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STEFAN ZAHORSKI (1933–1999)

OBITUARY

On 28th of August, 1999 passed away Professor Stefan Zahorski, PhD, DSci., director of Polymer Physics Group, Institute of Fundamental Technological Research (IFTR), Polish Academy of Sciences in Warsaw, active member of the Editorial Committee of *Archives of Mechanics*.

Stefan Zahorski, was graduated MS from the Department of Mechanical Engineering, Warsaw University of Technology. He worked in the University as a senior assistant between 1954 and 1961. In 1961 he joined IFTR, where he was worked until his last days. He obtained the PhD degree in Warsaw University of Technology (1961) and DSci. degree in the Institute of Fundamental Technological Research (1966). In 1972 S. Zahorski was promoted to the position of Associate Professor and in 1981 – Full Professor.

The main fields of research interests of Stefan Zahorski concentrated on continuum mechanics and nonlinear flows of viscoelastic fluids. His original research philosophy consisted in analyzing relatively *simple, well defined motions* applied to nonlinear viscoelastic fluids, often as general as the *Noll simple fluid*. Specific assumptions about the kinematics made possible obtaining analytical solutions and discussion of nonlinear effects in viscoelastic fluids without being too specific about the constitutive equations. S. Zahorski studied, also *flows with proportional stretch history* (generalization of the Coleman-Noll concept of flows with constant stretch history), *extensional and nearly extensional flows* (e.g. non-uniform elongational flows, elongational flows with superimposed shear waves, *convergent flows*), flows in *opposing jet rheometers, die swell effects*, effects of "*necking*" in isothermal and non-isothermal high-speed fibre spinning, etc.

S. Zahorski published over 100 research papers (mostly in the *Archives of Mechanics* and *J. Non-Newtonian Fluid Mechanics*) and the monograph "*Mechanika przepływów cieczy lepkosprężystych*" (PWN, Warszawa-Poznań 1978) whose English edition "*Mechanics of viscoelastic fluids*" was published in 1981 by PWN and Nijhoff, Warszawa-The Hague.

Professor Zahorski was a member of Polish Society for Theoretical and Applied Mechanics (PTMTS), GAMM and the International Society for Interaction between Mathematics and Mechanics (ISIMM). For many years he was a member of editorial committees of *Mechanika Teoretyczna i Stosowana* (Theoretical and Applied Mechanics), *Mekhanika Polimerov* (Polymer Mechanics, Riga), and *Archives of Mechanics* (Archiwum Mechaniki Stosowanej).

For his scientific achievements, Prof. Zahorski received the *M.T. Huber Prize* awarded by Division IV, Polish Academy of Sciences (1969), *Prize of the Scientific Secretary*, Polish Academy of Sciences (1977), and 1990 the *Polonia Restituta Commandor Cross.*

Professor Stefan Zahorski was a good and friendly man, respected and liked by his friends and collaborators. He will be missed by all of us.

Andrzej Ziabicki

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Multimode wave scattering problems in layered dissipative solids

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A GENERAL SCHEME within the frequency domain is elaborated for the scattering of obliquely-incident waves at a stratified, anisotropic, viscoelastic solid. Existence and uniqueness of the asymptotic wave propagator is established. Possible nonexistence of the scattering matrix at specific values of the incident field is then shown. Conditions for nonuniqueness or incompatibility of the direct scattering problem are provided.

1. Introduction

SCATTERING BY STRATIFIED viscoelastic media is of interest, e.g., in seismics and nondestructive testing. The linear viscoelastic model allows for the introduction of dispersion and attenuation effects that are of importance in realistic calculations. In this paper, a frequency-domain approach is elaborated for the determination of the scattering matrix, with the specific aim at establishing properties of the transmitted and backscattered wave fields generated by a wave obliquely incident from infinity. The scattering of horizontally-polarized shear waves, in isotropic viscoelastic solids and within the time domain, is investigated in [1] for oblique incidence and in [2] for normal incidence.

The material functions or parameters of the continuously stratified viscoelastic medium vary in one space dimension, z say. To fix ideas, we let z be directed vertically upward. The usual assumption is made that dependence on time and transverse coordinates is through the complex factor $\exp[i(\mathbf{k}_\parallel \cdot \mathbf{x} - \omega t)]$, where $\mathbf{k}_\parallel$ is a (possibly complex-valued) constant horizontal vector, ω is the (real, constant) frequency, $\mathbf{x}$ and t denote the position vector and time, respectively. The realness of $\mathbf{k}_\parallel$ amounts to considering the Fourier components of the unknown functions. It follows from these conditions that the dynamics of any stratified anisotropic solid may be modelled by a linear system of six first-order ordinary differential equations with complex-valued, z-dependent, coefficients in the unknown components of the displacement and traction vectors. Under the rather stringent assumption that the solid is isotropic, the equations of the system decouple and the usual formulations of scattering problems for the one-dimensional Helmholtz and Schrödinger equations is recovered [3 – 7].

We assume that the body is asymptotically homogeneous. Accordingly we express the unknown functions as superpositions, with z-dependent coefficients, of the inhomogeneous waves [8] associated with the eigenvectors and the eigenvalues of the governing system at $\pm\infty$. By looking at the differential equations for the pertinent coefficients, we can find a formal expression for the asymptotic wave propagator matrix $\mathbf{T}$ that provides the solution at $+\infty$ in terms of the solution at $-\infty$.

While the wave propagator matrix $\mathbf{T}$ is uniquely defined, existence and uniqueness or nonexistence and nonuniqueness of the solution to the direct scattering problem may occur, depending on the value of $\mathbf{k}_{\parallel}$. To our mind this feature has not been adequately investigated in the literature. Our approach in terms of the asymptotic eigenvectors allows a systematic treatment of the problem and a simple understanding of the conditions for existence and uniqueness of the solution.

Nonuniqueness seems to be related to the phenomenon of mode conversion and shows some analogy with the possible existence of interfacial waves at the common boundary between homogeneous solid half-spaces [9 – 11]. Yet nonuniqueness or nonexistence are not confined to the occurrence of interfacial waves. As an example, in the last section we consider horizontally-polarized waves in an inhomogeneous half-space, bonded to a homogeneous incidence half-space, and determine numerically the matrix $\mathbf{T}$. We then show that conditions on $\mathbf{T}$, such that the solution to the reflection-transmission problem does not exist, are realized by admissible values of the material parameters.

2. Time-harmonic waves in anisotropic media

Consider a body, in an unstressed configuration, occupying an unbounded region Ω which is described by the Cartesian coordinates $(x, y, z) =: \mathbf{x}$. Let V be the translational space associated with the three-dimensional Euclidean point space. Also, denote by $\mathbf{e}_3$ the unit vector of the z-axis. Let $\mathbf{u}$ denote the displacement vector with values in V; $\mathbf{u}(\mathbf{x}, t)$ maps $\Omega \times \mathbb{R}$ onto V and represents the displacement, at time t, of the point labelled by $\mathbf{x}$ [12].

Let $\boldsymbol{T}$ be the symmetric (Cauchy) stress tensor, $\boldsymbol{T} : V \to V$, and let the time-dependence be expressed through the common factor $\exp(-i\omega t)$. In the absence of body forces, the evolution of time-harmonic waves is governed by the equation

$$-\rho\omega^2\mathbf{u} = \nabla \cdot \boldsymbol{T}. \tag{2.1}$$

The tensor $\mathcal{T}$ depends linearly on the spatial derivatives of $\mathbf{u}$, namely

$$\mathcal{T} = \mathbf{C}(\nabla \otimes \mathbf{u}),$$

where $\mathbf{C}$ is a complex-valued fourth-order tensor. In components,

$$\mathcal{T}_{jh} = C_{jhkl}\partial_k u_l.$$

The standard symmetry properties

$$C_{jhkl} = C_{hjkl} = C_{hjlk} = C_{lkhj}$$

are taken to hold. The tensor $\mathbf{C}$ is real-valued in (linear) elasticity and is strictly complex-valued and dependent on ω in common dissipative models. For later convenience, for any two vectors $\mathbf{a}$ and $\mathbf{b}$ we let $\mathbf{aCb}$ be a second-order tensor defined by

$$(\mathbf{aCb})_{jk} = a_h C_{hjkl} b_l.$$

We assume that the material properties ρ and $\mathbf{C}$ depend only on the vertical coordinate z. Hence we look for solutions of the form

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}(z)\exp[i(\mathbf{k}_{\parallel} \cdot \mathbf{x} - \omega t)],$$

where $\mathbf{k}_{\parallel}$ is a horizontal, complex-valued, wave vector. Correspondingly the gradient takes the form

$$\nabla = i\mathbf{k}_{\parallel} + \mathbf{e}_3 \frac{d}{dz}.$$

Let $\mathbf{t} = \mathcal{T}\mathbf{e}_3 = \hat{\mathbf{t}}(z)\exp[i(\mathbf{k}_{\parallel} \cdot \mathbf{x} - \omega t)]$ be the traction at horizontal planes. Evaluation of $\nabla \otimes \mathbf{u}$ and substitution into the expression for $\mathcal{T}$ shows that the definition of $\hat{\mathbf{t}}$ and the equation of motion (2.1) become

$$\begin{aligned} \hat{\mathbf{t}} &= i(\mathbf{e}_3\mathbf{C}\mathbf{k}_{\parallel})\hat{\mathbf{u}} + (\mathbf{e}_3\mathbf{C}\mathbf{e}_3)\hat{\mathbf{u}}', \\ \rho\omega^2\hat{\mathbf{u}} &= -(\mathbf{k}_{\parallel}\mathbf{C}\mathbf{k}_{\parallel})\hat{\mathbf{u}} + i(\mathbf{k}_{\parallel}\mathbf{C}\mathbf{e}_3)\hat{\mathbf{u}}' + \hat{\mathbf{t}}', \end{aligned} \tag{2.2}$$

where a prime stands for d/dz. As shown in [11], thermodynamics implies that $\mathbf{e}_3\mathbf{C}\mathbf{e}_3$ is invertible. Hence, a comparison allows $\hat{\mathbf{u}}'$ and $\hat{\mathbf{t}}'$ to be expressed in terms of $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$ as

$$\hat{\mathbf{u}}' = -i(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{C}\mathbf{k}_{\parallel})\hat{\mathbf{u}} + (\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}\hat{\mathbf{t}}, \tag{2.3}$$

$$\hat{\mathbf{t}}' = [-\rho\omega^2\mathbf{1} + (\mathbf{k}_{\parallel}\mathbf{C}\mathbf{k}_{\parallel}) - (\mathbf{k}_{\parallel}\mathbf{C}\mathbf{e}_3)(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{C}\mathbf{k}_{\parallel})]\hat{\mathbf{u}} - i(\mathbf{k}_{\parallel}\mathbf{C}\mathbf{e}_3)(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}\hat{\mathbf{t}}. \tag{2.4}$$

To get a more compact notation, we let $\mathbf{w}$ be the column of the ordered set of components of $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$, i.e. $\mathbf{w} = [\hat{\mathbf{u}}, \hat{\mathbf{t}}]^T$ where the superscript T means transpose, whence

$$[\mathbf{u}, \mathbf{t}]^T = \mathbf{w}(z)\exp[i(\mathbf{k}_{\parallel} \cdot \mathbf{x} - \omega t)]. \tag{2.5}$$

In accordance with (2.2) and (2.3), let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix},$$

where the four 3×3 blocks $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are given by

$$\mathbf{A}_1 = -i(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{C}\mathbf{k}_\parallel), \quad \mathbf{A}_2 = (\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1} = \mathbf{A}_2^T, \quad \mathbf{A}_4 = \mathbf{A}_1^T.$$
$$\mathbf{A}_3 = -\rho\omega^2\mathbf{1} + (\mathbf{k}_\parallel\mathbf{C}\mathbf{k}_\parallel) - (\mathbf{k}_\parallel\mathbf{C}\mathbf{e}_3)(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{C}\mathbf{k}_\parallel) = \mathbf{A}_3^T.$$

The matrix $\mathbf{A}$ is neither symmetric nor Hermitian. Yet, by means of the 6×6 matrix

$$\mathbf{K} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{1}$ is the 3×3 identity matrix and $\mathbf{0}$ is the 3×3 zero matrix, we can write the symmetry condition

$$\mathbf{KA} = (\mathbf{KA})^T. \tag{2.6}$$

In terms of $\mathbf{A}$, Eqs. (2.2) and (2.3) can be given the form

$$\mathbf{w}' = \mathbf{A}\mathbf{w}. \tag{2.7}$$

where $\mathbf{w}$ and $\mathbf{A}$ depend only on the space variable z and the vector parameter $\mathbf{k}_\parallel$. This is the sought Stroh-like form of differential equations governing the behaviour of the body [10]. The general solution to an equation of the form (2.6) is sometimes termed as a multimode wave [13].

In electromagnetism, Stroh-like forms are more familiar (cf. [14]) also because Maxwell's equations as such are first-order equations. This is a further motivation for the investigation of (2.6) as a model of wave propagation.

For later purposes, we consider the mechanical model of isotropic bodies where

$$C_{hjkl}(z) = \mu(z)[\delta_{hk}\delta_{jl} + \delta_{hl}\delta_{jk}] + \lambda(z)\delta_{hj}\delta_{kl}. \tag{2.8}$$

If the body is a viscoelastic solid then the tensor (2.8) is complex-valued and μ and λ are parameterized by $\omega \in \mathbb{R}$ in the form

$$\mu = \mu_0 + \int_0^\infty \mu'(\eta)\exp(i\omega\eta)d\eta, \qquad \lambda = \lambda_0 + \int_0^\infty \lambda'(\eta)\exp(i\omega\eta)d\eta,$$

where $\mu_0, \lambda_0 \in \mathbb{R}$ are taken to be positive and $\mu', \lambda' \in L^1(\mathbb{R})$ are real-valued; the dependence of μ_0, λ_0 and μ', λ' on z is understood and not written. Further restrictions are due to thermodynamics. As shown in [15] (Eq. (3.2.11)), the dissipative character of the stress results in the inequalities

$$\mu'_s(\omega) < 0, \qquad 2\mu'_s(\omega) + \lambda'_s(\omega) < 0, \qquad \forall\omega > 0, \tag{2.9}$$

where the subscript s denotes the half-range Fourier-sine transform. Since

$$\mathbf{e}_3\mathbf{C}\mathbf{e}_3 = \mathrm{diag}[\mu, \mu, 2\mu + \lambda],$$

the invertibility of $\mathbf{e}_3\mathbf{C}\mathbf{e}_3$ amounts to the requirement that

$$\mu \neq 0, \qquad 2\mu + \lambda \neq 0,$$

which is a direct consequence of (2.9).

For formal simplicity we let $\mathbf{k}_\parallel$ have a real direction and choose the x-axis to be in the direction of $\mathbf{k}_\parallel$ so that $k_y = 0$. Upon (2.8), the system (2.7) then decouples in two subsystems,

$$\mathbf{w}_v' = \mathbf{A}_v \mathbf{w}_v, \qquad \mathbf{w}_h' = \mathbf{A}_h \mathbf{w}_h. \tag{2.10}$$

The vectors $\mathbf{w}_v, \mathbf{w}_h$ and matrices $\mathbf{A}_v, \mathbf{A}_h$ are given by

$$\mathbf{w}_v = \begin{bmatrix} \hat{u}_x \\ \hat{u}_z \\ \hat{t}_x \\ \hat{t}_z \end{bmatrix}, \qquad \mathbf{A}_v = \begin{bmatrix} 0 & -ik_x & 1/\mu & 0 \\ -i\gamma k_x & 0 & 0 & \gamma/\lambda \\ \zeta k_x^2 - \rho\omega^2 & 0 & 0 & -i\gamma k_x \\ 0 & -\rho\omega^2 & -ik_x & 0 \end{bmatrix},$$

where $\gamma = \lambda/(2\mu + \lambda)$, $\zeta = 4\mu(\mu + \lambda)/(2\mu + \lambda)$, and

$$\mathbf{w}_h = \begin{bmatrix} \hat{u}_y \\ \hat{t}_y \end{bmatrix}, \qquad \mathbf{A}_h = \begin{bmatrix} 0 & 1/\mu \\ \mu k_x^2 - \rho\omega^2 & 0 \end{bmatrix}. \tag{2.11}$$

The systems (2.10) describe vertically-polarized (with polarization in the xz-plane) waves and horizontally-polarized (y-polarized) waves [16]. The decoupling implies that the horizontal and vertical polarizations are conserved through the body. The system for $\mathbf{w}_h$ in (2.10) is equivalent to the ordinary differential equation

$$(\mu \hat{u}_y')' + (\rho\omega^2 - \mu k_x^2)\hat{u}_y = 0 \tag{2.12}$$

which is typical of scattering problems [3 – 7]; here, though, μ and k_x are complex-valued. Upon the transformation

$$\hat{u}_y = \sigma(z) \exp\Big[-(1/2) \int\limits_0^z (\mu'/\mu)(\zeta) d\zeta \Big],$$

Eq. (2.12) can be written in the normal form

$$\sigma''(z) + f(z, \omega, k_x)\sigma(z) = 0, \tag{2.13}$$

where

$$f(z, \omega, k_x) = \frac{\rho\omega^2}{\mu} - k_x^2 - \frac{1}{2}\frac{\mu''}{\mu} + \left(\frac{\mu'}{2\mu}\right)^2$$

is complex-valued and parameterized by ω and k_x.

3. Asymptotic wave propagator

Let $\mathbf{A}(z)$ be a matrix function on $\mathbb{R}$ with values in $M_6(\mathbb{C})$, namely the set of 6×6 matrices with complex entries. Let the matrix $\mathbf{A}$ be continuous on $\mathbb{R}$ except a point, say $z = 0$, where it may suffer a jump discontinuity, $\mathbf{0} \neq [\![\mathbf{A}]\!] := \mathbf{A}(0^+) - \mathbf{A}(0^-)$. Hence $\mathbf{w}(z) : \mathbb{R} \to \mathbb{C}^6$ satisfies the differential Eq. (2.7) as $z \in \mathbb{R} \setminus \{0\}$ and is continuous at $z = 0$, i.e.

$$\mathbf{w}(0^+) = \mathbf{w}(0^-).$$

This equality represents the welded contact condition such that $\mathbf{u}$ and $\mathbf{t}$ are continuous at any discontinuity surface for material parameters.

We now examine the asymptotic properties of the fields $\mathbf{u}$ and $\mathbf{t}$. It is convenient to introduce a representation in terms of a column vector $\mathbf{v}$ which is related to $\mathbf{w}$ through a wave-splitting technique (cf. [17]). To give evidence to the asymptotic properties, we apply the wave splitting in terms of the eigenvectors of $\mathbf{A}$ at $\pm\infty$.

Let $\mathbf{A}^-$ and $\mathbf{A}^+$ be the limit values of $\mathbf{A}$ as z approaches $-\infty$ and ∞. The matrices $\mathbf{A}^\pm$ are taken to be simple and the eigenvalues to be nonzero. Simplicity is a generic property; examples of non-simple matrices $\mathbf{A}$ correspond to peculiar values of k_x (cf. [16]). Let m^+ (m^-) be the maximal real part of the eigenvalues of $\mathbf{A}^+$ ($\mathbf{A}^-$). Also let $\|\cdot\|$ be a matrix norm in $M_6(\mathbb{C})$ like, e.g., $\|\mathbf{A}\| = \max |A_{ij}|, i, j = 1, ..., 6$. We assume that

$$\|\mathbf{A}(z) - \mathbf{A}^\pm\| = o([|z| \exp(2m^\pm z)]^{-1}), \quad \text{as } z \to \pm\infty. \tag{3.1}$$

For the sake of convenience we now restrict the analysis to the half-space $z > 0$ and hence denote by the superscript $+$ the pertinent asymptotic values. Strictly analogous relations hold for the half-space $z < 0$.

Denote by $i\sigma_\alpha^+$ and $\mathbf{p}_\alpha^+$, $\alpha = 1, ..., 6$, the eigenvalues and eigenvectors of $\mathbf{A}^+$. Let $\mathbf{P}^+$ be the matrix whose columns are the eigenvectors $\mathbf{p}_\alpha^+$, namely

$$\mathbf{P}^+ = [\mathbf{p}_1^+, ..., \mathbf{p}_6^+].$$

Since $\mathbf{A}^+$ is simple we have [18]

$$(\mathbf{P}^+)^{-1}\mathbf{A}^+\mathbf{P}^+ = \mathbf{S}^+, \tag{3.2}$$

where $\mathbf{S}^+$ is the diagonal matrix

$$\mathbf{S}^+ = \mathrm{diag}[i\sigma_1^+, ..., i\sigma_6^+].$$

It is convenient to consider the new variables $\mathbf{v}(z) : \mathbb{R} \to \mathbb{C}^6$ such that

$$\mathbf{w} = \mathbf{P}^+\mathbf{E}\mathbf{v},$$

where

$$\mathbf{E}(z) = \mathrm{diag}[\exp(i\sigma_1^+ z), ..., \exp(i\sigma_6^+ z)].$$

Incidentally, by definition $\mathbf{E}$ satisfies the differential equation

$$\mathbf{E}' = \mathbf{E}\mathbf{S}^+ = \mathbf{S}^+\mathbf{E}.$$

Substitution of $\mathbf{w}$ in (2.7) provides the differential equation for $\mathbf{v}$ in the form

$$\mathbf{v}' = \mathbf{M}\mathbf{v}, \tag{3.3}$$

where

$$\mathbf{M}(z) = \mathbf{E}^{-1}(z)(\mathbf{P}^+)^{-1}\mathbf{A}(z)\mathbf{P}^+\mathbf{E}(z) - \mathbf{S}^+.$$

Once the vector function $\mathbf{v}$ is determined we obtain the original unknown $\mathbf{w}(z) = [\mathbf{u}(z), \mathbf{t}(z)]^T$ in the form

$$\mathbf{w}(z) = \sum_{\alpha=1}^{6} v_\alpha(z)\mathbf{p}_\alpha^+ \exp[i(\mathbf{k}_\alpha^+ \cdot \mathbf{x} - \omega t)],$$

where $\mathbf{k}_\alpha^+ = \mathbf{k}_\parallel + \sigma_\alpha^+ \mathbf{e}_3$. The components v_α can be viewed as the amplitudes of inhomogeneous waves [8]. Indeed, since the functions v_α have a limit as $z \to \pm\infty$, asymptotically $\mathbf{w}$ is just a superposition of inhomogeneous waves.

We then investigate the existence and uniqueness of the solution $\mathbf{v}$ to (3.3) with a given initial value, e.g. $\mathbf{v}(0^+)$. By replacing $\mathbf{S}^+$ with (3.2) we can write $\mathbf{M}$ as

$$\mathbf{M} = \mathbf{E}^{-1}(\mathbf{P}^+)^{-1}[\mathbf{A}(z) - \mathbf{A}^+]\mathbf{P}^+\mathbf{E}.$$

Hence, because $-\Im\sigma_\alpha^+ \leqslant m^+$, $\alpha = 1, ..., 6$, we have the estimate

$$\|\mathbf{M}(z)\| \leqslant \|(\mathbf{P}^+)^{-1}\| \, \|\mathbf{A}(z) - \mathbf{A}^+\| \, \|\mathbf{P}^+\| \exp(2m^+ z).$$

Two consequences follow at once. First, if $\mathbf{A}$ is constant as $z \geqslant 0$ then $\mathbf{M}(z) = 0$, $\forall z \in (0, \infty)$. By (3.3) this implies that $\mathbf{v}$ is constant in homogeneous half-spaces. Second, the assumption (3.1) makes $\|\mathbf{M}\|$ to be integrable on $\mathbb{R}^+$ namely

$$\int_0^\infty \|\mathbf{M}(z)\| dz < \infty. \tag{3.4}$$

The integrability (3.4) and the continuity of $\mathbf{M} : \mathbb{R}^+ \to M_6(\mathbb{C})$ allows us to argue, step by step, as in [19], and conclude that there exists a fundamental matrix $\mathbf{U}(z)$ such that

$$\mathbf{U}' = \mathbf{M}\mathbf{U}, \qquad \mathbf{U}(0) = \mathbf{1}, \qquad \mathbf{v}(z) = \mathbf{U}(z)\mathbf{v}(0^+), \tag{3.5}$$

in $(0, \infty)$ and $\mathbf{v}(z)$ has a limit as $z \to \infty$. Letting

$$\mathbf{v}^+ = \lim_{z\to\infty} \mathbf{v}(z), \qquad \mathbf{U}^+ = \lim_{z\to\infty} \mathbf{U}(z),$$

we can write

$$\mathbf{v}^+ = \mathbf{U}^+\mathbf{v}(0^+). \tag{3.6}$$

Apart from purely formal changes, the same statements and results hold as $z \in (-\infty, 0)$. For instance,

$$\mathbf{E} = \mathrm{diag}[\exp(i\sigma_1^- z), ..., \exp(i\sigma_6^- z)],$$

and

$$\mathbf{v}^- = \mathbf{U}^- \mathbf{v}(0^-). \tag{3.7}$$

Owing to a possible discontinuity of $\mathbf{A}$, and hence of $\mathbf{v}$, at $z = 0$, we need a relation between $\mathbf{v}(0^-)$ and $\mathbf{v}(0^+)$. We know that since $\mathbf{w}$ is continuous, whence

$$\mathbf{w}(0^-) = \mathbf{w}(0^+),$$

that

$$\mathbf{E}(0^-) = \mathbf{E}(0^+) = \mathbf{1},$$

and that

$$\mathbf{w}(0^-) = \mathbf{P}^- \mathbf{v}(0^-), \qquad \mathbf{w}(0^+) = \mathbf{P}^+ \mathbf{v}(0^+).$$

As a consequence, $\mathbf{v}(0^-)$ and $\mathbf{v}(0^+)$ are connected by

$$\mathbf{v}(0^+) = (\mathbf{P}^+)^{-1} \mathbf{P}^- \mathbf{v}(0^-). \tag{3.8}$$

By means of (3.6), (3.7) and (3.8) we have

$$\mathbf{v}^+ = \mathbf{T}\mathbf{v}^-, \tag{3.9}$$

where

$$\mathbf{T} = \mathbf{U}^+ (\mathbf{P}^+)^{-1} \mathbf{P}^- (\mathbf{U}^-)^{-1} \tag{3.10}$$

may be viewed as the asymptotic wave-propagator matrix (cf. [20]). The matrix $\mathbf{T}$ is non-singular and is parameterized by $\mathbf{k}_\parallel$ through $\mathbf{P}^\pm$ and $\mathbf{U}^\pm$.

To sum up, the existence and uniqueness of the fundamental matrix $\mathbf{U}(z)$ implies the existence and uniqueness of the solution $\mathbf{v}(z)$ to the Cauchy problem for (3.3) in $(0, \infty)$ with initial value $\mathbf{v}(0^+)$, possibly through (3.8). In direct scattering problems, though, the values $\mathbf{v}(0^+)$ and $\mathbf{v}(0^-)$ are unknown. Rather, we consider an incident wave which comes, e.g., from $-\infty$; it is partially back-scattered and partially transmitted by the stratified medium. This means that neither $\mathbf{v}^+$ nor $\mathbf{v}^-$ in (3.9) can be regarded as known. The associated reflection-transmission problem for $\mathbf{v}^+$ and $\mathbf{v}^-$ is not a Cauchy problem and hence existence and uniqueness of the solution are not guaranteed. This is examined in the next section.

4. Existence and uniqueness of reflected and transmitted waves

The asymptotic limits of the matrix $\mathbf{A}$ allows the identification of incident, reflected and transmitted waves. The real part of σ_α^+ is the z-component of the phase speed of the corresponding asymptotic wave mode

$$\mathbf{w} = v_\alpha^+ \mathbf{p}_\alpha^+ \exp[i(\mathbf{k}_\alpha^+ \cdot \mathbf{x} - \omega t)].$$

Accordingly, we assume that the real part of σ_α is positive for three eigenvalues and negative for the three remaining ones. For definiteness we let $\Re\sigma_\alpha^+ > 0$ as $\alpha = 1, 2, 3$ and $\Re\sigma_\alpha^+ < 0$ as $\alpha = 4, 5, 6$. Hence we regard v_1, v_2, v_3 as being associated with forward-propagating waves, and v_4, v_5, v_6 with backward-propagating waves, in the z-direction. Alternative, non-equivalent partitions of the forward-backward propagating modes can be considered that are based on the direction of the energy flow or amplitude growth [21, 13]. They result in a different partition of the asymptotic wave modes. Yet the essence of the subsequent analysis holds, irrespective of the criterion adopted to select forward- and backward-propagating wave solutions.

For definiteness, at first let the incident wave come from $-\infty$. A reflected wave is going back to $-\infty$, a transmitted wave is going to $+\infty$ and no backward-propagating wave occurs at $+\infty$. Let $\mathbf{v}_i, \mathbf{v}_o$ be triples associated with input and output waves. The sixtuples $\mathbf{v}^-$ and $\mathbf{v}^+$ at $-\infty$ and $+\infty$ are given by

$$\mathbf{v}^- = \begin{bmatrix} \mathbf{v}_i^- \\ \mathbf{v}_o^- \end{bmatrix}, \qquad \mathbf{v}^+ = \begin{bmatrix} \mathbf{v}_o^+ \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{v}_o^-$ and $\mathbf{v}_o^+$ represent the reflected and transmitted waves. The matrix $\mathbf{T}$ can be viewed as given by 3×3 blocks. We then write (3.9) in the form

$$\begin{bmatrix} \mathbf{v}_o^+ \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{bmatrix} \begin{bmatrix} \mathbf{v}_i^- \\ \mathbf{v}_o^- \end{bmatrix},$$

or

$$\mathbf{T}_1 \mathbf{v}_i^- + \mathbf{T}_2 \mathbf{v}_o^- = \mathbf{v}_o^+, \qquad \mathbf{T}_3 \mathbf{v}_i^- + \mathbf{T}_4 \mathbf{v}_o^- = \mathbf{0}. \tag{4.1}$$

If, instead, the incident wave is coming from $+\infty$ then the reflected triple $\mathbf{v}_o^+$ and the transmitted triple $\mathbf{v}_o^-$ are given by

$$\mathbf{v}_o^+ = \mathbf{T}_2 \mathbf{v}_o^-, \qquad \mathbf{v}_i^+ = \mathbf{T}_4 \mathbf{v}_o^-, \tag{4.2}$$

where $\mathbf{v}_i^+$ is the incident triple. Equations (4.1) and (4.2) are to be solved in the unknowns $\mathbf{v}_o^-, \mathbf{v}_o^+$.

If $\mathbf{T}_4$ is non-singular then $\mathbf{v}_o^-$ and $\mathbf{v}_o^+$ are determined at once. By (4.1) we have

$$\mathbf{v}_o^- = -\mathbf{T}_4^{-1} \mathbf{T}_3 \mathbf{v}_i^-, \qquad \mathbf{v}_o^+ = (\mathbf{T}_1 - \mathbf{T}_2 \mathbf{T}_4^{-1} \mathbf{T}_3) \mathbf{v}_i^-.$$

Similarly, by (4.2) we determine $\mathbf{v}_o^+$ and $\mathbf{v}_o^-$ in terms of the incident wave $\mathbf{v}_i^+$. For generality, both $\mathbf{v}_i^-$ and $\mathbf{v}_i^+$ are allowed to occur. The output waves $\mathbf{v}_o^-$ and $\mathbf{v}_o^+$ are given in terms of the input waves $\mathbf{v}_i^-$ and $\mathbf{v}_i^+$ as

$$\begin{bmatrix} \mathbf{v}_o^+ \\ \mathbf{v}_o^- \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix} \begin{bmatrix} \mathbf{v}_i^- \\ \mathbf{v}_i^+ \end{bmatrix},$$

where

$$\mathbf{S}_1 = \mathbf{T}_1 - \mathbf{T}_2\mathbf{T}_4^{-1}\mathbf{T}_3, \quad \mathbf{S}_2 = \mathbf{T}_2\mathbf{T}_4^{-1}, \quad \mathbf{S}_3 = -\mathbf{T}_4^{-1}\mathbf{T}_3, \quad \mathbf{S}_4 = \mathbf{T}_4^{-1}.$$

The matrix

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix},$$

is called scattering matrix and is uniquely determined by $\mathbf{T}$. Of course it provides the reflected and transmitted waves in terms of the incident ones.

The matrix $\mathbf{T}_4$ depends on the complex vector parameter $\mathbf{k}_\parallel$ and hence may become singular by appropriate choices of $\mathbf{k}_\parallel$. Let $\det \mathbf{T}_4 = 0$. By (4.1) nonzero triples $\mathbf{v}_o^-$ in the nullspace of $\mathbf{T}_4$, $\mathcal{N}(\mathbf{T}_4)$, exist and make the case $\mathbf{v}_i^- = \mathbf{0}$ and $\mathbf{v}_o^- \neq \mathbf{0}$ to be possible while the remaining equation of (4.1) determines $\mathbf{v}_o^+$. Similar conclusions follow from (4.2). Hence scattering solutions $\mathbf{v}_s(z)$ can exist such that

$$\mathbf{v}_s^- = \begin{bmatrix} \mathbf{0} \\ \mathbf{v}_o^- \end{bmatrix}, \qquad \mathbf{v}_s^+ = \begin{bmatrix} \mathbf{v}_o^+ \\ \mathbf{0} \end{bmatrix}. \tag{4.3}$$

In the picture of plane waves, a wave solution satisfying (4.3) is regarded as an interfacial wave [9].

Now let the incident wave $\tilde{\mathbf{v}}_i^-$ meet the condition $\mathbf{T}_3\tilde{\mathbf{v}}_i^- \in \mathcal{R}(\mathbf{T}_4)$ where $\mathcal{R}(\mathbf{T}_4)$ is the range of $\mathbf{T}_4$. By (4.1) at least one pair of output triples exists, say $\tilde{\mathbf{v}}_o^-, \tilde{\mathbf{v}}_o^+$. Hence a solution $\tilde{\mathbf{v}}$ occurs subject to

$$\tilde{\mathbf{v}}^- = \begin{bmatrix} \tilde{\mathbf{v}}_i^- \\ \tilde{\mathbf{v}}_o^- \end{bmatrix}, \qquad \tilde{\mathbf{v}}^+ = \begin{bmatrix} \tilde{\mathbf{v}}_o^+ \\ \mathbf{0} \end{bmatrix}.$$

Since the system (3.3) is linear, the field $\mathbf{v}_s(z) + \tilde{\mathbf{v}}(z)$ is a solution such that

$$\left[\mathbf{v}_s + \tilde{\mathbf{v}}\right]^- = \begin{bmatrix} \tilde{\mathbf{v}}_i^- \\ \tilde{\mathbf{v}}_o^- + \mathbf{v}_o^- \end{bmatrix}, \qquad \left[\mathbf{v}_s + \tilde{\mathbf{v}}\right]^+ = \begin{bmatrix} \tilde{\mathbf{v}}_o^+ + \mathbf{v}_o^+ \\ \mathbf{0} \end{bmatrix}.$$

Hence $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{v}} + \mathbf{v}_s$ are two fields associated with the same incident field, which shows the nonuniqueness of the direct scattering problem.

Something else can also occur. The range of $\mathbf{T}_4$, $\mathcal{R}(\mathbf{T}_4)$, is two-dimensional, at most. Hence there are vectors $\mathbf{v}_i^+$ such that (4.2) is not compatible. Similarly, if $\mathbf{T}_3$ is non-singular, there are vectors $\mathbf{v}_i^-$ such that

$$\mathbf{T}_3\mathbf{v}_i^- \notin \mathcal{R}(\mathbf{T}_4).$$

Hence we say that, for such vectors $\mathbf{v}_i^+$ and $\mathbf{v}_i^-$, no vector $\mathbf{v}_o^-$ and $\mathbf{v}_o^+$ satisfies the second relation in (4.1) or (4.2) and then there is no solution for the scattering problem. In conclusion we can write

$$\det \mathbf{T}_4 \neq 0 \quad \Longrightarrow \quad \text{existence and uniqueness of } \mathbf{v}_o^-, \mathbf{v}_o^+, \text{ and } \mathbf{S},$$

$$\det \mathbf{T}_4 = 0,\ \mathbf{v}_i = \mathbf{v}_i^-, \quad \Longrightarrow \begin{cases} \text{no solution } \mathbf{v}_o^-, \mathbf{v}_o^+ \text{ if } \mathbf{T}_3\mathbf{v}_i \notin \mathcal{R}(\mathbf{T}_4), \\ \text{nonuniqueness of } \mathbf{v}_o^-, \mathbf{v}_o^+ \text{ if } \mathbf{T}_3\mathbf{v}_i \in \mathcal{R}(\mathbf{T}_4), \end{cases}$$

$$\det \mathbf{T}_4 = 0, \mathbf{v}_i = \mathbf{v}_i^+, \quad \Longrightarrow \begin{cases} \text{no solution } \mathbf{v}_o^-, \mathbf{v}_o^+ \text{ if } \mathbf{v}_i \notin \mathcal{R}(\mathbf{T}_4), \\ \text{nonuniqueness of } \mathbf{v}_o^-, \mathbf{v}_o^+ \text{ if } \mathbf{v}_i \in \mathcal{R}(\mathbf{T}_4). \end{cases}$$

It is of interest to compare this conclusion with existence and uniqueness results appearing in the literature. For example, [5] proves that there is at most one pair of reflection and transmission coefficients consistent with the wave reflection problem. Here we show how a less restrictive hypothesis makes nonuniqueness to be possible. Roughly, the differential equation

$$y'' + N(z)y = 0, \quad z \in \mathbb{R}, \tag{4.4}$$

is considered where $N(z) \geqslant \delta > 0$, $N' \in L(\mathbb{R})$ and N has finite limits as $z \to \pm\infty$. Also in view of (2.13), we observe that the qualitative difference with the approach in [5] is that we allow for a complex-valued coefficient N, which does not guarantee existence and uniqueness. A similar remark holds for [6]. Moreover, the nonexistence of $\mathbf{S}$, due to the vanishing of $\det \mathbf{T}_4$, is consistent with a remark made in [3] that $\mathbf{S}$ may not exist at some values of a suitable parameter. The equation examined in [3] is a particular case of (2.13).

As a comment on the condition $\det \mathbf{T}_4 = 0$, we observe that if the half-spaces are uniform then $\mathbf{U}^\pm$ is the identity. Hence $\mathbf{T} = (\mathbf{P}^+)^{-1}\mathbf{P}^-$. Now, by (3.10) we have

$$\mathbf{T} = \begin{bmatrix} (\mathbf{P}_3^+)^T\mathbf{P}_1^- + (\mathbf{P}_1^+)^T\mathbf{P}_3^- & (\mathbf{P}_3^+)^T\mathbf{P}_2^- + (\mathbf{P}_1^+)^T\mathbf{P}_4^- \\ (\mathbf{P}_4^+)^T\mathbf{P}_1^- + (\mathbf{P}_2^+)^T\mathbf{P}_3^- & (\mathbf{P}_4^+)^T\mathbf{P}_2^- + (\mathbf{P}_2^+)^T\mathbf{P}_4^- \end{bmatrix}.$$

Hence $\det \mathbf{T}_4 = 0$ takes the form

$$\det[(\mathbf{P}_4^+)^T\mathbf{P}_2^- + (\mathbf{P}_2^+)^T\mathbf{P}_4^-] = 0,$$

which corresponds to the condition for the occurrence of interfacial waves (see, e.g., the equivalent Eq. (2.11) of [9]). This is consistent with [11] where the condition for the existence of interfacial waves is related to possible nonexistence or nonuniqueness of the solution.

5. Application to horizontally-polarized waves

By (2.11), horizontally-polarized waves are described by

$$\mathbf{A} = \begin{bmatrix} 0 & 1/\mu \\ -\nu^2\mu & 0 \end{bmatrix},$$

where $\nu = \sqrt{\rho\omega^2/\mu - k_x^2}$, $\Re\nu > 0$. Accordingly we have

$$\mathbf{A}^+ = \begin{bmatrix} 0 & 1/\mu^+ \\ -(\nu^+)^2\mu^+ & 0 \end{bmatrix}, \qquad \mathbf{P}^+ = \begin{bmatrix} 1 & 1 \\ i\mu^+\nu^+ & -i\mu^+\nu^+ \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} i\nu^+ & 0 \\ 0 & -i\nu^+ \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} \exp(i\nu^+ z) & 0 \\ 0 & \exp(-i\nu^+ z) \end{bmatrix}, \qquad z > 0.$$

The matrix $\mathbf{M}$ is then given by

$$\mathbf{M}(z) = \begin{bmatrix} f(1+a) - i\nu^+ & -f(1-a)\exp(-2i\nu^+ z) \\ f(1-a)\exp(2i\nu^+ z) & -f(1+a) + i\nu^+ \end{bmatrix},$$

where

$$a(z) = \frac{\mu^2(z)\nu^2(z)}{(\mu^+\nu^+)^2}, \qquad f(z) = i\frac{\mu^+\nu^+}{2\mu(z)}.$$

In homogeneous media we have $\mu = \mu^+, \nu = \nu^+$ whence $a = 1$, $f = i\nu^+/2$ and $\mathbf{M}$ vanishes. This in turn implies that v_1 and v_2 are constant in homogeneous regions.

We now determine numerically a case where $\det \mathbf{T}_4 = 0$. First we look for the 2×2 fundamental matrix $\mathbf{U}^+$ such that

$$\begin{bmatrix} v_1^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} U_1^+ & U_2^+ \\ U_3^+ & U_4^+ \end{bmatrix} \begin{bmatrix} v_1(0^+) \\ v_2(0^+) \end{bmatrix}.$$

We regard $\mathbf{U}^+$ as given which means that (3.5) is taken to be solved.

We assume that the half-space $z < 0$ is homogeneous and hence

$$\begin{bmatrix} v_1^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} v_1(0^-) \\ v_2(0^-) \end{bmatrix}, \qquad \mathbf{U}^- = \mathbf{1}.$$

Since $\mathbf{T} = \mathbf{U}^+(\mathbf{P}^+)^{-1}\mathbf{P}^-$ and

$$\mathbf{P}^- = \begin{bmatrix} 1 & 1 \\ i\mu^-\nu^- & -i\mu^-\nu^- \end{bmatrix}$$

we obtain

$$\mathbf{T} = \frac{1}{2\mu^+\nu^+}\begin{bmatrix} \mu^+\nu^+(U_1^+ + U_2^+) + \mu^-\nu^-(U_1^+ - U_2^+) & \mu^+\nu^+(U_1^+ + U_2^+) - \mu^-\nu^-(U_1^+ - U_2^+) \\ \mu^+\nu^+(U_3^+ + U_4^+) + \mu^-\nu^-(U_3^+ - U_4^+) & \mu^+\nu^+(U_3^+ + U_4^+) - \mu^-\nu^-(U_3^+ - U_4^+) \end{bmatrix}.$$

Now we show that there is a value of μ^- such that

$$0 = T_4 = \mu^+\nu^+(U_3^+ + U_4^+) - \mu^-\nu^-(U_3^+ - U_4^+).$$

Such is the case if

$$\mu^-\nu^- = \frac{U_3^+ + U_4^+}{U_3^+ - U_4^+}\mu^+\nu^+ =: \alpha.$$

Upon substitution of the expression for ν we have

$$(k_x\mu^-)^2 - \rho^-\omega^2\mu^- + \alpha^2 = 0,$$

whence

$$\mu^- = \frac{\rho^-\omega^2}{2k_x^2}(1 \pm \sqrt{1 - (2\alpha k_x/\rho^-\omega^2)^2}).$$

By thermodynamics [15], only the solution such that $\Im\mu^- < 0$ is admissible.

