\theta^2 = 0.$$

On differentiating (4.7) with respect to θ and on substituting $F_\theta = cf$, the equation satisfied by f is (WHITHAM [8])

(4.8) $$cf_\xi + Mff_\theta = 0,$$

where

(4.9) $$M = (c_3/c)^2 - 1.5\,(c_H/c)^2.$$

The solution of the quasilinear equation (4.8) is

(4.10) $$f(\theta, \xi) = Z(\theta_1),$$

where $Z(\theta_1)$ is a function of θ_1 and

$$\theta_1 = \theta - M\xi Z(\theta_1)/c. \tag{4.11}$$

The main wave form $U_0(x,t,\xi) = F(\theta,\xi)$ propagates with a velocity c which is dependent on both the elastic and Alfvén wave velocities. It is also distorted for large x and a shock wave is formed (LARDNER [3]). The presence of the magnetic field changes the elastic non-linear effect. If the elastic field is linear, a non-linear effect due to the magnetic field persists.

A shock is formed for a value of θ for which $d\theta/d\theta_1 = 0$, i.e. when

$$Z'(\theta_1) = -c/(M\xi). \tag{4.12}$$

For an initially sinusoidal pulse $Z(\theta_1) = \sin(p\theta_1)$, the shock is formed if $\cos p\theta_1 = -c/(Mp\xi)$.

A shock is therefore formed in this case if $0 < (c/Mp\xi) \leqslant 1$ and the corresponding value of θ is

$$\theta = (1/p)\cos^{-1}(-c/Mp\xi) + (M\xi/c)(1 - c^2/M^2p^2\xi^2)^{1/2}. \tag{4.13}$$

To have an idea of the non-linear effect on the wave form, $f(\theta,\xi)$ is plotted against θ in Fig. 1 for different ξ, corresponding to the sinusoidal signal

$$f(\theta,0) = \sin(\pi\theta/5).$$

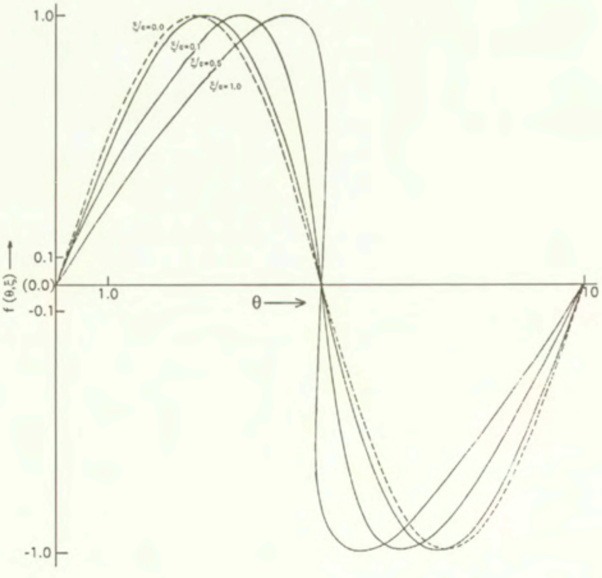

FIG. 1. $f(\theta,\xi)$ against θ for different values of ξ.

5. Conclusion

It is seen from the figure that, as the slow distance scale increases, the asymmetry grows and the possibility of shock increases.

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Corrigenda to two recent papers

In XIAO [Arch. Mech. **49**, 995–1039, 1997], the following corrigenda should be made:

In $(2.17)_2$ on page 1002, $(1+\delta)\pi/2$ is changed to $(1-\delta)\pi/2$.

The two surfaces $\mathbf{S}(\mathbf{X})$ given by (3.21) and (3.26) on pages 1007 and 1010 are changed to the forms

$$\mathbf{S}(\mathbf{X}) = (\mathbf{n}\otimes\mathbf{n}; \mathbf{n}\vee\mathbf{N}\eta_{2m-1}(\mathbf{v}^0_\alpha); \mathbf{N}\eta_{2m-1}(\mathbf{W}_\theta\mathbf{n}); \mathbf{N}\eta_{2m-1}((\mathbf{A}_\sigma\mathbf{n})^0), f_m(\mathbf{A}_\sigma)\mathbf{n},$$
$$\Phi_{2m-1}(\mathbf{q}(\mathbf{A}_\sigma)); (\mathbf{n}\cdot\mathbf{v}_\alpha)g_m(\mathbf{A}_\sigma)\mathbf{N}; (\mathbf{e}\cdot\mathbf{W}_\theta\mathbf{e}')g_m(\mathbf{A}_\sigma)\mathbf{n};$$
$$(\mathbf{q}(\mathbf{A}_\sigma)0\times\mathbf{q}(\mathbf{A}_\tau))g_m(\mathbf{A}_\sigma))$$

and

$$\mathbf{S}(\mathbf{X}) = (\mathbf{n}\otimes\mathbf{n}; \mathbf{n}\vee\mathbf{N}\eta_{2m-1}(\mathbf{W}_\theta\mathbf{n}); \mathbf{N}\eta_{2m-1}((\mathbf{A}_\sigma\mathbf{n})^0), f_m(\mathbf{A}_\sigma)\mathbf{n},$$
$$\Phi_{2m-1}(\mathbf{q}(\mathbf{A}_\sigma)); (\mathbf{n}\cdot\mathbf{v}_\alpha g_m(\mathbf{A}_\sigma)\mathbf{N}; (\mathbf{q}(\mathbf{A}_\sigma)\cdot\mathbf{q}(\mathbf{A}_\tau))g_m(\mathbf{A}_\sigma))$$

respectively. In the above and below, $\mathbf{N}=\mathbf{E}\mathbf{n}$.

In (3.30) and (3.32) on pages 1011 and 1012, $\mathbf{D}_1$ is changed to $\mathbf{D}_2$ and each $\eta_1(z)$ with $z=\mathbf{v}^0$, $\mathbf{W}_\theta\mathbf{n}$, $(\mathbf{A}_\sigma\mathbf{n})^0$, to $\mathbf{N}\eta_1(z)$.

In XIAO [Arch. Mech. **50**, 281–319, 1998], the following corrigenda should be introduced:

Line 16 on page 303: $V\ \{\overset{\circ}{\mathbf{u}}, ...\}$ is changed to $V\ \{\mathbf{u}, ...\}$.

Line 19 on page 303 and line 1 from the bottom on page 306: $\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{v}}$ is changed to $\mathbf{u}\cdot\mathbf{v}$.

Line 23 on page 303 and line 8 from the bottom on page 312: The invariant $(\mathbf{u}\cdot\mathbf{n})^2$ is added into the sets $I^1_m(\mathbf{u})$ and $I^3_m(\mathbf{u})$, respectively.

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