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Wave propagation in laminated plates of inextensible transversely isotropic elastic material

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THE PROPAGATOR method of GILBERT and BACKUS [1] is employed to obtain dispersion equations in a multi-ply laminated plate of finite depth but otherwise infinite extent. Each of the laminae is formed from the same transversely isotropic material, with the axis of transverse isotropy lying in the plane of the lamina and the material being inextensible in the direction of this axis. The material is intended to model a composite formed of an elastic matrix reinforced by a family of parallel fibres. The plate is constructed by arranging the laminae so that the directions of inextensibility in adjacent layers are mutually orthogonal, and attention is restricted to waves travelling along one of these directions. Results are obtained for two different symmetric arrangements of the laminae. For one of these configurations the dispersion equation is independent of the number of laminae which make up the plate, for the other configuration this is not the case. Numerical results are obtained using measured moduli for a specific fibre reinforced composite.

Zastosowano metodę „propagatorów" GILBERTA i BACKUSA [1] do wyprowadzenia równań dyspersyjnych wielowarstwowej płyty laminowanej o skończonej grubości i nieograniczonych wymiarach poprzecznych. Wszystkie warstwy wykonane są z tego samego materiału poprzecznie izotropowego, osie izotropii leżą w płaszczyznach warstw, a materiał jest nierozciągliwy w kierunku tych osi. Materiał ten ma modelować kompozyt złożony ze sprężystej matrycy wzmocnionej rodziną nierozciągliwych włókien. Płyta jest tak skonstruowana, że w sąsiadujących ze sobą warstwach kierunki nierozciągliwości są wzajemnie ortogonalne, a rozważa się propagację fal w tych właśnie kierunkach. Otrzymano wyniki dla dwóch różnych symetrycznych konfiguracji warstw. Dla jednej z nich równanie dyspersyjne jest niezależne od liczby warstw tworzących płytę, dla drugiej konfiguracji sytuacja jest odmienna. Wykonano obliczenia numeryczne dla pewnego konkretnego przypadku kompozytu zbrojonego włóknami.

Применен метод „пропагаторов" ГИЛЬБЕРТА и БАКУСА [1] для вывода дисперсионных уравнений многослоистой ламинированной плиты конечной длины и с неограниченными поперечными размерами. Все слои изготовлены из того же самого поперечно изотропного материала, оси изотропии лежат в плоскостях слоев, а материал нерастяжим в направлении этих осей. Этот материал должен моделировать композит, состоящий из упругой матрицы, упрочненной семейством нерастяжимых волокон. Плита так построена, что в соседствующих друг с другом слоях направления нерастяжимости взаимно ортогональны, а рассматривается распространение волн в этих именно направлениях. Получены результаты для двух разных симметричных конфигураций слоев. Для одной из них дисперсионное уравнение не зависит от количества слоев, образующих плиту, для второй конфигурации ситуация другая. Проведены численные расчеты для некоторого конкретного случая композита армированного волокнами.

1. Introduction

THE STUDY of elastic wave propagation in multi-layered wave-guides is conveniently carried out using matrix methods and there exists a considerable literature on the subject. References to the early work are contained in the paper by GILBERT and BACKUS [1] who develop the propagator matrix method originally attributed to Volterra. An alternative matrix method based on reflection and transmission matrices has been developed and exploited by KENNETT [2]. Most of the applications of these matrix methods have been in the field of seismology. There the waveguide is regarded as either a finite number of layers overlying a homogeneous half space, as in the work of KENNETT [2] and KERRY [3], or as a half space composed of a periodic arrangement of strata as considered by GILBERT [4] and SCHOENBERG [5].

In this paper we adopt the propagator matrix approach described in [1] to obtain dispersion equations for wave propagation in multilayered plates of finite thickness but of otherwise infinite extent. We consider specifically a plate constructed from laminae of a single transversely isotropic material which is inextensible in the direction of transverse isotropy, the laminae being arranged so that the axes of transverse isotropy in adjoining layers are orthogonal. We consider harmonic waves propagating in the plane of the plate and for simplicity here we restrict attention to waves whose direction of propagation is parallel to the axis of transverse isotropy for one set of layers (and therefore orthogonal to that in the other set). The method is not, however, restricted to propagation in the specific direction and results for the general direction of propagation will be presented in a further paper.