As an example, let

$$\mu(z) = \begin{cases} \mu_0 \exp(z^2), & z \in (0,2), \\ \mu_0 \exp(4) = \mu^+, & z \in [2,\infty), \end{cases}$$

where $\mu_0 = (12.10 - 0.40i)10^{10}$ g/cm s^2 is the value of μ in the model of Berkeley crust while

$$\rho^+ = 2.1 \text{ g/cm}^3,\ z \in (0,\infty), \qquad \rho^- = 1.5 \text{ g/cm}^3,\ z \in (-\infty, 0)$$

and

$$\omega = 30 \text{ s}^{-1}, \qquad k_x = 0.1 + 0.5i \text{ cm}^{-1}.$$

We find that

$$\mu^- = (.3676 - 2.468i)10^{11}\text{g/cm}^3$$

is the admissible solution. Accordingly, if such is the value of μ in the incidence half-space ($\mu^- \neq \mu(0^+)$) then $T_4 = 0$ and hence, since $T_3 \neq 0$, the problem (4.1) is incompatible.

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Merging and interacting wave fronts for reaction – diffusion equations

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In this paper the merging and interacting kink-type solution for general reaction diffusion equations are considered. Both the analytical and numerical investigations are presented. In particular, the existence of the complicated merging, elliptic and hyperbolic fronts in two dimensions is shown.

1. Introduction

In this paper the Cauchy problem for the reaction-diffusion equations is considered:

$$\frac{\partial u}{\partial t} = \Delta u + h^2(x) f(u), \qquad x \in R^n, \quad t \geqslant 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

where $f(u)$ and h^2 are smooth functions.

These equations play an important role in a number of applications such as the theory of combustion, phase transitions, polymer science, chemical kinetics and many others.

Here we shall study some special kind of solutions for problem (1.1) – (1.2) describing the motion, the interaction and merging of well localized wave fronts (interfaces). These fronts are involved in many mechanical, physical and biological phenomena.

The asymptotical theory of the interface propagation was developed in numerous papers, for example, [9, 10, 11, 6]. The first rigorous proof of existence of such fronts was given by [10] in the pioneering work on the scalar systems of Ginzburg-Landau type. Another approach for more general scalar equations, based on the comparison principle, was suggested in [1]. By this principle, Z. PERADZYŃSKI and B. KAŹMIERCZAK [12] have considered monotone parabolic systems of very general form. These results describe the localized fronts of a small curvature and do not pretend to describe completely the front interaction and the front merging.

At the present time, the theory of the front interaction in one-dimensional case is extensively developed (see, for example, [2, 7, 9]). However, all analytical results hold only when the travelling fronts (kinks) are separated by large distances (weak interaction). The first goal of this paper is to present an effective semi-analytic, semi-numerical method that allows to analyze the process of interaction and merging of two kinks completely, up to the final stage.

The theory of interaction of localized fronts in 2D is much less developed as compared to 1D. In fact, there exist only a few results in this field. The first of them was obtained by Z. PERADZYŃSKI [13] and concerned the 2D merging kink solutions. Physically, such solution describes a moving interface (with a constant velocity) between two phases. This interface has a form close to two lines intersecting at an angle $\alpha \neq \pi$. At the intersection point such an interface can have a large curvature, what makes the investigation especially difficult. In particular, the well known methods ([10, 11, 9]) cannot be applied. Let us also notice that the existence of some special odd 2D solutions was shown in [4]; however, it is well known that these solutions are unstable [14].

The second aim of the paper is to construct new *exact* analytic 2D solutions describing the complex effects of the interaction and merging of the kinks. This holds for some special coefficients h (for example, quadratic). These solutions can describe a compression of curved kinks of elliptic form; an interaction of two kinks with fronts of hyperbolic type; and at last, merging of two kinks with asymptotically planar fronts.

The paper is organized as follows. In Section 1 we describe the case of interaction of two one-dimensional kinks. In Section 2, we give the analytic approach that allows us to describe a class of exact 2D solutions.

2. Approximate solution of the kink – antikink collision problem for real Ginzburg-Landau model. Merging kinks in one dimension.

In this section we consider the kink - antikink collision problem for the real Ginzburg-Landau model. Mathematically, this problem can be written as the Cauchy problem

$$u_t = u_{xx} + u - u^3, \qquad x \in R, \qquad t > 0, \tag{2.1}$$

$$u_0 = 1 + \mathrm{tgh}(p_0(x - q_0)) - \mathrm{tgh}(p_0(x + q_0)), \tag{2.2}$$

where $|q_0| >> 1$, $p_0 = 1/\sqrt{2}$ are constants. This problem enables the following equivalent variational formulation

$$\frac{\delta D}{\delta u_t} = -\frac{\delta F}{\delta u}, \tag{2.3}$$

where D and F are functionals of dissipation and free energy:

$$D[u_t] = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2 dx, \tag{2.4}$$

$$F[u] = \int_{-\infty}^{\infty}\left(\frac{u_x^2}{2} + \frac{(u^2-1)^2}{4}\right) dx. \tag{2.5}$$

Such a formulation is rather important both for the mathematical description of solutions and the physical understanding of underlying phenomena. The aim of this section is to construct an approximate solution of Eq. (2.1) with initial data (2.2).

Firstly let us describe some physical effects appearing in (2.1) – (2.2). Suppose the Eq. (2.1) describes a state of bistable (two-phase) medium. The stable phases of this matter correspond to stationary solutions of Eq. (2.2) $u = u^+ = 1$ and $u = u^- = -1$. From this viewpoint, initial data (2.2) describe the case when a large interval of space $d \simeq 2q_0$ is filled by phase $u = u^-$ and surrounded on both sides by semi-infinite regions by the phase u^+, these phases being separated by the boundary layers (fronts). Although the initial data (2.2) almost exactly satisfy Eq. (2.1), Eq. (2.2) is an unstable solution. In fact, as time passes, the fronts approaching each other, and the distance between them reduces progressively, and when $t \to \infty$, the solution approach $u = u^+$ exponentially in time and uniformly in space. Physically, this process describes an absorption of phase u^- by phase u^+. When the distance between fronts is sufficiently large, one can describe its motion asymptotically. Then the principal part of the asymptotic solution is given by

$$u_0 = 1 + \operatorname{tgh}(p_0(x-q)) - \operatorname{tgh}(p_0(x+q)), \tag{2.6}$$

where $p_0 = 1/\sqrt{2}$ is a constant, $q = q(t)$ is the following function:

$$q(t) = \frac{1}{2\sqrt{2}}\ln\left(\exp(2\sqrt{2}q_0) - 48t\right). \tag{2.7}$$

This well – known result (see, for example, [2, 9, 7]) holds only when q_0 is sufficiently large. When the fronts approach each other at the distances comparable to 1, solution (2.6), (2.7) fails. In fact, the interaction of the fronts at these distances changes the character and analytical description of such interaction becomes more complicated.

In order to describe the entire process of fronts motion for the system (2.1) – (2.2) from weak interaction to collision, we apply the generalized Whitham principle [9, 18].

According to this principle, variational Eq. (2.3) can be approximated as follows:

$$\frac{\partial \bar{D}(\mathbf{s},\dot{\mathbf{s}})}{\partial \dot{s}_i} = -\frac{\partial \bar{F}(\mathbf{s})}{\partial s_i}, \tag{2.8}$$

where $\bar{D}$, $\bar{F}$ are the functionals D and F "Whitham averaged" on the approximation $U(x,\mathbf{s}(t))$. In other words,

$$\bar{D}(\mathbf{s},\dot{\mathbf{s}}) = \frac{1}{2}\int_{-\infty}^{\infty} U_t^2\,dx, \tag{2.9}$$

$$\bar{F}(\mathbf{s}) = \int_{-\infty}^{\infty}\left(\frac{U_x^2}{2} + \frac{(U^2-1)^2}{4}\right)dx, \tag{2.10}$$

where $\mathbf{s} = \mathbf{s}(t)$ is an unknown time – dependent vector function, which should be obtained from the system (1.8).

The approximation $U(x,\mathbf{s})$ depends explicitly on the space coordinate and implicitly, via $\mathbf{s}$, on time that allows us to make an approximate reduction from infinite - dimensional system (1.3) to finite-dimensional one (1.8).

Practically an approximation $U(x,\mathbf{s})$ will be as good as our prediction a real space shape of solution $u(x,t)$. The generalized Whitham principle helps us only to find the best approximation of exact solution $u(x,t)$ among the functions $U(x,\mathbf{s})$ where $\mathbf{s} = \mathbf{s}(t)$ is any time-dependent function. Let us note here that choice of the shape function U, number of components in $\mathbf{s}$ and dependence of $U(x,\mathbf{s})$ on $\mathbf{s}$ is a quite complicated problem. This is similar, in a sense, to the choice of a proper basis for the Galerkin approximation. In fact there exist no general rules to define these quantities and such a selection is more an art than a science.

Here we are going to describe a function U which is, according to the numerical grid solution of Eq. (2.1), a very good approximation of the true solution.

This approximation is as follows:

$$U(x,q,p) = 1 + \operatorname{tgh}(p(x-q)) - \operatorname{tgh}(p(x+q)), \qquad \mathbf{s}(t) = (q(t),p(t)). \tag{2.11}$$

The choice of this function (2.11) is mainly inspired by initial conditions and the physical nature of the problem. Both the unknown functions have a clear physical interpretation: q, as it was mentioned above, is the front coordinate and p is the slope of the front. This means that, in this approximation, the fronts can move and that all front distortions are reduced to the change of their slope. Direct numerical computation by the grid method shows that such a presumption is very close to the real front behaviour. According to the numerical results, the front generally moves according to our predictions but with a small distortion at the same time. Such a distortion level may reach about three percent of the solution and is strongly localized at the center of symmetry (Fig. 1).

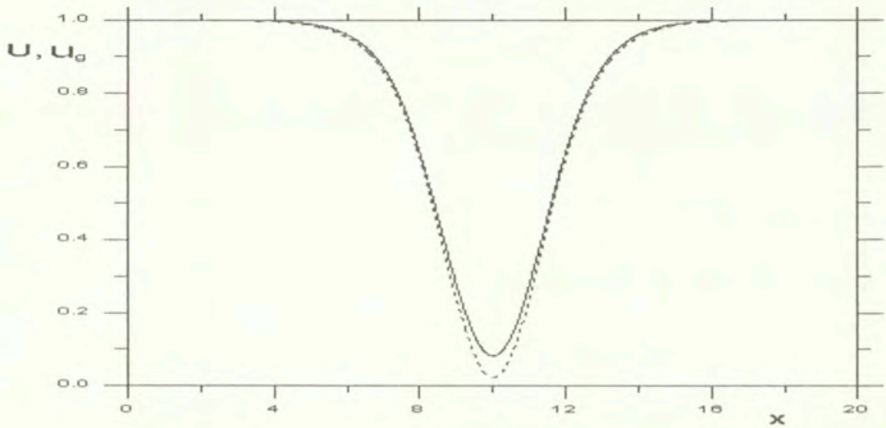

FIG. 1. Comparison of the grid solution u_g (dashed line) and approximate solution U, given by the formula (2.11) (solid line) at the moment corresponding to the final stage of the kink — antikink collision.

Let us describe now how the parameters q and p behave during the collision. At the initial stage of the front motion while the kink distance q is large (practically for $|q| > 3$), the motion goes as it is predicted by asymptotic formulas (1.7): the slope of the fronts remains constant and the coordinate q slowly decreases. When the fronts approaching each other are close enough, the slope starts to be involved in dynamics. Motion becomes faster and faster and the slope decreases making them wider. At the final stage, both parameters $p, q \to 0$ and the stable state $u = u^+$ appears.

The agreement between true and approximate solutions may be called quite good. Apparently the approximation preserves the main features of the front collision, nonetheless it would be helpful to find an algorithm to improve the accuracy of the approximation. In fact, such a refinement must give a more detailed description of the front distortion. Our numerical experiment shows that such an improved approximation can be given in the following form:

$$u_{ia} = \sum_{k=1}^{N} z_k(t)[\cosh^{-2}(p(t)(x - c_k q(t))) + \cosh^{-2}(p(t)(x + c_k q(t)))], \tag{2.12}$$

where c_k are some *a priori* given numbers and z_k are some time-dependent functions which can be defined by the Galerkin method from linearized at U Eq. (2.1). For example, if we take $N = 3$ and $\mathbf{c} = (c_1, c_2, c_3) = (0, 1, 2)$, the accuracy of solution will increase from three percent to 0.05 percent.

We should notice that a similar approximation has been used in [3] to understand qualitative behaviour of the kinks connected states for ϕ^4 model which is $u_{tt} - u_{xx} = u - u^3$. In this work such a approximation is called "parametric collective coordinate" (PCC). However, the agreement between numerical simulations and the result of PCC approach is only of a qualitative character.

3. Exact analytic solutions describing the interaction and merging of kinks in some inhomogeneous 2D-media

2.1. General approach

Consider the following equation:

$$u_t = \Delta u + h^2(x, y) f(u). \tag{3.1}$$

Suppose there exists a kink-type function satisfying the equation

$$-cU' = U'' + f(U). \tag{3.2}$$

The kink-type function is here a function $u(x)$ that is monotone and approaching exponentially its limits $u^{\pm}$, $u^+ \neq u^-$ as $x \to \pm\infty$. Under natural conditions of bistability (for example, see [5, 16, 17, 15] among others and references given therein), such a solution U exists and is globally stable.

Let us seek a solution of (3.1) in the following form:

$$u = U(\theta), \quad \theta = \phi(x, y) - vt. \tag{3.3}$$

Substitution of (3.3) to (3.1) leads to

$$-vU' = \Delta\phi U' + (\nabla\phi)^2 U'' + h^2(x, y) f(U). \tag{3.4}$$

Using condition (3.2) and equating coefficients at U' and U'' to zero, we obtain the following equations for ϕ

$$\Delta\phi + v - ch^2(x, y) = 0, \tag{3.5}$$

$$(\nabla\phi)^2 = h^2(x, y). \tag{3.6}$$

Certainly, the system (3.5) – (3.6) can not be solved for a general function h^2 because one function ϕ should satisfy two equations. Nevertheless, at least for some functions h^2 it is possible to construct phase ϕ which satisfies both Eqs. (3.5) and (3.6). It allows us to find a new class of exact solution describing the interesting physical effects.

2.2. Examples of curved fronts and merging kink solutions

In this section we consider two examples of Eq. (3.1) with different functions h^2 which lead to curved fronts and merging kink solutions.

Our first example concerns two simple terms h^2 which are quadratic functions of x and y. In this example, we intend to show that, even in such simple cases, one can observe a number of new non-trivial physical effects. In particular, we show that such a term generates solutions of (3.1) with moving curved fronts. These fronts are plane second-order curves.

The function h^2 is a sufficiently nonlocal function. In the second example we consider a case where h^2 is localized along some line. This means that asymptotically, at an exponential rate, h^2 tends to a constant as $x \to \infty$. This localization leads to merging front formation.

The first example deals with two types of Eq. (3.1) with polynomial h^2 functions. The first one (case A) leads to a kink-type solution with elliptic and hyperbolic fronts, the second type (case B) to a parabolic one.

Case A. Elliptic and hyperbolic fronts

As the first step let us consider Eq. (3.1) with

$$h^2 = (ax)^2 + (by)^2. \tag{3.7}$$

We suppose also that velocity of the initiated kink solution $c = 0$ (see Eq. (3.2)). It is easy to check that phase ϕ nucleated by such a term h^2 can be expressed in a way similar to h^2 as a quadratic function

$$\phi = \phi_0 + \alpha x^2 + \beta y^2, \tag{3.8}$$

where

$$\alpha = \pm\frac{a}{2}, \qquad \beta = \pm\frac{b}{2}, \qquad v = -2(\alpha + \beta). \tag{3.9}$$

Here we need to distinguish two different cases: $\mathrm{sign}(\alpha) = \mathrm{sign}(\beta)$, $\mathrm{sign}(\alpha) = -\mathrm{sign}(\beta)$.

The first case is especially simple and corresponds to the compressive elliptic front solution (see Fig. 2). Such a solution has a simple physical interpretation in the framework of the phase transition theory. Assume that Eq. (3.1) describes a two-phase nonlinear medium. In this sense, solution $u = u^+$ corresponds to one

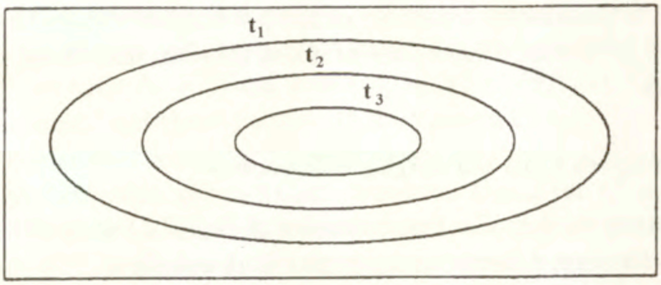

FIG. 2. Elliptic front behaviour.

phase, $u = u^-$ to another. At the initial moment, we have two regions consisting of different phases u^+ and u^-.

These regions are separated by a thin boundary layer (the phase front) which has an elliptic shape. In other words, the elliptic area consisting of one of the phases (say u^-) is surrounded by an infinite area with the u^+ phase. As time passes, the phase front is compressing which means that phase u^+ captures the region filled by phase u^- and as $t \to \infty$, the phase u^- vanishes and phase u^+ fills all the space. Such an effect is well known in the asymptotic theory of interface propagation [9].

The second case describes moving hyperbolic fronts and is slightly more complicated. Such fronts depending on a, b can behave in two different manners. Let us describe this motion by an example. Suppose that $a, b > 0$ and $\alpha > 0$, $\beta < 0$. Assume also ϕ_0 to be a large negative number. In this case the front positions are determined by:

$$\frac{a}{2}x^2 - \frac{b}{2}y^2 = -\phi_0 + (b-a)t.$$

At the initial stage the fronts are hyperbolic, symmetrical about the y - axis. The time evolution of this front is completely determined by the sign of difference $d = b - a$. When d is positive, the fronts are moving away from each other and the distance between them is increasing (see Fig. 3b). More complicated behaviour occurs when d is negative. In this case, the fronts are approaching each other and at a certain moment of time $t^* = d - \phi_0$, they degenerate into two crossing lines. After $t = t^*$, these fronts take a hyperbolic shape again but now symmetrical about the x-axis (Fig. 3a). The motion of these fronts is inverse compared to the initial stage and the fronts are moving away from each other (see Fig. 3).

Case B. Parabolic fronts.

Let us consider another polynomial function

$$h^2 = h_0^2 + (ax)^2. \tag{3.10}$$

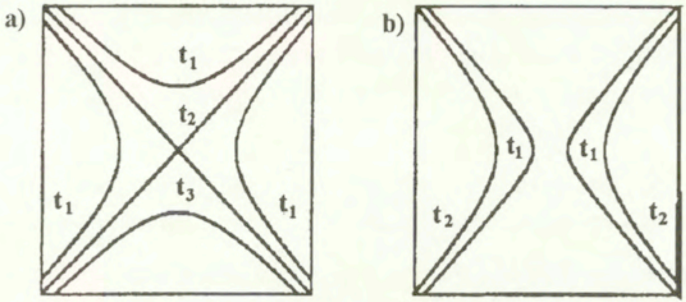

FIG. 3. Hyperbolic fronts behaviour.

Similarly to the previous case, we suppose the velocity c of initial solution (3.2) to be zero.

Substitution of (3.10) into conditions (3.5), (3.6) leads to the following expression for the phase ϕ:

$$\phi = \phi_0 + \alpha x^2 + \beta y, \tag{3.11}$$

where

$$\alpha = \pm\frac{a}{2}, \quad \beta = \pm h_0^2, \quad v = \pm a.$$

From expression (3.11) we see that the front has a parabolic shape moving towards large positive y values, see Fig 4.

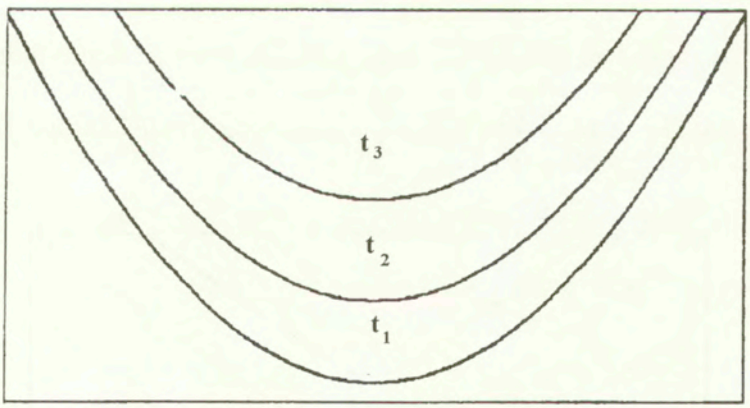

FIG. 4. Parabolic front behaviour.

In order to understand the physical nature of these phenomena, we again need to return to the phase transition theory. Consider a two-phase matter whose behaviour is governed by Eq. (3.1) with h^2 given by (3.10). At the initial time, both phases are present and divided by a parabolic interface (say, u^+ inside the interface and u^- outside it). In this interpretation, the front motion means that the phase u^+ located inside the parabolic area is driven out by the phase u_-.

Case C: Localized h and merging solutions

In the previous example function h^2 was strongly nonlocal. In the following example we consider a localized function h^2. This function can be defined as follows:

$$h^2 = A^2 + \left(\frac{a}{c}\right)^2 \operatorname{tgh}^2(ax), \tag{3.12}$$

where A and a are some constants and $c \neq 0$ is the initial velocity.

Let us notice that such an Eq. (3.1) can describe an almost homogenous medium, with linear inclusion (inhomogeneity) given by $\left(\frac{a}{c}\right)^2 \operatorname{tgh}^2(ax)$ where a is the width of inhomogeneity, $\left(\frac{a}{c}\right)^2$ is the amplitude.

Substitution of (3.12) into Eqs. (3.5) and (3.6) sets the following expression to the phase

$$\phi = \phi_0 + \alpha \ln(\cosh(ax)) + \beta y, \tag{3.13}$$

where

$$\alpha = -\frac{1}{c}, \qquad \beta = \pm A, \qquad v = \frac{(cA)^2 + a^2}{c}. \tag{3.14}$$

From Eq. (3.13) it is clear that the front is close to merging lines $y = \pm\frac{a}{cA}x + st$, as x, y tend to infinity (see Fig. 5). The angle γ of merging is equal to $\operatorname{arctg}(a/c)$. If the defect vanishes ($a = 0$), then the front becomes linear ($\gamma = \pi$) and moves in y-direction. If $a \to \infty$ (the defect of large amplitude), the angle γ tends to 0 and the front moves in y- direction along the defect and resembles a very narrow cone, almost a needle.

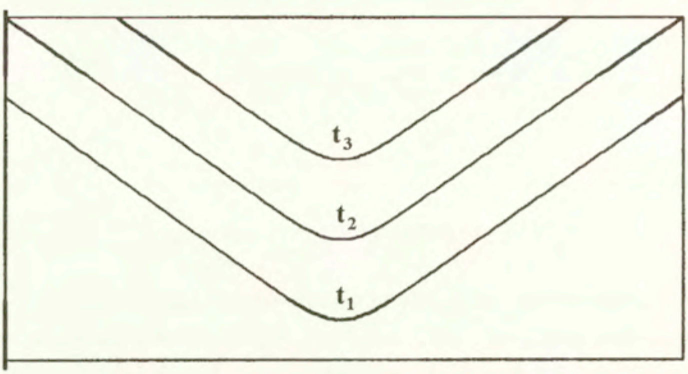

FIG. 5. The merging front behaviour.

The front moves as a whole with the speed v which also depends on the amplitude of the defect. The localization of the front grows as a increases.

One can give the following simple interpretation of these results. To move along the defect, the merging kink should have the appropriate velocity and the angle of merging. For higher and sharper defects, narrow and quick kinks move along the defect.

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Irreducible representations for constitutive equations of anisotropic solids I: crystal and quasicrystal classes D_{2mh}, D_{2m} and C_{2mv}

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A SIMPLE, UNIFIED PROCEDURE is applied to derive irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric second order tensor-valued anisotropic constitutive equations involving any finite number of vector variables and second order tensor variables. In the paper consisting of three parts, we consider all kinds of material symmetry groups as subgroups of the cylindrical group $D_{\infty h}$. This paper, together with a previous work, covers all kinds of material symmetric groups of solids, except for the five cubic crystal classes and the two icosahedral quasicrystal classes. In this part, our concern is with all crystal classes and quasicrystal classes D_{2mh}, D_{2m} and C_{2mv} for all integers $m \geqslant 2$.

1. Introduction

ANISOTROPIC SOLIDS, such as crystals, quasicrystals, composite materials and textured materials etc., manifest their macroscopic mechanical and physical behaviours in complicated and varied manners, e.g., elasticity, elastoplasticity, viscoelasticity, viscoplasticity, creep, damage, yielding, etc., as well as heat conductivity, electric and magnetic permeativity, piezoelectricity, electro- and magnetostrictions, photoelasticity, electromagnetic elasticity, etc. In continuum physics, such complicated and varied material behaviours are mathematically modelled by various forms of scalar-, vector- , skewsymmetric and symmetric second order tensor-valued functions of scalar and vector and second order tensor variables, which are commonly known as material constitutive equations, such as yield functions, elastic stored-energy functions and Helmholz free energy functions; Ohm's law of electric conduction, Fourier's law of heat conduction; electric field-stress relation, stress-electric field relation; stress-deformation relation, stress-strain rate relation, stress rate-strain rate relation, and evolution equations of internal state variables, etc. (see, e.g., NYE [18] for classical linear cases and TRUESDELL and NOLL [38] and ERINGEN and MAUGIN [12] for general cases). The principle of material objectivity and material symmetry require that constitutive equations of a solid obey a combined invariance restriction under the material symmetry group of this solid. As a rational basis of consistent mathematical modelling of complicated and varied material behaviours, it is desirable to obtain general reduced forms or representations of material constitutive equations under the just-stated invariance restriction. In the past decades, this aspect was extensively studied. Earlier, attention was concentrated on *polynomial representations* mainly for scalar-valued functions and for vector-valued and tensor-valued functions in some cases (see, e.g., the monographs by TRUESDELL and Noll [38], SPENCER [33], KIRAL and ERINGEN [14], BETTEN [4] and SMITH [26] for many related results). *Nonpolynomial representations* in a general sense were considered later for isotropic functions by WANG [40], SMITH [24], BOEHLER [6] and PENNISI and TROVATO [19] *et al.* and for anisotropic functions by LOKHIN and SEDOV [17], BOEHLER and RACLIN [10], BOEHLER [7 – 9] and LIU [16], *et al.*, and developed recently by RYCHLEWSKI [22], ZHANG and RYCHLEWSKI [54], ZHENG and SPENCER [59], as well as one of these authors (see XIAO [43 – 44, 47, 52]). Some results on polynomial representations can be found in the foregoing monographs and in ADKINS [1 – 2], SMITH and RIVLIN [28 – 29], PIPKIN and RIVLIN [20], SPENCER and RIVLIN [34 – 37], SPENCER [31 – 32], SMITH, SMITH and RIVLIN [30], SMITH and KIRAL [27], KIRAL and SMITH [15], SMITH [25], *et al.* Some recent results on nonpolynomial representations are given in ZHENG [55 – 56], ZHENG and BOEHLER [58], BISCHOFF-BEIERMANN and BRUHNS [5], JEMIOŁO and TELEGA [13], XIAO [42, 45 – 46, 48, 49 – 51], BRUHNS, XIAO and MEYERS [11], XIAO, BRUHNS and MEYERS [53]. Here it is not our intent to reproduce the huge body of literature. For details, refer to the monographs mentioned before and the recent reviews by BETTEN [3], RYCHLEWSKI and ZHANG [23] and ZHENG [57], as well as the relevant references therein.

Representations for material constitutive equations should be made as compact as possible. As compared with polynomial representations, nonpolynomial representations are not only more general both in notion and in scope, but may furnish more compact forms of reduced constitutive equations, as noted by WANG [40] for isotropic cases and by BOEHLER [7 – 10] for anisotropic cases. Although now many results in many cases are available, general aspects of tensor function representations, especially nonpolynomial representations, are still under investigation, which are concerned with any finite number of vector variables and tensor variables and all kinds of material symmetry groups including the 32 crystal classes and all denumerably infinitely many quasicrystal classes. In fact, except the well-known results for isotropic functions, until recently general results on irreducible nonpolynomial representations have been available only for such simple material symmetry groups as transverse isotropy groups and triclinic, monoclinic and rhombic (orthotropic) groups etc. (see ZHENG [55] and ZHENG and BOEHLER [58], JEMIOŁO and TELEGA [13]). General results for all kinds of material symmetry groups as subgroups of the transverse isotropy group $C_{\infty h}$, as well as some other particular results have been derived very recently by one of the authors (see the related references given before).

In a series of works, we aim to provide irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric second order tensor-valued anisotropic constitutive equations of any finite number of vector variables and second order tensor variables relative to all crystal and quasicrystal classes pertaining to the cylindrical group $D_{\infty h}$. The results for all crystal and quasicrystal classes as subgroups of the transverse isotropy group $C_{\infty h}$ are available in XIAO [50 – 51], as mentioned before. In a succeeding work consisting of three parts, we are concerned with all other crystal and quasicrystal classes. In the present part, we consider the crystal and quasicrystal classes D_{2mh}, D_{2m} and C_{2mv} for all integers $m \geqslant 2$, which will be given at suitable places respectively. In the other two parts that will appear, we shall treat the crystal and quasicrystal classes D_{2m+1d}, D_{2m+1}, C_{2m+1v}, D_{2m+1h} and D_{2md} for all integers $m \geqslant 1$, respectively. This series of works cover all kinds of anisotropic solids except cubic crystals and icosahedral quasicrystals. Throughout, we use the Schoenflies symbol to represent crystal and quasicrystal classes. For a detailed account of them, refer to, e.g., SPENCER [33] for crystal classes and VAINSHTEIN [39] for both crystal and quasicrystal classes.

2. Notations and prelimineries

Throughout, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{r}$, etc.; $\mathbf{W}$, $\mathbf{H}$, $\mathbf{\Omega}$, etc.; and $\mathbf{A}$, $\mathbf{B}$,$\mathbf{C}$, etc., are used to designate vectors, skewsymmetric second order tensors and symmetric second order tensors over a 3-dimensinal inner product space, respectively. R, V, Skw and Sym are used to denote the reals and the sets of all vectors, all skewsymmetric second order tensors and all symmetric second order tensors, respectively. Moreover, Orth (Orth^+) is used to represent the full (proper) orthogonal groups consisting of all orthogonal (proper orthogonal) tensors. The scalar product of two second order tensors $\mathbf{F}$ and $\mathbf{G}$ is denoted by $\mathrm{tr}\mathbf{F}\mathbf{G}^{\mathrm{T}} = \mathbf{F} : \mathbf{G} = F_{ij}G_{ij}$. For any two vectors $\mathbf{u}$, $\mathbf{v} \in V$, the notations $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\times\mathbf{v}$ and $\mathbf{u}\otimes\mathbf{v}$ are used to denote the scalar product, the vector product and the tensor product of the vectors $\mathbf{u}$ and $\mathbf{v}$, respectively; the mixed product of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{r}$ is signified by $[\mathbf{u}, \mathbf{v}, \mathbf{r}]$, i.e.

$$[\mathbf{u}, \mathbf{v}, \mathbf{r}] = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{r} = (\mathbf{v}\times\mathbf{r})\cdot\mathbf{u} = (\mathbf{r}\times\mathbf{u})\cdot\mathbf{v}\,; \tag{2.1}$$

and, finally, the notations $\mathbf{u}\wedge\mathbf{v}$ and $\mathbf{u}\vee\mathbf{v}$ are used to stand for the skewsymmetric and symmetric tensors defined by

$$\begin{cases} \mathbf{u}\wedge\mathbf{v} = -\mathbf{v}\wedge\mathbf{u} = \mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u}\,, \\ \mathbf{u}\vee\mathbf{v} = \mathbf{v}\vee\mathbf{u} = \mathbf{u}\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}\,. \end{cases} \tag{2.2}$$

2.1. Functional bases, generating sets and their irreducibility

Let g be a material symmetry group of solid materials, i.e. a subgroup of Orth. Scalar-valued function $f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$, vector-valued function $\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ and skewsymmetric or symmetric second order tensor-valued function $\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ of the a vector variables $\mathbf{u}_i \in V$, the b skewsymmetric tensor variables $\mathbf{W}_\sigma \in \mathrm{Skw}$ and the c symmetric tensor variables $\mathbf{A}_L \in \mathrm{Sym}$ are said to be invariant (for f) or form-invariant (for $\mathbf{h}$ and $\mathbf{F}$) under the group $g \subseteq \mathrm{Orth}$ if they, respectively, fulfil the invariance requirements

$$
\begin{aligned}
f(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L),\\
\mathbf{h}(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= \mathbf{Q}\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L),\\
\mathbf{F}(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= \mathbf{Q}\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)\mathbf{Q}^{\mathrm{T}},
\end{aligned}
$$

for any orthogonal tensor $\mathbf{Q} \in g$ and for any $(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L) \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$. In each scalar-valued function $f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ and each vector-valued or tensor-valued functiuon $\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ or $\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ fulfilling the above invariance requirement will be called an *invariant* of $(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ under the group g and a form-invariant vector-valued or tensor-valued function under the group g, separately. In particular, the commonly-known isotropic functions refer to the case when $g = \mathrm{Orth}$. Every other case results in anisotropic functions.

The foregoing invariance requirements place restrictions on the forms of the tensor functions $f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$, $\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ and $\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$. Finding general reduced forms or representations of tensor functions under such restrictions constitutes the central topic in the theory of representations for tensor functions as applied to material constitutive modelling. One of the main general facts in this field is as follows (see PIPKIN and WINEMAN [21, 41]; see also SPENCER [33]): There is a finite set of invariants under the group $g \subseteq \mathrm{Orth}$, $\{I_1, \cdots, I_r\}$, such that every invariant under g is expressible as a single-valued function of this set of invariants. A set of invariants with the just-mentioned property is known as a *functional basis* for the invariants under the group g. On the other hand, there is a finite set of form-invariant vector-valued or second order tensor-valued functions under the group $g \subseteq Orth$, $\{\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_s\}$, such that every form-invariant vector-valued or second order tensor-valued function $\boldsymbol{\psi}$ is expressible as a linear combination of this set of form-invariant vector-valued or second order tensor-valued functions with scalar coefficients that are invariants under g. A set of form-invariant vector-valued or tensor-valued functions with the just-mentioned property is known as a *generating set* for the form-invariant vector-valued or tensor-valued functions under the group g, each element of which is accordingly called a vector generator or a tensor generator.

Thus, finding general reduced forms or representations of the invariant f and the form-invariant vector-valued and tensor-valued functions $\mathbf{h}$ and $\mathbf{F}$ under a given material symmetry group $g \subseteq$ Orth is equivalent to determining a functional basis (for f) and a *generating set* (for $\mathbf{h}$ and $\mathbf{F}$ separately) under the group g. Moreover, both functional bases and generating sets to be employed are further required to be *irreducible* in order to arrive at compact representations. A functional basis (resp. a generating set) under the group $g \subseteq$ Orth is said to be irreducible if none of its proper subsets is again a functional basis (resp. a generating set) under the group g.

Let X be a set of vectors and second order tensors and let M be one of the spaces V, Skw and Sym. We use the notation $\Gamma(X)$ to denote the intersection of the symmetry groups of all vectors and second order tensors in the set X, called the symmetry group of X. Moreover, we use the notation $M(g)$, where $g \subseteq$ Orth is an orthogonal subgroup, to designate the set of all vectors or tensors in M, each of which is invariant under the action of the group g. The former is a subgroup of Orth, while the latter is a g-invariant subspace of M. A criterion for generating sets is as follows (see XIAO [44]).

CRITERION 1. The form-invariant vector-valued or tensor-valued functions under the group $g \subseteq$ Orth, $\mathbf{G}_1(X), \cdots, \mathbf{G}_s(X)$, where the variables X pertain to a g-invariant domain $D \subseteq V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$, form a generating set for the form-invariant vector-valued or tensor-valued functions under g defined on D if and only if the inequality

$$\mathrm{rank}\{\mathbf{G}_1(X), \cdots, \mathbf{G}_r(X)\} \geqslant \mathrm{dim} M(g \cap \Gamma(X)) \tag{2.3}$$

is satisfied for each $X \in D$, where $M = V$, Skw, Sym, respectively, when the vector-valued, skewsymmetric tensor-valued and symmetric tensor-valued functions are involved, respectively.

A useful property for generating sets is: Let $G(X) = \{\mathbf{G}_1(X), \cdots, \mathbf{G}_r(X)\}$ be any given generating set for form-invariant vector-valued or tensor-valued functions under the group $g \subseteq$ Orth, where the variables X pertain to a g-invariant domain $D \subseteq V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$. Then the set $G(X)$ generates the *admissible range subspace* $M(\Gamma(X) \cap g)$ at each point $X \in D$, i.e.

$$\mathrm{span} G(X) = M(\Gamma(X) \cap g) \text{ for each } X \in D\,, \tag{2.4}$$

where M is the range of the tensor function in question. The above formula can be derived from THEOREMS 2.2 – 2.3 in XIAO [44].

Moreover, the following criterion for functional bases is well-known [21, 41, 33].

Criterion 2. A set $I(X)$ of invariants of the variables $X \in D$ under the group $g \subseteq$ Orth is a functional basis for the invariants of the variables $X \in D$ under the group $g \subseteq$ Orth iff the variables $X \in D$ is determined to within an orthogonal tensor pertaining to g by this set, i.e. the condition $I(X') = I(X)$ for $X, X' \in D$ implies that X and X' pertain to the same g-orbit: $X' = \mathbf{Q} \star X$, $\mathbf{Q} \in g$.

In the above, the symbols rank$\mathcal{S}$ and dim$\tilde{\mathcal{S}}$, where $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are a set of vectors or tensors and a vector or a tensor subspace, are used to designate the number of linearly independent elements in the set $\mathcal{S}$ and the dimension of the subspace $\tilde{\mathcal{S}}$, respectively.

We shall apply the aforementioned criterion to verify that a given set of vector-valued or tensor-valued functions defined on a given domain is a generating set required and, moreover, to check the irreducibility of this generating set. Towards the latter goal, it suffices to show that if each vector or tensor generator $\mathbf{G}_0$ is removed from a generating set $\mathcal{S}$ in question, then the proper subset $\mathcal{S}\backslash\{\mathbf{G}_0\}$ fails to fulfil the presented criterion at a point X_0. We shall call a generator $\mathbf{G}_0 \in \mathcal{S}$ to be irreducible if it has the property just indicated. Evidently, a generating set $\mathcal{S}$ is irreducible iff every generator $\mathbf{G}_0 \in \mathcal{S}$ is irreducible.

2.2. Symmetry groups of vectors and second order tensors

To apply the criterion given before, we need to evaluate symmetry groups of various kinds of sets of vectors and/or second order tensors and the values of the dimension dim$M(g)$ for $M = V$, Skw, Sym and for all subgroups $g \subseteq D_{\infty h}$. In this subsection, we shall provide some basic facts and results for future use.

Henceforth, the notation $\mathbf{R}_{\mathbf{u}}^{\theta}$ will be used to represent the right-handed rotation through the angle θ about an axis in the direction of the vector $\mathbf{u} \neq \mathbf{0}$.

The symmetry group of vector $\mathbf{0} \neq \mathbf{u} \in V$ and tensors $\mathbf{O} \neq \mathbf{W} \in$ Skw and $\mathbf{A} \in$ Sym are as follows:

$$\Gamma(\mathbf{u}) = \{\mathbf{R}_{\mathbf{u}}^{\theta}, -\mathbf{R}_{\mathbf{a}}^{\pi} \mid \theta \in R;\ \mathbf{a} \neq \mathbf{0},\ \mathbf{a} \cdot \mathbf{u} = 0\} \equiv C_{\infty v}(\mathbf{u})\,. \tag{2.5}$$

$$\Gamma(\mathbf{W}) = \{\pm\mathbf{R}_{\mathbf{w}}^{\theta} \mid \theta \in R\} \equiv C_{\infty h}(\mathbf{w}), \quad \mathbf{w} = \mathbf{E} : \mathbf{W}\,. \tag{2.6}$$

$$\Gamma(\mathbf{A}) = \begin{cases} \text{Orth} \ \text{ if } \mathbf{A} = x\mathbf{I}\,, \\ D_{\infty h}(\mathbf{a}) \text{ if } \exists \mathbf{0} \neq \mathbf{a} \in V:\ \mathbf{A} = x\mathbf{I} + y\mathbf{a} \otimes \mathbf{a},\ y \neq 0\,, \\ D_{2h}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \text{ otherwise}\,. \end{cases} \tag{2.7}$$

In the last expression, $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are used to denote three orthonormal eigenvectors of $\mathbf{A}$. From (2.7) it follows:

(2.8) $$C_{2h}(\mathbf{a}) \equiv \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{a}}\} \subset \Gamma(\mathbf{A}) \Longleftrightarrow \mathbf{a}$$

is an eigenvector of the symmetric tensor $\mathbf{A}$.

Throughout, $\mathbf{E}$ is used to denote Levi-Civita tensor, i.e. the third order permutation tensor, and $(\mathbf{E} : \mathbf{W})_i = E_{ijk}W_{jk}$ is the axial vector of $\mathbf{W}$. Besides, $\mathbf{I}$ is used to denote the second order identity tensor, $D_{\infty h}(\mathbf{a})$ the cylindrical group with the preferred axis $\mathbf{a}$, i.e.

(2.9) $$D_{\infty h}(\mathbf{a}) = \Gamma(\mathbf{a}\otimes\mathbf{a}) = \{\pm\mathbf{R}^{\theta}_{\mathbf{a}}, \pm\mathbf{R}^{\pi}_{\mathbf{l}} \mid \theta \in R;\ \mathbf{l} \neq \mathbf{0},\ \mathbf{l}\cdot\mathbf{a} = 0\},$$

and moreover

(2.10) $$D_{2h}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{a}_1}, \pm\mathbf{R}^{\pi}_{\mathbf{a}_2}, \pm\mathbf{R}^{\pi}_{\mathbf{a}_3}\}.$$

Besides the above groups, the following groups will be used:

(2.11) $$C_{\infty}(\mathbf{n}) = C_{\infty h}(\mathbf{n}) \cap \mathrm{Orth}^{+} = \{\mathbf{R}^{\theta}_{\mathbf{n}} \mid \theta \in R\},$$

(2.12) $$D_2(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = D_{2h}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \cap \mathrm{Orth}^{+} = \{\mathbf{I}, \mathbf{R}^{\pi}_{\mathbf{a}_1}, \mathbf{R}^{\pi}_{\mathbf{a}_2}, \mathbf{R}^{\pi}_{\mathbf{a}_3}\},$$

(2.13) $$C_{2v}(\mathbf{a}_3, \mathbf{a}_1, \mathbf{a}_2) = \{\mathbf{I}, -\mathbf{R}^{\pi}_{\mathbf{a}_1}, -\mathbf{R}^{\pi}_{\mathbf{a}_2}, \mathbf{R}^{\pi}_{\mathbf{a}_3}\}, \quad S_2 = \{\pm\mathbf{I}\}, \quad C_1 = \{\mathbf{I}\},$$

(2.14) $$C_{2h}(\mathbf{r}) = \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{r}}\}, \quad C_2(\mathbf{r}) = \{\mathbf{I}, \mathbf{R}^{\pi}_{\mathbf{r}}\}, \quad C_{1h}(\mathbf{r}) = \{\mathbf{I}, -\mathbf{R}^{\pi}_{\mathbf{r}}\}.$$

Henceforth, we shall cite these subgroups with their defining vector(s) dropped, if no confusion arises.

Next, for all subgroups $g \subseteq D_{\infty h}$, we provide the values of the dimension $\dim M(g)$ for $M = V$, Skw, Sym respectively, by the following tables.

Table 1. $M = V$ **and** $g_{\infty v} = \{$ **All subgroups of** $C_{\infty v}$ **except** C_1 **and** $C_{1h}\}$

g	C_1	C_{1h}	$g_{\infty v}$	others
$\dim M(g)$	3	2	1	0

Table 2. $M = Skw$ **and** $g_{\infty h} = \{$**All subgroups of** $C_{\infty h}$ **except** C_1 **and** $S_2\}$

g	C_1	S_2	$g_{\infty h}$	others
$\dim M(g)$	3	3	1	0

Table 3. $M =$ **Sym**

g	C_1	S_2	C_2	C_{1h}	C_{2h}	D_2	C_{2v}	D_{2h}	others
$\dim M(g)$	6	6	4	4	4	3	3	3	2

2.3. $D_{\infty h}$**-invariant decompositions of a vector and a second order tensor**

Let $\mathbf{n}$ and $\mathbf{e}$ be two given orthonormal vectors. For vector $\mathbf{u} \in V$ and skew-symmetric and symmetric tensors $\mathbf{W} \in$ Skw and $\mathbf{A} \in$ Sym, the following decomposition formulas hold:

$$
\begin{aligned}
\mathbf{u} &= (\mathbf{u}\cdot\mathbf{n})\mathbf{n} + \overset{\circ}{\mathbf{u}} \\
\overset{\circ}{\mathbf{u}} &= \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\,;
\end{aligned} \tag{2.15}
$$

$$
\begin{aligned}
\mathbf{W} &= \frac{1}{2}(\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{N} - \mathbf{n}\wedge\mathbf{W}\mathbf{n}\,, \\
\mathbf{N} &= \mathbf{E}\mathbf{n}\,;
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
\mathbf{A} &= \overset{\circ}{\mathbf{A}} + \mathbf{A}_0\,, \\
\overset{\circ}{\mathbf{A}} &= \mathbf{A} - \mathbf{A}_0\,, \\
\mathbf{A}_0 &= \frac{1}{2}(3\mathbf{n}\cdot\mathbf{A}\mathbf{n} - \mathrm{tr}\mathbf{A})\mathbf{n}\otimes\mathbf{n} + \frac{1}{2}(\mathrm{tr}\mathbf{A} - \mathbf{n}\cdot\mathbf{A}\mathbf{n})\mathbf{I}\,.
\end{aligned} \tag{2.17}
$$

Let $\mathbf{D}_1, \cdots, \mathbf{D}_4$ be the four traceless tensors given by

$$
\mathbf{D}_1 = \mathbf{e}\otimes\mathbf{e} - \mathbf{e}'\otimes\mathbf{e}', \quad \mathbf{D}_2 = \mathbf{e}\vee\mathbf{e}'; \quad \mathbf{D}_3 = \mathbf{n}\vee\mathbf{e}, \quad \mathbf{D}_4 = \mathbf{n}\vee\mathbf{e}', \tag{2.18}
$$

where

$$
\mathbf{e}' = \mathbf{n}\times\mathbf{e}\,, \tag{2.19}
$$

i.e., the triplet $(\mathbf{e}, \mathbf{e}', \mathbf{n})$ constitutes a right-handed orthonormal system. Then each symmetric tensor $\mathbf{A} \in \mathrm{Sym}$ has the further decomposition

$$
\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_\mathbf{n} + \mathbf{A}_\mathbf{e}\,, \tag{2.20}
$$

where the three tensors $\mathbf{A}_0$, $\mathbf{A}_\mathbf{n}$ and $\mathbf{A}_\mathbf{e}$ are mutually orthogonal and take forms

$$
\mathbf{A}_\mathbf{n} = \mathbf{n}\vee\overset{\circ}{\mathbf{A}}\mathbf{n} = |\overset{\circ}{\mathbf{A}}\mathbf{n}|(\mathbf{D}_3\cos\phi(\mathbf{A}) + \mathbf{D}_4\sin\phi(\mathbf{A}))\,, \tag{2.21}
$$

$$
\mathbf{A}_\mathbf{e} = |\mathbf{q}(\mathbf{A})|(\mathbf{D}_1\cos\psi(\mathbf{A}) + \mathbf{D}_2\sin\psi(\mathbf{A}))\,. \tag{2.22}
$$

Here and henceforth, $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$ is used to denote the norm of the vector $\mathbf{u}$, and $\mathbf{q}(\mathbf{A})$ to represent a vector depending on $\mathbf{A}$, defined by

$$
\mathbf{q}(\mathbf{A}) = \frac{1}{2}((\mathrm{tr}\mathbf{A}\mathbf{D}_1)\mathbf{e} + (\mathrm{tr}\mathbf{A}\mathbf{D}_2)\mathbf{e}')\,. \tag{2.23}
$$

Moreover, $\phi(\mathbf{A})$ and $\psi(\mathbf{A})$ are used to designate the *two angles formed by the two vectors* $\overset{\circ}{\mathbf{A}}\mathbf{n}$ *and* $\mathbf{e}$ *and by the two vectors* $\mathbf{q}(\mathbf{A})$ *and* $\mathbf{e}$ respectively, i.e.,

$$
\phi(\mathbf{A}) = \langle\overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{e}\rangle, \quad \psi(\mathbf{A}) = \langle\mathbf{q}(\mathbf{A}), \mathbf{e}\rangle\,. \tag{2.24}
$$

The decomposition formula (2.20) with $(2.17)_2$ and (2.21) - (2.22), which is invariant under the cylindrical group $D_{\infty h}(\mathbf{n})$, is a slight variant of that introduced by BISCHOFF-BEIERMANN and BRUHNS [5] (see also BRUHNS, XIAO and MEYERS [11] and XIAO [48 - 49]).

Finally, for each integer r and each vector $\mathbf{z}$ in the $\mathbf{n}$-plane we define two scalar functions $\alpha_r(\mathbf{z})$ and $\beta_r(\mathbf{z})$, a vector-valued function $\boldsymbol{\eta}_r(\mathbf{z})$ and a symmetric tensor-valued function $\boldsymbol{\Phi}_r(\mathbf{z})$ by

$$\alpha_r(\mathbf{z}) = |\mathbf{z}|^r \cos r\langle\mathbf{z},\mathbf{e}\rangle, \qquad \beta_r(\mathbf{z}) = |\mathbf{z}|^r \sin r\langle\mathbf{z},\mathbf{e}\rangle\,, \tag{2.25}$$

$$\boldsymbol{\eta}_r(\mathbf{z}) = \alpha_r(\mathbf{z})\mathbf{e} - \beta_r(\mathbf{z})\mathbf{e}'\,, \tag{2.26}$$

$$\boldsymbol{\Phi}_r(\mathbf{z}) = \alpha_r(\mathbf{z})\mathbf{D}_1 - \beta_r(\mathbf{z})\mathbf{D}_2\,. \tag{2.27}$$

The following formulas are useful:

$$\begin{aligned} \boldsymbol{\eta}_r(\mathbf{z}) &= \boldsymbol{\Phi}_{r-1}(\mathbf{z})\mathbf{z} = |\mathbf{z}|^{-2}(\alpha_{r+1}(\mathbf{z})\mathbf{z} - \beta_{r+1}(\mathbf{z})(\mathbf{n}\times\mathbf{z}))\,,\\ \alpha_r(\mathbf{z}) &= \boldsymbol{\eta}_{r-1}(\mathbf{z})\cdot\mathbf{z}\,,\\ \beta_r(\mathbf{z}) &= -[\mathbf{n},\mathbf{z},\boldsymbol{\eta}_{r-1}(\mathbf{z})]\,. \end{aligned} \tag{2.28}$$

For $\mathbf{A}\in\mathrm{Sym}$, $J(\mathbf{A})$ is used to denote a $C_{\infty h}(\mathbf{n})$-invariant given by

$$J(\mathbf{A}) = [\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}{}^2\mathbf{n}] = |\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^2|\mathbf{q}(\mathbf{A})|\sin(2\phi(\mathbf{A}) - \psi(\mathbf{A}))\,. \tag{2.29}$$

A useful fact concerning $J(\mathbf{A})$ is as follows:

$$J(\mathbf{A}) = 0 \iff \mathbf{n}\times\overset{\circ}{\mathbf{A}}\,\mathbf{n} \text{ is an eigenvector of } \mathbf{A}\,. \tag{2.30}$$

It should be noted that the two orthonormal vectors $\mathbf{n}$ and $\mathbf{e}$ will always be arranged to be in the directions of the principal symmetry axis and a two-fold symmetry axis of the material symmetry group $g\subseteq D_{\infty h}(\mathbf{n})$ concerned.

3. A unified procedure for constructing both functional bases and generating sets

Usually, it is not easy to derive representations for anisotropic functions of vectors and tensors, even for the case when only one vector or one tensor variable is involved, except for some simple anisotropy groups. This situation may be improved by recent results obtained by one of the authors. It has been proved (see XIAO [44, 52]) that irreducible generating sets for arbitrary-order tensor-valued anisotropic and isotropic functions of any finite number of vector variables and second order tensor variables, can be formed by union of generating sets for the same type of anisotropic or isotropic functions of certain subsets of not more than three variables. Therefore, the representation problem for the former may be reduced to that for the latter. Moreover, by developing the isotropic extention method for anisotropic functions by LOKHIN and SEDOV [17], BOEHLER *et al.* [7 – 10], LIU [16] and RYCHLEWSKI [22], *et al.*, it has been shown (see XIAO [43, 47]) that any type of rth-order tensor-valued anisotropic function of a set of vector variables and second order tensor variables for $r = 0, 1, 2$, may be presented as an rth-order tensor-valued isotropic function of an extended set of vector variables and second order tensor variables. Hence, complete representations for the former are obtainable from those for the latter by applying the well-known results for isotropic functions of vectors and second order tensors. Further, a unified procedure for constructing both functional bases and generating sets has been established recently by incorporating these facts and others (see XIAO [50 – 51]). For any given subgroup $g \subset D_{\infty h}$, this unified procedure is outlined as follows.

STEP 1: *Representations involving single variables* $\mathbf{x} = \mathbf{u}, \mathbf{W}, \mathbf{A}$. Determine irreducible representations (functional basis and generating sets) for invariants and form-invariant vector-valued and skewsymmetric and symmetric tensor-valued functions of each variable $\mathbf{x} \in \{\mathbf{u}, \mathbf{W}, \mathbf{A}\}$ under the subgroup g. Then, form the scalar products $\mathbf{r} \cdot \mathbf{h}_r(\mathbf{x})$ and $\mathbf{H} : \boldsymbol{\psi}_s(\mathbf{x}) = \mathrm{tr}\mathbf{H}\boldsymbol{\psi}_s(\mathbf{x})$ and $\mathbf{C} : \mathbf{F}_t(\mathbf{x}) = \mathrm{tr}\mathbf{C}\mathbf{F}_t(\mathbf{x})$ of a vector variable $\mathbf{r} \in V$ and each presented vector generator $\mathbf{h}_r(\mathbf{x})$, of a skewsymmetric tensor variable $\mathbf{H} \in \mathrm{Skw}$ and each presented skewsymmetric tensor generator $\boldsymbol{\psi}_s(\mathbf{x})$, and of a symmetric tensor variable $\mathbf{C} \in \mathrm{Sym}$ and each presented symmetric tensor generator $\mathbf{F}_t(\mathbf{x})$, respectively;

STEP 2: *Representations involving g-irreducible sets of two variables*, $(\mathbf{x}, \mathbf{y}) = (\mathbf{u}, \mathbf{v}), (\mathbf{u}, \mathbf{W}), (\mathbf{u}, \mathbf{A}), (\mathbf{W}, \boldsymbol{\Omega}), (\mathbf{W}, \mathbf{A}), (\mathbf{A}, \mathbf{B})$. The process is the same as Step 1, except for the fact that the single variable $\mathbf{x}$ therein is replaced by the two variables $(\mathbf{x}, \mathbf{y})$ here. By a g-irreducible set of two variables $(\mathbf{x}, \mathbf{y})$ we mean a set $(\mathbf{x}, \mathbf{y})$ with the property

$$\Gamma(\mathbf{x}, \mathbf{y}) \cap g \neq \Gamma(\mathbf{x}') \cap g, \qquad \mathbf{x}' = \mathbf{x},\ \mathbf{y}\,; \tag{3.1}$$

The above condition and a similar condition below will be explained shortly.

STEP 3: *Representations involving g-irreducible sets of three variables*, $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{u}, \mathbf{v}, \mathbf{W}), (\mathbf{u}, \mathbf{v}, \mathbf{A}), (\mathbf{u}, \mathbf{v}, \mathbf{r}), (\mathbf{u}, \mathbf{W}, \boldsymbol{\Omega}), (\mathbf{u}, \mathbf{W}, \mathbf{A})$ *and* $(\mathbf{u}, \mathbf{A}, \mathbf{B})$. The process is the same as STEP 1, except for the fact that the single variable $\mathbf{x}$ therein is replaced by the set $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ of three variables here. By a g-irreducible set of three variables $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ we mean a set $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ with the property

$$\Gamma(\mathbf{x}, \mathbf{y}, \mathbf{z}) \cap g \neq \Gamma(\mathbf{x}', \mathbf{y}') \cap g \ \text{ for any } \ \mathbf{x}', \mathbf{y}' \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}\,. \tag{3.2}$$

STEP 4. Collect all invariants and generators obtained in STEP 1 – STEP 3 and let each variable involved run over the general set of variables, $(\mathbf{u}_1, \cdots, \mathbf{u}_a, \mathbf{W}_1, \cdots, \mathbf{W}_b, \mathbf{A}_1, \cdots, \mathbf{A}_c) \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$. Then finally we obtain the desired general representations for invariants and form-invariant vector-valued and skewsymmetric and symmetric tensor-valued functions of any finite number of vector variables and second order tensor variables under the subgroup g.

The introduction of the g-irreducibility conditions (3.1) and (3.2) is mainly based on the fact that is given below. Consider any given symmetry group g and any given set of two variables of interest, $(\mathbf{x}, \mathbf{y})$. For the case when there is one of $\mathbf{x}$ and $\mathbf{y}$, i.e. $\mathbf{x}' \in \{\mathbf{x}, \mathbf{y}\}$, such that

$$g \cap \mathit{\Gamma}(\mathbf{x}') = g \cap \mathit{\Gamma}(\mathbf{x}, \mathbf{y}),$$

we have (see CRITERION 1 in Sec. 2)

$$\mathrm{rank} G(\mathbf{x}') \geqslant \dim M(g \cap \mathit{\Gamma}(\mathbf{x}')) = \dim M(g \cap \mathit{\Gamma}(\mathbf{x}, \mathbf{y}))$$

for a generating set $G(\mathbf{x}')$ for the g-form-invariant tensor functions of a single variable $\mathbf{x}'$ taking values in a g-invariant domain M. If a generating set $G(\mathbf{x}')$ for a single variable $\mathbf{x}'$, given at STEP 1, is incorporated into a generating set $G(\mathbf{x}, \mathbf{y})$ for two variables $(\mathbf{x}, \mathbf{y})$, i.e. $G(\mathbf{x}') \subset G(\mathbf{x}, \mathbf{y})$, then using the foregoing inequality we infer

$$\mathrm{rank} G(\mathbf{x}, \mathbf{y}) \geqslant \mathrm{rank} G(\mathbf{x}') \geqslant \dim M(g \cap \mathit{\Gamma}(\mathbf{x}, \mathbf{y})).$$

This shows that for the foregoing case concerning the two variables $(\mathbf{x}, \mathbf{y})$, a generating set for the single variable $\mathbf{x}$ or $\mathbf{y}$ is already sufficient in order to fulfil CRITERION 1, and no generators dependent on both $\mathbf{x}$ and $\mathbf{y}$ are needed. Thus, in constructing a generating set for two variables $(\mathbf{x}, \mathbf{y})$, the case indicated before is trivial and can be ignored, and hence it is enough to consider the case specified by (3.1). Similarly, in constructing a generating set for three variables $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, it is enough to consider the case specified by (3.2).

It will be seen that the *g-irreducibility conditions* (3.1) and (3.2), which specify particular forms of sets of two or three variables, may result in a considerable simplification of fulfilling the related steps, as has been shown in XIAO [50 – 51]. In contrast with this, it seems difficult to deal directly with the sets of two or three variables that are set free.

Moreover, when forming the scalar products of each variable $\mathbf{z} \in \{\mathbf{r}, \mathbf{H}, \mathbf{C}\}$ and the presented generators, some reduction can be made. Let X_0 be a g-irreducible set of two variables considered in the second step in the foregoing unified procedure, and let $\mathbf{z}_0 \in X_0$ and $\mathbf{z} \in \{\mathbf{r}, \mathbf{H}, \mathbf{C}\}$. Then $(\mathbf{z}_0, \mathbf{z})$ is also a set of two variables. If this set has been covered in fulfilling step 2, then $\mathbf{z}$ may be treated as being subjected to the condition:

$$(3.3) \qquad \Gamma(\mathbf{z}) \supset \Gamma(X_0) \cap g, \qquad \Gamma(\mathbf{z}_0, \mathbf{z}) \cap g \neq \Gamma(X_0, \mathbf{z}) \cap g\,.$$

The reason is as follows. The unified procedure is mainly based on the fact: The general domain $D = V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$ can be decomposed into union of certain *g-symmetry-reduced* subdomains: $D = D_1 \cup \cdots \cup D_s$ (see Theorem 3.1 in XIAO [44] and the related results in XIAO [52]). Each D_α has the property: for each set $X \in D_\alpha$ of vector and second order tensor variables, there is a subset $Z_0 \subset X$ with not more than three variables, such that $\Gamma(Z_0) \cap g = \Gamma(X) \cap g$. Thus, generating sets for $X \in D_\alpha$ are given by those for Z_0. At the same time, a functional basis for $X \in D_\alpha$ is given by

$$(3.4) \qquad I(X) = I(Z_0) \cup (\mathbf{r} \cdot V(Z_0)) \cup (\mathbf{H} : \mathrm{Skw}(Z_0)) \cup (\mathbf{C} : \mathrm{Sym}(Z_0))\,,$$

where $\mathbf{r}$, $\mathbf{H}$ and $\mathbf{C}$ are three generic variables running over all vectors, all skewsymmetric tensors and all symmetric tensors in the set $X \backslash Z_0$, respectively; $I(Z_0)$, $V(Z_0)$, $\mathrm{Skw}(Z_0)$ and $\mathrm{Sym}(Z_0)$ are, respectively, a functional basis and vector, skewsymmetric tensor and symmetric tensor generating sets for Z_0 under the group g; and $\mathbf{r} \cdot V(Z_0)$, $\mathbf{H} : Skw(Z_0)$ and $\mathbf{C} : Sym(Z_0)$ are three sets of the invariants formed by the inner products between the forgoing generic variables and the generators in the foregoing three generating sets, respectively. As step 4 indicates, the results for all subdomains D_α collectively supply the desired general results for the whole domain D. Now we explain the reduction indicated before. In the just-mentioned process, let $Z_0 = X_0 \subset X$ be a set of two variables. Then $(3.3)_1$ is evidently true for each $\mathbf{z} \in X$. Suppose that $(3.3)_2$ is not true. Then, by using $\Gamma(X_0) \cap g = \Gamma(X) \cap g$ we infer

$$\Gamma(\mathbf{z}_0, \mathbf{z}) \cap g = \Gamma(X) \cap g,$$

where $\mathbf{z}_0 \in X_0$ and $\mathbf{z} \in \{\mathbf{r}, \mathbf{H}, \mathbf{C}\} \subset X$. As a result, by replacing Z_0 with $(\mathbf{z}_0, \mathbf{z})$ in (3.4) we obtain a functional basis for $X \in D_\alpha$. Thus, if the set $(\mathbf{z}_0, \mathbf{z})$ has been covered, the just-mentioned basis makes the basis $I(X)$ given by (3.4) redundant. It is the just-shown fact that results in the reduction condition (3.3).

It should be pointed out that the reduction condition (3.3) is not necessary for the aforementioned unified procedure. However, for some of the sets of two variables, taking this condition into consideration will be helpful to remove some redundant invariants from the scalar products, as will be seen (e.g., see Sec. 4.2(**vi**)).

Applying the aforementioned unified procedure, in this part and the other two parts we shall derive irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of any finite number of vector variables and second order tensor variables under all crystal and quasicrystal classes as subgroups of the cylindrical group $D_{\infty h}$.

It should be pointed out that the generating sets thus obtained are irreducible, while the functional bases thus obtained need not be so. The irreducibility of the latter will be examined elsewhere.

Here, we confine ourselves to material symmetries of solids, which are characterized by finite and continuously infinite subgroups of the 3-dimensional full orthogonal group. Other kinds of material symmetries are possible, such as those of liquid crystals etc., which are characterized by subgroups of the 3-dimensional unimodular group $\mathcal{U}(3)$. The above procedure based upon the notion of isotropic extension may be extended to cover the latter kinds of material symmetry groups. The main basis in this more general aspect has been laid down by RYCHLEWSKI [22], in which the existence and reality of isotropic extension in the most general form has been proved with an arbitrary group acting on an arbitrary set, not necessarilly restricted to the subgroups of the 3-dimensional full orthogonal group.

4. Crystal and quasicrystal classes D_{2mh} for $m \geqslant 2$

The classes D_{2mh} are of the form

$$D_{2mh}(\mathbf{n},\mathbf{e}) = \{\pm\mathbf{R}_{\mathbf{n}}^{2k\pi/2m},\ \pm\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/2m}\mathbf{e},\ k=0,1,\cdots,2m-1\}. \tag{4.1}$$

These classes include the crystal classes D_{4h} and D_{6h} as the particular cases when $m = 2,\ 3$. For the sake of simplicity, henceforth we shall use $\mathbf{l}$ to represent one of the *two-fold axis vectors* $\mathbf{l}_0, \mathbf{l}_1, \cdots, \mathbf{l}_{2m-1}$. The latter constitute an equipartition of a unit circle on the $\mathbf{n}$-plane.

4.1. Single variables

(i) A single vector $\mathbf{u}$

Each anisotropic function of the vector variable $\mathbf{u}$ under D_{2mh} may be extended as an isotropic function of the extended set of three variables, $(\mathbf{u}, \mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}}), \mathbf{n}\otimes\mathbf{n})$ (see THEOREM 2 in XIAO [47]). Applying this fact and the related results for isotropic functions and following the unified procedure in Sec. 3, we construct the following table.

V	$\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n},\ \mathring{\mathbf{u}},\ \boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{u}})\}\ (\equiv V_{2m}(\mathbf{u}))$.
Skw	$\{\beta_{2m}(\mathring{\mathbf{u}})\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{u}})\}\ (\equiv \mathrm{Skw}_{2m}(\mathbf{u}))$.
Sym	$\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}}, \mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{u}})\}$ $(\equiv \mathrm{Sym}_{2m}(\mathbf{u}))$.

$$R \quad (\mathbf{r}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n}), \overset{\circ}{\mathbf{r}}\cdot\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{r}}\cdot\eta_{2m-1}(\overset{\circ}{\mathbf{u}});$$

$$(\mathrm{tr}\mathbf{HN})\beta_{2m}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\mathbf{Hn}, (\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Hn};$$

$$\mathrm{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{Cn}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, \mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{n},$$

$$(\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{n};$$

$$\{(\mathbf{u}\cdot\mathbf{n})^2, |\overset{\circ}{\mathbf{u}}|^2, \ \alpha_{2m}(\overset{\circ}{\mathbf{u}})\}\ (\equiv I_{2m}(\mathbf{u})).$$

Throughout Parts I-III, we replace the scalar product $\mathbf{r}\cdot\boldsymbol{\psi}$ and $\mathrm{tr}\mathbf{CG}$, where $\boldsymbol{\psi}$ and $\mathbf{G}$ are a vector generator and a symmetric tensor generator, with $\overset{\circ}{\mathbf{r}}\cdot\boldsymbol{\psi}$ and $\mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{G}$ respectively, if the invariant $(\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\psi}$ and the two invariants $\mathbf{n}\cdot\mathbf{Gn}$ and $\mathrm{tr}\mathbf{G}$ are redundant. In fact, by using the decomposition formula (2.15) and (2.17) we have

$$\mathbf{r}\cdot\boldsymbol{\psi} = \overset{\circ}{\mathbf{r}}\cdot\boldsymbol{\psi} + (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\psi}, \qquad \mathrm{tr}\mathbf{CG} = \mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{G} + p_1(\mathbf{C})\mathbf{n}\cdot\mathbf{Gn} + p_2(\mathbf{C})\mathrm{tr}\mathbf{G},$$

where $p_{1,2}(\mathbf{C}) = \dfrac{1}{2}(\mathrm{tr}\mathbf{C} \pm \mathbf{n}\cdot\mathbf{Cn})$ are two $D_{\infty h}(\mathbf{n})$-invariants of $\mathbf{C}$.

We need to show that the presented sets $V_{2m}(\mathbf{u})$, $\mathrm{Skw}_{2m}(\mathbf{u})$, $\mathrm{Sym}_{2m}(\mathbf{u})$ and $I_{2m}(\mathbf{u})$ furnish the desired generating sets and functional basis. In fact, by applying the related results for isotropic functions, we derive complete representations for vector-, 2nd order tensor- and scalar-valued isotropic functions of the extended set of variables, $(\mathbf{u}, \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\otimes\mathbf{n})$, as follows:

$$\mathbf{u}, (\mathbf{n}\otimes\mathbf{n})\mathbf{u}, \quad \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})\mathbf{u};$$

$$\mathbf{u}\wedge(\mathbf{n}\otimes\mathbf{n})\mathbf{u}, \ \mathbf{u}\wedge(\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})\mathbf{u}), \quad \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})\mathbf{u}\wedge(\mathbf{n}\otimes\mathbf{n})\mathbf{u};$$

$$\mathbf{I}, \ \mathbf{n}\otimes\mathbf{n}, \ \ \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}), \ \ \mathbf{u}\vee(\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})\mathbf{u}), \ \ \mathbf{u}\otimes\mathbf{u}, \ \mathbf{u}\vee(\mathbf{n}\otimes\mathbf{n})\mathbf{u};$$

$$|\mathbf{u}|^2, \ \mathbf{u}\cdot(\mathbf{n}\otimes\mathbf{n})\mathbf{u}, \quad \mathbf{u}\cdot\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})\mathbf{u}.$$

In deriving the above results, many obviously redundant generators and invariants have been removed by using the facts

$$(\mathbf{n}\otimes\mathbf{n})^2 = \mathbf{n}\otimes\mathbf{n}, \ \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})^2 = |\overset{\circ}{\mathbf{u}}|^{4m-4}(\mathbf{I}-\mathbf{n}\otimes\mathbf{n}), \ \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})\mathbf{n} = \mathbf{0}.$$

By using the formulas (2.15) and (2.28), from the results given above one may easily derive the results listed in the aforementioned table.

It can readily be proved that the functional basis given is irreducible. Moreover, it is evident that the three presented generating sets are irreducible. In fact, each of them is minimal.

(ii) A single skewsymmetric tensor $\mathbf{W}$

Every vector-valued anisotropic function of the variable $\mathbf{W}\in\mathrm{Skw}$ under D_{2mh} vanishes. Each scalar-valued or second order tensor-valued anisotropic function of

$\mathbf{W}$ under D_{2mh} may be extended as a scalar-valued or second order tensor-valued isotropic function of the extended set of variables, $(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n})$ (see Theorem 2 in XIAO [47]). Applying this fact and the related results for isotropic functions, we construct the following table.

$$\text{Skw} \quad \{\mathbf{W}, \mathbf{n}\wedge\boldsymbol{\eta}_{2m-1}(\mathbf{Wn}), \mathbf{\Phi}_{2m-2}(\mathbf{Wn})\mathbf{W}^2 - \mathbf{W}^2\mathbf{\Phi}_{2m-2}(\mathbf{Wn})\} \quad (\equiv \mathrm{Skw}_{2m}(\mathbf{W})).$$

$$\text{Sym} \quad \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{n}\vee\mathbf{Wn}, \mathbf{W}^2, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{\Phi}_{2m-2}(\mathbf{Wn})\mathbf{W} - \mathbf{W}\mathbf{\Phi}_{2m-2}(\mathbf{Wn})\} \quad (\equiv Sym_{2m}(\mathbf{W})).$$

$$R \quad \mathrm{tr}\mathbf{HW}, \boldsymbol{\eta}_{2m-2}(\mathbf{Wn})\cdot\mathbf{Hn}, \mathrm{tr}\mathbf{HW}^2\mathbf{\Phi}_{2m-2}(\mathbf{Wn});$$
$$\mathrm{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{Cn}, (\overset{\circ}{\mathbf{C}}\,\mathbf{n})\cdot(\mathbf{Wn}), \mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{W}^2, \mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{\Phi}_{2m-2}(\mathbf{Wn}),$$
$$\mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{W}\mathbf{\Phi}_{2m-2}(\mathbf{Wn});$$
$$\{(\mathrm{tr}\mathbf{WN})^2,\ |\mathbf{Wn}|^2, \alpha_{2m}(\mathbf{Wn}),\ (\mathrm{tr}\mathbf{WN})\beta_{2m}(\mathbf{Wn})\} \ (\equiv I_{2m}(\mathbf{W})).$$

We show that the sets $I_{2m}(\mathbf{W})$, $\mathrm{Skw}_{2m}(\mathbf{W})$ and $\mathrm{Sym}_{2m}(\mathbf{W})$ provide the desired functional basis and generating sets. First, we consider the skewsymmetric tensor-valued function. Suppose $\mathbf{Wn} = \mathbf{0}$. Then the symmetry group $\Gamma(\mathbf{W})$ is given by $C_{\infty h}(\mathbf{n})$ if $\mathbf{W} \neq \mathbf{O}$ or by Orth if $\mathbf{W} = \mathbf{O}$. From Table 2 given in Sec. 2, it is evident that the presented set $\mathrm{Skw}_{2m}(\mathbf{W})$ obeys the criterion (2.3). Suppose $\mathbf{Wn} \neq \mathbf{0}$. Then we have $\Gamma(\mathbf{\Phi}_{2m-2}(\mathbf{Wn})) \subseteq D_{\infty h}(\mathbf{n})$, and hence

$$\Gamma(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n}) = \Gamma(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn})).$$

From the latter and the criterion (2.3), we infer that a generating set for the two variables $(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}))$ offers a generating set for the three variables $(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n})$. By using the related result for isotropic functions we know that the former is just given by the set $\mathrm{Skw}_{2m}(\mathbf{W})$.

Next, we consider the scalar-valued function. By using the related result for isotropic functions, we know that a functional basis for the three variables $(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n})$ is given by $|\mathbf{Wn}|^2$, $\mathrm{tr}\mathbf{W}^2$, $\mathrm{tr}\mathbf{W}^2\mathbf{\Phi}$, $\mathrm{tr}\mathbf{W}^2\mathbf{\Phi}^2$, $\mathrm{tr}\mathbf{\Phi}^2\mathbf{W}^2\mathbf{\Phi}\mathbf{W}$ with $\mathbf{\Phi} = \mathbf{\Phi}_{2m-2}(\mathbf{Wn})$. In deriving the above results, many obviously redundant invariants have been removed by using

$$(\mathbf{n}\otimes\mathbf{n})^2 = \mathbf{n}\otimes\mathbf{n},\ \mathbf{\Phi}_{2m-2}(\mathbf{Wn})\mathbf{n} = \mathbf{0},\ \mathbf{\Phi}_{2m-2}(\mathbf{Wn})^2 = |\mathbf{Wn}|^{4m-4}(\mathbf{I} - \mathbf{n}\otimes\mathbf{n}).$$

Again by using the above facts and the formula (2.28) and the identity (see (2.16))

$$\mathbf{W}^2 = \frac{1}{2}(\mathrm{tr}\mathbf{WN})\mathbf{n}\vee(\mathbf{n}\times\mathbf{Wn}) - \mathbf{Wn}\otimes\mathbf{Wn} - |\mathbf{Wn}|^2\mathbf{n}\otimes\mathbf{n} - \frac{1}{4}(\mathrm{tr}\mathbf{WN})^2(\mathbf{I} - \mathbf{n}\otimes\mathbf{n}), \tag{4.2}$$

we further deduce that, of the five invariants given before, the first three yield the three invariant $|\mathbf{Wn}|^2$, $(\mathrm{tr}\mathbf{WN})^2$ and $\alpha_{2m}(\mathbf{Wn})$, the fourth is redundant, and the last yields the invariant $(\mathrm{tr}\mathbf{WN})\beta_{2m}(\mathbf{Wn})$.

Finally, we prove that the set $\mathrm{Sym}_{2m}(\mathbf{W})$ is a desired generating set by showing that this set obeys the criterion (2.3). It can easily be proved that the set $\mathrm{Sym}_{2m}(\mathbf{W})$ obeys (2.3) when $\mathbf{Wn} = \mathbf{0}$. In what follows we suppose $\mathbf{Wn} \neq \mathbf{0}$. Then the three vectors

$$\mathbf{n},\ \mathbf{e}_1 = \mathbf{Wn},\ \mathbf{e}_2 = \mathbf{n} \times \mathbf{Wn}\,, \tag{4.3}$$

form an orthogonalized basis of the space V, and hence the six symmetric tensors $\mathbf{n}\otimes\mathbf{n}$, $\mathbf{I}-\mathbf{n}\otimes\mathbf{n}$, $\mathbf{n}\vee\mathbf{e}_1$, $\mathbf{C}_1 = \mathbf{n}\vee\mathbf{e}_2$, $\mathbf{C}_2 = \mathbf{e}_1\vee\mathbf{e}_2$ and $\mathbf{C}_3 = \mathbf{e}_1\otimes\mathbf{e}_1 - \mathbf{e}_2\otimes\mathbf{e}_2$ form an orthogonalized basis of the space Sym. Of the six generators in the set $\mathrm{Sym}_{2m}(\mathbf{W})$, the first three yield the first three tensors in the just-mentioned basis. Now consider the latter three generators in the set $\mathrm{Sym}_{2m}(\mathbf{W})$, denoted by $\mathbf{G}_1 = \mathbf{\Phi}_{2m-2}(\mathbf{W})$, $\mathbf{G}_2 = \mathbf{W}^2$, $\mathbf{G}_2 = \mathbf{W}\mathbf{\Phi}_{2m-2}(\mathbf{W}) - \mathbf{\Phi}_{2m-2}(\mathbf{W})\mathbf{W}$. We have

$$\Delta = \begin{vmatrix} \mathrm{tr}\mathbf{G}_1\mathbf{C}_1 & \mathrm{tr}\mathbf{G}_1\mathbf{C}_2 & \mathrm{tr}\mathbf{G}_1\mathbf{C}_3 \\ \mathrm{tr}\mathbf{G}_2\mathbf{C}_1 & \mathrm{tr}\mathbf{G}_2\mathbf{C}_2 & \mathrm{tr}\mathbf{G}_2\mathbf{C}_3 \\ \mathrm{tr}\mathbf{G}_3\mathbf{C}_1 & \mathrm{tr}\mathbf{G}_3\mathbf{C}_2 & \mathrm{tr}\mathbf{G}_3\mathbf{C}_3 \end{vmatrix} = \begin{vmatrix} 0 & -2\beta_{2m}(\mathbf{Wn}) & 2\alpha_{2m}(\mathbf{Wn}) \\ xy^2 & 0 & -y^4 \\ 2\beta_{2m} & 2x\alpha_{2m}(\mathbf{Wn}) & 2x\beta_{2m}(\mathbf{Wn}) \end{vmatrix},$$

i.e., $\Delta = 4y^4(x^2y^{4m-2} + (\beta_{2m}(\mathbf{Wn}))^2)$ with $x = \mathrm{tr}\mathbf{WN}$ and $y = |\mathbf{Wn}|$. Hence, we infer

$$\mathrm{rankSym}_{2m}(\mathbf{W}) = \begin{cases} 4 \text{ if } \Delta = 0\,, \\ 6 \text{ if } \Delta \neq 0\,, \end{cases}$$

for $\mathbf{Wn} \neq \mathbf{0}$. From the latter and

$$\Delta = 0 \Longrightarrow \mathbf{W} = c\mathbf{El}_k \Longrightarrow \Gamma(\mathbf{W}) \cap D_{2mh}(\mathbf{n},\mathbf{e}) = C_{2h}(\mathbf{l}_k),$$

as well as from Table 3 given in Sec. 2, we conclude that the set $\mathrm{Sym}_{2m}(\mathbf{W})$ obeys the criterion (2.3) when $\mathbf{Wn} \neq \mathbf{0}$.

Both the generating sets $\mathrm{Skw}_{2m}(\mathbf{W})$ and $\mathrm{Sym}_{2m}(\mathbf{W})$ are minimal and hence irreducible.

(iii) A single symmetric tensor $\mathbf{A}$

$$\begin{aligned} \text{Skw}\quad & \{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{N}, \mathbf{n}\wedge \mathring{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\wedge\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}) + |\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-4}J(\mathbf{A})\mathbf{N}, \\ & |\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-2}\mathbf{n}\wedge \mathring{\mathbf{A}}^{\,2}\mathbf{n} + \beta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{N}\}\ (\equiv \mathrm{Skw}_{2m}(\mathbf{A})). \\ \text{Sym}\quad & \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{A}}, \mathbf{\Phi}_{2m-2}(\mathring{\mathbf{A}}\,\mathbf{n}), \mathring{\mathbf{A}}\,\mathbf{n}\otimes\mathring{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}), \\ & \mathbf{\Phi}_{m-1}(\mathbf{q}(\mathbf{A})), \mathbf{n}\vee\mathbf{A}_{\mathbf{e}}\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})\}\ (\equiv \mathrm{Sym}_{2m}(\mathbf{A})). \end{aligned}$$

$$R \quad (\mathrm{tr}\mathbf{HN})\beta_m(\mathbf{q}(\mathbf{A})), (\mathbf{Hn})\cdot \overset{\circ}{\mathbf{A}}\mathbf{n}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\cdot\mathbf{Hn}$$
$$-|\overset{\circ}{\mathbf{A}}\mathbf{n}|^{2m-4}(\mathrm{tr}\mathbf{HN})J(\mathbf{A}),$$
$$|\overset{\circ}{\mathbf{A}}\mathbf{n}|^{2m-2}(\mathbf{Hn})\cdot\overset{\circ}{\mathbf{A}}{}^2\mathbf{n} - (\mathrm{tr}\mathbf{HN})\beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n});$$
$$\{\mathbf{n}\cdot\mathbf{An}, \mathrm{tr}\mathbf{A}, |\overset{\circ}{\mathbf{A}}\mathbf{n}|^2, |\mathbf{q}(\mathbf{A})|^2, \alpha_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \alpha_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\cdot\overset{\circ}{\mathbf{A}}{}^3\mathbf{n},$$
$$\mathrm{tr}\mathbf{A}_\mathbf{n}^2\mathbf{\Phi}_{m-1}(\mathbf{q}(\mathbf{A}))\} \ (\equiv I_{2m}(\mathbf{A}))\,.$$

In the above table and in the tables given in (iv)-(xiv), we do not supply the invariants depending on two or three symmetric tensors that are derived from the scalar products of the symmetric tensor variable $\mathbf{C} \in \mathrm{Sym}$ and the presented symmetric tensor generators. In the final general result that will be given by Theorem 1 we shall directly quote the result established by Theorem 1 in XIAO, BRUHNS and MEYERS [53], which is simpler and more compact than the foregoing invariants from the scalar products.

It is known (see XIAO [48 – 49]) that the sets $\mathrm{Sym}_{2m}(\mathbf{A})$ and $I_{2m}(\mathbf{A})$ given above are an irreducible generating set and an irreducible functional basis for symmetric tensor-valued and scalar-valued anisotropic functions of a symmetric tensor $\mathbf{A}$ under D_{2mh} for each $m \geqslant 2$. Hence, in what follows we only need to prove that the set $\mathrm{Skw}_{2m}(\mathbf{A})$ given is an irreducible generating set for skewsymmetric tensor-valued functions of $\mathbf{A}$ under D_{2mh}, i.e. it obeys the criterion (2.3). In fact, by using (2.28) – (2.29) we derive the equalities

$$\mathbf{G}_1 = -|\mathbf{z}|^{-2}\beta_{2m}(\mathbf{z})\mathbf{n}\wedge(\mathbf{n}\times\mathbf{z}) + |\mathbf{z}|^{2m-4}J(\mathbf{A})\mathbf{N} + |\mathbf{z}|^{-2}\alpha_{2m}(\mathbf{z})\mathbf{n}\wedge\mathbf{z},$$

$$\mathbf{G}_2 = |\mathbf{z}|^{2m-4}J(\mathbf{A})\mathbf{n}\wedge(\mathbf{n}\times\mathbf{z}) + \beta_{2m}\mathbf{N} + |\mathbf{z}|^{2m-4}(\mathbf{n}\cdot\overset{\circ}{\mathbf{A}}{}^3\mathbf{n})\mathbf{n}\wedge\mathbf{z}, \quad \mathbf{z} = \overset{\circ}{\mathbf{A}}\mathbf{n}.$$

Here and below, $\mathbf{G}_1$ and $\mathbf{G}_2$ are used to denote the last two generators in the set $\mathrm{Skw}_{2m}(\mathbf{A})$, respectively. Observing that the coefficient determinant of the two generators with respect to the two tensors $\mathbf{N}$ and $\mathbf{n}\wedge(\mathbf{n}\times\mathbf{z})$ is given by $\Delta = J(\mathbf{A})^2|\mathbf{z}|^{4m-8} + |\mathbf{z}|^{-2}(\beta_{2m}(\mathbf{z}))^2$, we deduce

$$(4.4) \qquad \mathrm{rank}\ \ \mathrm{Skw}_{2m}(\mathbf{A}) \geqslant \begin{cases} \mathrm{rank}\{\mathbf{n}\wedge\mathbf{z}, \mathbf{G}_1, \mathbf{G}_2\} & \\ = \mathrm{rank}\{\mathbf{N}, \mathbf{n}\wedge\mathbf{z}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{z})\} = 3 & \\ & \text{if } \ \Delta \neq 0, \\ \mathrm{rank}\{\mathbf{n}\wedge\mathbf{z}\} = 1 & \text{if } \ \Delta = 0, \ \mathbf{z}\neq\mathbf{0}, \\ \mathrm{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{N}\} = 1 & \text{if } \ \mathbf{z}=\mathbf{0}, \ \ \beta_m(\mathbf{q}(\mathbf{A}))\neq 0, \\ 0 & \text{if } \ |\mathbf{z}| = \beta_m(\mathbf{q}(\mathbf{A})) = \mathbf{0}, \end{cases}$$

$$\text{(4.5)}\qquad \Gamma(\mathbf{A})\cap D_{2mh}(\mathbf{n},\mathbf{e})\supset\begin{cases} C_{2h}(\mathbf{l}_k) & \text{if } \Delta=0,\ \mathbf{z}\neq\mathbf{0},\\ & \text{i.e. } \beta_{2m}(\mathbf{z})=J(\mathbf{A})=0,\ \mathbf{z}\neq\mathbf{0},\\ C_{2h}(\mathbf{n}) & \text{if } \mathbf{z}=\mathbf{0},\ \beta_m(\mathbf{q}(\mathbf{A}))\neq 0,\\ D_{2h}(\mathbf{n},\mathbf{l}_k,\mathbf{n}\times\mathbf{l}_k) & \text{if } |\mathbf{z}|=\beta_m(\mathbf{q}(\mathbf{A}))=0. \end{cases}$$

In deriving $(4.5)_1$, (2.30) is used. Thus, from (4.4) – (4.5) and Table 2 in Sec. 2 we infer that the set $\mathrm{Skw}_{2m}(\mathbf{A})$ obeys (2.3). Further, by considering $\mathbf{A}_1=\mathbf{n}\vee(\mathbf{e}+\mathbf{l}_1)$ and $\mathbf{A}_2=\mathbf{e}\vee\mathbf{l}_1$, we deduce that the last three generators (for $\mathbf{A}_1$) and the first generator (for $\mathbf{A}_2$) in the set $\mathrm{Skw}_{2m}(\mathbf{A})$ are irreducible.

4.2. D_{2mh}-irreducible sets of two variables

(iv) The D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v})$ of two vectors

$$V \qquad V_{2m}(\mathbf{u})\cup V_{2m}(\mathbf{v})\,.$$

$$\mathrm{Skw}\qquad \mathrm{Skw}_{2m}(\mathbf{u})\cup Skw_{2m}(\mathbf{v})\cup\{\mathbf{u}\wedge\mathbf{v},|\mathbf{u}|^{2m-2}\mathbf{u}\wedge\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})+|\mathbf{v}|^{2m-2}\mathbf{v}\wedge\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\}\ (\equiv\mathrm{Skw}_{2m}(\mathbf{u},\mathbf{v})).$$

$$\mathrm{Sym}\qquad \mathrm{Sym}_{2m}(\mathbf{u})\cup\mathrm{Sym}_{2m}(\mathbf{v})\cup\{\mathbf{u}\vee\mathbf{v},|\mathbf{u}|^{2m-2}\mathbf{u}\vee\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})+|\mathbf{v}|^{2m-2}\mathbf{v}\vee\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\}\ \ (\equiv\mathrm{Sym}_{2m}(\mathbf{u},\mathbf{v}))\,.$$

$$R\qquad \mathbf{r}\cdot V_{2m}(\mathbf{z}),\mathbf{H}:\mathrm{Skw}_{2m}(\mathbf{z}),\mathbf{C}:\mathrm{Sym}_{2m}(\mathbf{z}),\mathbf{z}=\mathbf{u},\ \mathbf{v};$$

$$\mathbf{u}\cdot\mathbf{H}\mathbf{v},|\mathbf{u}|^{2m-2}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})\cdot\mathbf{H}\mathbf{u}+|\mathbf{v}|^{2m-2}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{H}\mathbf{v};$$

$$\mathbf{u}\cdot\overset{\circ}{\mathbf{C}}\mathbf{v},|\mathbf{u}|^{2m-2}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})\cdot\overset{\circ}{\mathbf{C}}\mathbf{u}+|\mathbf{v}|^{2m-2}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{C}}\mathbf{v};$$

$$I_{2m}(\mathbf{u})\cup I_{2m}(\mathbf{v})\cup\{(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}),\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{v}},\overset{\circ}{\mathbf{u}}\cdot\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}}),\overset{\circ}{\mathbf{v}}\cdot\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\}.$$

To prove the above results, we first work out the D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v})$, which is specified by (see (3.1)) $\Gamma(\mathbf{u},\mathbf{v})\cap D_{2mh}\neq\Gamma(\mathbf{z})\cap D_{2mh}$, $\mathbf{z}=\mathbf{u},\ \mathbf{v}$. Evidently, $\Gamma(\mathbf{z})\cap D_{2mh}(\mathbf{n},\mathbf{e})\neq C_1$, i.e. $\mathrm{rank}V_{2m}(\mathbf{z})\neq 3$ for $\mathbf{z}=\mathbf{u},\ \mathbf{v}$. Hence, we have $(\mathbf{z}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{z}})=0$. The latter produces the following three disjoint cases for $\mathbf{z}$:

$$\text{(4.6)}\qquad a\mathbf{n},\ a\neq 0;\quad a\mathbf{e}+b\mathbf{e}',\ a^2+b^2\neq 0;\quad a\mathbf{n}+b\mathbf{l},\ ab\neq 0\,.$$

Considering the combinations of the above forms and excluding the cases

$$\mathbf{u}=a\mathbf{n},\ \mathbf{v}=b\mathbf{n};\ \mathbf{u}=c\mathbf{n},\ \mathbf{v}=a\mathbf{n}+b\mathbf{l};\mathbf{u}=c\mathbf{l},\mathbf{v}=a\mathbf{n}+b\mathbf{l};$$

$$\mathbf{u}=a\mathbf{e}+b\mathbf{e}',\ \mathbf{v}=c\mathbf{e}+d\mathbf{e}',\ \beta_{2m}(\mathbf{z})\neq 0,\ \mathbf{z}=\mathbf{u}\text{ or }\mathbf{z}=\mathbf{v};$$

which violate the D_{2mh}-irreducibility condition for $(\mathbf{u}, \mathbf{v})$, we derive the following four cases for the D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v})$:

(c1) $\mathbf{u} = a\mathbf{n}, \quad \mathbf{v} = c\mathbf{e} + d\mathbf{e}', \; a(c^2 + d^2) \neq 0;$

(c2) $\mathbf{u} = a\mathbf{e}, \; \mathbf{v} = c\mathbf{l}, \; \mathbf{l} \neq \mathbf{e}, \; ac \neq 0;$

(c3) $\mathbf{u} = a\mathbf{e} + b\mathbf{e}', \quad \mathbf{v} = c\mathbf{n} + d\mathbf{e}, \; bcd \neq 0;$

(c4) $\mathbf{u} = a\mathbf{n} + b\mathbf{e}, \; \mathbf{v} = c\mathbf{n} + d\mathbf{l}, \; \mathbf{l} \neq \mathbf{e}, \; abcd \neq 0.$

For most cases for g-irreducible sets considered here and later, there will be one or two unit vectors that can be chosen among the two-fold axis vectors of the group g. For the sake of simplicity and without losing generality we can fix one of them as we wish. For instance, in cases (c2)–(c4) above, one of the two-fold axis vectors involved is fixed as $\mathbf{e}$ $(= \mathbf{l}_0)$.

For case (c1), we have

$$\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2mh} = C_{1h}(\mathbf{l}) \text{ if } \beta_{2m}(\mathring{\mathbf{v}}) = 0,$$

$$\operatorname{rank}(V_{2m}(\mathbf{u}) \cup V_{2m}(\mathbf{v})) \geqslant \begin{cases} \operatorname{rank}\{\mathbf{u}, \mathring{\mathbf{v}}, \boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{v}})\} = 3 & \text{if } \beta_{2m}(\mathring{\mathbf{v}}) \neq 0\,, \\ \operatorname{rank}\{\mathbf{u}, \mathring{\mathbf{v}}\} = 2 & \text{if } \beta_{2m}(\mathring{\mathbf{v}}) = 0\,, \end{cases}$$

$$\operatorname{rank}\ \operatorname{Skw}_{2m}(\mathbf{u}, \mathbf{v}) \geqslant \begin{cases} \operatorname{rank}\{\mathbf{N}, \mathbf{u} \wedge \mathbf{v}, \mathbf{u} \wedge \boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{v}})\} = 3 & \\ & \text{if } \beta_{2m}(\mathring{\mathbf{v}}) \neq 0\,, \\ \operatorname{rank}\{\mathbf{u} \wedge \mathbf{v}\} = 1 & \text{if } \beta_{2m}(\mathring{\mathbf{v}}) = 0\,, \end{cases}$$

$$\operatorname{rank}\ \operatorname{Sym}_{2m}(\mathbf{u}, \mathbf{v}) \geqslant \begin{cases} \operatorname{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{v}} \otimes \mathring{\mathbf{v}}, \boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{v}}), & \\ \quad \mathbf{u} \vee \mathbf{v}, \mathbf{u} \vee \boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{v}})\} = 6 & \text{if } \beta_{2m}(\mathring{\mathbf{v}}) \neq 0\,, \\ \operatorname{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{v}} \otimes \mathring{\mathbf{v}}, \mathbf{u} \vee \mathbf{v}\} = 4 & \\ & \text{if } \beta_{2m}(\mathring{\mathbf{v}}) = 0\,. \end{cases}$$

For case (c2) we have $\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2mh} = C_{1h}(\mathbf{n})$ and

$$\operatorname{rank}(V_{2m}(\mathbf{u}) \cup V_{2m}(\mathbf{v})) \geqslant \operatorname{rank}\{\mathring{\mathbf{u}}, \mathring{\mathbf{v}}\} = 2,$$

$$\operatorname{rank}\ \operatorname{Skw}_{2m}(\mathbf{u}, \mathbf{v}) \geqslant \operatorname{rank}\{\mathbf{u} \wedge \mathbf{v}\} = 1,$$

$$\operatorname{rank}\ \operatorname{Sym}_{2m}(\mathbf{u}, \mathbf{v}) \geqslant \operatorname{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{u}} \otimes \mathring{\mathbf{u}}, \mathbf{u} \vee \mathbf{v}\} = 4.$$

For case (c3)–(c4) we have

$$\text{rank}(V_{2m}(\mathbf{u}) \cup V_{2m}(\mathbf{v})) \geqslant \text{rank}\{\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\} = 3,$$

$$\text{rank}\ \ \text{Skw}_{2m}(\mathbf{u},\mathbf{v}) \geqslant \text{rank}\{(\mathbf{v}\cdot\mathbf{n})\mathbf{n}\wedge \overset{\circ}{\mathbf{v}}, \mathbf{u}\wedge\mathbf{v}, \mathbf{u}\wedge \eta_{2m-1}(\overset{\circ}{\mathbf{v}})\} = 3,$$

$$\text{rank}\ \ \text{Sym}_{2m}(\mathbf{u},\mathbf{v}) \geqslant \text{rank}\{\mathbf{I}, \mathbf{n}\cdot\mathbf{n}, \overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{v}}, \mathbf{u}\vee\mathbf{v}, \mathbf{u}\vee\eta_{2m-1}(\overset{\circ}{\mathbf{v}})\} = 6.$$

From the above results and Tables 1 – 3 in Sec. 2, we infer that the three presented sets of generators obey the criterion (2.3), and therefore they provide the desired vector, skewsymmetric tensor and symmetric tensor generating sets. Moreover, by means of case (c1)–(c4) it can easily be proved that the presented set $I_{2m}(\mathbf{u},\mathbf{v})$ is a functional basis for the D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v})$.

Finally, by considering the pair $\mathbf{u}_0 = \mathbf{n}$ and $\mathbf{v}_0 = \mathbf{e} + \mathbf{l}_1$ we deduce that the respective last two generators in the two sets $\text{Skw}_{2m}(\mathbf{u},\mathbf{v})$ and $\text{Sym}_{2m}(\mathbf{u},\mathbf{v})$ are irreducible.

(v) The D_{2mh}-irreducible set $(\mathbf{W},\boldsymbol{\Omega})$ of two skewsymmetric tensors

Skw $\{\mathbf{W}, \boldsymbol{\Omega}, \mathbf{W}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{W}\}\ (\equiv \text{Skw}_{2m}(\mathbf{W},\boldsymbol{\Omega}))$.

Sym $\text{Sym}_{2m}(\mathbf{W}) \cup \text{Sym}_{2m}(\boldsymbol{\Omega}) \cup \{\mathbf{W}\boldsymbol{\Omega} + \boldsymbol{\Omega}\mathbf{W}, |\text{tr}\boldsymbol{\Omega}\mathbf{N}|(\text{tr}\boldsymbol{\Omega}\mathbf{N})\mathbf{W}\mathbf{n}\vee\mathbf{N}\mathbf{W}\mathbf{n} + |\text{tr}\mathbf{W}\mathbf{N}|(\text{tr}\mathbf{W}\mathbf{N})\boldsymbol{\Omega}\mathbf{n}\vee\mathbf{N}\boldsymbol{\Omega}\mathbf{n}\}$ $(\equiv \text{Sym}_{2m}(\mathbf{W},\boldsymbol{\Omega}))$.

R $\text{tr}\mathbf{H}\mathbf{W}, \text{tr}\mathbf{H}\boldsymbol{\Omega}, \text{tr}\mathbf{H}\mathbf{W}\boldsymbol{\Omega}; \mathbf{C} : Sym_{2m}(\mathbf{W}), \mathbf{C} : Sym_{2m}(\boldsymbol{\Omega}), \text{tr}\overset{\circ}{\mathbf{C}}\ \mathbf{W}\boldsymbol{\Omega}, |\text{tr}\boldsymbol{\Omega}\mathbf{N}|(\text{tr}\boldsymbol{\Omega}\mathbf{N})[\mathbf{n}, \mathbf{W}\mathbf{n}, \overset{\circ}{\mathbf{C}}\ \mathbf{W}\mathbf{n}] + |\text{tr}\mathbf{W}\mathbf{N}|(\text{tr}\mathbf{W}\mathbf{N})[\mathbf{n}, \boldsymbol{\Omega}\mathbf{n}, \overset{\circ}{\mathbf{C}}\ \boldsymbol{\Omega}\mathbf{n}];$ $I_{2m}(\mathbf{W}) \cup I_{2m}(\boldsymbol{\Omega}) \cup \{\text{tr}\mathbf{W}\boldsymbol{\Omega}\}\ (\equiv I_{2m}(\mathbf{W},\boldsymbol{\Omega}))$.

To prove the above results, we first work out the D_{2mh}-irreducible set $(\mathbf{W},\boldsymbol{\Omega})$, which is specified by (see (3.1)) $\Gamma(\mathbf{W},\boldsymbol{\Omega})\cap D_{2mh} \neq \Gamma(\mathbf{z})\cap D_{2mh}$, $\mathbf{z} = \mathbf{W},\ \boldsymbol{\Omega}$. It is evident that $\Gamma(\mathbf{z})\cap D_{2mh} \neq S_2$ for $\mathbf{z} = \mathbf{W},\ \boldsymbol{\Omega}$. The latter implies that either $\mathbf{R}^{\pi}_{\mathbf{n}}$ or $\mathbf{R}^{\pi}_{\mathbf{l}}$ pertains to the symmetry group $\Gamma(\mathbf{z})$. Hence, the skewsymmetric tensor $\mathbf{z} \in \{\mathbf{W},\boldsymbol{\Omega}\}$ takes one of the forms:

$$a\mathbf{E}\mathbf{n},\ a \neq 0,; \quad a\mathbf{E}\mathbf{l},\ a \neq 0\,. \tag{4.7}$$

Considering the combinations of the above forms, we derive the following two cases for the D_{2mh}-irreducible set $(\mathbf{W},\boldsymbol{\Omega})$:

(c1) $\mathbf{W} = a\mathbf{E}\mathbf{n},\ \boldsymbol{\Omega} = b\mathbf{E}\mathbf{e},\ ab \neq 0;$

(c2) $\mathbf{W} = a\mathbf{E}\mathbf{e},\ \boldsymbol{\Omega} = b\mathbf{E}\mathbf{l},\ \mathbf{l} \neq \mathbf{e},\ ab \neq 0.$

For the above two cases, it may easily be understood that the variables $(\mathbf{W}, \mathbf{\Omega})$ can be determined to within an orthogonal tensor $\mathbf{Q} \in D_{2mh}$ by the presented set $I_{2m}(\mathbf{W}, \mathbf{\Omega})$, and hence the latter is a desired functional basis. Moreover, for the sets $(\mathbf{W}, \mathbf{\Omega})$ at issue, we have $\Gamma(\mathbf{W}, \mathbf{\Omega}) \cap D_{2mh} = \Gamma(\mathbf{W}, \mathbf{\Omega}) = S_2$. Hence, a generating set for skewsymmetric tensor-valued isotropic functions of $(\mathbf{W}, \mathbf{\Omega})$ offers a desired generating set. The former is just the presented set $\mathrm{Skw}_{2m}(\mathbf{W}, \mathbf{\Omega})$. This set is obviously irreducible.

We show that the presented set $\mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{\Omega})$ supplies a desired symmetric tensor generating set. In fact, we have

$$\begin{aligned}\text{rank}\ \ \mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{\Omega}) \geqslant \text{rank}\{&\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathbf{\Omega}^2, \mathbf{n} \vee \mathbf{\Omega n}, \mathbf{W\Omega} + \mathbf{\Omega W},\\ &\mathbf{\Omega n} \vee \mathbf{N\Omega n}\} = 6,\end{aligned}$$

$$\begin{aligned}\text{rank}\ \ \mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{\Omega}) \geqslant \text{rank}\{&\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathbf{W}^2, \mathbf{n} \vee \mathbf{Wn}, \mathbf{n} \vee \mathbf{\Omega n},\\ &\mathbf{W\Omega} + \mathbf{\Omega W}\} = 6,\end{aligned}$$

for cases (c1) and (c2) separately. Since dim Sym $= 6$, these indicate that the set $\mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{\Omega})$ obeys the criterion (2.3). Further, by considering case (c1), we deduce that the last two generators in the set $\mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{\Omega})$ are irreducible.

(vi) The D_{2mh}-irreducible set $(\mathbf{W}, \mathbf{A})$ of a skewsymmetric tensor and a symmetric tensor

$$\text{Skw} \quad \{\mathbf{W}, \mathring{\mathbf{A}}\,\mathbf{W} + \mathbf{W}\,\mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{W}^2 - \mathbf{W}^2\,\mathring{\mathbf{A}}\}\ (\equiv \mathrm{Skw}_{2m}(\mathbf{W}, \mathbf{A})).$$

$$\text{Sym} \quad \mathrm{Sym}_{2m}(\mathbf{W}) \cup \mathrm{Sym}_{2m}(\mathbf{A}) \cup \{\mathring{\mathbf{A}}\,\mathbf{W} - \mathbf{W}\,\mathring{\mathbf{A}}, (\mathrm{tr}\mathbf{WN})\,\mathring{\mathbf{A}}\,\mathbf{n} \vee \mathbf{N}\,\mathring{\mathbf{A}}\,\mathbf{n}\}\ (\equiv \mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{A})).$$

$$\begin{aligned}R \quad &\mathrm{tr}\mathbf{HW}; \mathbf{C} : \mathrm{Sym}_{2m}(\mathbf{W}), \mathrm{tr}\mathring{\mathbf{C}}\mathring{\mathbf{A}}\,\mathbf{W}, (\mathrm{tr}\mathbf{WN})[\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{C}}\mathring{\mathbf{A}}\,\mathbf{n}];\\ &I_{2m}(\mathbf{W}) \cup I_{2m}(\mathbf{A}) \cup \{(\mathbf{Wn}) \cdot (\mathring{\mathbf{A}}\,\mathbf{n}), \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W}^2,\\ &|\,\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-2}(\mathbf{Wn})\cdot\,\mathring{\mathbf{A}}\,^2\mathbf{n} - (\mathrm{tr}\mathbf{WN})\beta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\}\ (\equiv I_{2m}(\mathbf{W}, \mathbf{A}))\ .\end{aligned}$$

In the above table, the skewsymmetric tensor variable $\mathbf{H}$ is regarded as being subjected to the condition: $\mathbf{H} = c\mathbf{W}$. In fact, from cases (c1)–(c3) derived later and the condition (see (3.3))

$$\Gamma(\mathbf{W}, \mathbf{H}) \cap D_{2mh} \neq \Gamma(\mathbf{W}, \mathbf{A}, \mathbf{H}) \cap D_{2mh}\ (= S_2),$$

we derive $\mathbf{H} = c\mathbf{W}$. The other case for $\mathbf{H}$ has been covered by (v).

The proof for the presented results is as follows. First, we work out the D_{2mh}-irreducible set $(\mathbf{W}, \mathbf{A})$, which is specified by (see (3.1)) $\Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2mh} \neq \Gamma(\mathbf{z}) \cap$

D_{2mh}, $\mathbf{z} = \mathbf{W}$, $\mathbf{A}$. It is evident that $\Gamma(\mathbf{z}) \cap D_{2mh} \neq S_2$, $\mathbf{z} = \mathbf{W}$, $\mathbf{A}$. The latter implies that either $\mathbf{R}_{\mathbf{n}}^{\pi}$ or $\mathbf{R}_{\mathbf{l}}^{\pi}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ for each $\mathbf{z} \in \{\mathbf{W}, \mathbf{A}\}$. Hence, the skewsymmetric tensor $\mathbf{W}$ takes one of the forms given by (4.7), and $\overset{\circ}{\mathbf{A}}$ takes one of the forms:

$$a\mathbf{D}_1 + b\mathbf{D}_2,\ a^2 + b^2 \neq 0; \quad a(\mathbf{l} \otimes \mathbf{l} - \mathbf{l}' \otimes \mathbf{l}') + b\mathbf{n} \vee \mathbf{l}',\ b \neq 0\,. \tag{4.8}$$

Here and henceforth, for each vector $\mathbf{u}$, the notation $\mathbf{u}'$ is used to represent a vector given by

$$\mathbf{u}' = \mathbf{n} \times \mathbf{u}.$$

Considering the combinations of the forms given by (4.7) and (4.8) and excluding the cases

$$\mathbf{W} = c\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \beta_m(\mathbf{q}(\mathbf{A})) \neq 0;$$

$$\mathbf{W} = c\mathbf{E}\mathbf{l},\ \overset{\circ}{\mathbf{A}} = a(\mathbf{l} \otimes \mathbf{l} - \mathbf{l}' \otimes \mathbf{l}');$$

$$\mathbf{W} = c\mathbf{E}\mathbf{l},\ \overset{\circ}{\mathbf{A}} = a(\mathbf{l} \otimes \mathbf{l} - \mathbf{l}' \otimes \mathbf{l}') + b\mathbf{n} \vee \mathbf{l}';$$

which violate the D_{2mh}-irreducibility condition for $(\mathbf{W}, \mathbf{A})$, we derive the following four cases for the D_{2mh}-irreducible set $(\mathbf{W}, \mathbf{A})$:

$$\text{(c1)}\ \mathbf{W} = f\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1,\ fa \neq 0;$$

$$\text{(c2)}\ \mathbf{W} = f\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_4,\ fb \neq 0;$$

$$\text{(c3)}\ \mathbf{W} = f\mathbf{E}\mathbf{e},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ fb \neq 0;$$

$$\text{(c4)}\ \mathbf{W} = f\mathbf{E}\mathbf{e},\ \overset{\circ}{\mathbf{A}} = a(\mathbf{l} \otimes \mathbf{l} - \mathbf{l}' \otimes \mathbf{l}') + b\mathbf{n} \vee \mathbf{l}',\ \mathbf{l} \neq \mathbf{e},\ fb \neq 0.$$

For cases (c1)–(c4), we have $\Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2mh} = \Gamma(\mathbf{W}, \overset{\circ}{\mathbf{A}})$. From this fact and the criterion (2.3) it follows that generating sets for tensor-valued anisotropic functions of the D_{2mh}-irreducible set $(\mathbf{W}, \mathbf{A})$ are obtainable from those for tensor-valued isotropic functions of $(\mathbf{W}, \overset{\circ}{\mathbf{A}})$. As a result, by applying the related result for isotropic functions we know that the presented set $\mathrm{Skw}_{2m}(\mathbf{W}, \mathbf{A})$ supplies a desired irreducible skewsymmetric tensor generating set.

Now we show that the presented set $\mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{A})$ obeys the criterion (2.3). Case (c1) can be treated easily. For case (c2) we have

$$\begin{aligned}\text{rank}\ \ \mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{A}) \geqslant \text{rank}\{&\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{A}}, \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),\\ &\overset{\circ}{\mathbf{A}}\,\mathbf{W} - \mathbf{W}\,\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\,\mathbf{n} \vee \mathbf{N}\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}\} = 6,\end{aligned}$$

and for cases (c3)–(c4) we have

$$\text{rank } \ \mathrm{Sym}_{2m}(\mathbf{W},\mathbf{A}) \geqslant \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{W}^2, \mathbf{n}\vee\mathbf{Wn}, \mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{W}-\mathbf{W}\,\mathring{\mathbf{A}}\} = 6.$$

Thus, we infer that the presented set $\mathrm{Sym}_{2m}(\mathbf{W},\mathbf{A})$ obeys the criterion (2.3), and hence it provides a desired symmetric tensor generating set. Moreover, by considering case (c2) we deduce that the last two generators in the set $\mathrm{Sym}_{2m}(\mathbf{W},\mathbf{A})$ are irreducible.

Next, we are concerned with the presented set $I_{2m}(\mathbf{W},\mathbf{A})$ of invariants. Let

$$I'(\mathbf{W},\mathbf{A}) = \{(\mathrm{tr}\mathbf{WN})^2, |\mathbf{Wn}|^2, \mathrm{tr}\mathbf{A}, \mathbf{n}\cdot\mathbf{An}, |\,\mathring{\mathbf{A}}\,\mathbf{n}|^2, \mathbf{n}\cdot\,\mathring{\mathbf{A}}^{\,3}\mathbf{n}, |\mathbf{q}(\mathbf{A})|^2, (\mathbf{Wn})\cdot(\mathring{\mathbf{A}}\,\mathbf{n}), \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W}^2, (\mathbf{Wn})\cdot\,\mathring{\mathbf{A}}^{\,2}\mathbf{n}\}.$$

We shall prove that the latter offers a functional basis of the D_{2mh}-irreducible set $(\mathbf{W},\mathbf{A})$ under the group $D_{\infty h}(\mathbf{n})$ and this basis is determined by the presented set $I_{2m}(\mathbf{W},\mathbf{A})$. In fact, the just-mentioned basis is obtainable from an isotropic functional basis of $(\mathbf{W},\mathring{\mathbf{A}},\mathbf{n}\otimes\mathbf{n})$ (see BOEHLER [8]), plus the two invariants $\mathrm{tr}\mathbf{A}$ and $\mathbf{n}\cdot\mathbf{An}$. Applying the related result for isotropic functions we know that the just-mentioned isotropic functional basis is given by

$$\mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W}^2, (\mathbf{Wn})\cdot\,\mathring{\mathbf{A}}\,\mathbf{n},\ (\mathbf{Wn})\cdot\,\mathring{\mathbf{A}}^{\,2}\mathbf{n},\ \mathrm{tr}\mathring{\mathbf{A}}^{\,2}\mathbf{W}^2, \mathrm{tr}\mathbf{W}^2\,\mathring{\mathbf{A}}^{\,2}\mathbf{W}\,\mathring{\mathbf{A}}, \mathrm{tr}\mathbf{W}^2\,\mathring{\mathbf{A}}\,\mathbf{W}(\mathbf{n}\otimes\mathbf{n}),$$

as well as by the invariants of a single variable $\mathbf{W}$ or $\mathbf{A}$ in $I'(\mathbf{W},\mathbf{A})$. Each of the latter has been covered or can be determined by the bases $I_{2m}(\mathbf{W})$ or $I_{2m}(\mathbf{A})$. The first three invariants above yield the last three invariants in the set $I'(\mathbf{W},\mathbf{A})$. The last three invariants above are redundant. In fact, the just-mentioned fact can be proved easily for cases (c1)–(c3). For case (c4) we have

$$I = (\mathbf{Wn})\cdot\,\mathring{\mathbf{A}}\,\mathbf{n} = fb(\mathbf{l}\cdot\mathbf{e}),\ \mathrm{tr}\mathring{\mathbf{A}}^{\,2}\mathbf{W}^2 = -I^2 - f^2(a^2+b^2), \quad \mathrm{tr}\mathbf{W}^2\,\mathring{\mathbf{A}}\,\mathbf{W}(\mathbf{n}\otimes\mathbf{n}) = -f^2 I,$$

$$\mathrm{tr}\mathbf{W}^2\,\mathring{\mathbf{A}}^{\,2}\mathbf{W}\,\mathring{\mathbf{A}} = b^{-2}(b^2-2a^2)(f^2b^2-I^2)I,$$

with $f^2 = |\mathbf{Wn}|^2$, $a^2 = |\mathbf{q}(\mathbf{A})|^2$, $b^2 = |\,\mathring{\mathbf{A}}\,\mathbf{n}|^2$ and $fb \neq 0$. Then, we deduce that the foregoing fact is also true for case (c4).

Hence, we conclude that the set $I'(\mathbf{W},\mathbf{A})$ given before is a functional basis of the D_{2mh}-irreducible set $(\mathbf{W},\mathbf{A})$ under the cylindrical group $D_{\infty h}(\mathbf{n})$. Moreover, for the D_{2mh}-irreducible set $(\mathbf{W},\mathbf{A})$, the last invariant in the basis $I'(\mathbf{W},\mathbf{A})$ is given by the last invariant in the set $I_{2m}(\mathbf{W},\mathbf{A})$ (note here that $(\mathrm{tr}\mathbf{WN})\beta_{2m}$ $(\overset{\circ}{\mathbf{A}}\,\mathbf{n})=0$).

Then, applying the just-proved fact, for two D_{2mh}-irreducible sets $(\mathbf{W},\mathbf{A})$ and $(\mathbf{W}',\mathbf{A}')$ we deduce

$$I_{2m}(\mathbf{W}',\mathbf{A}')=I_{2m}(\mathbf{W},\mathbf{A})\Longrightarrow I'(\mathbf{W}',\mathbf{A}')=I'(\mathbf{W},\mathbf{A})$$
$$\Longrightarrow \exists\mathbf{Q}\in D_{\infty h}(\mathbf{n}):\ \mathbf{W}'=\mathbf{QWQ}^{\mathrm{T}},\ \mathbf{A}'=\mathbf{QAQ}^{\mathrm{T}}.$$

Moreover, we have

$$I_{2m}(\mathbf{W}',\mathbf{A}')=I_{2m}(\mathbf{W},\mathbf{A})\Longrightarrow I_{2m}(\mathbf{W}')=I_{2m}(\mathbf{W}),\ I_{2m}(\mathbf{A}')=I_{2m}(\mathbf{A}),$$
$$\Longrightarrow \exists\mathbf{R}_1,\mathbf{R}_2\in D_{2mh}:\ \mathbf{W}'=\mathbf{R}_1\mathbf{W}\mathbf{R}_1^{\mathrm{T}},\ \mathbf{A}'=\mathbf{R}_2\mathbf{A}\mathbf{R}_2^{\mathrm{T}}.$$

From these facts we derive

$$\mathbf{R}_1^{\mathrm{T}}\mathbf{Q}\in\Gamma(\mathbf{W})\cap D_{\infty h}(\mathbf{n}),\ \mathbf{R}_2^{\mathrm{T}}\mathbf{Q}\in\Gamma(\mathbf{A})\cap D_{\infty h}(\mathbf{n}).$$

From the latter and the facts: $\Gamma(\mathbf{A})\cap D_{\infty h}(\mathbf{n})=D_{2h}(\mathbf{n},\mathbf{e},\mathbf{e}')$ and

$$\Gamma(\mathbf{A})\cap D_{\infty h}(\mathbf{n})=C_{2h}(\mathbf{e}),\qquad \Gamma(\mathbf{W})\cap D_{\infty h}(\mathbf{n})=C_{2h}(\mathbf{e}),$$
$$\Gamma(\mathbf{W})\cap D_{\infty h}(\mathbf{n})=C_{2h}(\mathbf{l}),$$

for cases (c1)–(c4) respectively, we infer that $\mathbf{Q}\in D_{2mh}$ for cases (c1)–(c4).

Thus, we conclude that $I_{2m}(\mathbf{W},\mathbf{A})$ is a functional basis of the D_{2mh}-irreducible $(\mathbf{W},\mathbf{A})$ under the group D_{2mh}.

REMARK. The above procedure can be used to deal with functional bases for other kinds of g-irreducible sets $(\mathbf{x},\mathbf{y})$ of two variables in future. Accordingly, henceforth for each similar case we need only to show that a presented set of invariants for a g-irreducible set $(\mathbf{x},\mathbf{y})$ determines a functional basis of $(\mathbf{x},\mathbf{y})$ under the transverse isotropy group $C_{\infty v}(\mathbf{n})$ (for $g\subset C_{\infty v}(\mathbf{n})$) or $D_{\infty h}(\mathbf{n})$ (for other g).

(vii) The D_{2mh}-irreducible set $(\mathbf{A},\mathbf{B})$ of two symmetric tensors

$$\mathrm{Skw}\quad \mathrm{Skw}_{2m}(\mathbf{A})\cup\mathrm{Skw}_{2m}(\mathbf{B})\cup\{\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}},$$
$$\overset{\circ}{\mathbf{B}}\,\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\,\mathbf{n},\overset{\circ}{\mathbf{A}}\,\mathbf{n}\wedge\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\,\mathbf{n}\}\ (\equiv\mathrm{Skw}_{2m}(\mathbf{A},\mathbf{B})).$$

Sym $\quad \mathrm{Sym}_{2m}(\mathbf{A}) \cup \mathrm{Sym}_{2m}(\mathbf{B}) \cup \{\mathring{\mathbf{A}}\mathring{\mathbf{B}} + \mathring{\mathbf{B}}\mathring{\mathbf{A}}\}$ $(\equiv \mathrm{Sym}_{2m}(\mathbf{A},\mathbf{B}))$.

$R \quad I_{2m}(\mathbf{A}) \cup I_{2m}(\mathbf{B}) \cup \{\mathrm{tr}\mathbf{A}_\mathbf{n}\mathbf{B}_\mathbf{n}, \mathrm{tr}\mathbf{A}_\mathbf{e}\mathbf{B}_\mathbf{e}, \mathrm{tr}\mathring{\mathbf{A}}^{\,2}\,\mathring{\mathbf{B}}, \mathrm{tr}\mathring{\mathbf{B}}^{\,2}\,\mathring{\mathbf{A}}\}$ $(\equiv I_{2m}(\mathbf{A},\mathbf{B}))$.

In the above table, the scalar product between each presented skewsymmetric tensor generator and the skewsymmetric tensor variable $\mathbf{H}$ has been omitted. In fact, for any $\mathbf{O} \neq \mathbf{H} \in \mathrm{Skw}$ and any $\mathbf{A}$, $\mathbf{B} \in \mathrm{Sym}$ we have $\Gamma(\mathbf{z}_0, \mathbf{H}) = \Gamma(\mathbf{A},\mathbf{B},\mathbf{H})$, $\mathbf{z}_0 \in \{\mathbf{A},\mathbf{B}\}$, which violates the condition (3.3) with $X_0 = (\mathbf{A},\mathbf{B})$ and $\mathbf{z} = \mathbf{H}$ and $g = D_{2mh}$. The case when $\mathbf{H} \neq \mathbf{O}$ has been covered by (**vi**).

To prove the presented results, we first work out the D_{2mh}-irreducible set $(\mathbf{A},\mathbf{B})$, specified by (see (3.1)) $\Gamma(\mathbf{A},\mathbf{B}) \cap D_{2mh} \neq \Gamma(\mathbf{z}) \cap D_{2mh}$, $\mathbf{z} = \mathbf{A}$, $\mathbf{B}$. It is evident that $\Gamma(\mathbf{z}) \cap D_{2mh} \neq S_2$. The latter implies that each symmetric tensor $\mathbf{z} \in \{\mathbf{A},\mathbf{B}\}$ take one of the forms given by (4.8). Thus, considering the combinations of the forms given by (4.8) and excluding the cases

$$\mathring{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \mathring{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_2,\ \beta_m(\mathbf{q}(\mathbf{z})) \neq 0,\ \mathbf{z} = \mathbf{A} \text{ or } \mathbf{z} = \mathbf{B};$$

$$\mathring{\mathbf{A}} = a(\mathbf{l}\otimes\mathbf{l} - \mathbf{l}'\otimes\mathbf{l}'),\ \mathring{\mathbf{B}} = c(\mathbf{l}\otimes\mathbf{l} - \mathbf{l}'\otimes\mathbf{l}') + d\mathbf{n}\vee\mathbf{l}';$$

which violate the D_{2mh}-irreducibility condition for $(\mathbf{A},\mathbf{B})$, we derive the following three disjoint cases for the D_{2mh}-irreducible set $(\mathbf{A},\mathbf{B})$:

(c1) $\mathring{\mathbf{A}} = a\mathbf{D}_1,\ \mathring{\mathbf{B}} = b(\mathbf{l}\otimes\mathbf{l} - \mathbf{l}'\otimes\mathbf{l}'),\ \mathbf{l} \neq \mathbf{e},\ \mathbf{e}',\ ab \neq 0;$

(c2) $\mathring{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \mathring{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_4,\ bd \neq 0;$

(c3) $\mathring{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_4,\ \mathring{\mathbf{B}} = c(\mathbf{l}\otimes\mathbf{l} - \mathbf{l}'\otimes\mathbf{l}') + d\mathbf{n}\vee\mathbf{l}',\ \mathbf{l} \neq \mathbf{e},\ bd \neq 0.$

Then, for case (c1) we have $\Gamma(\mathbf{A},\mathbf{B}) \cap D_{2mh} = C_{2h}(\mathbf{n})$ and

$$\mathrm{rank}\ \ \mathrm{Skw}_{2m}(\mathbf{A},\mathbf{B}) \geqslant \mathrm{rank}\{\mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}}\} = 1,$$

$$\mathrm{rank}\ \ \mathrm{Sym}_{2m}(\mathbf{A},\mathbf{B}) \geqslant \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{A}}, \mathring{\mathbf{B}}\} = 4.$$

For case (c2) we have

$$\mathrm{rank}\ \ \mathrm{Skw}_{2m}(\mathbf{A},\mathbf{B}) \geqslant \mathrm{rank}\{\mathbf{n}\wedge\mathring{\mathbf{B}}\,\mathbf{n}, \mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}}, \mathbf{B}\mathbf{n}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n}\} = 3,$$

$$\mathrm{rank}\ \ \mathrm{Sym}_{2m}(\mathbf{A},\mathbf{B}) \geqslant \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{A}}, \mathring{\mathbf{B}}, \mathring{\mathbf{A}}\,\mathbf{n}\otimes\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{A}}\mathring{\mathbf{B}} + \mathring{\mathbf{B}}\mathring{\mathbf{A}}\} = 6.$$

For case (c3), we have

$$\mathrm{rank}\ \ \mathrm{Skw}_{2m}(\mathbf{A},\mathbf{B}) \geqslant \mathrm{rank}\{\mathbf{n}\wedge\mathring{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\wedge\mathring{\mathbf{B}}\,\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}\wedge\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}}\} = 3,$$

$$\mathrm{rank}\ \ \mathrm{Sym}_{2m}(\mathbf{A},\mathbf{B}) \geqslant \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}\otimes\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{A}}, \mathring{\mathbf{B}}, \mathring{\mathbf{A}}\mathring{\mathbf{B}} + \mathring{\mathbf{B}}\mathring{\mathbf{A}}\} = 6.$$

From the above results and Tables 2–3 in Sec. 2, we infer that the two sets $\mathrm{Skw}_{2m}(\mathbf{A},\mathbf{B})$ and $\mathrm{Sym}_{2m}(\mathbf{A},\mathbf{B})$ obey the criterion (2.3). Further, by considering the pair $\mathbf{A}_1 = \mathbf{D}_1$ and $\mathbf{B}_1 = \mathbf{n}\vee\mathbf{l}_1$ we infer that the generators $\mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}}$, $\mathring{\mathbf{B}}\ \mathbf{n}\wedge\ \mathring{\mathbf{A}}\mathring{\mathbf{B}}\ \mathbf{n}$ and $\mathring{\mathbf{A}}\mathring{\mathbf{B}} + \mathring{\mathbf{B}}\mathring{\mathbf{A}}$ are irreducible. Moreover, by considering the pair $\mathbf{A}_2 = \mathbf{n}\wedge\mathbf{l}_1$ and $\mathbf{B}_2 = \mathbf{D}_1$ we deduce that the generator $\mathring{\mathbf{A}}\ \mathbf{n}\wedge\ \mathring{\mathbf{B}}\mathring{\mathbf{A}}\ \mathbf{n}$ is also irreducible.

Next, we show that the presented set $I_{2m}(\mathbf{A},\mathbf{B})$ of invariants determines a functional basis of $(\mathbf{A},\mathbf{B})$ under the cylindrical group $D_{\infty h}(\mathbf{n})$. Indeed, the latter is obtainable from the four invariants $\mathrm{tr}\mathbf{C}$, $\mathbf{n}\cdot\mathbf{Cn}$, with $\mathbf{C}=\mathbf{A},\ \mathbf{B}$, as well as an isotropic functional basis of $(\mathring{\mathbf{A}},\mathring{\mathbf{B}},\mathbf{n}\otimes\mathbf{n})$ (see, e.g., BOEHLER [8]). By applying the related result for isotropic functions we know that the latter basis is given by

$$\mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}},\ \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,(\mathbf{n}\otimes\mathbf{n}),\ \mathrm{tr}\mathring{\mathbf{A}}^{\,2}\,\mathring{\mathbf{B}},\ \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}^{\,2},\ \mathrm{tr}\mathring{\mathbf{A}}^{\,2}\,\mathring{\mathbf{B}}^{\,2},$$

as well as certain invariants of a single tensor $\mathbf{A}$ or $\mathbf{B}$. Each of the latter is covered or determined by the basis $I_{2m}(\mathbf{A})$ or $I_{2m}(\mathbf{B})$. The first four invariants above yield the last four invariants in the set $I_{2m}(\mathbf{A},\mathbf{B})$. Moreover, it is readily verified that the invariant $\mathrm{tr}\mathring{\mathbf{A}}^{\,2}\,\mathring{\mathbf{B}}^{\,2}$ is redundant for each of cases (c1)–(c3).

(viii) The D_{2mh}-irreducible set $(\mathbf{u},\mathbf{W})$ of a vector and a skewsymmetric tensor

V	$V_{2m}(\mathbf{u})\cup\{\mathbf{Wu},\mathbf{W}^2\mathbf{u},(\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\mathbf{Wn})+(\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathbf{Wn}))\mathbf{n}\}$ $(\equiv V_{2m}(\mathbf{u},\mathbf{W}))$.
Skw	$\mathrm{Skw}_{2m}(\mathbf{u})\cup\{\mathbf{W},\mathbf{u}\wedge\mathbf{Wu},\mathbf{u}\wedge\mathbf{W}^2\mathbf{u}\}$.
Sym	$\mathrm{Sym}_{2m}(\mathbf{u})\cup\mathrm{Sym}_{2m}(\mathbf{W})\cup\{\mathbf{u}\vee\mathbf{Wu},(\mathrm{tr}\mathbf{WN})\,\mathring{\mathbf{u}}\vee(\mathbf{n}\times\mathring{\mathbf{u}})\}$.
R	$\mathbf{r}\cdot V_{2m}(\mathbf{u});\mathbf{H}:\mathrm{Skw}_{2m}(\mathbf{u});\mathbf{C}:\mathrm{Sym}_{2m}(\mathbf{u}),\mathbf{C}:\mathrm{Sym}_{2m}(\mathbf{A});$ $\mathbf{r}\cdot\mathbf{Wu},\mathbf{r}\cdot\mathbf{W}^2\mathbf{u},(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{r}}\cdot\eta_{2m-1}(\mathbf{Wn})+(\mathbf{r}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathbf{Wn});$ $\mathrm{tr}\mathbf{HW},\mathbf{u}\cdot\mathbf{HWu},\mathbf{u}\cdot\mathbf{HW}^2\mathbf{u};\mathbf{u}\cdot\mathring{\mathbf{C}}\,\mathbf{Wu},(\mathrm{tr}\mathbf{WN})[\mathbf{n},\mathring{\mathbf{u}},\mathring{\mathbf{C}}\mathring{\mathbf{u}}];$
	$I_{2m}(\mathbf{u})\cup I_{2m}(\mathbf{W})\cup\{(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathbf{Wn},\mathbf{u}\cdot\mathbf{W}^2\mathbf{u}\}$.

Since $2r$th-order tensor-valued anisotropic functions of the variables $(\mathbf{u},\mathbf{W})$ under the group D_{2mh} are equivalent to those of the variables $(\mathbf{W},\mathbf{u}\otimes\mathbf{u})$ under the same group, where $r\geqslant 0$, in the above table the results except those for the vector generators and their related invariants, can be derived by setting $\mathbf{A}=\mathbf{u}\otimes\mathbf{u}$ in the table for the variables $(\mathbf{W},\mathbf{A})$ in (**vi**). In what follows we need only to show that the presented set $V_{2m}(\mathbf{u},\mathbf{W})$ supplies a desired vector generating set.

The D_{2mh}-irreducibility condition for $(\mathbf{u},\mathbf{W})$ is given by (see (3.1)):

$$\Gamma(\mathbf{u},\mathbf{W})\cap D_{2mh}\neq\Gamma(\mathbf{z})\cap D_{2mh},\ \mathbf{z}=\mathbf{u},\ \mathbf{W}.$$

It is evident that $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{W} \neq \mathbf{O}$, and, moreover, $\Gamma(\mathbf{u}) \cap D_{2mh} \neq C_1$. The latter yields $(\mathbf{u} \cdot \mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{u}}) = 0$. Three cases will be discussed.

First, let $\mathbf{u} \cdot \mathbf{n} = 0$ and $\beta_{2m}(\overset{\circ}{\mathbf{u}}) \neq 0$. Then $\mathbf{u} = \overset{\circ}{\mathbf{u}}$, and the two vectors $\overset{\circ}{\mathbf{u}}$ and $\eta_{2m-1}(\overset{\circ}{\mathbf{u}})$ are linearly independent. Hence, we have

$$\operatorname{rank} V_{2m}(\mathbf{u}, \mathbf{W}) \geqslant \operatorname{rank}\{\overset{\circ}{\mathbf{u}}, \eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{W}\mathbf{u}, (\overset{\circ}{\mathbf{u}} \cdot \eta_{2m-1}(\mathbf{W}\mathbf{n}))\mathbf{n}\} = 3.$$

In the above, we have $\mathbf{W}\mathbf{n} \neq \mathbf{0}$. The case when $\mathbf{W}\mathbf{n} = \mathbf{0}$ violates the D_{2mh}-irreducibility condition for $(\mathbf{u}, \mathbf{W})$ (see (3.1)) and hence is excluded.

Second, let $\beta_{2m}(\overset{\circ}{\mathbf{u}}) = 0$ and $(\mathbf{u} \cdot \mathbf{n})|\,\overset{\circ}{\mathbf{u}}\,| \neq 0$, i.e. $\mathbf{u} = a\mathbf{n} + b\mathbf{e}$ with $ab \neq 0$, and let

$$\mathbf{W} = x\mathbf{e} \wedge \mathbf{e}' + y\mathbf{e} \wedge \mathbf{n} + z\mathbf{e}' \wedge \mathbf{n}, \; x^2 + z^2 \neq 0.$$

Then we have

$$\operatorname{rank} V_{2m}(\mathbf{u}, \mathbf{A}) \geqslant \{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{W}\mathbf{u}, \mathbf{W}^2\mathbf{u}, (\mathbf{u} \cdot \mathbf{n})\eta_{2m-1}(\mathbf{W}\mathbf{n})$$
$$+(\overset{\circ}{\mathbf{u}} \cdot \eta_{2m-1}(\mathbf{W}\mathbf{n}))\mathbf{n}\} = \operatorname{rank}\{\mathbf{n}, \mathbf{e}, (az - bx)\mathbf{e}', y(bz + ax)\mathbf{e}',$$
$$\beta_{2m-1}(\mathbf{W}\mathbf{n})\mathbf{e}'\} = 3$$

for

$$az - bx \neq 0 \text{ or } y(bz + ax) \neq 0 \text{ or } \beta_{2m-1}(\mathbf{W}\mathbf{n}) \neq 0.$$

In the above, the case $az - bx = y(bz + ax) = \beta_{2m-1}(\mathbf{W}\mathbf{n}) = 0$ has been excluded, since this case yields $x = z = 0$ and $y \neq 0$, i.e. $\mathbf{W} = y\mathbf{e} \wedge \mathbf{n}$, which violates the D_{2mh}-irreducibility condition for $(\mathbf{u}, \mathbf{W})$.

Third, let $\overset{\circ}{\mathbf{u}} = \mathbf{0}$, i.e. $\mathbf{u} = a\mathbf{n} \neq \mathbf{0}$. According to Theorem 2 in XIAO [47], we know that an isotropic vector generating set of the extended variables $(\mathbf{u}, \mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{W}\mathbf{n}), \mathbf{n} \otimes \mathbf{n})$ offers a desired anisotropic vector generating set of $(\mathbf{u}, \mathbf{W})$. Applying this fact and the related result for isotropic functions we infer that the former is included in the set $V_{2m}(\mathbf{u}, \mathbf{W})$.

Finally, let $\mathbf{u} \cdot \mathbf{n} = \beta_{2m}(\overset{\circ}{\mathbf{u}}) = 0$ and $\overset{\circ}{\mathbf{u}} \neq \mathbf{0}$, i.e. $\mathbf{u} = a\mathbf{e} \neq \mathbf{0}$, and let

$$\mathbf{W} = x\mathbf{e} \wedge \mathbf{e}' + y\mathbf{e} \wedge \mathbf{n} + z\mathbf{e}' \wedge \mathbf{n}, \; x^2 + y^2 + z^2 \neq 0.$$

Then we have (note $x^2 + y^2 + z^2 \neq 0$)

$$\operatorname{rank} V_{2m}(\mathbf{u}) \geqslant \operatorname{rank}\{\overset{\circ}{\mathbf{u}}, \mathbf{W}\mathbf{u}, \mathbf{W}^2\mathbf{u}, (\mathbf{u} \cdot \mathbf{n})\eta_{2m-1}(\mathbf{W}\mathbf{n})$$
$$+ (\overset{\circ}{\mathbf{u}} \cdot \eta_{2m-1}(\mathbf{W}\mathbf{n}))\mathbf{n}\} = \operatorname{rank}\{\mathbf{e}, \alpha_{2m-1}(\mathbf{W}\mathbf{n})\mathbf{n}, x\mathbf{e}' + y\mathbf{n}, zy\mathbf{e}' - zx\mathbf{n}\}$$
$$= \begin{cases} 3 \text{ if } z(x^2 + y^2) \neq 0 \text{ or } z = 0, \; xy \neq 0, \\ 2 \text{ if } z = y = 0 \text{ or } z = x = 0, \\ 1 \text{ if } x = y = 0, \end{cases}$$

and

$$\Gamma(\mathbf{u},\mathbf{W})\cap D_{2mh}=\begin{cases} C_{1h}(\mathbf{e}') \text{ if } z=x=0, & y\neq 0\,,\\ C_{1h}(\mathbf{n}) \text{ if } z=y=0, & x\neq 0\,,\\ C_2(\mathbf{e}) \text{ if } x=y=0, & z\neq 0\,.\end{cases}$$

From the above facts and Table 1 in Sec 2, we conclude that the set $V_{2m}(\mathbf{u},\mathbf{W})$ is a desired vector generating set. Further, by considering the two pairs: $\mathbf{u}_1=\mathbf{e}$ and $\mathbf{W}_1=\mathbf{N}+\mathbf{n}\wedge\mathbf{e}'$, $\mathbf{u}_2=\mathbf{n}+\mathbf{e}$ and $\mathbf{W}_2=\mathbf{E}(\mathbf{n}+\mathbf{e})$, we infer the last three generators in $V_{2m}(\mathbf{u},\mathbf{W})$ are irreducible.

(ix) The D_{2mh}-irreducible set $(\mathbf{u},\mathbf{A})$ of a vector and a symmetric tensor

V	$V_{2m}(\mathbf{u})\cup\{\mathring{\mathbf{A}}\,\mathbf{u},\mathring{\mathbf{A}}^{\,2}\mathbf{u},(\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}),(\mathring{\mathbf{A}}\,\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}})+\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}})\,\mathring{\mathbf{A}})\mathbf{u}\}\ (\equiv V_{2m}(\mathbf{u},\mathbf{A}))$.
Skw	$\mathrm{Skw}_{2m}(\mathbf{u})\cup Skw_{2m}(\mathbf{A})\cup\{\mathbf{u}\wedge\mathring{\mathbf{A}}\,\mathbf{u},\mathring{\mathbf{u}}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{u}},(\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n})\,\mathring{\mathbf{u}}\wedge\mathring{\mathbf{A}}\,\mathbf{n}\}$.
Sym	$\mathrm{Sym}_{2m}(\mathbf{u})\cup Sym_{2m}(\mathbf{A})\cup\{\mathbf{u}\vee\mathring{\mathbf{A}}\,\mathbf{u}\}$.
R	$\mathbf{r}\cdot V_{2m}(\mathbf{u});\mathbf{H}:\mathrm{Skw}_{2m}(\mathbf{u}),\mathbf{H}:\mathrm{Skw}_{2m}(\mathbf{A});\mathbf{C}:\mathrm{Sym}_{2m}(\mathbf{u})$; $\mathbf{r}\cdot\mathring{\mathbf{A}}\,\mathbf{u},\mathbf{r}\cdot\mathring{\mathbf{A}}^{\,2}\mathbf{u},(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{r}}\cdot\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})+(\mathbf{r}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})$, $\mathbf{r}\cdot(\mathring{\mathbf{A}}\,\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}})+\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}})\,\mathring{\mathbf{A}})\mathbf{u}$; $\mathbf{u}\cdot\mathbf{H}\,\mathring{\mathbf{A}}\,\mathbf{u},\mathring{\mathbf{u}}\cdot\mathbf{H}\,\mathring{\mathbf{A}}\mathring{\mathbf{u}},(\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathbf{H}\,\mathring{\mathbf{A}}\,\mathbf{n};\mathbf{u}\cdot\mathring{\mathbf{C}}\mathring{\mathbf{A}}\,\mathbf{u}$; $I_{2m}(\mathbf{u})\cup I_{2m}(\mathbf{A})\cup\{\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{u}},(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n},\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}^{\,2}\,\mathring{\mathbf{u}}\}$.

Since $2r$th-order tensor-valued anisotropic functions of the variables $(\mathbf{u},\mathbf{A})$ under the group D_{2mh} are equivalent to those of the variables $(\mathbf{A},\mathbf{u}\otimes\mathbf{u})$ under the same group, where $r\geqslant 0$, in the above table the results except those for the vector generators and their related invariants are derived by setting $\mathbf{B}=\mathbf{u}\otimes\mathbf{u}$ in the table for the variables $(\mathbf{A},\mathbf{B})$ in **(vii)**. In deriving the functional basis in the above table, some redundant invariants have been removed.

In what follows we prove that the presented set $V_{2m}(\mathbf{u},\mathbf{A})$ supplies a desired vector generating set.

For $\mathring{\mathbf{u}}\neq\mathbf{0}$ we have

$$\Gamma(\mathbf{u})\cap D_{2mh}=\Gamma(\mathbf{u},\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}}))=\begin{cases} C_{1h}(\mathbf{n}) \text{ if } \mathbf{u}\cdot\mathbf{n}=0,\ \beta_{2m}(\mathring{\mathbf{u}})\neq 0\,,\\ C_{2v}(\mathbf{l},\mathbf{n},\mathbf{n}\times\mathbf{l}) \text{ if } \mathbf{u}=a\mathbf{l},\ a\neq 0\,,\\ C_{1h}(\mathbf{l}) \text{ if } \mathbf{u}=a\mathbf{n}+b\mathbf{n}\times\mathbf{l},\ ab\neq 0\,,\\ C_1 \text{ if } (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{u}})\neq 0\,.\end{cases}$$

Hence, for $\overset{\circ}{\mathbf{u}}\neq \mathbf{0}$ we infer

$$\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh} = (\Gamma(\mathbf{u})\cap D_{2mh})\cap\Gamma(\mathbf{A}) = \Gamma(\mathbf{u},\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}),\overset{\circ}{\mathbf{A}}).$$

From the latter equality and the criterion (2.3) we know that, when $\overset{\circ}{\mathbf{u}}\neq \mathbf{0}$, isotropic generating sets for the variables $(\mathbf{u},\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}),\overset{\circ}{\mathbf{A}})$ provide anisotropic generating sets for the variables $(\mathbf{u},\mathbf{A})$ under the group D_{2mh}. Moreover, when $\overset{\circ}{\mathbf{u}}= \mathbf{0}$, i.e. $\mathbf{u}=a\mathbf{n}$, isotropic generating sets for the variables $(\mathbf{u},\overset{\circ}{\mathbf{A}},\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{A}}\ \mathbf{n}),\mathbf{\Phi}_{m-1}(\mathbf{q}(\mathbf{A})),\mathbf{n}\otimes\mathbf{n})$ supply anisotropic generating sets for the variables $(\mathbf{u},\mathbf{A})$ under D_{2mh} (see Theorem 2 in XIAO [47]). As a result, by applying the related result for isotropic functions, when $\overset{\circ}{\mathbf{u}}\neq \mathbf{0}$, we know that the desired vector generating set is formed by the generators in the set $V_{2m}(\mathbf{u},\mathbf{A})$ except $(\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})$. When $\overset{\circ}{\mathbf{u}}= \mathbf{0}$, i.e. $\mathbf{u}=a\mathbf{n}$, the desired vector generating set is formed by the five generators $(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$, $\overset{\circ}{\mathbf{A}}\ \mathbf{u}$, $\overset{\circ}{\mathbf{A}}{}^2\mathbf{u}$, $(\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})$ and $\boldsymbol{\phi}(\mathbf{u},\mathbf{A})=(\mathbf{u}\cdot\mathbf{n})\mathbf{\Phi}_{m-1}(\mathbf{q}(\mathbf{A}))\ \overset{\circ}{\mathbf{A}}\ \mathbf{n}$. The first four generators are included in the presented set $V_{2m}(\mathbf{u},\mathbf{A})$.

We show that the generator $\boldsymbol{\phi}(\mathbf{u},\mathbf{A})$ is redundant. In fact, when $\overset{\circ}{\mathbf{A}}\ \mathbf{n}\neq\mathbf{0}$, the three vectors $(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$, $\overset{\circ}{\mathbf{A}}\ \mathbf{u}$ and $\mathbf{r}=\mathbf{n}\times\overset{\circ}{\mathbf{A}}\ \mathbf{u}$ constitute an orthogonalized basis of V (note $\mathbf{u}=a\mathbf{n}\neq\mathbf{0}$). The components of the last three of the foregoing five generators with respect to $\hat{\mathbf{u}}$ are of the forms

$$\hat{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}{}^2\mathbf{u}=\alpha x^2 y\sin(2\phi(\mathbf{A})-\psi(\mathbf{A})),\ \hat{\mathbf{u}}\cdot\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n}) = \beta x^{2m}\sin 2m\phi(\mathbf{A}),$$

$$\hat{\mathbf{u}}\cdot\boldsymbol{\phi}(\mathbf{u},\mathbf{A})=\gamma x^2 y^{m-1}\sin(2\phi(\mathbf{A})+(m-1)\psi(\mathbf{A})).$$

Here $x=|\overset{\circ}{\mathbf{A}}\ \mathbf{n}|$, $y=|\mathbf{q}(\mathbf{A})|$ and α, β and γ are nonvanishing. From these and the identity

$$\sin(2\phi(\mathbf{A})+(m-1)\psi(\mathbf{A}))=\sin 2m\phi(\mathbf{A})\cos(m-1)(2\phi(\mathbf{A})-\psi(\mathbf{A}))-\cos 2m\phi(\mathbf{A})\sin(m-1)(2\phi(\mathbf{A})-\psi(\mathbf{A}))\,,$$

we deduce that the last one of the foregoing three components is determined by the other two. Hence, the generator $\boldsymbol{\phi}(\mathbf{u},\mathbf{A})$ is redundant.

Thus, we conclude that the presented set $V_{2m}(\mathbf{u},\mathbf{A})$ is a desired generating set. Moreover, from the property of this set concerning $\overset{\circ}{\mathbf{u}}\neq\mathbf{0}$ and $\overset{\circ}{\mathbf{u}}=\mathbf{0}$, indicated in the above proof, we know that the invariant $(\mathbf{u}\cdot\mathbf{n})\mathbf{r}\cdot\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})$ given by the scalar product of the variable $\mathbf{r} \in V$ and the generator $(\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$, may be replaced by the third invariant at line 6 in the table given before.

Finally, from the criterion (2.3) and the facts

$$\mathbf{u}=\mathbf{n},\ \mathbf{A}=\mathbf{D}_1+\mathbf{D}_3:\ V_{2m}(\mathbf{u})=\{\mathbf{n}\},\ \overset{\circ}{\mathbf{u}}=\mathbf{0},$$

$$\overset{\circ}{\mathbf{A}}\,\mathbf{u}=(\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})=\mathbf{e};$$

$$\mathbf{u}=\mathbf{e},\ \mathbf{A}=\mathbf{D}_2+\mathbf{D}_3:\ (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})=\mathbf{0},$$

$$\overset{\circ}{\mathbf{A}}{}^2\mathbf{u}=2\mathbf{e},\ V_{2m}(\mathbf{u})=\{\mathbf{e}\};$$

$$\mathbf{u}=\mathbf{n},\ \mathbf{A}=\mathbf{n}\vee(\mathbf{e}+\mathbf{l}_1):\overset{\circ}{\mathbf{u}}=\mathbf{0},\ \overset{\circ}{\mathbf{A}}{}^2\mathbf{n}=\mathbf{n},\ V_{2m}(\mathbf{u})=\{\mathbf{n}\},$$

where $\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh}=C_1$ for each pair $(\mathbf{u},\mathbf{A})$ given, we deduce that the generator $\overset{\circ}{\mathbf{A}}{}^2\mathbf{u}$, the generator $\overset{\circ}{\mathbf{A}}\,\mathbf{u}$, the last generator in the set $V_{2m}(\mathbf{u},\mathbf{A})$ and the generator $(\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})$, are irreducible, respectively.

4.3. D_{2mh}-irreducible sets of three variables

(x) The D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{W})$ of two vectors and a skewsymmetric tensor

V	$\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n},\overset{\circ}{\mathbf{u}},(\mathbf{v}\cdot\mathbf{n})\mathbf{n},\overset{\circ}{\mathbf{v}},\mathbf{W}\mathbf{u},\mathbf{W}\mathbf{v}\}\ (\equiv V(\mathbf{u},\mathbf{v},\mathbf{W}))$.
Skw	$\{\mathbf{u}\wedge\mathbf{v},\mathbf{W},\mathbf{u}\wedge\mathbf{W}\mathbf{v}+\mathbf{v}\wedge\mathbf{W}\mathbf{u}\}\ (\equiv \mathrm{Skw}(\mathbf{u},\mathbf{v},\mathbf{W}))$.
Sym	$\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}},\mathbf{u}\vee\mathbf{v},\mathbf{u}\vee\mathbf{W}\mathbf{u},\mathbf{v}\vee\mathbf{W}\mathbf{v},\mathbf{u}\vee\mathbf{W}\mathbf{v}+\mathbf{v}\vee\mathbf{W}\mathbf{u}\}$ $(\equiv \mathrm{Sym}(\mathbf{u},\mathbf{v},\mathbf{W}))$.
R	$\mathbf{r}\cdot V(\mathbf{u},\mathbf{v},\mathbf{W});\mathbf{u}\cdot\mathbf{H}\mathbf{v},\mathrm{tr}\mathbf{H}\mathbf{W},\mathbf{u}\cdot(\mathbf{H}\mathbf{W}-\mathbf{W}\mathbf{H})\mathbf{v};$ $\mathrm{tr}\mathbf{C},\mathbf{n}\cdot\mathbf{C}\mathbf{n},\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}},\mathbf{u}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\mathbf{u},\overset{\circ}{\mathbf{v}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{v}},\mathbf{v}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\mathbf{v},\mathbf{u}\cdot(\overset{\circ}{\mathbf{C}}\,\mathbf{W}-\mathbf{W}\,\overset{\circ}{\mathbf{C}})\mathbf{v};$ $\{(\mathbf{u}\cdot\mathbf{n})^2,\mid\overset{\circ}{\mathbf{u}}\mid^2,(\mathbf{v}\cdot\mathbf{n})^2,\mid\overset{\circ}{\mathbf{v}}\mid^2\}\ (\equiv I(\mathbf{u},\mathbf{v},\mathbf{W}))$.

The proof for the above results is as follows. From the condition (3.2) with $\mathbf{x}=\mathbf{u}$, $\mathbf{y}=\mathbf{v}$ and $\mathbf{z}=\mathbf{A}$ and $g=D_{2mh}$, it is evident that the two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent and $\mathbf{W}\neq\mathbf{O}$, and $\Gamma(\mathbf{u},\mathbf{v})\cap D_{2mh}\neq C_1$, $\Gamma(\mathbf{z},\mathbf{W})\cap D_{2mh}\neq C_1$, $\mathbf{z}=\mathbf{u},\ \mathbf{v}$. From the first expression above and $\Gamma(\mathbf{u},\mathbf{v})=C_{1h}(\mathbf{u}\times\mathbf{v})$, we infer that the vector $\mathbf{u}\times\mathbf{v}$ must be in the direction of one of the symmetry axis vectors $\mathbf{n}$ and $\mathbf{l}_1,\cdots,\mathbf{l}_{2m}$. Since the set $(\mathbf{u},\mathbf{v})$ should be a D_{2mh}-irreducible set, we further deduce that the two vectors $\mathbf{u}$ and $\mathbf{v}$ must be in directions of two of the symmetry axis vectors $\mathbf{n}$ and $\mathbf{l}_k$, $k=0,1,\cdots,2m-1$. In what follows we shall prove that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal and either $\mathbf{W}\mathbf{u}=\mathbf{0}$ or $\mathbf{W}\mathbf{v}=\mathbf{0}$ holds.

In fact, the second expression for $(\mathbf{z}, \mathbf{W})$ given before implies that for each vector $\mathbf{z} \in \{\mathbf{u}, \mathbf{v}\}$, the axis vector $\mathbf{w} = \mathbf{E} : \mathbf{W}$ of $\mathbf{W}$ must be either normal to or in the direction of $\mathbf{z}$. This fact leads to the two possibilitie for $(\mathbf{u}, \mathbf{v}, \mathbf{W})$: (a) The vector $\mathbf{E} : \mathbf{W}$ is orthogonal to one of the vectors $\mathbf{u}$ and $\mathbf{v}$ and in the direction of the other, and (b) the vector $\mathbf{E} : \mathbf{W}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. The latter is excluded, since it results in $\Gamma(\mathbf{u}, \mathbf{v}, \mathbf{W}) = \Gamma(\mathbf{u}, \mathbf{v}) = C_{1h}(\mathbf{u} \times \mathbf{v})$, which violates the condition (3.2). Hence, we conclude that the fact stated before is true.

Thus, the D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{W})$ is given by

$$\text{(c1) } \mathbf{u} = a\mathbf{n},\ \mathbf{v} = b\mathbf{e},\ \mathbf{W} = c\mathbf{N},\ abc \neq 0;$$

$$\text{(c2) } \mathbf{u} = a\mathbf{n},\ \mathbf{v} = b\mathbf{e},\ \mathbf{W} = c\mathbf{n} \wedge \mathbf{e}',\ abc \neq 0;$$

$$\text{(c3) } \mathbf{u} = a\mathbf{e},\ \mathbf{v} = b\mathbf{e}',\ \mathbf{W} = c\mathbf{n} \wedge \mathbf{e},\ abc \neq 0.$$

With the above cases one may readily verify that the four sets $V(\mathbf{u}, \mathbf{v}, \mathbf{W})$, $\mathrm{Skw}(\mathbf{u}, \mathbf{v}, \mathbf{W})$, $\mathrm{Sym}(\mathbf{u}, \mathbf{v}, \mathbf{W})$ and $I(\mathbf{u}, \mathbf{v}, \mathbf{W})$ providet desired generating sets and a functional basis for the D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{W})$ under the group D_{2mh}. Further, by considering the set $(\mathbf{u}, \mathbf{v}, \mathbf{W})$ given by case (c1), we infer that the two tensor generators $\mathbf{u} \wedge \mathbf{W}\mathbf{v} + \mathbf{v} \wedge \mathbf{W}\mathbf{u}$ and $\mathbf{u} \vee \mathbf{W}\mathbf{v} + \mathbf{v} \vee \mathbf{W}\mathbf{u}$ are irreducible.

(xi) The D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{A})$ of two vectors and a symmetric tensor

$$V \quad \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \mathring{\mathbf{u}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}, \mathring{\mathbf{v}}, \mathring{\mathbf{A}}\,\mathbf{u}, \mathring{\mathbf{A}}\,\mathbf{v}\}\ (\equiv V(\mathbf{u}, \mathbf{v}, \mathbf{A})).$$

$$\mathrm{Skw} \quad \{\mathbf{u} \wedge \mathbf{v}, \mathbf{u} \wedge \mathring{\mathbf{A}}\,\mathbf{u}, \mathbf{v} \wedge \mathring{\mathbf{A}}\,\mathbf{v}, \mathbf{u} \wedge \mathring{\mathbf{A}}\,\mathbf{v} + \mathbf{v} \wedge \mathring{\mathbf{A}}\,\mathbf{u}\}\ (\equiv \mathrm{Skw}(\mathbf{u}, \mathbf{v}, \mathbf{A})).$$

$$\mathrm{Sym} \quad \{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{u}} \otimes \mathring{\mathbf{u}}, \mathring{\mathbf{v}} \otimes \mathring{\mathbf{v}}, \mathbf{u} \vee \mathbf{v}, \mathbf{u} \vee \mathring{\mathbf{A}}\,\mathbf{u}, \mathbf{v} \vee \mathring{\mathbf{A}}\,\mathbf{v},$$
$$\mathbf{u} \vee \mathring{\mathbf{A}}\,\mathbf{v} + \mathbf{v} \vee \mathring{\mathbf{A}}\,\mathbf{u}\}\ (\equiv \mathrm{Sym}(\mathbf{u}, \mathbf{v}, \mathbf{A})).$$

$$R \quad \mathbf{r} \cdot V(\mathbf{u}, \mathbf{v}, \mathbf{A}); \mathbf{u} \cdot \mathbf{H}\mathbf{v}, \mathbf{u} \cdot \mathbf{H}\mathring{\mathbf{A}}\,\mathbf{u}, \mathbf{v} \cdot \mathbf{H}\mathbf{v}, \mathbf{v} \cdot \mathbf{H}\mathring{\mathbf{A}}\,\mathbf{v},$$
$$\mathbf{u} \cdot (\mathbf{H}\mathring{\mathbf{A}} - \mathring{\mathbf{A}}\mathbf{H})\mathbf{v};$$
$$\mathrm{tr}\mathbf{C}, \mathbf{n} \cdot \mathbf{C}\mathbf{n}, \mathring{\mathbf{u}} \cdot \mathring{\mathbf{C}}\mathring{\mathbf{u}}, \mathbf{u} \cdot \mathring{\mathbf{C}}\mathring{\mathbf{A}}\,\mathbf{u}, \mathring{\mathbf{v}} \cdot \mathring{\mathbf{C}}\mathring{\mathbf{v}}, \mathbf{v} \cdot \mathring{\mathbf{C}}\mathring{\mathbf{A}}\,\mathbf{v}, \mathbf{u} \cdot (\mathring{\mathbf{C}}\mathring{\mathbf{A}} - \mathring{\mathbf{A}}\mathring{\mathbf{C}})\mathbf{v};$$
$$\{(\mathbf{u}\cdot\mathbf{n})^2, |\mathring{\mathbf{u}}|^2, (\mathbf{v}\cdot\mathbf{n})^2, |\mathring{\mathbf{v}}|^2, \mathbf{n} \cdot \mathbf{A}\mathbf{n}, \mathrm{tr}\mathbf{A}, |\mathring{\mathbf{A}}|^2, |\mathbf{q}(\mathbf{A})|^2,$$
$$\mathring{\mathbf{u}} \cdot \mathring{\mathbf{A}}\mathring{\mathbf{u}}, \mathring{\mathbf{v}} \cdot \mathring{\mathbf{A}}\mathring{\mathbf{v}}\}\ (\equiv I(\mathbf{u}, \mathbf{v}, \mathbf{A})).$$

To prove the above results, we work out the D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{A})$ specified by the condition (3.2) with $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{u}, \mathbf{v}, \mathbf{A})$ and $g = D_{2mh}$. The latter yields

$$(4.9) \qquad \Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh}\neq C_1,\ \Gamma(\mathbf{v},\mathbf{A})\cap D_{2mh}\neq C_1\,.$$

According to the first half of the proof in (x), the vectors $\mathbf{u}$ and $\mathbf{v}$ are in directions of two of the symmetry axis vectors $\mathbf{n}$ and $\mathbf{l}_k$. Moreover, if one of the symmetric axis vectors is an eigenvector of the symmetric tensor $\mathbf{A}$, say $\mathbf{a}$, then $\mathbf{u}$ and $\mathbf{v}$ can not be normal to $\mathbf{a}$ simultaneously, since otherwise we would have $\Gamma(\mathbf{u},\mathbf{v},\mathbf{A})=\Gamma(\mathbf{u},\mathbf{v})\cap D_{2mh}=C_{1h}(\mathbf{a})$.

Consider the symmetric tensor $\mathbf{A}$. Since the centrosymmetric subgroups of the group $\Gamma(\mathbf{A})\cap D_{2mh}$ are given by S_2, $C_{2h}(\mathbf{a})$, $D_{2h}(\mathbf{n},\mathbf{a}_1,\mathbf{a}_2)$ and D_{2mh}, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are two mutually orthogonal two-fold axis vectors of the group D_{2mh}, and $\mathbf{a}$ is one of the symmetry axis vectors $\mathbf{l}_k$ and $\mathbf{n}$. It is evident that the following two cases for the symmetric tensor $\mathbf{A}$ can be excluded: $\Gamma(\mathbf{A})\cap D_{2mh}=S_2,\ D_{2mh}$. Hence, there are two cases for $\mathbf{A}$ left, which are discussed below.

Let $\Gamma(\mathbf{A})\cap D_{2mh}=C_{2h}(\mathbf{a})$, $\mathbf{a}\in\{\mathbf{n},\mathbf{l}_1,\cdots,\mathbf{l}_{2m}\}$. Then the two conditions given by (4.9) imply that either of the vectors $\mathbf{u}$ and $\mathbf{v}$ is normal to $\mathbf{a}$ or in the direction of $\mathbf{a}$, since the nontrivial proper subgroups of the group $C_{2h}(\mathbf{a})$ are merely $C_{1h}(\mathbf{a})$ and $C_2(\mathbf{a})$. From this fact and the foregoing fact concerning $\mathbf{u}$ and $\mathbf{v}$ we derive the three cases: (a) $\mathbf{a}=\mathbf{l}$, $\mathbf{u}=a\mathbf{l}$ and $\mathbf{v}=b\mathbf{n}$; (b) $\mathbf{a}=\mathbf{l}$, $\mathbf{u}=a\mathbf{l}$ and $\mathbf{v}=b\mathbf{n}\times\mathbf{l}$; and (c) $\mathbf{a}=\mathbf{n}$, $\mathbf{u}=a\mathbf{n}$ and $\mathbf{v}=b\mathbf{l}$. Here and henceforth, $ab\neq 0$ and $\mathbf{l}\in\{\mathbf{l}_1,\cdots,\mathbf{l}_{2m}\}$.

Let $\Gamma(\mathbf{A})\cap D_{2mh}=D_{2h}(\mathbf{n},\mathbf{a}_1,\mathbf{a}_2)$, $\mathbf{a}_1,\ \mathbf{a}_2\in\{\mathbf{l}_1,\cdots,\mathbf{l}_{2m}\}$. Then, from (2.5) and the two conditions given by (4.9) we deduce that either of the vectors $\mathbf{u}$ and $\mathbf{v}$ is normal to or in the direction of one of the vectors $\mathbf{n}$, $\mathbf{a}_1$ and $\mathbf{a}_2$. From this fact and the aforementioned fact concerning $\mathbf{u}$ and $\mathbf{v}$, we derive the only one case: (d) $\mathbf{u}=a\mathbf{n}$ and $\mathbf{v}=b\mathbf{l}$ with $\mathbf{l}\times\mathbf{a}_i\neq\mathbf{0}$, $i=1,\ 2$.

From the above analysis, we know that the D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{A})$ is specified by the four cases (a)-(d) for $\mathbf{A}$ above. Without loss of generality, we set $\mathbf{l}=\mathbf{e}$ in these cases. Then we have

(c1) $\mathbf{u}=a\mathbf{e},\ \mathbf{v}=b\mathbf{n},\ \overset{\circ}{\mathbf{A}}=x\mathbf{D}_1+y\mathbf{D}_4,\ aby\neq 0;$

(c2) $\mathbf{u}=a\mathbf{e},\ \mathbf{v}=b\mathbf{e}',\ ab\neq 0,\ \overset{\circ}{\mathbf{A}}=x\mathbf{D}_1+y\mathbf{D}_4,\ aby\neq 0;$

(c3) $\mathbf{u}=a\mathbf{n},\ \mathbf{v}=b\mathbf{e},\ ab\neq 0,\ \overset{\circ}{\mathbf{A}}=x\mathbf{D}_1+y\mathbf{D}_2,\ aby\neq 0.$

In the above, we would mention that the cases (c) and (d) derived before have been combined into case (c3).

With cases (c1)–(c3), one may readily verify that the four sets $V(\mathbf{u},\mathbf{v},\mathbf{A})$, $\mathrm{Skw}(\mathbf{u},\mathbf{v},\mathbf{A})$, $\mathrm{Sym}(\mathbf{u},\mathbf{v},\mathbf{A})$ and $I(\mathbf{u},\mathbf{v},\mathbf{A})$ provide the desired generating sets and a functional basis for the D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{A})$. Further, by considering the set $(\mathbf{u},\mathbf{v},\mathbf{A})$ given by case (c1), we infer that the tensor generators $\mathbf{u}\wedge\overset{\circ}{\mathbf{A}}\mathbf{v}+\mathbf{v}\wedge\overset{\circ}{\mathbf{A}}\mathbf{u}$ and $\mathbf{u}\vee\overset{\circ}{\mathbf{A}}\mathbf{v}+\mathbf{v}\vee\overset{\circ}{\mathbf{A}}\mathbf{u}$ are irreducible.

(xii) The D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of three vectors

If the three vectors $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ are linearly dependent, i.e. they lie on the same plane, then there are two of them, say $\mathbf{u}$ and $\mathbf{v}$, such that $\Gamma(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \Gamma(\mathbf{u}, \mathbf{v})$, which violates the g-irreducibility condition (3.2) with $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{u}, \mathbf{v}, \mathbf{w})$ and any given subgroup $g \subseteq$ Orth. Thus, we deduce that the three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent for each g-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ with any given subgroup $g \subseteq$ Orth. Hence we construct the following table.

V	$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ $(\equiv V(\mathbf{u}, \mathbf{v}, \mathbf{w}))$.
Skw	$\{\mathbf{u} \wedge \mathbf{v}, \mathbf{v} \wedge \mathbf{w}, \mathbf{w} \wedge \mathbf{u}\}$ $(\equiv \mathrm{Skw}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.
Sym	$\{\mathbf{u} \otimes \mathbf{u}, \mathbf{v} \otimes \mathbf{v}, \mathbf{w} \otimes \mathbf{w}, \mathbf{u} \vee \mathbf{v}, \mathbf{v} \vee \mathbf{w}, \mathbf{w} \vee \mathbf{u}\}$ $(\equiv \mathrm{Sym}(\mathbf{u}, \mathbf{v}, \mathbf{w}))$.
R	$\mathbf{r} \cdot V(\mathbf{u}, \mathbf{v}, \mathbf{w}); \mathbf{H} : \mathrm{Skw}(\mathbf{u}, \mathbf{v}, \mathbf{w}); \mathbf{C} : \mathrm{Sym}(\mathbf{u}, \mathbf{v}, \mathbf{w});$ $\{\lvert\mathbf{u}\rvert^2, \lvert\mathbf{v}\rvert^2, \lvert\mathbf{w}\rvert^2, \mathbf{u} \cdot \mathbf{v}, \mathbf{v} \cdot \mathbf{w}, \mathbf{w} \cdot \mathbf{u}\}$ $(\equiv I(\mathbf{u}, \mathbf{v}, \mathbf{w}))$.

Evidently, each invariant and each generator in the four presented sets $M(\mathbf{u}, \mathbf{v}, \mathbf{w})$, where $M = I, V$, Skw, Sym, are isotropic and involve not more than two vector variables, and therefore they are determined by the functional basis and the generating sets for one and two vector variables under the group $g \subseteq$ Orth. As a result, for any given subgroup $g \subseteq$ Orth, the isotropic vector generating set $V(\mathbf{u}, \mathbf{v}, \mathbf{w})$ can be omitted. Further, if the set $(\mathbf{u}, \mathbf{v})$ of two vector variables has been covered before, as is the case treated here, all the isotropic invariants and generators listed in the above table can be omitted.

(xiii) The D_{2mh}-irreducible sets $(\mathbf{u}, \mathbf{W}, \mathbf{\Omega})$ and $(\mathbf{u}, \mathbf{W}, \mathbf{A})$

According to Theorem 3.2 in XIAO [52], it suffices to supply generating sets for vector-valued functions for the above two sets of variables and the set of variables given later. The desired results are given as follows.

$$V \quad \{\mathbf{u}, \mathbf{W}\mathbf{u}, \mathbf{\Omega}\mathbf{u}, \mathbf{W}^2\mathbf{u}, \mathbf{\Omega}^2\mathbf{u}, (\mathbf{W}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{W})\mathbf{u}\}.$$

$$V \quad \{\mathbf{u}, \mathbf{W}\mathbf{u}, \overset{\circ}{\mathbf{A}}\,\mathbf{u}, \mathbf{W}^2\mathbf{u}, \overset{\circ}{\mathbf{A}}{}^2\mathbf{u}, (\mathbf{W}\,\overset{\circ}{\mathbf{A}} - \overset{\circ}{\mathbf{A}}\,\mathbf{W})\mathbf{u}\}.$$

The proof is as follows. Let X_0 be either of the two sets $(\mathbf{u}, \mathbf{W}, \mathbf{\Omega})$ and $(\mathbf{u}, \mathbf{W}, \overset{\circ}{\mathbf{A}})$. We first prove

$$\Gamma(X_0) \cap D_{2mh} = \Gamma(X_0)\,. \tag{4.10}$$

In fact, if $\Gamma(X_0) = C_1$, then it is evident that (4.10) holds. The other case, i.e. $\Gamma(X_0) \neq C_1$, implies that there is a unit vector $\mathbf{a}$ such that either $C_{1h}(\mathbf{a}) \subseteq \Gamma(X_0)$ or $C_2(\mathbf{a}) \subseteq \Gamma(X_0)$ holds. If $C_{1h}(\mathbf{a}) \subseteq \Gamma(X_0)$, then from (2.5) – (2.6) we deduce that $\mathbf{u}$ is normal to $\mathbf{a}$ and $\mathbf{W}\mathbf{a} = \mathbf{0}$. Hence we have $\Gamma(X_0) = \Gamma(\mathbf{u}, \mathbf{W}) = C_{1h}(\mathbf{a})$, which violates the condition (3.2) and should be excluded. On the other hand, if $C_2(\mathbf{a}) \subseteq \Gamma(X_0)$, then again from (2.5) and (2.6) we deduce that $\mathbf{u} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{Wa} = \mathbf{0}$. Hence we have $\Gamma(X_0) \subseteq \Gamma(\mathbf{u}, \mathbf{W}) = C_\infty(\mathbf{a})$. Since the symmetry group of any vector or second-order tensor has nothing but 2-fold and ∞-fold symmetry axes, we infer that $\Gamma(X_0) = \Gamma(\mathbf{u}, \mathbf{W}) = C_\infty(\mathbf{a})$ or $\Gamma(X_0) = C_2(\mathbf{a})$. The former violates the condition (3.2) and is excluded. For the latter, we infer that (4.10) holds, if $\mathbf{a}$ is a symmetry axis vector of the group D_{2mh}. If the latter is not true, then we have $\Gamma(X_0) \cap D_{2mh} = \Gamma(\mathbf{u}, \mathbf{W}) \cap D_{2mh} = C_1$, which violates the condition (3.2) and is excluded.

Thus, we infer that (4.10) holds. Then, from (4.10) and criterion (2.3) it follows that a generating set for vector-valued isotropic functions of each D_{2mh}-irreducible set X_0 supplies a generating set for vector-valued form-invariant functions of X_0 under D_{2mh}. The former can be derived by applying the related results for isotropic functions, as given before. Moreover, the irreducibility of the generators $(\mathbf{W\Omega} - \mathbf{\Omega W})\mathbf{u}$ and $(\mathbf{W}\ \mathring{\mathbf{A}} - \mathring{\mathbf{A}}\ \mathbf{W})\mathbf{u}$ can be deduced by considering: $\mathbf{u}_0 = \mathbf{n}$, $\mathbf{W}_0 = \mathbf{En}$, $\mathbf{\Omega}_0 = \mathbf{n} \wedge \mathbf{e}$ and $\mathbf{u}_0 = \mathbf{e}$, $\mathbf{W}_0 = \mathbf{En}$, $\mathbf{A}_0 = \mathbf{n} \vee \mathbf{e}$.

(xiv) The D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{A}, \mathbf{B})$ of a vector and two symmetric tensors

A desired generating set for vector-valued functions is given by

$$V \qquad V_{2m}(\mathbf{u}, \mathbf{A}) \cup V_{2m}(\mathbf{u}, \mathbf{B}) \cup \{(\mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{u}\}.$$

The proof is as follows. From the condition (3.2) with $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{u}, \mathbf{A}, \mathbf{B})$ and $g = D_{2mh}\ \Gamma(\mathbf{u}) \cap D_{2mh} \neq C_1$, it is evident that $\Gamma(\mathbf{u}) \cap D_{2mh} \neq C_1$, D_{2mh} and

$$\Gamma(\mathbf{u}, \mathbf{C}) \cap D_{2mh} \neq C_1, \ \mathbf{C} = \mathbf{A},\ \mathbf{B}\,. \tag{4.11}$$

From the former and (2.5) we infer that $\mathbf{u}$ is normal to one of the symmetry axis vectors of the group D_{2mh}. Hence we derive the three cases for $\mathbf{u}$: (a) $\mathbf{u} = c\mathbf{a} \neq \mathbf{0}$ with $\mathbf{a} \in \{\mathbf{n}, \mathbf{l}_0, \cdots, \mathbf{l}_{2m-1}\}$, (b) $\mathbf{u} = a\mathbf{n} + b\mathbf{l}$ with $ab \neq 0$, and (c) $\mathbf{u} \cdot \mathbf{n} = 0$ with $\mathbf{u} \times \mathbf{l}_k \neq \mathbf{0}$, $k = 1, \cdots, 2m$. For the latter two cases we have $\Gamma(\mathbf{u}) \cap D_{2mh} = C_{1h}(\mathbf{a})$ with $\mathbf{a} = \mathbf{n} \times \mathbf{l}$ and $\mathbf{a} = \mathbf{n}$, respectively. Hence we deduce

$$\Gamma(\mathbf{u}, \mathbf{C}) \cap D_{2mh} = C_{1h}(\mathbf{a}) \cap \Gamma(\mathbf{A}) = \begin{cases} C_1 = \Gamma(\mathbf{u}, \mathbf{A}, \mathbf{B}) \cap D_{2mh} \\ \quad \text{if } \mathbf{a} \text{ is not an eigenvector of } \mathbf{A}\,, \\ C_{1h}(\mathbf{a}) = \Gamma(\mathbf{u}) \cap D_{2mh} \\ \qquad\qquad \text{if } \mathbf{a} \text{ is an eigenvector } \mathbf{A}\,. \end{cases}$$

From the above we know that the condition (3.2) is violated and hence the case at issue is excluded.

In what follows we are concerned with case (a) for $\mathbf{u}$ indicated before. From (4.11) and $\Gamma(\mathbf{u},\mathbf{C})\cap D_{2mh}\subseteq\Gamma(\mathbf{u})\cap D_{2mh}$ we derive

$$\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh}=C_{1h}(\mathbf{a}'),\ C_2(\mathbf{a})\subseteq\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh},$$

$$\Gamma(\mathbf{u},\mathbf{B})\cap D_{2mh}=C_{1h}(\mathbf{a}''),\ C_2(\mathbf{a})\subseteq\Gamma(\mathbf{u},\mathbf{B})\cap D_{2mh},$$

where $\mathbf{a}'$ and $\mathbf{a}''$ are symmetry axis vectors of D_{2mh} normal to $\mathbf{u}=c\mathbf{a}$. Evidently, $\mathbf{a}'$ and $\mathbf{a}''$ are not coincident, or else the condition (3.2) will be violated. Then, we derive the three cases:

(c1) $C_2(\mathbf{a})\subseteq\Gamma(\mathbf{u},\mathbf{C}),\ \mathbf{C}=\mathbf{A},\ \mathbf{B};$

(c2) $\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh}=C_{1h}(\mathbf{a}'),\ \Gamma(\mathbf{u},\mathbf{B})\cap D_{2mh}=C_{1h}(\mathbf{a}''),\ \mathbf{a}'\times\mathbf{a}''\neq\mathbf{0};$

(c3) $\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh}=C_{1h}(\mathbf{a}'),\ C_2(\mathbf{a})\subseteq\Gamma(\mathbf{u},\mathbf{B})\cap D_{2mh}.$

For case (c1), we have $C_2(\mathbf{a})\subseteq\Gamma(\mathbf{u},\mathbf{A},\mathbf{B})\cap D_{2mh}$ and $\mathbf{u}=c\mathbf{a}\neq\mathbf{0}$, it is easy to show that the subset $V_{2m}(\mathbf{u})$ obeys the criterion (2.3).

For case (c2), using the formula (2.4) we have

$$\begin{aligned}\operatorname{rank}(V_{2m}(\mathbf{u},\mathbf{A})\ &\cup\ V_{2m}(\mathbf{u},\mathbf{B}))\\ &=\operatorname{rank}(V(\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh})\cup V(\Gamma(\mathbf{u},\mathbf{B})\cap D_{2mh}))\\ &=\operatorname{rank}(V(C_{1h}(\mathbf{a}'))\cup V(C_{1h}(\mathbf{a}'')))\\ &=\operatorname{rank}\{\mathbf{a},\mathbf{a}\times\mathbf{a}',\mathbf{a}\times\mathbf{a}''\}=3\,.\end{aligned}$$

For case (c3), $\mathbf{u}=c\mathbf{a}$ is an eigenvector of $\mathbf{B}$ (see (2.8)). Moreover, $(\mathbf{a},\mathbf{a}',\mathbf{r}\equiv\mathbf{a}\times\mathbf{a}')$ is an orthonormal basis of V. In terms of this basis, we have the expressions

$$\mathbf{A}\mathbf{u}=\alpha\mathbf{a}+\beta\mathbf{r},\ \mathbf{B}=x\mathbf{a}\otimes\mathbf{a}+y\mathbf{a}'\otimes\mathbf{a}'+z\mathbf{r}\otimes\mathbf{r}+w\mathbf{a}'\vee\mathbf{r},\ \beta z\neq 0.$$

For the former, the following facts are used: the vector $\mathbf{A}\mathbf{u}\in V(C_{1h}(\mathbf{a}))$ is normal to $\mathbf{a}$, and $\mathbf{a}$ is not an eigenvector of $\mathbf{A}$ (hence $z\neq 0$). Moreover, $w=0$ has been excluded, since otherwise we have $C_{2h}(\mathbf{a}')\subseteq\Gamma(\mathbf{B})$ and hence

$$\Gamma(\mathbf{u},\mathbf{A},\mathbf{B})\cap D_{2mh}=\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh}=C_{1h}(\mathbf{a}'),$$

which violates the condition (3.2). Thus, for case (c3) we have

$$\operatorname{rank}(V_{2m}(\mathbf{u},\mathbf{A})\cup\{(\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{u}\})=\operatorname{rank}\{\mathbf{a},\mathbf{r},\beta w\mathbf{a}'\}=3.$$

From the above we conclude that the set of vector generators offers a desired generating set. The irreducibility of the generator $(\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{u}$ can be deduced by considering

$$\mathbf{u}_0=\mathbf{n},\ \mathbf{A}_0=\mathbf{n}\vee\mathbf{e},\ \mathbf{B}_0=\mathbf{n}\vee\mathbf{e}'.$$

4.4 The general results

THEOREM 1. *The four sets given by*

$$I_{2m}(\mathbf{u}); I_{2m}(\mathbf{W}); I_{2m}(\mathbf{A}); (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, \mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{v}}), \mathring{\mathbf{v}}\cdot\eta_{2m-1}(\mathring{\mathbf{u}});$$

$$\mathrm{tr}\mathbf{W}\boldsymbol{\Omega}, \eta_{2m-1}(\mathbf{Wn})\cdot\boldsymbol{\Omega}\mathbf{n}, \eta_{2m-1}(\boldsymbol{\Omega}\mathbf{n})\cdot\mathbf{Wn}, \mathrm{tr}\mathbf{W}\boldsymbol{\Omega}^2\boldsymbol{\Phi}_{2m-2}(\boldsymbol{\Omega}\mathbf{n}),$$
$$\mathrm{tr}\boldsymbol{\Omega}\mathbf{W}^2\boldsymbol{\Phi}_{2m-2}(\mathbf{Wn});$$

$$(\mathring{\mathbf{A}}\,\mathbf{n})\cdot\mathring{\mathbf{B}}\,\mathbf{n}, \mathrm{tr}\mathbf{A_e}\mathbf{B_e}, \mathrm{tr}\mathring{\mathbf{A}}^{\,2}\mathring{\mathbf{B}}, \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}^{\,2}, \mathrm{tr}\mathbf{A_e}\boldsymbol{\Phi}_{m-1}(\mathbf{q}(\mathbf{B})),$$
$$\mathrm{tr}\mathbf{A_n}\mathbf{B_n}\boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{A}}\,\mathbf{n}), \mathrm{tr}\mathbf{A_e}\boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{B}}\,\mathbf{n}), \mathrm{tr}\mathbf{B_e}\boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{A}}\,\mathbf{n});$$

$$(\mathring{\mathbf{A}}\,\mathbf{n})\cdot\mathbf{Wn}, \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W}^2, \mathrm{tr}\mathring{\mathbf{A}}\,\boldsymbol{\Phi}_{2m-2}(\mathbf{Wn}), \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W}\boldsymbol{\Phi}_{2m-2}(\mathbf{Wn}),$$
$$(\mathrm{tr}\mathbf{WN})\beta_m(\mathbf{q}(\mathbf{A})),$$
$$\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})\cdot\mathbf{Wn} - |\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-4}(\mathrm{tr}\mathbf{WN})J(\mathbf{A}),$$
$$|\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-2}(\mathbf{Wn})\cdot\mathring{\mathbf{A}}^{\,2}\mathbf{n} - (\mathrm{tr}\mathbf{WN})\beta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n});$$

$$(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathbf{Wn}, \mathbf{u}\cdot\mathbf{W}^2\mathbf{u}, (\mathrm{tr}\mathbf{WN})\beta_{2m}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathbf{Wn};$$

$$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}^{\,2}\,\mathring{\mathbf{u}}, \mathrm{tr}\mathring{\mathbf{A}}\,\boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathring{\mathbf{A}}\,\mathbf{n};$$

$$\mathrm{tr}\mathbf{W}\boldsymbol{\Omega}\mathbf{H}; \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}\mathring{\mathbf{C}}; \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{W}, (\mathrm{tr}\mathbf{WN})[\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n}],$$
$$(\mathrm{tr}\mathbf{WN})[\mathbf{n}, \mathring{\mathbf{B}}\,\mathbf{n}, \mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n}];$$

$$\mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W}\boldsymbol{\Omega}, |\mathrm{tr}\boldsymbol{\Omega}\mathbf{N}|(\mathrm{tr}\boldsymbol{\Omega}\mathbf{N})[\mathbf{n}, \mathbf{Wn}, \mathring{\mathbf{A}}\,\mathbf{Wn}] + |\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})[\mathbf{n}, \boldsymbol{\Omega}\mathbf{n}, \mathring{\mathbf{A}}\,\boldsymbol{\Omega}\mathbf{n}];$$

$$\mathbf{u}\cdot\mathbf{Wv}, \mathbf{u}\cdot\mathbf{W}^2\mathbf{v}, (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{v}}\cdot\eta_{2m-1}(\mathbf{Wn}) + (\mathbf{v}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathbf{Wn}),$$
$$|\mathbf{u}|^{2m-2}\eta_{2m-1}(\mathring{\mathbf{v}})\cdot\mathbf{Wu} + |\mathbf{v}|^{2m-2}\eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathbf{Wv};$$

$$\mathbf{u}\cdot\mathring{\mathbf{A}}\,\mathbf{v}, \mathbf{u}\cdot\mathring{\mathbf{A}}^{\,2}\mathbf{v}, (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{v}}\cdot\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}) + (\mathbf{v}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}),$$
$$\mathbf{u}\cdot(\mathring{\mathbf{A}}\,\boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{u}}) + \boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{u}})\,\mathring{\mathbf{A}})\mathbf{v}, \mathbf{u}\cdot(\mathring{\mathbf{A}}\,\boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{v}}) + \boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{v}})\,\mathring{\mathbf{A}})\mathbf{v},$$
$$|\mathbf{u}|^{2m-2}\eta_{2m-1}(\mathring{\mathbf{v}})\cdot\mathring{\mathbf{A}}\,\mathbf{u} + |\mathbf{v}|^{2m-2}\eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathring{\mathbf{A}}\,\mathbf{v};$$

$$\mathbf{u}\cdot\mathbf{W}\boldsymbol{\Omega}\mathbf{u}, \mathbf{u}\cdot\mathbf{W}\boldsymbol{\Omega}^2\mathbf{u}, \mathbf{u}\cdot\mathbf{W}^2\boldsymbol{\Omega}\mathbf{u}; \mathbf{u}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{u}; \mathbf{u}\cdot\mathring{\mathbf{A}}\,\mathbf{Wu}, (\mathrm{tr}\mathbf{WN})[\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{A}}\mathring{\mathbf{u}}],$$

$$\mathring{\mathbf{u}}\cdot\mathbf{W}\,\mathring{\mathbf{A}}\mathring{\mathbf{u}},\ (\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathbf{W}\,\mathring{\mathbf{A}}\,\mathbf{n};$$

$$\mathbf{u}\cdot(\mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W})\mathbf{v};\ \mathbf{u}\cdot(\mathbf{W}\,\mathring{\mathbf{A}}-\mathring{\mathbf{A}}\,\mathbf{W})\mathbf{v};\ \mathbf{u}\cdot(\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{v};$$

and

$$V_{2m}(\mathbf{u});\ \mathbf{W}\mathbf{u},\ \mathbf{W}^2\mathbf{u},\ (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\mathbf{W}\mathbf{n})+(\mathring{\mathbf{u}}\cdot\boldsymbol{\eta}_{2m-1}(\mathbf{W}\mathbf{n}))\mathbf{n};$$

$$\mathring{\mathbf{A}}\,\mathbf{u},\ \mathring{\mathbf{A}}^{\,2}\mathbf{u},\ (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}),\ (\mathring{\mathbf{A}}\,\boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{u}})+\boldsymbol{\Phi}_{2m-2}(\mathring{\mathbf{u}})\,\mathring{\mathbf{A}})\mathbf{u};$$

$$(\mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W})\mathbf{u};\ (\mathbf{W}\,\mathring{\mathbf{A}}+\mathbf{W}\,\mathring{\mathbf{A}})\mathbf{u};\ (\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{u};$$

and

$$\mathrm{Skw}_{2m}(\mathbf{u}),\ \mathrm{Skw}_{2m}(\mathbf{W}),\ \mathrm{Skw}_{2m}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v},\ |\mathbf{u}|^{2m-2}\mathbf{u}\wedge\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{v}})+|\mathbf{v}|^{2m-2}\mathbf{v}\wedge\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{u}});$$

$$\mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W};\ \mathring{\mathbf{A}}\,\mathbf{W}+\mathbf{W}\,\mathring{\mathbf{A}},\ \mathring{\mathbf{A}}\,\mathbf{W}^2-\mathbf{W}^2\,\mathring{\mathbf{A}};$$

$$\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}},\ \mathring{\mathbf{A}}\,\mathbf{n}\wedge\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n},\ \mathring{\mathbf{B}}\,\mathbf{n}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n};$$

$$\mathbf{u}\wedge\mathbf{W}\mathbf{u},\ \mathbf{u}\wedge\mathbf{W}^2\mathbf{u};\ \mathbf{u}\wedge\mathring{\mathbf{A}}\,\mathbf{u},\ \mathring{\mathbf{u}}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{u}},\ (\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n})\,\mathring{\mathbf{u}}\wedge\mathring{\mathbf{A}}\,\mathbf{n};$$

$$\mathbf{u}\wedge\mathbf{W}\mathbf{v}+\mathbf{v}\wedge\mathbf{W}\mathbf{u};\ \mathbf{u}\wedge\mathring{\mathbf{A}}\,\mathbf{v}+\mathbf{v}\wedge\mathring{\mathbf{A}}\,\mathbf{u};$$

and

$$\mathrm{Sym}_{2m}(\mathbf{u}),\ \mathrm{Sym}_{2m}(\mathbf{W}),\ \mathrm{Sym}_{2m}(\mathbf{A});$$

$$\mathbf{u}\vee\mathbf{v},\ |\mathbf{u}|^{2m-2}\mathbf{u}\vee\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{v}})+|\mathbf{v}|^{2m-2}\mathbf{v}\vee\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{u}});$$

$$\mathbf{W}\boldsymbol{\Omega}+\boldsymbol{\Omega}\mathbf{W},$$

$$|\mathrm{tr}\boldsymbol{\Omega}\mathbf{N}|(\mathrm{tr}\boldsymbol{\Omega}\mathbf{N})\mathbf{W}\mathbf{n}\vee\mathbf{N}\mathbf{W}\mathbf{n}+|\mathrm{tr}\mathbf{W}\mathbf{N}|(\mathrm{tr}\mathbf{W}\mathbf{N})\boldsymbol{\Omega}\mathbf{n}\vee\mathbf{N}\boldsymbol{\Omega}\mathbf{n};$$

$$\mathring{\mathbf{A}}\,\mathbf{W}-\mathbf{W}\,\mathring{\mathbf{A}},\ (\mathrm{tr}\mathbf{W}\mathbf{N})\,\mathring{\mathbf{A}}\,\mathbf{n}\vee\mathbf{N}\,\mathring{\mathbf{A}}\,\mathbf{n};\ \mathring{\mathbf{A}}\mathring{\mathbf{B}}+\mathring{\mathbf{B}}\mathring{\mathbf{A}};$$

$$\mathbf{u}\vee\mathbf{W}\mathbf{u},\ (\mathrm{tr}\mathbf{W}\mathbf{N})\,\mathring{\mathbf{u}}\vee(\mathbf{n}\times\mathring{\mathbf{u}});$$

$$\mathbf{u}\vee\mathring{\mathbf{A}}\,\mathbf{u};\ \mathbf{u}\vee\mathbf{W}\mathbf{v}+\mathbf{v}\vee\mathbf{W}\mathbf{u};\ \mathbf{u}\vee\mathring{\mathbf{A}}\,\mathbf{v}+\mathbf{v}\vee\mathring{\mathbf{A}}\,\mathbf{u};$$

where $(\mathbf{u},\mathbf{v})=(\mathbf{u}_i,\mathbf{u}_j)$, $(\mathbf{W},\boldsymbol{\Omega},\mathbf{H})=(\mathbf{W}_\sigma,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B},\mathbf{C})=(\mathbf{A}_L,\mathbf{A}_M,\mathbf{A}_N)$, $j>i=1,\cdots,a$, $\theta>\tau>\sigma=1,\cdots,b$, $N>M>L=1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1, \cdots, \mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1, \cdots, \mathbf{A}_c$ *under the group* D_{2mh} *for each* $m \geqslant 2$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold axis of the group* D_{2mh}.

In the above theorem, the nine invariants of two or three symmetric tensors are quoted from the established results (see Theorem 1 in XIAO, BRUHNS and MEYERS [53]).

5. Crystal and quasicrystal classes D_{2m} for $m \geqslant 2$

The classes at issue take forms

$$D_{2m}(\mathbf{n},\mathbf{e}) = D_{2mh}(\mathbf{n},\mathbf{e}) \cap \mathrm{Orth}^+ = \{\mathbf{R}_{\mathbf{n}}^{k\pi/m}, \mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/2m}\mathbf{e},\ k = 1, \cdots, 2m\}. \tag{5.1}$$

They include the crystal classes D_4 and D_6 as particular cases when $m = 2, 3$.

Let $I^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$, $\mathrm{Skw}^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$ and $\mathrm{Sym}^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$ be, respectively, an irreducible functional basis and irreducible generating sets for scalar-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of $(a + b)$ skewsymmetric tensor variables and c symmetric tensor variables under a centrosymmetrical orthogonal subgroup g containing the central inversion $-\mathbf{I}$. Then, according to Theorems 2.1-2.2 in XIAO [43], the four sets

$$I^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),$$

$$\mathbf{E} : \mathrm{Skw}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),$$

$$\mathrm{Skw}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),$$

$$\mathrm{Sym}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),$$

supply, respectively, an irreducible functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of a vector variables, b skewsymmetric tensor variables and c symmetric tensor variables under the rotation subgroup of g, i.e. $g \cap \mathrm{Orth}^+$. Here, the second set above is obtained by forming the double dot product between each skewsymmetric tensor generator and the third order Levi-Civita tensor $\mathbf{E}$.

From the above facts and Theorem 1, we obtain the following result.

THEOREM 2. *The four sets given by*

$$I_{2m}(\mathbf{u}), (\mathbf{u} \cdot \mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{u}}); I_{2m}(\mathbf{W}); I_{2m}(\mathbf{A}); I_{2m}(\mathbf{W}, \mathbf{\Omega}, \mathbf{H}, \mathbf{A}, \mathbf{B}, \mathbf{C});$$

$$\mathbf{u}\cdot\mathbf{v}, \overset{\circ}{\mathbf{u}}\cdot\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}}), \overset{\circ}{\mathbf{v}}\cdot\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}}), [\mathbf{u},\mathbf{v},\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})], [\mathbf{v},\mathbf{u},\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})]; \quad [\mathbf{u},\mathbf{v},\mathbf{r}];$$

$$\mathrm{tr}\mathbf{W}(\mathbf{Eu}), [\mathbf{n},\overset{\circ}{\mathbf{u}},\boldsymbol{\eta}_{2m-1}(\mathbf{Wn})], [\mathbf{n},\mathbf{Wn},\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})],$$

$$\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Wu}, \mathrm{tr}(\mathbf{Eu})\mathbf{W}^2\boldsymbol{\Phi}_{2m-2}(\mathbf{Wn});$$

$$[\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{A}}\mathbf{n}], \mathbf{u}\cdot\overset{\circ}{\mathbf{A}}\mathbf{u}, \mathrm{tr}\overset{\circ}{\mathbf{A}}\boldsymbol{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}), \mathrm{tr}\overset{\circ}{\mathbf{A}}(\mathbf{Eu})\boldsymbol{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\beta_m(\mathbf{q}(\mathbf{A})),$$

$$[\mathbf{n},\overset{\circ}{\mathbf{u}},\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n})] - |\overset{\circ}{\mathbf{A}}\mathbf{n}|^{2m-4}(\mathbf{u}\cdot\mathbf{n})J(\mathbf{A}),$$

$$|\overset{\circ}{\mathbf{A}}\mathbf{n}|^{2m-2}[\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{A}}{}^2\mathbf{n}] - (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n});$$

$$\mathbf{u}\cdot\mathbf{Wv}; \mathbf{u}\cdot\overset{\circ}{\mathbf{A}}\mathbf{v}, |\mathbf{v}\cdot\mathbf{n}|(\mathbf{v}\cdot\mathbf{n})[\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}] + |\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\overset{\circ}{\mathbf{v}},\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{v}}];$$

$$\mathrm{tr}(\mathbf{Eu})\mathbf{W}\boldsymbol{\Omega}; \mathrm{tr}(\mathbf{Eu})\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}},$$

$$(\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\overset{\circ}{\mathbf{A}}\mathbf{n},\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\mathbf{n}], (\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\overset{\circ}{\mathbf{B}}\mathbf{n},\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\mathbf{n}];$$

$$\mathrm{tr}(\mathbf{Eu})\mathbf{W}\overset{\circ}{\mathbf{A}}, |\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\mathbf{Wn},\overset{\circ}{\mathbf{A}}\mathbf{Wn}] - |\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})[\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}];$$

and

$$\mathbf{u}, \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{u}\times\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}}); \mathbf{E}:\mathrm{Skw}_{2m}(\mathbf{W}); \mathbf{E}:Skw_{2m}(\mathbf{A});$$

$$\mathbf{u}\times\mathbf{v}; \mathbf{Wu}; \overset{\circ}{\mathbf{A}}\mathbf{u}, \mathbf{u}\times\overset{\circ}{\mathbf{A}}\mathbf{u}; \mathbf{E}:(\mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W});$$

$$\overset{\circ}{\mathbf{A}}(\mathbf{E}:\mathbf{W}), \mathbf{E}:(\overset{\circ}{\mathbf{A}}\mathbf{W}^2-\mathbf{W}^2\overset{\circ}{\mathbf{A}});$$

$$\mathbf{E}:(\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}), \overset{\circ}{\mathbf{A}}\mathbf{n}\times\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{B}}\mathbf{n}\times\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\mathbf{n};$$

and

$$\mathbf{Eu}, \mathbf{E}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{u}\wedge\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}}); \mathrm{Skw}_{2m}(\mathbf{W}); \mathrm{Skw}_{2m}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v}; \mathbf{E}(\mathbf{Wu}); \mathbf{E}(\overset{\circ}{\mathbf{A}}\mathbf{u}), \mathbf{u}\wedge\overset{\circ}{\mathbf{A}}\mathbf{u};$$

$$\mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W}; \overset{\circ}{\mathbf{A}}\mathbf{W}+\mathbf{W}\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\mathbf{W}^2-\mathbf{W}^2\overset{\circ}{\mathbf{A}};$$

$$\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\mathbf{n}\wedge\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{B}}\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\mathbf{n};$$

and

$$\mathrm{Sym}_{2m}(\mathbf{Eu}); \mathrm{Sym}_{2m}(\mathbf{W}); \mathrm{Sym}_{2m}(\mathbf{A});$$

$$\mathbf{u}\vee\mathbf{v},\ |\mathbf{v}\cdot\mathbf{n}|(\mathbf{v}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}})+|\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{v}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}});$$

$$\mathbf{W\Omega}+\mathbf{\Omega W},\ |\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn}\vee\mathbf{NWn}$$

$$+|\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n}\vee\mathbf{N\Omega n};$$

$$\overset{\circ}{\mathbf{A}}\mathbf{W}-\mathbf{W}\overset{\circ}{\mathbf{A}},\ (\mathrm{tr}\mathbf{WN})\overset{\circ}{\mathbf{A}}\mathbf{n}\vee\mathbf{N}\overset{\circ}{\mathbf{A}}\mathbf{n};\ \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}+\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}};$$

$$(\mathbf{Eu})\mathbf{W}+\mathbf{W}(\mathbf{Eu}),\ |\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})\mathbf{Wn}\vee\mathbf{NWn}$$

$$+|\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}});$$

$$\overset{\circ}{\mathbf{A}}(\mathbf{Eu})-(\mathbf{Eu})\overset{\circ}{\mathbf{A}},\ (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{A}}\mathbf{n}\vee\mathbf{N}\overset{\circ}{\mathbf{A}}\mathbf{n};$$

where $(\mathbf{u},\mathbf{v},\mathbf{r}) = (\mathbf{u}_i,\mathbf{u}_j,\mathbf{u}_k)$, $(\mathbf{W},\mathbf{\Omega},\mathbf{H}) = (\mathbf{W}_\sigma,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B},\mathbf{C}) = (\mathbf{A}_L,\mathbf{A}_M,\mathbf{A}_N)$, $k > j > i = 1,\cdots,a$, $\theta > \tau > \sigma = 1,\cdots,b$, $N > M > L = 1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* D_{2m} *for each* $m \geqslant 2$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold axis of the group* D_{2m}.

Here and henceforth, $I_{2m}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C})$ is used to represent the invariants of two or three second order tensors given in THEOREM 1.

6. Crystal and quasicrystal classes C_{2mv} for $m \geqslant 2$

The classes at issue are of the form

$$C_{2mv}(\mathbf{n},\mathbf{e}) = \{\mathbf{R}_{\mathbf{n}}^{k\pi/m}, -\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/2m}\mathbf{e},\ k = 1,\cdots,2m\}. \tag{6.1}$$

They include the crystal classes C_{4v} and C_{6v} as particular cases when $m = 2$, 3.

For anisotropic functions under any subgroup $g \subseteq C_{\infty v}$, the general cases involving any number of vector variables and tensor variables may be reduced to the cases involving not more than two variables (see THEOREM 2.2 in XIAO [52]). As a result, the third step in the procedure outlined in Sec. 3 can be omitted. Further reduction is possible. Let X_0 represent any of the five sets of variables, $\mathbf{W}$, $\mathbf{A}$, $(\mathbf{W},\mathbf{\Omega})$, $(\mathbf{W},\mathbf{A})$ and $(\mathbf{A},\mathbf{B})$. Then each scalar-valued or tensor-valued anisotropic function of X_0 under the group $C_{2mv}(\mathbf{n},\mathbf{e})$ is a scalar-valued or tensor-valued anisotropic function of X_0 under the larger group $D_{2mh}(\mathbf{n},\mathbf{e})$ ($\supset C_{2mv}(\mathbf{n},\mathbf{e})$). Thus, in the general results for the group C_{2mv} (THEOREM 3 below), we can directly cite the invariants and the tensor generators depending on skewsymmetric and/or symmetric tensor variables in THEOREM 1. Moreover, let Y_0 be any set of a single variable or two variables. Then each anisotropic function of Y_0 under the group C_{2mv} is an anisotropic function of $(Y_0, \mathbf{n})$ under the larger group D_{2mh} ($\supset C_{2mv}$). Thus, for all sets of variables, $(\mathbf{u})$, $(\mathbf{u}, \mathbf{W})$, $(\mathbf{u}, \mathbf{A})$ and $(\mathbf{u}, \mathbf{v})$, the desired results for the group C_{2mv} can be obtained by setting $\mathbf{v} = \mathbf{n}$ in the tables given in Sec. 4 (**iv**), (**x**), (**xi**) and setting $\mathbf{w} = \mathbf{n}$ in the table given in Sec. 4 (**xii**), respectively. In addition, for each of the sets $\mathbf{W}$, $\mathbf{A}$, $(\mathbf{W}, \mathbf{\Omega})$, $(\mathbf{W}, \mathbf{A})$ and $(\mathbf{A}, \mathbf{B})$, the desired vector generating set under the group C_{2mv} and the invariants from the scalar products related to this generating set can be derived by taking $\mathbf{u} = \mathbf{n}$ in the corresponding results in the tables given in Sec. 4 (**viii**), (**ix**), (**xiii**), (**xiv**), respectively. Combining these facts, we arrive at the general result for the group C_{2mv} as follows.

THEOREM 3. *The four sets given by*

$$\mathbf{u}\cdot\mathbf{n}, |\,\mathring{\mathbf{u}}\,|^2, \alpha_{2m}(\mathring{\mathbf{u}}); I_{2m}(\mathbf{W}); I_{2m}(\mathbf{A}); I_{2m}(\mathbf{W}, \mathbf{\Omega}, \mathbf{H}, \mathbf{A}, \mathbf{B}, \mathbf{C});$$

$$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, \mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{v}}), \mathring{\mathbf{v}}\cdot\eta_{2m-1}(\mathring{\mathbf{u}});$$

$$\mathring{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathbf{W}^2\mathbf{n}, (\mathrm{tr}\mathbf{W}\mathbf{N})\beta_{2m}(\mathring{\mathbf{u}}), \eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathbf{W}\mathbf{n}, \mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathbf{W}\mathbf{n});$$

$$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}^{\,2}\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{u}}, \mathrm{tr}\mathring{\mathbf{A}}\;\Phi_{2m-2}(\mathring{\mathbf{u}}), \eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathring{\mathbf{A}}\,\mathbf{n},$$

$$\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n});$$

$$\mathbf{u}\cdot\mathbf{W}\mathbf{v}; \mathbf{u}\cdot\mathring{\mathbf{A}}\,\mathbf{v}; \mathbf{u}\cdot(\mathbf{W}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{W})\mathbf{n}; \mathbf{u}\cdot(\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{n}, \mathbf{u}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{u};$$

$$\mathbf{u}\cdot(\mathbf{W}\mathring{\mathbf{A}}-\mathring{\mathbf{A}}\mathbf{W})\mathbf{n}, \mathbf{u}\cdot\mathbf{W}\mathring{\mathbf{A}}\,\mathbf{u};$$

and

$$\mathbf{n}, \mathring{\mathbf{u}}, \eta_{2m-1}(\mathring{\mathbf{u}}); \mathbf{W}\mathbf{n}, \mathbf{W}^2\mathbf{n}, \eta_{2m-1}(\mathbf{W}\mathbf{n});$$

$$\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{A}}^{\,2}\mathbf{n}, \eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}); \mathbf{W}\,\mathring{\mathbf{u}}; \mathring{\mathbf{A}}\mathring{\mathbf{u}};$$

$$(\mathbf{W}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{W})\mathbf{n}; (\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{n}; (\mathbf{W}\mathring{\mathbf{A}}-\mathring{\mathbf{A}}\mathbf{W})\mathbf{n};$$

and

$$\mathbf{n}\wedge\mathring{\mathbf{u}}, \mathbf{n}\wedge\eta_{2m-1}(\mathring{\mathbf{u}}), \beta_{2m}(\mathring{\mathbf{u}})\mathbf{N}; \mathrm{Skw}_{2m}(\mathbf{W}); \mathrm{Skw}_{2m}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v}; \mathbf{W}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{W}; \mathring{\mathbf{A}}\,\mathbf{W}+\mathbf{W}\,\mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{W}^2-\mathbf{W}^2\,\mathring{\mathbf{A}};$$

$$\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{n}\wedge\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{B}}\,\mathbf{n}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n};$$

$$\mathbf{n}\wedge\mathbf{W}\mathbf{u}+\mathbf{u}\wedge\mathbf{W}\mathbf{n}; \mathbf{u}\wedge\mathring{\mathbf{A}}\,\mathbf{u}, \mathbf{n}\wedge\mathring{\mathbf{A}}\,\mathbf{u}+\mathbf{u}\wedge\mathring{\mathbf{A}}\,\mathbf{n};$$

and

$$\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}});$$

$$\mathrm{Sym}_{2m}(\mathbf{W}); \mathrm{Sym}_{2m}(\mathbf{A});$$

$$\mathbf{u}\vee\mathbf{v}; \mathbf{W\Omega}+\mathbf{\Omega W}, |\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn}\vee\mathbf{NWn}$$

$$+|\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n}\vee\mathbf{N\Omega n};$$

$$\overset{\circ}{\mathbf{A}}\,\mathbf{W}-\mathbf{W}\,\overset{\circ}{\mathbf{A}}, (\mathrm{tr}\mathbf{WN})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}\vee\mathbf{N}\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}+\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}};$$

$$\mathbf{u}\vee\mathbf{Wu}, \mathbf{n}\vee\mathbf{Wu}+\mathbf{u}\vee\mathbf{Wn}; \mathbf{u}\vee\overset{\circ}{\mathbf{A}}\,\mathbf{u}, \mathbf{n}\vee\overset{\circ}{\mathbf{A}}\,\mathbf{u}+\mathbf{u}\vee\overset{\circ}{\mathbf{A}}\,\mathbf{n};$$

where $(\mathbf{u},\mathbf{v}) = (\mathbf{u}_i,\mathbf{u}_j)$, $(\mathbf{W},\mathbf{\Omega},\mathbf{H}) = (\mathbf{W}_\sigma,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B},\mathbf{C}) = (\mathbf{A}_L,\mathbf{A}_M,\mathbf{A}_N)$, $j > i = 1,\cdots,a$, $\theta > \tau > \sigma = 1,\cdots,b$, $N > M > L = 1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* C_{2mv} *for each* $m \geqslant 2$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold axis of the group* C_{2mv}.

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Comparison of two entropy principles and their applications in granular flows with/without fluid

Dedicated to Prof. Y.-H. Pao
on the occasion of his seventieth birthday

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TWO ENTROPY PRINCIPLES that are commonly used are: (i) the Clausius-Duhem inequality with the procedure of exploitation due to Coleman-Noll (CD-CN), and (ii) the entropy principle of Müller-Liu (ML). CD-CN makes *a priori* postulates about the entropy flux and entropy supply and assumes external source terms in (most) balance laws. ML postulate the entropy flux to be a general constitutive variable and treat all field equations as constraints for the exploitation of the entropy principle. These and further differences are explained, and results are presented with the use of both principles for (i) a granular solid with a scalar structure equation, and (ii) for a saturated mixture of granular/fluid constituents with scalar structure equations for each constituent. It is shown that the two entropy principles yield different results. It is further indicated which theories are likely to be problematic when the CD-CN approach is used. These theories are then applied to analyses of steady fully-developed gravity flows down an inclined plane.

Key words: entropy principle, granular flow, solid-fluid mixture, constitutive equations, gravitational flow.

1. Introduction

TO SOME EXTENT, modern continuum thermodynamics amounts to a collection of "thermodynamical theories" sharing common premises and common methodology. There are the theories of elastic materials, of viscous materials, of materials with memory, of mixtures, and so on. It is generally the case that, in the context of each theory, one considers all processes (compatible with classical conservation laws) that bodies composed of the prescribed material might admit. Moreover, there exist for the theory some universal physical principles that have been abstracted from experience. Therefore one can reduce the generality of the constitutive relations of dependent material variables by relying upon these principles. The most important of these principles is the second law of thermodynamics.

Mathematicians interested in continuum thermodynamics are generally not aware of the differences in the various postulations of the second law of thermodynamics. Virtually the same is true for many continuum mechanicians; in particular it is surprising how shallowly and mechanically many continuum mechanicians handle the second law. It appears that they have superfically learned how Coleman-Noll apply the Clausius-Duhem inequality and use it as a machine to generate inferences with little contemplation whether the deduced results make physically sense. In this paper we will make an attempt to explain how the basic postulates of two forms of the entropy principle differ from one another and then demonstrate that they yield different results. It is these results which allow us to favour one set of basic postulates over the other. The two entropy principles are the

(i) generalized Clausius-Duhem – Coleman-Noll approach (CD-CN),

(ii) Müller-Liu entropy principle (ML).

We will make clear below what we mean by "generalized Clausius-Duhem approach". Our demonstration of the essential steps in these two principles will include only the most important mathematical steps and omit significant details that would detract from the main ideas. The reader can fill in these details himself by reading the pertinent literature.

In Section 2, we present the two approaches of the entropy principles according to a set of generalized field equations and constitutive relations and compare their differences. Section 3 is devoted to the representation of the constitutive equations of both (i) a dry granular material and (ii) a multiphase mixture from thermodynamic considerations of the Müller-Liu approach. Furthermore, these are compared with those of CD-CN. In order to assess the implications of the theories, we consider in Sec. 4 a specific boundary-value problem, namely gravity-flow down an inclined plane for the dry granular material and the solid-fluid mixture. In Section 5, this paper is summarized.

2. On entropy principles

In this section we will explain how the two exploitations of the entropy principle are made and what postulates are underlying them. We then demonstrate how they differ from one another. The constitutive class for which this comparison is implemented is a restricted one in which constitutive relations express a dependent variable as a function of its independent variables (and not a functional), or for which a constitutive relation may be expressed as a differential equation among some variables. The constitutive class fathomed by this assumption is still very large and covers most solid, fluid and mixture theories including many dealing with hereditary effects.

2.1. Basic equations with source terms

Consider a field theory for a number of field variables $\mathbf{u} = (\mathbf{u}_i, \mathbf{u}_d)$ defined over the body. Let u_i be the independent fields, i.e., those field variables for which the theory provides field equations. Let, moreover, $\mathbf{u}_d$ be the dependent field variables which are functionally expressed in terms of the independent fields. Let $\mathbf{s}$, s^{ε} be source terms, arbitrary known functions defined over the body and over time.

Any continuum mechanical field theory consists of the following statements:

• Balance laws

$$\mathcal{F}(\mathbf{u}_i, \mathbf{u}_d) - \mathbf{s} = \mathbf{0}, \qquad f^{\varepsilon}(u_i, u_d) - s^{\varepsilon} = 0. \tag{2.1}$$

These are for instance the balance laws of momentum, angular momentum and energy, but in electromagneto-mechanical applications they can also include some of the Maxwell equations. In (2.1) we have singled out one scalar-type equation – the one with the superscript ε – from the others; this is the energy equation. $\mathcal{F}$ and f denote functional differential operators involving differentiations of space and time.

• Constraint relations and source-free balance laws

It is often so that the field variables are subjected to constraint conditions which are either of kinematic or thermomechanical nature. These constraint conditions are also expressible as functional relations between the field variables $(\mathbf{u}_i, \mathbf{u}_d)$,

$$\mathcal{C}(\mathbf{u}_i, \mathbf{u}_d) = \mathbf{0}. \tag{2.2}$$

For example, an incompressible material is kinematically constrained by the equation $\det \mathbf{F} = 1$ where $\mathbf{F}$ is the deformation gradient or $\det \mathbf{C} = 1$, where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. The corresponding Eq. (2.2) is

$$\operatorname{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1}) = 0 \qquad \text{or} \qquad \dot{\gamma} = 0, \tag{2.3}$$

where γ is the material mass density. Another example of a constraint condition is the *saturation condition* in a porous mixture of a soild and a fluid. It states that the water fills the entire pore space. If ν_f and ν_s are the fluid and solid volume fractions, then the constraint condition requires $\nu_f + \nu_s = 1$ or

$$\frac{d}{dt}(\nu_f + \nu_s) = 0. \tag{2.4}$$

Source-free balance relations are also of the form expressed by Eq. (2.2). The conservation equation of mass is of such a form, or the constituent balance laws of mass in a mixture of a finite number of constituents. In these latter laws production terms can enter due to phase changes or chemical reactions. These are no source terms as their origin is within the body and not external.

It is often so that authors introduce external source terms in evolution equations of the type (2.2) to make them of type (2.1). In most situations the reason is mathematical, but there is no justification on physical grounds to do so. For instance, to add an arbitrary source term to a balance law of mass is physically not justifiable. Neither can balance relations for hidden variables have such external source terms simply because they express something about the microstructure of the body which is entirely internal to the body. Equations like this are the equilibrated force balances in granular theories, the balance laws for configurational forces used in connection with phase changes, the spin balances in polar theories such as micropolar, micromorphic and liquid crystal theories etc.

• Constitutive relations

The constitutive relations are functional relations between the dependent fields $\mathbf{u}_d$ and the independent fields $\mathbf{u}_i$. When $\mathbf{u}_d$ are expressed as functional relations of $\mathbf{u}_i$, they read

(2.5) $$\mathbf{u}_d = \mathcal{M}(\mathbf{u}_i).$$

It is for these relations that we have divided the field variables into $\mathbf{u}_i$ and $\mathbf{u}_d$. Examples are an equation of the stress tensor in terms of the strain tensor in an elastic constitutive relation, or the heat flux expressed as being affine to the temperature gradient.

In most continuum thermodynamic theories it is stipulated that the balance laws and constitutive relations together define a well-posed problem; in other words, with appropriate initial and boundary conditions these equations are supposed to yield unique functions of space and time for the field variables, at least for some finite non-zero interval of time. When constraint conditions are added, additional variables enter the theory, which represent the constraint stresses or forces that must be applied to guarantee the maintainance of the constraint conditions. These additional fields are not contained in $\mathbf{u}_i$ and $\mathbf{u}_d$ of (2.5).

Combining (2.1), (2.2) and (2.5) yields

(2.6) $$\left.\begin{aligned} &\mathcal{F}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) - \mathbf{s} = \mathbf{0}, \\ &f^{\varepsilon}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) - s^{\varepsilon} = 0, \end{aligned}\right\} \mathbb{F}(\mathbf{u}_i) - \mathbf{s} = \mathbf{0},$$
$$\mathbf{C}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) = \mathbf{0}.$$

These equations are called *field equations.* Any set of $\mathbf{u}_i$ that satisfies the Eq. (2.6) is called a *thermodynamic process*. In reality the constitutive functions (2.5) are not arbitrary, they should obey universal physical principles, i.e., one can reduce the generality of these functions by relying upon these physical principles. The most important of these principles is the second law of thermodynamics, which we now introduce in the form of the *entropy principle*.

There exists an entropy density η, entropy flux $\boldsymbol{\phi}$, entropy production density π^η and entropy supply density s^η, which obey a balance law. The second law of thermodynamics requires that the following inequality is satisfied,

$$\pi^\eta := \mathcal{H}(\eta, \boldsymbol{\phi}) - s^\eta \geqslant 0. \tag{2.7}$$

Now, any process which satifies (2.7) (via the constitutive relations) represents a so-called physically admissible process. The entropy inequality, however, must not hold for arbitray fields $\mathbf{u}_i$, but only for thermodynamic processes, i.e. solutions of the field equations. The *working principle* is therefore that all thermodynamic processes must satisfy (2.7) or all fields which satisfy (2.6) must in addition satisfy (2.7). We must point out that as long as η, $\boldsymbol{\phi}$, s^η are not related to any of the quantities in (2.6), the second law is an empty statement. *Various second laws differ by the method how this link is made.* Here, we will shortly present two evaluation methods of the second law.

2.2. Generalized Coleman-Noll evaluation of the Clausius-Duhem inequality

It is assumed that

- there exists an absolute temperature θ,
- there exist *a priori postulates* by which the entropy supply rate density s^η and the entropy flux $\boldsymbol{\phi}$ are connected to some field variables of Eqs. (2.1) and (2.2). For instance, in the classical Clausius-Duhem inequality one postulates the relations

$$s^\eta = \frac{s^\varepsilon}{\theta}, \qquad \boldsymbol{\phi} = \boldsymbol{\phi}^{\mathcal{M}}(\mathbf{u}_i) = \frac{\mathbf{q}}{\theta}(\mathbf{u}_i), \tag{2.8}$$

where s^ε, $\mathbf{q}$ represent the energy supply density and energy flux density vector. Most authors use (2.8) if they apply the CD-CN procedure for the exploitation of the entropy inequality. In mixture theories, however, supporters of the CD-approach generally recognize that entropy flux and heat flux need not necessarily be collinear. In those instances entropy and energy balance statements are formally written down for each constituent, and (2.8) is replaced by

$$s_a^\eta = \frac{s_a^\varepsilon}{\theta_a}, \qquad \boldsymbol{\phi}_a = \boldsymbol{\phi}_a^{\mathcal{M}}(\mathbf{u}_i) = \frac{\mathbf{q}_a(\mathbf{u}_i)}{\theta_a}, \qquad a \in (1, ..., N), \tag{2.9}$$

where a is a counting index for the number of constituents. The second law of thermodynamics is then here expressed as a statement concerning the entropy balance as a whole; it requires its entropy production to be non-negative. We will encounter this example below. Finally, notice that in (2.9) we have assumed each constituent to possess its own temperature. Of course we may also specialize these relations to constituents having the same temperatures

- η is a constitutive quantity with constitutive relation

$$\eta = \eta^{\mathcal{M}}(\mathbf{u}_i). \tag{2.10}$$

Combination of the energy Eq. $(2.1)_2$ and the entropy inequality (2.7) by use of the postulate (2.8) yields

$$\underbrace{\mathcal{H}\left(\eta^{\mathcal{M}}(\mathbf{u}_i), \boldsymbol{\phi}^{\mathcal{M}}(\mathbf{u}_i)\right) - \frac{1}{\theta} f_\varepsilon\left(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)\right)}_{\mathbb{H}(\mathbf{u}_i)} \geqslant 0 \quad \text{or} \quad \mathbb{H}(\mathbf{u}_i) \geqslant 0, \tag{2.11}$$

which should be satisfied for all thermodynamic processes. This form of the entropy inequality no longer contains any source terms.

A clear formulation of the fundamental approach to the exploitation of the Second Law is due to COLEMAN and NOLL [1]. It is as follows: The "universe" is such that there can always be found a neighbourhood of a material point such that the sources $\mathbf{s}$ may have any value. In the CN-CD approach one assumes all balance Eqs. (2.1) to contain free source terms, therefore only relations (2.2) constrain the independent $\mathbf{u}_i$ in the exploitation of the inequality (2.11). It has been shown by LIU [7] that for constitutive relations of the class restricted above, satisfaction of (2.11) for all fields constrained by (2.2) is equivalent to satisfying the inequality

$$\mathbb{H}(\mathbf{u}_i) - \boldsymbol{\lambda}_{\mathcal{C}} \cdot \mathcal{C}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) \geqslant 0, \quad \forall \mathbf{u}_i \tag{2.12}$$

for unconstrained fields u_i, where $\boldsymbol{\lambda}_{\mathcal{C}}$ represent the corresponding Lagrange multipliers. When the constitutive Eqs. (2.5) are introduced into inequality (2.12) and all the indicated differentiations are performed, this inequality can be written in the form

$$\mathbf{a}(\mathbf{u}_i) \cdot (D\mathbf{u}_i) + b(\mathbf{u}_i) \geqslant 0, \tag{2.13}$$

where $D\mathbf{u}_i$ represent new emerging temporal and spatial derivatives of the independent variables $\mathbf{u}_i$ [1], which are not included in the constitutive relations (2.5), hence inequality (2.13) is linear in $D\mathbf{u}_i$. Since the inequality must hold for all fields $\mathbf{u}_i$ and the variables $D\mathbf{u}_i$ can hence take any values, the inequality could be violated unless

$$\mathbf{a}(\mathbf{u}_i) = 0, \; b(\mathbf{u}_i) \geqslant 0. \tag{2.14}$$

[1]) The variables $D\mathbf{u}_i$ may and generally do arise in the balance laws (2.1), but since source terms are present, these equations can always be fulfilled by selecting the external sources accordingly. Thus these equations do not influence inequality (2.13).

We recall that the main purpose of the entropy principle is to derive restrictions upon the constitutive relation (2.5). With relations (2.14) the following results can be obtained:

- reduced dependences of constitutive relations,
- thermostatic equilibrium relations for constitutive quantities,
- thermodynamic potential relations,
- Gibbs relation.

In particular, entropy, internal energy and free energies depend in general only on a reduced number of variables, always those of thermostatic equilibrium. Thus, these variables have dependences in non-equilibrium as if the non-equilibrium states would correspond to an equilibrium. This is a disadvantage and perhaps also questionable, because statistical mechanics shows that the non-equilibrium entropy should depend on non-equilibrium variables, such as strain rate and temperature gradient, if the Enskog procedure is pushed to second iterates. Incidentally, non-collinearity of the entropy flux to the heat flux is also shown by the same Enskog procedure.

Some important points relating to the CD-CN approach should be made:

• When there are no constraints, only the energy equation has an influence on the result (2.14).

• To preserve the property that all balance equations contain free source terms, authors often invent source terms without physical motivation, e.g., for mass balances, structure balance laws, etc. In such cases the results obtained from the Coleman-Noll exploitation of the entropy inequality are dubious.

• When besides θ also $\dot{\theta}$ is an independent constitutive variable in the constitutive relations (2.5), this approach is *a priori* in doubt because the existence of absolute temperature is questionable under those circumstances except in equilibrium.

• When mixtures with distinct constituent temperatures are considered, the method is equally in doubt.

2.3. Müller-Liu's entropy principle

In the CD-CN approach, the flux and the supply of entropy are related *a priori* to the flux and supply of heat. And, free sources are assumed for all balance equations except perhaps the balance of mass. In order to relax these assumptions, MÜLLER [9] proposed an entropy principle in which the entropy and its flux are both *a priori* unrestricted constitutive quantities. LIU [7] introduced Lagrange multipliers to consider the influences of all balance laws on the entropy inequality, by which the exploitation of the general entropy inequality is much facilitated.

It is assumed that

- θ is an empirical temperature,

• the entropy density η and the entropy flux $\boldsymbol{\phi}$ are general constitutive relations and no *a priori* postulates are introduced,

$$\eta = \eta^{\mathcal{M}}(\mathbf{u}_i), \qquad \boldsymbol{\phi} = \boldsymbol{\phi}^{\mathcal{M}}(\mathbf{u}_i), \tag{2.15}$$

• source terms do not affect the material behaviour.

To satisfy the entropy inequality (2.7) for all thermodynamic processes, all field Eqs. (2.6) serve as constraints for the inequality (2.7). It follows that

$$\mathcal{H}\left(\eta^{\mathcal{M}}(\mathbf{u}_i), \boldsymbol{\phi}^{\mathcal{M}}(\mathbf{u}_i)\right) - \boldsymbol{\Lambda}\cdot\mathbb{F}(\mathbf{u}_i) - \boldsymbol{\lambda}_{\mathcal{C}}\cdot\mathcal{C}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) + (\boldsymbol{\Lambda}\cdot\mathbf{s} - s^{\eta}) \geqslant 0, \tag{2.16}$$

where $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}, \lambda^{\varepsilon})$, $\boldsymbol{\lambda}_{\mathcal{C}}$ represent Lagrange multipliers. The above third assumption requires

$$s^{\eta} = \boldsymbol{\Lambda}\cdot\mathbf{s}, \tag{2.17}$$

so that the entropy supply s^{η} is known as soon as $\boldsymbol{\Lambda}$ is determined. By evaluation of the entropy inequality (2.16) for a given constitutive class, the following variables or relations can be obtained:

• Lagrange multipliers $\boldsymbol{\Lambda}$, $\boldsymbol{\lambda}_{\mathcal{C}}$,

• reduced dependences of constitutive relations,

• thermostatic equilibrium relations for constitutive quantities,

• Gibbs relation,

• thermodynamic potential relations.

It is important to emphasize that these results differ from those of the classical evaluation of the entropy inequality of Coleman-Noll in the following respects:

• This second law holds for open and for closed systems.

• Results are in many cases the same as for the CD-CN approach, but not when the theories are complex. **As a rule**: Differences are likely to occur when structural variables enter the formulation such as for the Cosserat continua, liquid crystals, gradient theories, porous media.

• Experience shows that when results between the two entropy principles differ, those obtained by the Müller-Liu principle are generally physically better founded.

In particular we note that entropy, internal and free energies may depend on non-equilibrium variables yielding a different Gibbs relation than that obtained with the CD-CN approach. As a rule, the differences occur primarily in thermodynamic non-equilibrium, but not exclusively. For instance, in the theory of liquid crystals the orientation field of the rodlike molecules in equilibrium is determined by the entropy flux contribution that is not collinear with the heat flux vector. If the Clausius-Duhem inequality were true, the orientation field in thermodynamic equilibrium would be arbitrary, and hence chaotic. We would never be able to read on our laptop screen what we write if the screen is a liquid crystal display.

In granular media of elongated particles (rice) the situation must be very much the same.

3. Consequences of the entropy principles

In this section, results are presented that are obtained with the use of the entropy principle of Müller-Liu and compared with those of the classical evaluation of the entropy inequality of Coleman-Noll for a granular solid with a scalar structure equation, and a saturated mixture of granular/fluid constituents with scalar structure equations for each constituent.

The necessary thermal and mechanical field variables are introduced as primitive quantities. Specifically, there exists a kinematic variable, the volume fraction or volume distribution function ν (see e.g. GOODMAN and COWIN [4], WANG and HUTTER [17] for a granular material and PASSMAN *et al.* [13], WANG and HUTTER [18] for a solid-fluid mixture). It is complemented by the distributed mass density (true mass density) γ, the stress tensor $\mathbf{T}$, body force $\mathbf{b}$, specific internal energy ε, heat flux vector $\mathbf{q}$ and heat supply r. In addition, to account for the energy flux and energy supply associated with the time rate of change of the volume distribution, a higher order stress and body force were introduced by GOODMAN and COWIN [4]. An equilibrated inertia k, equilibrated stress vector $\mathbf{h}$ and intrinsic equilibrated body force f are introduced. For a solid-fluid mixture, the above listed variables should be denoted with an added subscript a for each constituent a, with $a = s$ for the solid and $a = f$ for the fluid, respectively.

3.1. Granular material

For a granular material, the distributed solid body must satisfy the basic laws of motion of continuum mechanics. Accordingly, the following balance equations must be satisfied:

$$
\begin{aligned}
\mathcal{R} &:= \dot{\overline{\gamma\nu}} + \gamma\nu \operatorname{div} \mathbf{v} = 0, \\
\mathbf{M} &:= \gamma\nu\dot{v} - \operatorname{div} \mathbf{T} - \gamma\nu \mathbf{b} = \mathbf{0}, \\
\mathcal{N} &:= \gamma\nu k\ddot{\nu} - \operatorname{div} \mathbf{h} - \gamma\nu f = 0, \\
\mathcal{E} &:= \gamma\nu\dot{\varepsilon} - \mathbf{T}\cdot\mathbf{D} - \mathbf{h}\cdot \operatorname{grad}\dot{\nu} + \gamma\nu f\dot{\nu} + \operatorname{div}\mathbf{q} - \gamma\nu r = 0,
\end{aligned} \tag{3.1}
$$

where $\dot{\overline{(\bullet)}}$ indicates the material time derivative. The balance Eqs. $(3.1)_{1,2}$ are analogous to the classical balance equations of mass and linear momentum. The third equation is a scalar structure equation, which describes the balance of equilibrated force (see GOODMAN and COWIN, [4]). The conservation of energy $(3.1)_4$ differs from the traditional statements by considering the works of equilibrated force. $(3.1)_{1,3}$ are source-free equations and thus belong to the class (2.2); $(3.1)_{2,4}$ do have source terms, $\mathbf{b}$ and r. We also point out that GOODMAN and COWIN [4] and PASSMAN *et al.* [13] also introduce a source term in $(3.1)_3$, thus making this equation to have no influence in the CD-CN entropy principle.

For the granular material, the following independent constitutive variables are postulated:

$$\mathcal{C} = \hat{\mathcal{C}}(\nu,\, \operatorname{grad}\nu, \dot{\nu}, \gamma, \theta,\, \operatorname{grad}\theta, \mathbf{D}) \tag{3.2}$$

for the dependent constitutive variables $\mathcal{C} = \{\psi, \eta, \mathbf{T}, \mathbf{h}, f, \mathbf{q}, \boldsymbol{\phi}\}$. The forms of these constitutive relations are reduced by the entropy inequality (2.7), which here can be written as

$$\Pi = \gamma\nu\dot{\eta} + \operatorname{div}\boldsymbol{\phi} - \gamma\nu s \geqslant 0. \tag{3.3}$$

According to Müller-Liu's entropy principle, the following inequality must be satified for all physical processes

$$\Pi = \rho\dot{\eta} + \operatorname{div}\boldsymbol{\phi} - \rho s - 1/\theta\left\{\lambda^{\nu}\mathcal{R} + \boldsymbol{\lambda}^{v}\cdot\mathbf{M} + \lambda^{k}\mathcal{N}\right\} - \lambda^{\varepsilon}\mathcal{E} \geqslant 0, \tag{3.4}$$

in which the balance relations (3.1) appear as constraints on the entropy inequality, where λ^{ν}, $\boldsymbol{\lambda}^{v}$, λ^{k} and λ^{ε} represent the Lagrange multipliers. For convenience, a factor $1/\theta$ has been extracted above from λ^{ν}, $\boldsymbol{\lambda}^{v}$ and λ^{k}.

Substituting (3.2) into (3.4) and assuming material isotropy, the corresponding restrictions on forms such as (3.2) have been obtained elsewhere (WANG and HUTTER, [17]). By assumptions that the inner free energy, which is defined by $\psi = \varepsilon - \theta\eta$, does not depend on $\dot{\nu}$, $\psi \neq \hat{\psi}(\bullet, \dot{\nu})$, and supposing the Lagrange multiplier for the energy equation to be $\lambda^{\varepsilon} = 1/\theta$ [2], we can obtain the expressions for the Lagrange multipliers

$$\boldsymbol{\lambda}^{v} = \mathbf{0}, \qquad \lambda^{k} = 0, \qquad \lambda^{\nu} = -\gamma\frac{\partial\psi}{\partial\gamma} \tag{3.5}$$

and the reduced constitutive relations

$$\begin{aligned} &\psi = \hat{\psi}(\nu,\, \operatorname{grad}\nu\cdot\operatorname{grad}\nu, \gamma, \theta), \qquad \boldsymbol{\phi} = \mathbf{q}/\theta, \\ &\mathbf{h} = \gamma\nu\frac{\partial\psi}{\partial\operatorname{grad}\nu} = \mathcal{A}\operatorname{grad}\nu \quad \text{with } \mathcal{A} = 2\gamma\nu\frac{\partial\psi}{\partial(\operatorname{grad}\nu\cdot\operatorname{grad}\nu)}. \end{aligned} \tag{3.6}$$

In thermodynamic equilibrium, which is defined by $(\dot{\nu}, \operatorname{grad}\theta, \mathbf{D}) = \mathbf{0}$ and denoted by the superscript E, the stress $\mathbf{T}$, the heat flux $\mathbf{q}$ and the intrinsic equilibrated body force f can be expressed as

$$\mathbf{T}^{E} = -\nu p\mathbf{I} - \mathcal{A}\operatorname{grad}\nu\otimes\operatorname{grad}\nu, \qquad \mathbf{q}^{E} = \mathbf{0}, \qquad f^{E} = \frac{p-\beta}{\gamma\nu}, \tag{3.7}$$

[2]) This assumption is not reasonable in cases when $\dot{\theta}$ should also be an independent constitutive variable. Since we will not include such a dependence, the a priori assignment $\lambda^{\varepsilon} = 1/\theta$ is justifiable on the basis that Müller and Liu have proved it in LIU and MÜLLER [8].

where p represents the thermodynamic pressure and β the configuration pressure,

$$p = \gamma^2 \frac{\partial \psi}{\partial \gamma}, \qquad \beta = \gamma \nu \frac{\partial \psi}{\partial \nu}. \tag{3.8}$$

For incompressible granular grains $\gamma = \text{const}$, so p is an independent field variable and can no longer be determined by the free energy ψ as expressed in $(3.8)_1$. One can prove this point by means of two different methods. *One* is: We return to the constitutive assumptions (3.2) and note that, in view of the restriction $\gamma = \text{const}$, the list of variables appearing in (3.2) is no longer independent. We delete γ from the constitutive equations and repeat the above analysis. *The other method* is based on the method of Lagrange multipliers. We begin with the same constitutive postulates (3.2), but consider $\dot{\gamma} = 0$ as a new constraint, which can be combined to the entropy inequality (3.4) with a new Lagrange multiplier, and then repeat the above evaluation of the entropy inequality. The two approaches yield the same results.

Some results are summarized in the following points:

• The entropy flux $\boldsymbol{\phi}$ is in general not collinear with the heat flux with $(1/\theta)$ as a factor. Only when the Helmholtz free energy is assumed to be not a function of $\dot{\nu}$ $[\psi \neq \psi(\bullet, \dot{\nu})$ – note the rule of equipresence may be violated in this case], the classical result $\boldsymbol{\phi} = \mathbf{q}/\theta$ does hold.

• If a free source term in the equilibrated force balance would have been permitted, one would have proved $\boldsymbol{\phi} = \mathbf{q}/\theta$ under all circumstances.

• $\mathcal{A} \neq 0$ gives rise to Mohr-Coulomb yield stresses in thermodynamic equilibrium provided the volume fraction is *non-uniform.*

• CD-CN and M-L yield different results *under dynamic, but not static conditions.*

3.2. Granular-fluid mixture theory

Similar to the process in Subsec. 3.1 for a granular material, we can obtain reduced constitutive relations for a saturated solid-fluid mixture. Details can be found in WANG and HUTTER [18]. The corresponding balance equations are

$$\begin{aligned}
\mathcal{R}_a &:= \acute{\rho}_a + \rho_a \operatorname{div} \mathbf{v}_a = 0, \\
\mathbf{M}_a &:= \rho_a \acute{\mathbf{v}}_a - \operatorname{div} \mathbf{T}_a - \rho_a \mathbf{b}_a - \mathbf{m}_a^+ = 0, \\
\mathcal{N}_a &:= \rho_a k_a \acute{\acute{\nu}}_a - \operatorname{div} \mathbf{h}_a - \rho_a f_a = 0, \\
\mathcal{E} &:= \rho \dot{\varepsilon} + \operatorname{div} \mathbf{q} - \mathbf{T} \cdot \mathbf{D} - \textstyle\sum \mathbf{h}_a \cdot \operatorname{grad} \acute{\nu}_a + \textstyle\sum \rho_a f_a \acute{\nu}_a - \rho r = 0
\end{aligned} \tag{3.9}$$

for the solid phase ($a = 1 = s$) and the fluid phase ($a = 2 = f$), respectively. Here, $\acute{f}_a = \partial f / \partial t + (\operatorname{grad} f_a) \cdot \mathbf{v}_a = \dot{f}_a + (\operatorname{grad} f_a) \cdot \mathbf{u}_a$ ($\mathbf{u}_a = \mathbf{v}_a \cdot \mathbf{v}$) is the material time derivative with respect to $\mathbf{v}_a$, while $\dot{f}_a = \partial f/\partial t + (\operatorname{grad} f_a)\cdot\mathbf{v}$ is the material time derivative with respect to the mixture velocity, $\mathbf{v}\cdot\mathbf{m}_a^+$ is internal growth of linear momentum of the constituent a with the condition $\mathbf{m}_s^+ + \mathbf{m}_f^+ = 0$. It is assumed that the energy exchange between the fluid and the solid constituents is so efficient that the mixture can be characterized by a single temperature θ, and we need then work only with the energy balance of the mixture $(3.9)_4$ and the entropy inequality of the mixture

$$\Pi = \rho\dot{\eta} + \operatorname{div}\boldsymbol{\phi} - \rho s \geqslant 0. \tag{3.10}$$

The constituent and mixture fields and fluxes are connected by the sum relations

$$\begin{gathered}\rho = \sum \rho_a, \quad \mathbf{v} = \sum \xi_a \mathbf{v}_a, \quad \varepsilon = \varepsilon_I + \frac{1}{2}\Sigma\xi_a\mathbf{u}_a\cdot\mathbf{u}_a, \quad r = \sum\xi_a r_a,\\ \eta = \sum\xi_a\eta_a, \quad s = \sum\xi_a s_a, \quad \Pi = \sum\eta_a^+, \quad \mathbf{T} = \sum(\mathbf{T}_a - \rho_a\mathbf{u}_a\otimes\mathbf{u}_a),\\ \boldsymbol{\phi} = \sum(\boldsymbol{\phi}_a + \rho_a\eta_a\mathbf{u}_a), \quad \mathbf{q} = \sum\left\{\mathbf{q}_a - \left[\mathbf{T}_a - \rho_a(\varepsilon_a + \frac{1}{2}\mathbf{u}_a\cdot\mathbf{u}_a)\mathbf{I}\right]\mathbf{u}_a\right\}\end{gathered} \tag{3.11}$$

with $\sum = \sum_{a=1}^{2}$ and $\xi_a = \rho_a/\rho$.

A physical process must simultaneously satisfy (3.9) and (3.10) as well as other possible additional constraint relations, such as that of saturation

$$\nu_s + \nu_f = 1 \quad \longrightarrow \quad \mathcal{S} := \sum(\acute{\nu}_a - \mathbf{u}_a\cdot\operatorname{grad}\nu_a) = 0. \tag{3.12}$$

According to the Müller-Liu approach of the entropy inequality, one can account for all these requirements by requesting

$$\Pi = \rho\dot{\eta} + \operatorname{div}\boldsymbol{\phi} - \rho s - 1/\theta\left\{\sum\lambda_a^\nu\mathcal{R}_a + \sum\boldsymbol{\lambda}_a^v\cdot\mathbf{M}_a + \sum\lambda_a^k\mathcal{N}_a + \pi\mathcal{S}\right\} - \lambda^\varepsilon\mathcal{E} \geqslant 0. \tag{3.13}$$

This entropy inequality (3.13) applies to the general class of two-phase media. Each class is characterized by particular constitutive postulates. For the fluid-saturated granular material, the following independent constitutive variables are postulated

$$\mathcal{C}_s = \hat{\mathcal{C}}_s(\mathcal{S}_s), \qquad \mathcal{C}_f = \hat{\mathcal{C}}_f(\mathcal{S}_f), \qquad \mathbf{m}_s^+ = \hat{\mathbf{m}}_s^+(\mathcal{F}_s, \mathcal{F}_f) = -\mathbf{m}_f^+ \tag{3.14}$$

for $\mathcal{C}_a \in \{\varepsilon_a, \eta_a, \mathbf{T}_a, \mathbf{h}_a, \mathbf{q}_a, \boldsymbol{\phi}_a\}$, as well as the mechanical interactions $\mathbf{m}_s^+$, with

$$\begin{gathered}\mathcal{S}_a = (\nu_a, \operatorname{grad}\nu_a, \acute{\nu}_a, \gamma_a, \operatorname{grad}\gamma_a, \theta, \operatorname{grad}\theta, \mathbf{D}_a),\\ \mathcal{F}_a = (\nu_a, \operatorname{grad}\nu_a, \acute{\nu}_a, \gamma_a, \operatorname{grad}\gamma_a, \theta, \operatorname{grad}\theta, \mathbf{u}_a, \mathbf{D}_a, \mathbf{W}_a),\end{gathered} \tag{3.15}$$

where $\mathbf{W}_a$ is the skew-symmetric part of grad $\mathbf{v}_a$, $\mathcal{W}_a$ the difference $\mathcal{W}_a = \mathbf{W}_a - \mathbf{W}$, with $\mathbf{W} = \operatorname{skw}(\operatorname{grad}\mathbf{v})$, respectively. Here, the principles of phase separation and material objectivity have been assumed.

Substituting (3.14) into (3.13) and assuming material isotropy, the corresponding restrictions on the constitutive relations (3.14) have been obtained elsewhere (WANG and HUTTER, [18]. They are expressions for the constituent entropy flux $\boldsymbol{\Phi}_a$ and the equilibrated stress vector $\mathbf{h}_a$ as well as the dependence of the constituent inner specific free energy ψ_a, which is defined by $\psi_a = \varepsilon_a - \theta\eta$, viz.,

(3.16)
$$\begin{aligned} &\boldsymbol{\Phi}_a = \mathbf{q}_a/\theta, \qquad \psi_a = \hat{\psi}_a(\nu_a, \operatorname{grad}\nu_a \cdot \operatorname{grad}\nu_a, \gamma_a, \theta),\\ &\mathbf{h}_a = \rho_a \frac{\partial\psi_a}{\partial \operatorname{grad}\nu_a} = \mathcal{A}_a \operatorname{grad}\nu_a, \text{ with } \mathcal{A}_a = 2\rho_a \frac{\partial\psi_a}{\partial(\operatorname{grad}\nu_a \cdot \operatorname{grad}\nu_a)} \end{aligned}$$

and the expressions for the heat flux $\mathbf{q}$, the intrinsic equilibrated body force f_a, the stress $\mathbf{T}_a$ and internal growths of linear momentum $\mathbf{m}_a^+$ in thermodynamic equilibrium (denoted by the superscript E, defined by $(\acute{\nu}_s, \acute{\nu}_f, \operatorname{grad}\theta, \mathbf{v}_s, \mathbf{v}_f, \mathbf{D}_s, \mathbf{D}_N) = \mathbf{0}$)

(3.17)
$$\begin{aligned} \mathbf{q}^E &= \mathbf{0}, \qquad f_a^E = \frac{p_a - \beta_a}{\gamma_a\nu_a} - \frac{\pi}{\gamma_a\nu_a},\\ \mathbf{T}_a^E &= -\nu_a\left(p_a + \gamma_a(\psi_I - \psi_a)\right)\mathbf{I} - \mathcal{A}_a \operatorname{grad}\nu_a \otimes \operatorname{grad}\nu_a,\\ \mathbf{m}_s^{+E} &= \pi \operatorname{grad}\nu_s + (\psi_I - \psi_s)(1-\xi_s)\operatorname{grad}(\nu_s\gamma_s)\\ &\qquad - (\psi_I - \psi_f)\xi_f \operatorname{grad}(\nu_f\gamma_f) = -\mathbf{m}_f^{+E}, \end{aligned}$$

where $\psi_I = \xi_s\psi_s + \xi_f\psi_f$ is the mixture inner free energy. The variables π, β_a and p_a all have the meaning of pressure. As the Lagrange multiplier associated with the saturation constraint π is called the *saturation pressure*, which is a new independent variable for a saturation mixture. β_a is the *configuration pressure* and p_a the *thermodynamic pressure*, respectively, which can be expressed as

(3.18)
$$\beta_a := \rho_a \frac{\partial\psi_a}{\partial\nu_a}, \qquad p_a := \gamma_a^2 \frac{\partial\psi_a}{\partial\gamma_a},$$

where the expression for p_a is only suitable for compressible constituents; otherwise, for incompressible constituents, i.e. constituents whose true mass density does not change, p_a is an independent field variable and can no longer be determined by the free energy ψ_a as expressed in $(3.18)_2$.

It is important to emphasize that these results differ from those obtained by evaluation of the entropy inequality using the Coleman-Noll approach for a solid-fluid mixture (PASSMAN *et al.*, [13]).

• The partial entropy flux $\boldsymbol{\phi}_a$ is in general not collinear with the heat flux with $(1/\theta)$ as a prefactor. Only in the case the Helmholtz free energy is assumed *not* to be a function of $\acute{\nu}_a$, $\forall a$, the result $\boldsymbol{\phi}_a = \mathbf{q}_a/\theta$ is obtained. But even then $\boldsymbol{\phi} \neq \mathbf{q}/\theta$, as

$$\boldsymbol{\phi} = \frac{\mathbf{q}}{\theta} - \frac{1}{\theta}\sum_a \left\{\rho_a\psi_a\mathbf{u}_a - \left(\mathbf{T}_a - \frac{\rho_a}{2}(\mathbf{u}_a\cdot\mathbf{u}_a)\mathbf{1}\right)\mathbf{u}_a\right\}.$$

• If the CD-CN approach is used *with* a source term in the equilibrated force balances, $\boldsymbol{\phi}_a = \mathbf{q}_a/\theta$ would have been obtained even with $\acute{\nu}_a$ as independent constitutive variables. This has been done so by PASSMANN *et al.* [13].

• If the same theory is developed using the CD-CN approach, i.e. by *a priori* setting $\boldsymbol{\phi}_a = \mathbf{q}_a/\theta$ and introducing a source term in the balance of equilibrated forces, then the equilibrium results are

$$\begin{aligned} &\mathbf{q}^E = 0, \qquad f_a^E = \frac{1}{\gamma_a\nu_a}(p_a - \beta_a - \pi), \\ &\mathbf{T}_a^E = -\nu_a p_a\mathbf{I} - \mathcal{A}_a \operatorname{grad}\nu_a \otimes \operatorname{grad}\nu_a, \qquad \mathbf{m}_s^{+E} = \pi \operatorname{grad}\nu_s = -\mathbf{m}_f^{+E}, \end{aligned} \tag{3.19}$$

and they are different from (3.17).

• When $\mathcal{A}_a = 0$, $\forall a \in (1, ..., N)$, no shear stresses can be supported in equilibrium. Therefore, volume fraction gradients as independent constitutive variables are important.

• Formally, for $\mathcal{A}_a = 0$, $\forall a \in (1, ..., N)$, the above formulas for the stresses and the interaction forces do not agree with the corresponding formulas obtained by SVENDSEN and HUTTER [16], even though with $\mathbf{h}_a = \mathbf{0}$ and $\breve{\nu}_a = 0$, $\forall a \in (1, ..., N)$, the two formulations are the same.

This result is no surprise: It says that "the limit of a theory need not be the theory of the limit".

Finally it should be stated that the reduced entropy production rate is independent of the saturation pressure. So, the constraint variable does not produce entropy, as it should for physical reasons, for details see WANG and HUTTER [18].

4. Application in inclined gravity-flow problem

In Sec. 3 we derived the equilibrium expressions of the stresses, heat fluxes and intrinsic equilibrated body forces by the restrictions of the entropy principle on the constitutive equations. We assumed that these quantities may be decomposed into the thermodynamic equilibrium parts (denoted by the superscript E) and the dynamic parts (denoted by the superscript D). Furthermore, for the dynamic parts a linear theory is considered, in which the dynamic parts of these quantities are linear in the dynamic variables. By substituting these expressions into the field equations we can obtain a system of the field equations to analyse the cases of steady isothermal flows of a layer of uniform thickness L of a granular material, as well as a granular-fluid mixture down a rough plate inclined at an angle θ to the horizontal (in our computations we take $\theta = 40°$), as shown in Fig. 1.

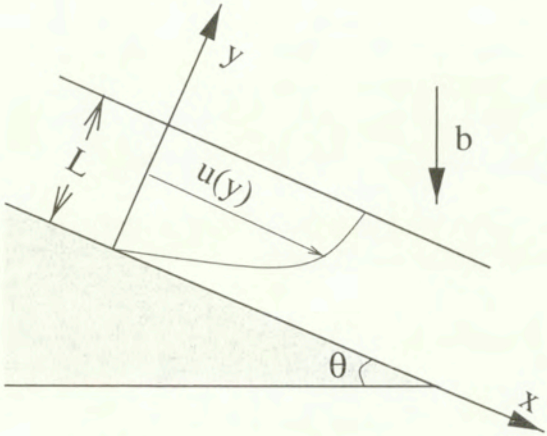


FIG. 1. Inclined gravity-flow and coordinate system.

4.1. Granular materials

Under the assumptions

$$\mathbf{T} = \mathbf{T}^E + \mathbf{T}^D, \qquad \mathbf{q} = \mathbf{q}^E + \mathbf{q}^D, \qquad f = f^E + f^D \tag{4.1}$$

and a linear theory for the dynamic parts

$$\mathbf{q}^D = -\kappa \operatorname{grad}\theta, \qquad \mathbf{T}^D = \xi\dot{\nu}\mathbf{I} + \lambda(\mathrm{tr}\mathbf{D})\mathbf{I} + 2\mu\mathbf{D}, \qquad f^D = -\zeta\dot{\nu} - \delta\mathrm{tr}\mathbf{D}, \tag{4.2}$$

we obtain with the use of (3.7), (4.2) in (4.1) the constitutive equations

$$\begin{aligned} \mathbf{T} &= [-p + \lambda\mathrm{tr}\mathbf{D} + \xi\dot{\nu}]\mathbf{I} - \mathcal{A}\operatorname{grad}\nu \otimes \operatorname{grad}\nu + 2\mu\mathbf{D}, \\ \mathbf{q} &= -\kappa \operatorname{grad}\theta, \qquad f = \frac{p}{\gamma\nu} - \frac{\beta}{\gamma\nu} - \delta\mathrm{tr}\mathbf{D} - \zeta\dot{\nu}. \end{aligned} \tag{4.3}$$

Substituting (4.3) into the field Eq. (3.1), using the expressions for the inner free energy ψ (PASSMAN *et al.*, [12]) and the viscosity μ (PASSMAN *et al.*, [12, 14]; SAVAGE, [15]),

$$\gamma\nu\psi = a_0(\nu - \nu_c)^2 + \alpha_0\left(\frac{\nu_c}{\nu_\infty - \nu}\right)^2 \operatorname{grad}\nu \cdot \operatorname{grad}\nu, \tag{4.4}$$

$$\mu = \mu_0\left(\frac{\nu_c}{\nu_\infty - \nu}\right)^4,$$

where ν_∞ is the volume fraction corresponding to the densest possible packing of the material, and ν_c is the critical volume fraction (GOODMAN and COWIN, [3]), assuming $\mathbf{v} = [u(y), 0, 0]$ and introducing the following dimensionless variables

$$\bar{y} = \frac{y}{\lambda}, \quad \bar{p} = \frac{p}{a_0}, \quad \bar{u} = u \Big/ \left(\frac{\gamma_0 b \lambda^2}{\mu_0}\right), \tag{4.5}$$

where λ is an internal length scale

$$\lambda = \sqrt{\frac{\alpha_0}{a_0}}, \tag{4.6}$$

we can conveniently formulate the granular flow problem down an inclined plane in terms of the dimensionless equations

$$\frac{d}{d\bar{y}}\left[\nu\bar{p} + 2\left(\frac{\nu_c}{\nu_\infty - \nu}\right)^2 \left(\frac{d\nu}{d\bar{y}}\right)^2\right] + S\cos\theta\nu = 0, \tag{4.7}$$

$$\frac{d}{d\bar{y}}\left[\left(\frac{\nu_c}{\nu_\infty - \nu}\right)^4 \frac{d\bar{u}}{d\bar{y}}\right] + \nu\sin\theta = 0, \tag{4.8}$$

$$\frac{d}{d\bar{y}}\left[2\left(\frac{\nu_c}{\nu_\infty - \nu}\right)^2 \frac{d\nu}{d\bar{y}}\right] + \bar{p} - \frac{1}{\nu}\left[(\nu - \nu_c)^2 - \left(\frac{\nu_c}{\nu_\infty - \nu}\right)^2 \left(\frac{3\nu - \nu_\infty}{\nu - \nu_\infty}\right)\left(\frac{d\nu}{d\bar{y}}\right)^2\right] = 0 \tag{4.9}$$

with the non-dimensional boundary conditions

$$\nu(0) = \nu_0,\ \bar{u}(0) = 0, \quad \frac{d\nu}{d\bar{y}}(\bar{L}) = 0;\ \frac{\partial\bar{u}}{\partial\bar{y}}(\bar{L}) = 0;\ \bar{p}(\bar{L}) = 0, \tag{4.10}$$

where the dimensionless parameter S represents the grain size, and $\bar{L}$ the dimensionless flow thickness, defined as

$$S = \frac{\gamma b \lambda}{a_0}, \qquad \bar{L} = \frac{L}{\lambda}, \tag{4.11}$$

respectively. Evidently, the problem is characterized by five dimensionless parameters ν_0, ν_∞, ν_c, S and $\bar{L}$. The variation of the first three is rather narrow and will not be studied here, but the remaining two typify the grain size and the flow depth in terms of the internal length scale. We choose

$$\nu_\infty = 0.644, \quad \nu_c = 0.555, \quad \nu_0 = 0.51, \tag{4.12}$$

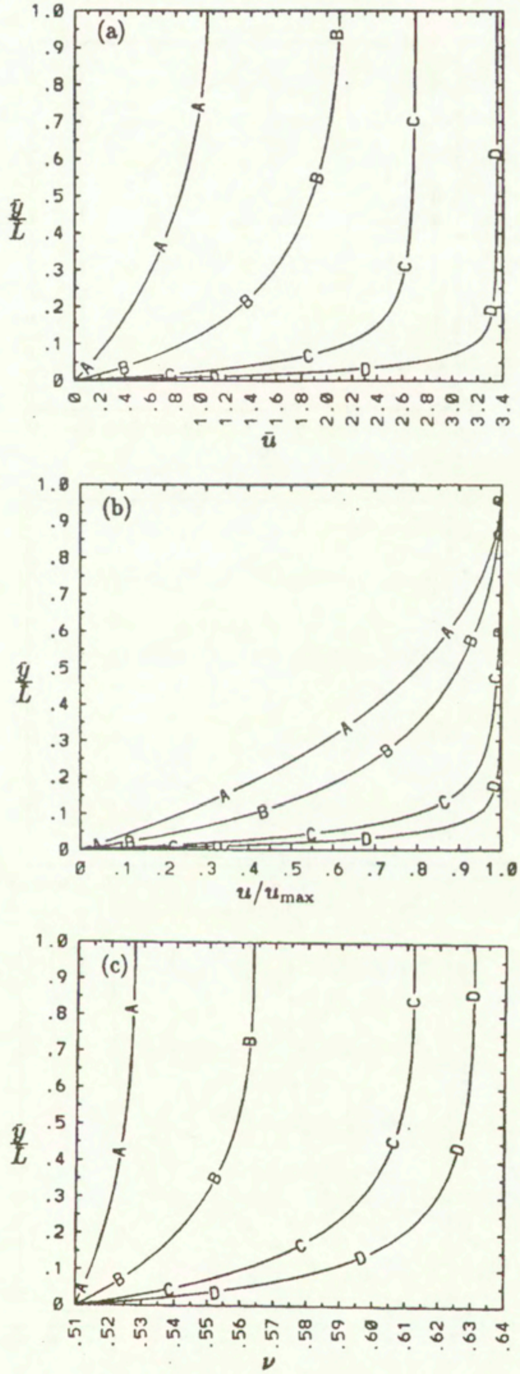


FIG. 2. Non-dimensional velocity profiles (a), normalized velocity profiles (b) and volume fraction profiles (c) for a fixed value of the parameter $S = 0.1$ and various values of $\bar{L}$: $\bar{L} = 5\,(A)$; $10\,(B)$; $20\,(C)$ and $30\,(D)$.

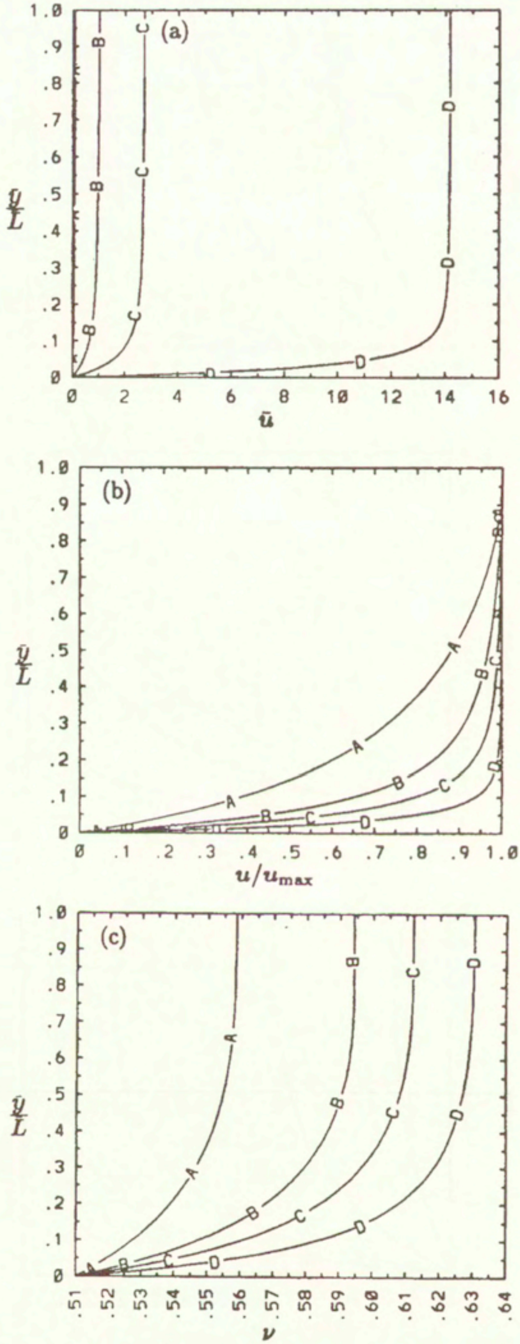


FIG. 3. Non-dimensional velocity profiles (a), normalized velocity profiles (b) and volume fraction profiles (c) for a fixed value of $\bar{L} = 20$ and various values of S: $S = 0.01\,(A)$; $0.05\,(B)$; $0.1\,(C)$ and $0.3\,(D)$.

values as given by SAVAGE [15], appropriate to natural angular beach sand with diameters of particles from 0.318 mm to 0.414 mm. Some typical results calculated for the volume distribution, dimensionless velocity for this gravity-flow problem, are shown in Figs. 2, 3 for various values of the dimensionless granular flow thickness $\bar{L}$ and the parameter S, indicating the grain size, respectively.

From Figs. 2, 3 the following results can be deduced:

• The granular velocity and the volume fraction depend strongly on the flow layer thickness and the grain size.

• For a fixed grain size (a fixed S), if the layer thickness is small, the shear can extend from the bottom to the free surface, which behaves much like an incompressible fluid, and the volume fraction has only a small change across the depth, whereas for thicker grain flow the flow structure is far from an incompressible fluid, in which in a large region near the free surface of the grain flow is similar to that of a plug flow, with a nearly constant velocity and less changed volume fraction, the shear layers close to the bottom, where dilatation has occurred, may be very thin.

• For a fixed $\bar{L}$ [3], indicating the ratio of the flow thickness to the grain size, fine grains (small values of S) show a velocity profile similar to that of a fluid. Increasing the grain size increases the manifestation of the granular character of the material, a plug flow near the free surface and a large shear with large dilatation near the bottom.

• The granular velocity increases with increasing flow thickness.

4.2. Granular-fluid mixture

Similarly, we assume that $\mathbf{T}_a$, f_a and $\mathbf{m}_a^+$ may be decomposed according to

$$\mathbf{T}_a = \mathbf{T}_a^E + \mathbf{T}_a^D, \qquad f_a = f_a^E + f_a^D, \qquad \mathbf{m}_a^+ = \mathbf{m}_a^{+E} + \mathbf{m}_a^{+D}, \tag{4.13}$$

where $\mathbf{T}_a^E$, f_a^E and $\mathbf{m}_a^{+E}$ represent the thermodynamic equilibrium parts, as displayed in (3.17), while $\mathbf{T}_a^D$, f_a^D and $\mathbf{m}_a^{+D}$ are their dynamic contributions. For the dynamic parts, scalar-, vector-, and tensor-valued quantities are assumed to depend explicitly and linearly on scalar-, vector-, and tensor-valued independent dynamic variables, respectively, by the forms

$$\mathbf{T}_a^D = 2\mu_a \mathbf{D}_a, \qquad f_a^D = \lambda_a \acute{\nu}_a, \qquad \mathbf{m}_s^{+D} = -m_D(\mathbf{v}_s - \mathbf{v}_f) = -\mathbf{m}_f^{+D}, \tag{4.14}$$

where μ_a, λ_a, m_D are the functions (PASSMAN *et al.*, [14]; WANG and HUTTER, [18])

$$\mu_s = \bar{\mu}_s \left(\frac{\nu_c}{\nu_\infty - \nu_s} \right)^4, \qquad \mu_f = \nu_f^2 \bar{\mu}_f, \qquad m_D = \nu_s (1 - \nu_s) D \tag{4.15}$$

[3]) Here, a fixed $\bar{L}$ means, if the grain size varies, that the flow thickness should be changed accordingly.

with $\bar{\mu}_s$, $\bar{\mu}_f$ and D being positive constants. To obtain the explicit expressions of $\mathbf{T}_a$, $\mathbf{m}_a^+$ and f_a, a representation for the specific free energy ψ_a for each constituent a is needed. Following PASSMAN *et al.* [14], we choose the expressions

$$\nu_s\gamma_s\psi_s = a_s[\nu_s - \nu_c]^2 + \bar{\alpha}_s\left(\frac{\nu_c}{\nu_\infty - \nu_s}\right)^2(\operatorname{grad}\nu_s\cdot\operatorname{grad}\nu_s), \tag{4.16}$$

$$\nu_f\gamma_f\psi_f = a_f[\nu_f - (1-\nu_c)]^2.$$

For the steady motion down the inclined plate, by assuming

$$\begin{aligned} &\mathbf{v}_s = [u_s(y),0,0], \qquad v_f = [u_f(y),0,0], \quad \nu_s = \nu_s(y), \qquad \nu_f = \nu_f(y),\\ &p_s = p_s(y), \quad p_f = p_f(y), \qquad \pi = \pi(y), \end{aligned} \tag{4.17}$$

$$\mathbf{b}_s = \mathbf{b}_f = [b\sin\theta, -b\cos\theta, 0]$$

and introducing the following dimensionless variables:

$$\bar{y} = \frac{y}{\lambda_s}, \qquad \bar{p}_s = \frac{p_s}{a_s}, \qquad \bar{p}_f = \frac{p_f}{a_s}, \qquad \bar{\beta}_s = \frac{\beta_s}{a_s}, \quad \bar{\beta}_f = \frac{\beta_f}{a_s}, \tag{4.18}$$

$$\bar{\pi} = \frac{\pi}{a_s}, \quad \bar{u}_s = u_s\Big/\left(\frac{\gamma_s g\lambda_s^2}{\bar{\mu}_s}\right), \bar{u}_f = u_f\Big/\left(\frac{\gamma_s g\lambda_s^2}{\bar{\mu}_s}\right),$$

where λ_s is an internal length scale of grains

$$\lambda_s = \sqrt{\frac{\bar{\alpha}_s}{a_s}}, \tag{4.19}$$

we can obtain the following dimensionless governing differential equations:

$$\nu_s + \nu_f = 1, \tag{4.20}$$

$$\begin{aligned} \frac{d}{d\bar{y}}\Bigg\{\nu_s\bar{p}_s + \xi_f\left[\left(\frac{\nu_s\zeta_a}{\nu_f\zeta_\gamma} - 1\right)(\nu_s-\nu_c)^2 - \frac{1}{(\nu_\infty-\nu_s)^2}\left(\frac{d^2\nu_s}{d\bar{y}^2}\right)\right] \\ +2\left(\frac{\nu_c}{\nu_\infty-\nu_s}\right)^2\left(\frac{d\nu_s}{d\bar{y}}\right)^2\Bigg\} - \Bigg\{\pi + \frac{(\xi_f^2-\zeta_\gamma\xi_s^2)\xi_f}{\nu_s} \\ \left[\left(\frac{\nu_s\zeta_a}{\nu_f\zeta_\gamma} - 1\right)(\nu_s-\nu_c)^2 - \frac{1}{(\nu_\infty-\nu_s)^2}\left(\frac{d^2\nu_s}{d\bar{y}^2}\right)\right]\Bigg\}\frac{d\nu_s}{d\bar{y}} + S\nu_s\cos\theta = 0, \end{aligned} \tag{4.21}$$

$$\frac{d}{d\bar{y}}\left\{\nu_f\bar{p}_f+\xi_s\frac{\nu_f}{\nu_s}\zeta_\gamma\left[\left(\frac{\nu_s\zeta_a}{\nu_f\zeta_\gamma}-1\right)(\nu_s-\nu_c)^2-\frac{1}{(\nu_\infty-\nu_s)^2}\left(\frac{d^2\nu_s}{d\bar{y}^2}\right)\right]\right\} \\ -\left\{\pi+\frac{(\xi_f^2-\zeta_\gamma\xi_s^2)\xi_f}{\nu_s}\left[\left(\frac{\nu_s\zeta_a}{\nu_f\zeta_\gamma}-1\right)(\nu_s-\nu_c)^2 \right.\right. \\ \left.\left. -\frac{1}{(\nu_\infty-\nu_s)^2}\left(\frac{d^2\nu_s}{d\bar{y}^2}\right)\right]\right\}\frac{d\nu_f}{d\bar{y}}+\zeta_\gamma S\nu_f\cos\theta=0, \tag{4.22}$$

$$\frac{d}{d\bar{y}}\left[\left(\frac{\nu_c}{\nu_\infty-\nu_s}\right)^4\frac{d\bar{u}_s}{d\bar{y}}\right]-\bar{D}\nu_s(1-\nu_s)(\bar{u}_s-\bar{u}_f)+\nu_s\sin\theta=0, \tag{4.23}$$

$$\frac{d}{d\bar{y}}\left[\zeta_\mu\nu_f^2\frac{d\bar{u}_f}{d\bar{y}}\right]+\bar{D}\nu_s(1-\nu_s)(\bar{u}_s-\bar{u}_f)+\zeta_\gamma\nu_f\sin\theta=0, \tag{4.24}$$

$$\bar{\pi}=\bar{p}_f-\bar{\beta}_f, \tag{4.25}$$

$$\bar{\beta}_s-\bar{\beta}_f=\bar{p}_s-\bar{p}_f+\frac{d}{d\bar{y}}\left[2\left(\frac{\nu_c}{\nu_\infty-\nu_s}\right)^2\frac{d\nu_s}{d\bar{y}}\right] \tag{4.26}$$

with the expressions for the dimensionless configuration pressures $\bar{\beta}_s$ and $\bar{\beta}_f$ as follows:

$$\nu_s\bar{\beta}_s=(\nu_s^2-\nu_c^2)+\frac{3\nu_s-\nu_c}{(\nu_\infty-\nu_s)^3}\left(\frac{d\nu_s}{d\bar{y}}\right)^2, \qquad \nu_f\bar{\beta}_f=\zeta_a(\nu_s^2-\nu_c^2),$$

where the dimensionless parameters S, $\bar{L}$, $\bar{D}$, ζ_γ, ζ_μ and ζ_a are defined as

$$S=\frac{\gamma b\lambda_s}{a_s}, \qquad \bar{L}=\frac{L}{\lambda_s}, \qquad \bar{D}=D\Big/\left(\frac{\bar{\mu}_s}{\lambda_s^2}\right), \qquad \zeta_\gamma=\frac{\gamma_f}{\gamma_s}, \\ \zeta_\mu=\frac{\mu_f}{\bar{\mu}_s}, \qquad \zeta_a=\frac{a_f}{a_s}, \tag{4.27}$$

respectively. Together with ν_0, ν_c, ν_∞ these are nine dimensionless quantities. Here S represents a dimensionless scale of the grain size, $\bar{L}$ the dimensionless flow thickness, $\bar{D}$ the dimensionless drag coefficient, ζ_γ the ratio of the fluid material density to that of the solid grain, ζ_μ the ratio of the viscosities and ζ_a the ratio of the "energy storage capacities" of two constituents. (4.20) – (4.27) is a system of seven equations with seven unknowns ν_s, ν_f, $\bar{\pi}$, $\bar{p}_s$, $\bar{p}_f$, $\bar{u}_s$, $\bar{u}_f$, which will be solved subjected to the boundary conditions

$$\nu_s(0)=\nu_0, \qquad \nu_f(0)=1-\nu_0, \qquad \bar{u}_s(0)=0, \qquad \bar{u}_f(0)=0, \\ \frac{d\nu_s}{d\bar{y}}(\bar{L})=0, \qquad \frac{d\nu_f}{d\bar{y}}(\bar{L})=0, \qquad \frac{\partial\bar{u}_s}{\partial\bar{y}}(\bar{L})=0, \qquad \frac{\partial\bar{u}_f}{\partial\bar{y}}(\bar{L})=0, \\ \bar{p}_s(\bar{L})=0, \qquad \bar{p}_f(\bar{L})=0. \tag{4.28}$$

We choose to investigate the case with estimated parameters corresponding to a mixture of water with natural beach sand with the given non-dimensional parameters listed in Table 1 (see SAVAGE, [15]; PASSMAN *et al.*, [14]).

Table 1. Values of non-dimensional parameters arising in the field equations

ζ_γ	ζ_a	ζ_μ	$\bar{D}$	ν_∞	ν_c	ν_0
0.45	1.0	0.01	20	0.644	0.555	0.51

Profiles of the solid volume fraction ν_s and the solid, fluid velocities are illustrated in Figs. 4, 5 for various values of the two remaining dimensionless quantities: the non-dimensional flow thickness $\bar{L}$ and the parameter S. These figures illustrate the following behaviour of the solid-fluid mixture flow:

• The solid, fluid velocities and the solid volume fraction are strongly dependent on the flow layer thickness and the grain size.

• A relatively thin layer thickness shows a marked variation of the solid volume fraction ν_s across the whole depth, while thicker flow shows an increasing tendency of ν_s to increase quickly to an asymptotic value toward the free surface. In a relatively narrow zone near the bottom, where the shear is the largest, dilatation occurs. But the absolute thickness of this dilatation layer seems to be less dependent on the flow thickness.

• For small flow thickness and small grain size the shear can extend to the whole flow region. As $\bar{L}$ increases or S increases (at a fixed $\bar{L}$, this means that both the grain size and the flow layer thickness increase), there is an increasing tendency that the shear is limited to a narrow layer near the bottom with high dilatancy; above the layer the velocity can be regarded as constant. In this case for such a granular solid-fluid mixture flow down an inclined plane, one can often assume that only a portion of the flow which is close to the base is fluidized, while the upper portion is passively moving with the speed of the particles at the upper edge of this fluidized layer.

• In Table 2 the dimensionless solid and fluid velocities in the mixture and the solid velocities in the dry granular material at the free surface (e.g. maximum velocities) are displayed. Evidently, in the solid-fluid mixture, if the thickness is relatively small, the surface velocities increase by increasing the flow thickness, while for sufficiently large stream thickness, they depend only slightly on the stream thickness. On the contrary, in the dry granular solid, the solid velocity increases monotonously by increasing the flow thickness.

• Comparison of Figs. 2 – 5 shows that in the solid-fluid mixture the fluid velocity is always much larger than that of the solid; here the fluid is pulling the solid via the drag force interacting between them. Furthermore, this solid velocity in the mixture is always larger than that in a dry granular flow. These properties can be more easily inferred from Table 2.

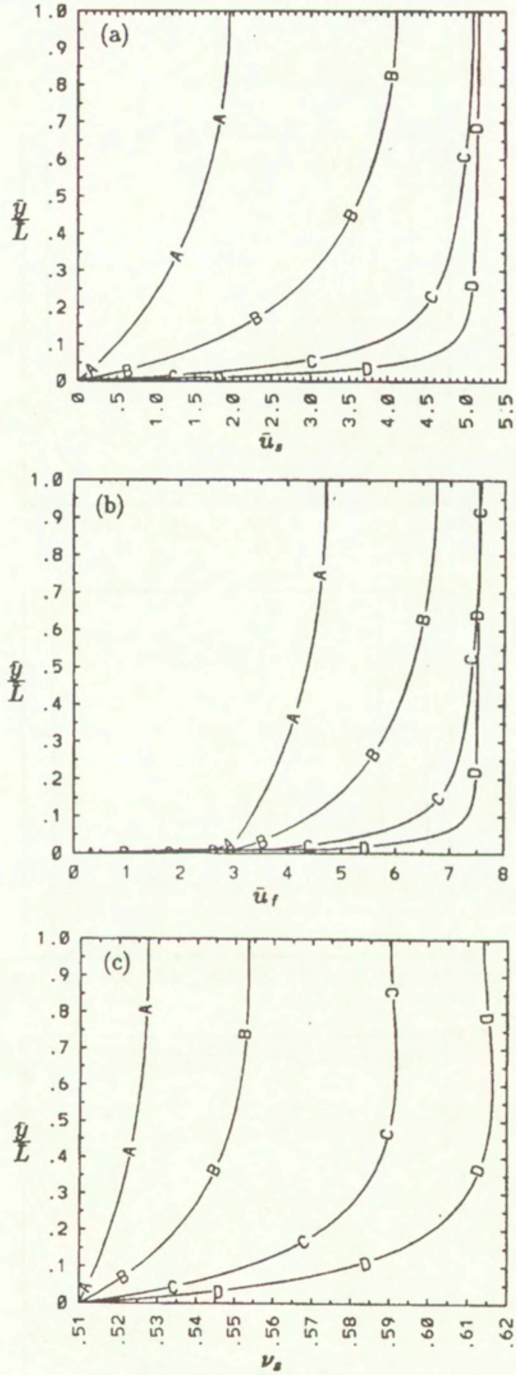
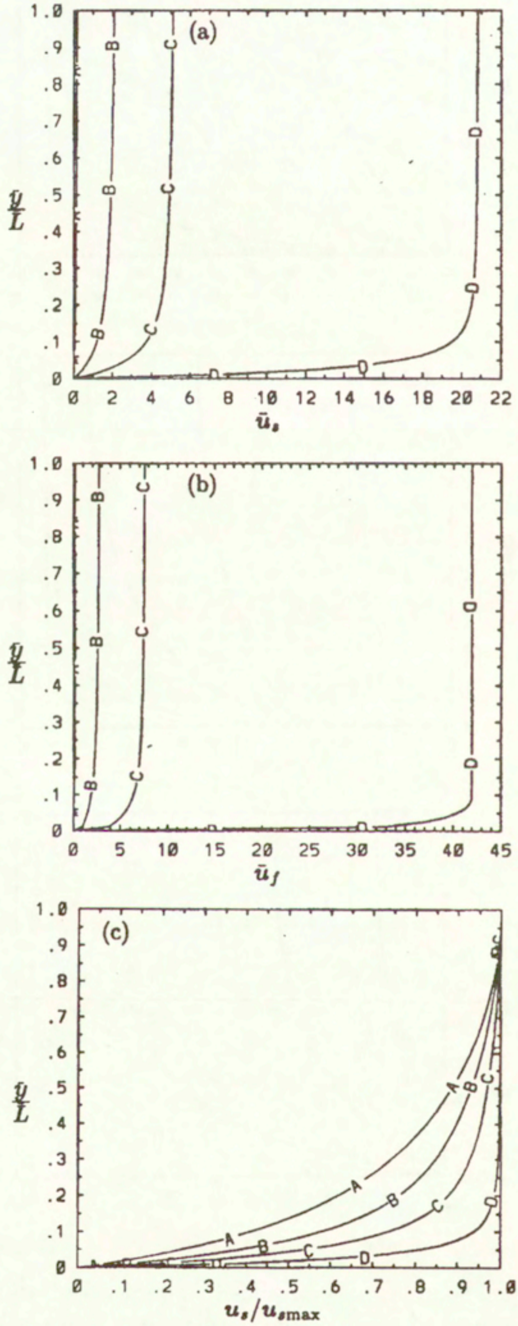
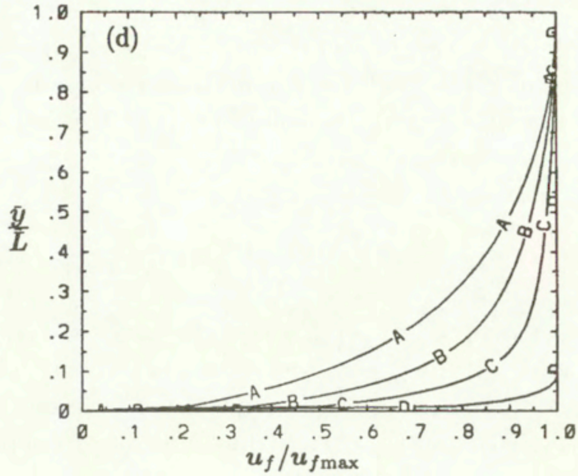


FIG. 4. Non-dimensional solid velocity profiles (a), fluid velocity profiles (b) and volume fraction profiles (c) for a fixed value of the parameter $S = 0.1$ and various values of $\bar{L}$: $\bar{L} = 5\,(A)$; $10\,(B)$; $20\,(C)$ and $30\,(D)$.



[FIG. 5]



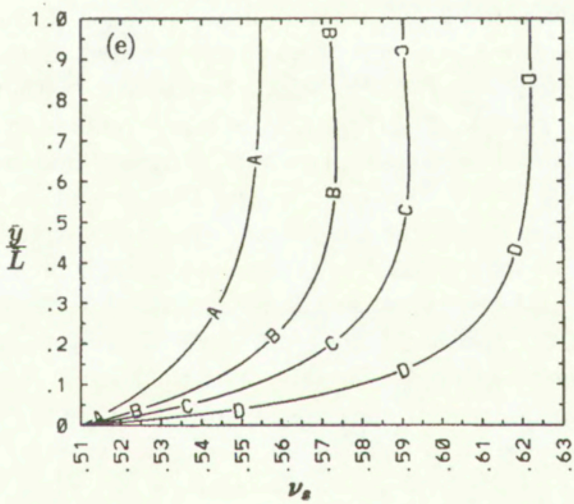


FIG. 5. Non-dimensional solid, fluid velocity profiles (a) (b), normalized solid, fluid velocity profiles (c) (d) and volume fraction profiles (e) for a fixed value of $\bar{L} = 20$ and various values of S: $S = 0.01\,(A)$; $0.05\,(B)$; $0.1\,(C)$ and $0.3\,(D)$.

Table 2. Non-dimensional granular velocities in a dry granular flow, granular and fluid velocities in a granular-fluid mixture at the free surface for various values of the non-dimensional flow thickness $\bar{L}$

	$\bar{L}$	5	10	20	30
In a granular flow	Granular velocity $\bar{u}$	1.05	2.10	2.70	3.36
In a granular-fluid	Granular velocity $\bar{u}_s$	1.95	4.11	5.09	5.16
mixture	Fluid velocity $\bar{u}_f$	4.72	6.75	7.57	7.55

• It might be mentioned that for the drag coefficient $D = 0$ (the corresponding results are not displayed in this paper) the fluid phase behaves very similar to a viscous fluid; the shear of the fluid can extend to the whole flow region. On the other hand, the dominant shear of the solid flow may still be mainly restricted to the regions near the bottom.

5. Concluding remarks

In this paper we attempted to explain how the basic postulates of two forms of the entropy principle: (i) the generalized Clausius-Duhem – Coleman-Noll approach and (ii) the Müller-Liu entropy principle differ from one another. CD-CN makes *a priori* postulates about the entropy flux and entropy supply and assumes external source terms in (most) balance laws. ML postulate the entropy flux to be a general constitutive variable and treat all field equations as constraints for the exploitation of the entropy principle. It was demonstrated that they yielded different constitutive relations for the granular material with/without fluid. Results were presented with the use of both principles for a granular solid with a scalar structure equation and a saturated mixture of granular/fluid constituents with scalar structure equations for each constituent. These results allow us to favour one set of basic postulates over the other. These theories were then applied to the analysis of steady fully-developed gravity flows down an inclined plane. A series of non-dimensional field equations were derived. Numerical results show that for a large thickness of the flow and large grain sizes, dilatant shearing layers exist only near the bottom. In the zones far away from the bottom the shearing nearly vanishes, where each constituent moves as an entire body in a plug-flow manner, while for small thickness of the granular flow and fine grains, the behaviour of the granular flows is similar to that of a viscous fluid, the shear can extend from the bottom to the free surface.

Finally, some points should be emphasized:

• HUTTER *et al.* [5] demonstrated that the emerging solutions of a constitutive mixture theory, if it is obtained by the CD-CN exploitation of the entropy principle (EHLERS, [2]), are extremely restricted. There exists no solution for a simple gravity-driven shearing flow of viscous constituents. However, in this case, if the mixture theory is derived from the Müller-Liu approach of the entropy principle (SVENDSEN and HUTTER, [16]), this nonexistence of the solution can be avoided. This is tempting to favour the Müller-Liu approach of the entropy principle over the CD-CN on another account, not just according to basic postulates.

• Different authors do not unanimously agree upon the form of the scalar structure equations to describe the constituent volume fractions ν_a. SVENDSEN and HUTTER [16], HUTTER *et al.* [16] treated the solid volume fraction as an internal variable and write an evolution equation balancing its time rate of change with its production π_a in the form

$$\acute{\nu}_a = \pi_a.$$

A disadvantage of this form is that if the constituents are incompressible, this equation is no longer independent. In this case, it is the same as the mass balance. WILMAŃSKI [19], on the other hand, using statistical arguments on the microscale demonstrated that the Svendsen-Hutter equation needed to be complemented by a flux term, thus arriving at a complete balance law

$$\acute{\nu}_a = \operatorname{div} \mathbf{h}_a + \pi_a.$$

The two different entropy principles applied to the different granular mixture models give different results. Differences between these models should to be studied.

• CD-CN should be abandoned in the following classes of models: *Polar continua* (solids, anisotropic fluids, liquid crystals) because the spin balance has no free source terms; *Structured continua*; *Mixtures* (as is long known); *Coupled field theories*, etc.

• Many of these theories suffer from the necessity of formulating the boundary conditions which are physically not transparent.

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BRIEF NOTES

Electrification capillary stability of a hollow jet

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THE PRESENT WORK extends Chandrasekhar's theory CHANDRASEKHAR [2] of axisymmetric capillary instability of a hollow jet. Here the instability of this model is investigated for all (non)-axisymmetric perturbation modes under the combined effect of the capillary and electrification forces. The electrification dispersion relation has been derived, studied analytically and numerically, and the (un-) stable domains are identified. Some reported results are recovered as limiting cases. The principle of the exchange of instability is valid. The capillary instability of a hollow jet becomes worse in the presence of the electrification forces.

1. Introduction

THE IDEA OF A HOLLOW JET model which is a gas cylinder (of zero inertia) submerged in an infinite liquid, is due to RAYLEIGH [1]. CHANDRASEKHAR [2] reported the capillary dispersion relation of such model for axisymmetric perturbation mode $m = 0$ (m is the azimuthal wavenumber) only, see also DRAZIN and REID [3]. CHENG [4] analyzed the capillary instability of this model, taking into account (or not) the gas inertia force. Concerning more detailed studies of pure hydrodynamic stability for $m = 0$, we may refer to the complete analysis of LIN and LAIN [5] and LEE and WANG [6]. The hydromagnetic stability of a hollow jet has been developed by RADWAN [7]. The latter author in [8] has examined the rotating forces effects on the capillary instability of a hollow jet. The model of a hollow jet will describe the phenomena observed in nature, such as e.g. gas escaping from below an oil layer or a jet formed up when gas is pumped into a fluid.

The purpose of the present paper is to examine the effect of the electrification forces on the capillary instability of a hollow cylinder. This will be carried out for general cylindrical wave propagation upon using the energy conservation principle in the form different from that used in our previous papers.

The most interesting issue in this work is that both the potential energy of surface tension and that of electrification have the surface area of the gas-liquid interface as an extensive variable, but the sign of intensive variables of each potential energy are opposite. Hence, it is expected that the competition between the capillary and the electrification instabilities may show a variety of stability characteristics.

To our best knowledge, the present electrification problem has not been treated or even approached, up to now, in the literature.

2. Basic state

Consider a circular gas cylinder (of radius a) dispersed in an infinite liquid. Following CHANDRASEKHAR [2] we assume that the liquid inertia force predominat over that of the gas cylinder, i.e. the gas motion could be ignored relative to that of the liquid in the perturbed state. But at the same time we have to be sure that the constant gas pressure in the unperturbed state is of considerable value, otherwise the model will collapse and this is not our case, see Eq. (2.12) below. The liquid could be water, water solutions containing salt or even glycerin while the gas could be air, helium or freon 12, see KENDALL [9]. The liquid is assumed to be non-viscous and incompressible. An electric potential V_0 is applied along the dielectric gas-liquid interface. We shall use the cylindrical polar coordinates (r, ϕ, z) with the z-axis coinciding with the axis of the gas cylinder. The present model of a hollow jet is acting upon the capillary, pressure gradient and electrification forces. The gravitation force effects are not considered here.

The basic equations required for investigating the stability of the present problem (using the SI unit system) inside of the liquid are:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

$$\nabla^2 W = D; \tag{2.3}$$

along the gas-liquid interface

$$p_s = T\left(\mathbf{r}_1^{-1} + \mathbf{r}_2^{-1}\right), \tag{2.4}$$

$$\left(\mathbf{r}_1^{-1} + \mathbf{r}_2^{-1}\right) = \nabla\cdot\mathbf{n}. \tag{2.5}$$

Here $\rho, \mathbf{u}$ and p are the liquid mass density, velocity vector and kinetic pressure, p_s is the curvature pressure due to the capillary force, r_1 and r_2 are the principal radii of curvature, W is the electrification potential, D is the electric charge density term due to the electrification at the gas-liquid interface and will be zero here, T is the surface tension coefficient and $\mathbf{n}$ is a unit outward vector normal to the gas-liquid interface, given by

$$\mathbf{n} = \frac{\nabla f}{\sqrt{\nabla f \cdot \nabla f}}, \tag{2.6}$$

where

$$-f(r, \phi, z; t) = 0 \tag{2.7}$$

is the equation of motion of the gas-liquid boundary surface.

The unperturbed state, characterized by $\mathbf{u} = 0$, $\partial/\partial\phi = 0$ and $\partial/\partial z = 0$, has been studied. The unperturbed basic quantities are given by

$$p_0 = \text{const}, \tag{2.8}$$

$$p_{0s} = -T/a, \tag{2.9}$$

$$W_0 = V_0(\ln r/\ln a). \tag{2.10}$$

Upon applying the balance of the total pressures across the gas-liquid interface at $r = a$, we obtain

$$p_0 = p_g + \varepsilon_0 \left(V_0/(a \ln a)\right)^2 - T/a, \tag{2.11}$$

where p_g is the gas constant pressure in the unperturbed state.
For $p_0 > 0$, it must be

$$p_g + \varepsilon_0 \left(V_0/(a \ln a)\right)^2 > T/a \tag{2.12}$$

otherwise the model under consideration will collapse towards a hollow jet of a radius smaller than a.

3. Eigenvalue problem

For small departure from the initial unperturbed state, the perturbation equations describing the oscillation of the hollow jet model are obtained by solving Eqs. (2.1) – (2.5). The constants of integration are determined upon applying appropriate boundary conditions across the perturbed interface. The kinetic and potentials energies of the fluids are computed. Moreover, upon using the energy conservation principle, the following dispersion relation is derived:

$$\omega^2 = \left(\frac{T}{\rho a^3}(m^2 + x^2 - 1) + \frac{\varepsilon_0 V_0^2}{\rho a^4 (\ln a)}\left(1 + \frac{x K_m'(x)}{K_m(x)}\right)\right)\left(\frac{x K_m'(x)}{K_m(x)}\right). \tag{3.1}$$

Equation (3.1) is the desired stability criterion of the present model. It relates the growth rate ω with the longitudinal and azimuthal wavenumbers x and m, the second kind of the modified Bessel function $K_m(x)$ and its derivative of order m, the permittivity ε_0 of the fluid medium and other parameters T, a, V_0 and ρ of the problem.

In absence of the electrification effects ($V_0 = 0$) and simultaneously assuming that the fluid disturbance is longitudinally axisymmetric $m = 0$, dispersion relation (3.1) reduces to that indicated by RAYLIEGH [1] and just given by CHANDRASEKHAR [2].

If we assume that the perturbation of the gas-liquid interface could be axisymmetric and non-axisymmetric $m \geqslant 0$, and at the same time $V_0 = 0$, Eq. (3.1) degenerates to RADWAN'S result [8] if the magnetic field effects were neglected in reference [8].

It is recommended that all quantities can be expressed in dimensionless form using the radius a of the jet, the surface tension coefficient T, the mass density ρ of the liquid and the electrification potential V_0 as scalar values. Taking into account that the quantity $(T/\rho a^3)$ as well as $(\varepsilon_0 V_0^2/(\rho a^4 \ln a))$ has a dimension of $(\text{time})^{-2}$, we introduce

$$N^2 = \frac{\omega^2}{T/\rho a^3}, \tag{3.2}$$

$$\Gamma = \frac{\varepsilon_0 V_0^2}{aT(\ln a)} \tag{3.3}$$

so that the eigenvalue relation (3.1) can be written in the dimensionless form

$$N^2 = (m^2 + x^2 - 1\left(\frac{xK_m'(x)}{K_m(x)}\right) + \Gamma\left(1 + \frac{xK_m'(x)}{K_m(x)}\right)\left(\frac{xK_m'(x)}{K_m(x)}\right). \tag{3.4}$$

In the axisymmetric perturbation mode $m = 0$, the relation (3.4) reduces to

$$N^2 = (1 - x^2)\left(\frac{xK_1(x)}{K_0(x)}\right) + \Gamma\left(1 - \frac{xK_1(x)}{K_0(x)}\right)\left(\frac{xK_1(x)}{K_0(x)}\right). \tag{3.5}$$

4. Stability discussion

4.1. Hydrodynamic stability

This is the case in which the hollow fluid jet is uncharged. The stability criterion which describes such a case is given in its general form, from Eq. (3.1) at $V_0 = 0$, by

$$L^2 = (m^2 + x^2 - 1)\left(\frac{xK_m'(x)}{K_m(x)}\right), \tag{4.1}$$

with

$$(4.1)' \qquad L^2 = \frac{\omega^2}{T/\rho a^3}, \qquad \text{as} \quad V_0 = 0.$$

In order to discuss the stability here and in other sections, it is found more convenient to write down certain propertiess of I_m and K_m and their derivatives.

For each non-zero real value of x and $m \geqslant 0$, cf. ABRAMOWITZ and STEGUN [11], we have

$$(4.2) \qquad I_m(x) \succ 0,$$

$$(4.3) \qquad K_m(x) \succ 0,$$

where $I_m(x)$ is always positive and monotonically increasing while $K_m(x)$ is monotonic decreasing but never negative. The recurrence relations of the modified Beesel functions are

$$(4.4) \qquad 2I'_m(x) = I_{m-1}(x) + I_{m+1}(x),$$

$$(4.5) \qquad 2K'_m(x) = -K_{m-1}(x) - K_{m+1}(x).$$

Using Eqs. (4.4), (4.5) and (4.2), (4.3) we obtain

$$(4.6) \qquad I'_m(x) \succ 0,$$

$$(4.7) \qquad K'_m(x) \prec 0,$$

for each $x \neq 0$ and $m \geqslant 0$. In view of the inequalities (4.3) and (4.7), for $x \neq 0$ we obtain

$$(4.8) \qquad x\left(K'_m(x)/K_m(x)\right) < 0.$$

Now, let us return to the dispersion relation (3.5). By taking into account the inequality (4.8), Eq. (3.5) yield that

$$(4.9) \qquad L^2 < 0 \qquad \text{for} \qquad m \geqslant 1 \qquad \text{as} \quad x \neq 0,$$

while for $m = 0$, we have

$$(4.10) \qquad L^2 > 0 \qquad \text{as} \qquad -1 \prec x \prec 1,$$

$$(4.11) \qquad L^2 \leqslant 0 \qquad \text{as} \qquad x \geqslant 1 \quad \text{or} \quad x \leqslant -1.$$

This means that a hollow cylindrical jet is stable for all non-axisymmetric modes m of all short and long wavelengths, and also for sausage mode $m = 0$ whose wavelength $\lambda(= 2\pi/k)$ is shorter than the circumference $2\pi a$ of the gas core jet. The hollow jet is capillary unstable only for axisymmetric mode $m = 0$ whose wavelength λ is longer than $2\pi a$ where the case when $\lambda = 2\pi a$ is that of marginal stability.

In order to verify the foregoing analytical results, the capillary dispersion relation (3.5) is calculated numerically for the most unstable mode $m = 0$.

The values of the quadratic dimensionless temporal amplification L are tabulated and presented graphically as a function of x, see Fig. 1. The data reveal the following conclusions. The capillary unstable domain of the hollow jet is the only interval $0 < x < 1$. The maximum mode of instability occurs at $x = 0.484$. The numerical values of L^2 increase rapidly for very small values of x and reach the maximum at $x = 0.50$; then they are fastly decreasing and change the sign at $x = 1$. The values of L^2 are negative for all values of $x \geqslant 1$. The point $x = 1$ represents a transition from stable to unstable domains.

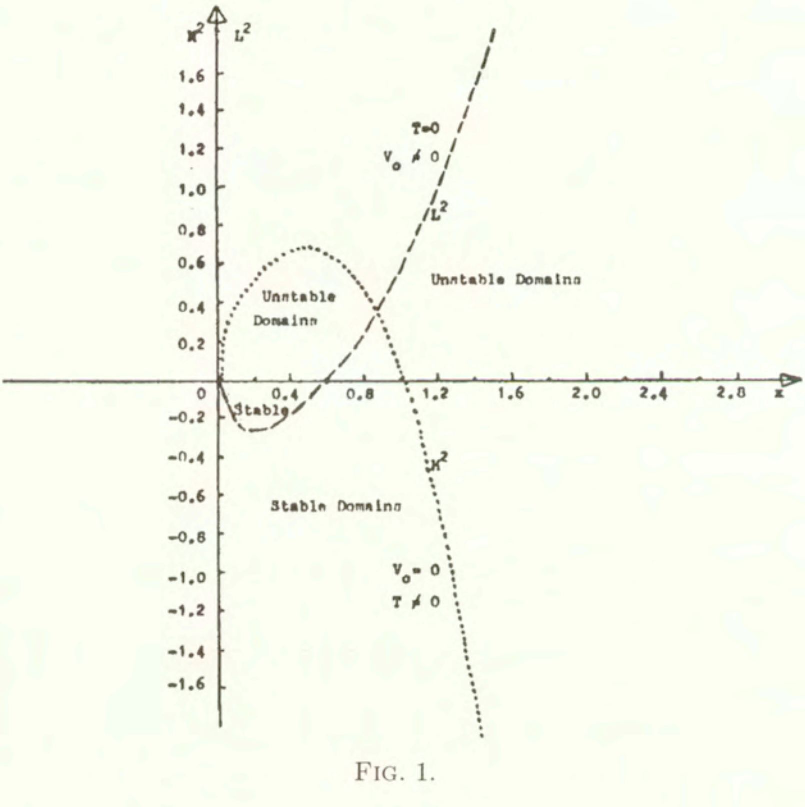

FIG. 1.

4.2. Electrification stability

In this case we assume that the hollow jet is acting upon the electrification force only and we neglect the capillary force influence. The dispersion relation follows from the general characteristic Eq. (3.1) with $T = 0$ in the form

$$M^2 = \left(1 + \frac{xK'_m(x)}{K_m(x)}\right)\left(\frac{xK'_m(x)}{K_m(x)}\right), \tag{4.12}$$

with

$$M^2 = \varepsilon_0 V_0^2/(\rho a^4 \ln a). \tag{4.12)'$$

Applying the inequality (4.8) to the relation (4.12) we find that M^2 is negative, hence the electrification hollow jet is stable iff the inequality

$$\left(1 + \frac{xK_m'(x)}{K_m(x)}\right) \geqslant 0 \tag{4.13}$$

is satisfied, and vice versa where the equality in (4.13) corresponds to the neutral stability states. From the viewpoint of the inequalities (4.3) and (4.7), the condition (4.13) may be rewritten as

$$K_m(x) \geqslant x\left|K_m'(x)\right|. \tag{4.14}$$

It is difficult to identify analytically whether the condition (4.13) is satisfied or not. However, in order to avoid these difficulties, it is recommended that we should analyse (4.12) numerically.

In the axisymmetric sausage mode $m = 0$, it is found that the inequality (4.13) is satisfied in the ranges $0 < x < 0.595088$ and so as $0.59509 < x < \infty$. Therefore the electrification force is stabilizing in the domain $0 < x \leqslant 0.595088$ and destabilizing in the neughboring domains $0.59509 < x < \infty$. The point at which $x = 0.595088$ is that of a transition from a stability state to one of instability, see Fig. 1. In the non-axisymmetric perturbation mode $m = 1$, it is found that the inequality (4.13) is not satisfied for all short and long wavelengths. This means that the electrification force has a strong destabilizing influence on the charged hollow jet.

4.3. Hydro-electrification stability discussions

In this general case the model of a hollow jet is acting upon the combined effects of the electrification and capillary forces. The stability criterion required for investigation of such a case is given by the characteristic Eq. (3.4) in its general form. The latter could be discussed with the aid of the Subsec. (4.1) as $(T \neq 0, V_0 = 0)$ and (4.2) as $(T = 0, V_0 \neq 0)$. The first case of hydrodynamic investigations reveal that the capillary stable domains are

$$1 \leqslant x < \infty, \qquad \text{as} \qquad m = 0, \tag{4.15}$$

$$0 < x < \infty \qquad \text{as} \qquad m \geqslant 1, \tag{4.16}$$

while the only unstable domain is

$$0 < x < 1, \qquad \text{as} \qquad m = 0. \tag{4.17}$$

The second case of electrification analysis indicates that the model is stable in the domain

$$(4.18)\qquad 0 < x \leqslant 0.59452, \qquad \text{as} \qquad m = 0,$$

while it is unstable in the domains

$$(4.19)\qquad 0.5945 < x < \infty, \qquad \text{as} \qquad m = 0,$$

$$(4.20)\qquad 0 < x < \infty, \qquad \text{as} \qquad m \geqslant 1.$$

In the case of consideration of the combined effects of both the capillary and electrification forces, it is found that the unstable results are not as it could be expected.

The dispersion relation (3.4) as $m = 0$ has been investigated numerically for different values of the dimensionless electrification factor

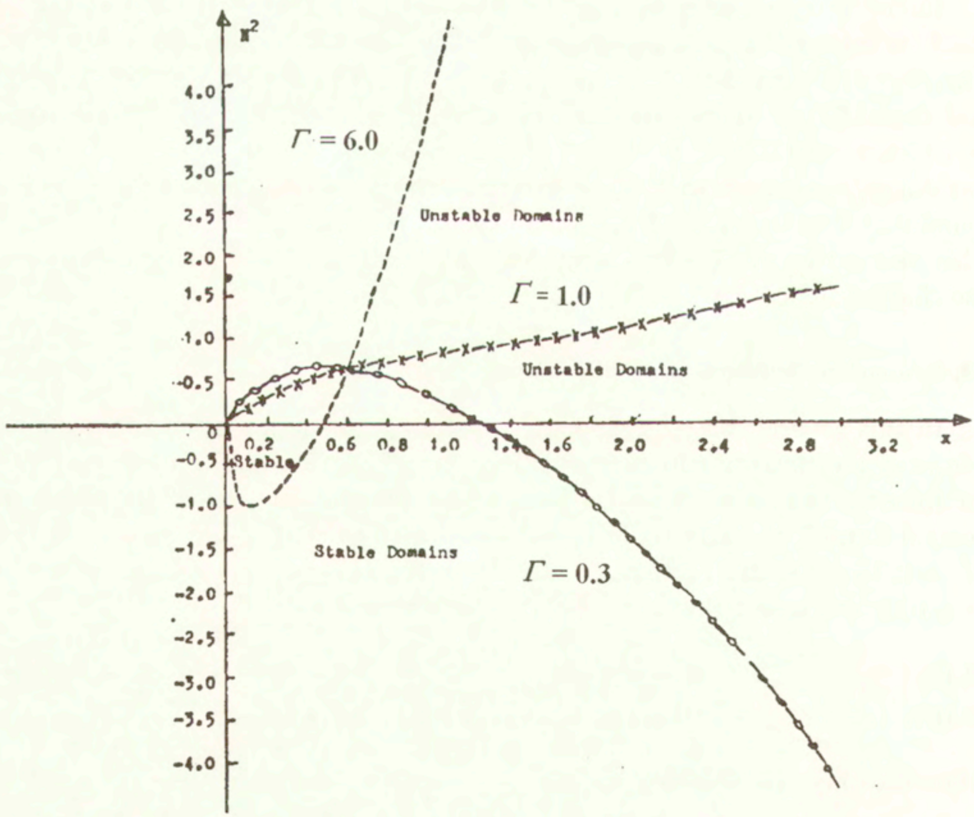

FIG. 2.

$$\Gamma = \left(\frac{\varepsilon_0 V_0^2}{\rho a^4 (\ln a)}\right) \Big/ \left(\frac{T}{\rho a^3}\right) = \left(\frac{\varepsilon_0 V_0^2}{T a (\ln a)}\right). \tag{4.21}$$

The numerical data are illustrated in the range $0 \leqslant x \leqslant 3.0$ of short and long wavelengths, see Fig. 2. It is found that the model is completely unstable for $\Gamma = 1$ for all short and long wavelengths. As $\Gamma = 0.3$, the model is unstable for small values of x, i.e. for very long wavelengths as $0 < x \leqslant 1.1$, while it is stable in the neighboring domain $1.2 \leqslant x \leqslant 3.0$. The latter domain increases with increasing x values. For $\Gamma = 6.0$ the situation has been reversed: the model is stable for very long wavelengths as $0 < x \leqslant 0.4$ while it is unstable for short wavelengths $0.5 \leqslant x < 3.0$. This change is due to the influence of the electrification force. This indicates that the principle of exchange of instability is valid in the present case.

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