The inextensible transversely isotropic material is intended to model a composite material consisting of a single family of parallel strong fibres embedded in an elastic matrix. SPENCER [6] has pointed out that for many static stress analysis problems the idealized material which is inextensible in the direction of transverse isotropy can provide an adequate simple model of such a composite. GREEN [7] and GREEN and MILOSAVLJEVIC [8] have shown that this idealized material does *not* provide a good model for wave propagation in a *single* plate of fibre reinforced material in either the long wavelength or the short wavelength limit. On the other hand, BAYLIS and GREEN [9] show that for laminated plates of the type to be considered here, the idealized material can give an acceptable approximation to the short-wavelength (high frequency) behaviour of a laminated composite, with a considerable simplification in the analysis involved. Aside from these considerations, the results obtained here give an exact solution to the problem of wave propagation in a laminate subject to internal constraints.

Elastic wave propagation in laminated plates has been examined by a number of authors. Amongst these, JONES [10] has obtained the exact dispersion equation for a two-ply laminate of orthotropic material, whilst KULKARNI and PAGANO [11] report exact results for 2, 3, 4 and 5-ply laminates of orthotropic materials at a variety of fibre orientations and undergoing dynamic bending deformations. For multi-ply laminates with a large number of plies, formed from a periodic array of basic units, results have been obtained using approximate theories based on some form of averaging procedure for an infinite body composed of the periodically repeating elements. Examples of these are the mixture theory of MURAKAMI and HEGEMIER [12], the effective stiffness theory of SUN, ACHENBACH and HERRMANN [13] and the new quotient method of NEMAT-NASSER [14]. An exact solution for a multi-layer plate consisting of a finite sequence of unit cells composed of two alternating layers, each of which is homogeneous and isotropic, has been obtained by HERRMANN, BEAUPRE and AULD [15]. This solution relates to *S.H.* waves propagating in the plane of the plate and the dispersion equations relating phase-velocity (or frequency) to wavenumber are found to be independent of the number of repeating unit cells contained in the plate. HERRMANN *et al.* [15] obtained their results using Floquet theory for the infinite periodic composite. SCHOENBERG [5] uses propagator matrices to obtain the dispersion equation for coupled P and SV waves in a multilayer body consisting of a single isotropic material with slip at the interfaces between layers. He derives the reflection and transmission coefficient for a half space of such a material. Our approach here closely follows that of GILBERT [4] and SCHOENBERG [5]. In Sect. 2 we present a general description of the method. Detailed propagator matrices are derived in Sect. 3 and the dispersion equations for two different symmetric configurations are obtained in Sect. 4. Section 5 contains numerical results for a specific composite and the paper closes with a discussion of these results.

2. The propagator matrix method

The propagator matrix approach can be used to obtain exact solutions to the problem of wave propagation in a multilayered medium in which each layer is a homogeneous elastic material. In this Section we shall give an outline of the method and its application to laminated plates, leaving the details of specific problems to the following Sections.

Consider a laminated plate composed of n parallel layers of depth $h_1, h_2, \ldots, h_n$, respectively and choose a Cartesian system of axes $Ox_1x_2x_3$ such that the layers are parallel to the x_2x_3-plane. The plate is assumed to be of infinite extent in the x_2 and x_3 directions and each of the displacement components $u_i^{(r)}(x_j, t)$ $(i, j = 1, 2, 3)$ in the r^{th} layer is expressed as the product of a function $U_i^{(r)}(x_1)$ of x_1 only and either $\sin\phi$ or $\cos\phi$, where $\phi = (k_2x_2+k_3x_3-\omega t)$ and t is the time. These displacements represent a plane wave of angular frequency ω, propagating in the direction of the vector k whose components are (O, k_2, k_3). When these displacements are substituted into the anisotropic elastic stress strain relations of the material of the r^{th} layer, each of the stress components $t_{ij}^{(r)}(x_k, t)$ will also have the form of the product of a function of x_1 only, $T_{ij}^{(r)}(x_1)$, with either $\sin\phi$ or $\cos\phi$. The stress functions $T_{ij}^{(r)}(x_1)$ are linear combinations of the displacement functions $U_k^{(r)}(x_1)$ and their first derivatives. Consequently, the equations of motion in the r^{th} layer reduce to a coupled system of three second-order ordinary differential equations for $U_k^{(r)}(x_1)$ $(k = 1, 2, 3)$ and the general solutions involve 6 arbitrary constants. These constants can be expressed in terms of the three traction components $T_{11}^{(r)}$, $T_{12}^{(r)}$, $T_{13}^{(r)}$ and the three displacement components $U_1^{(r)}$, $U_2^{(r)}$, $U_3^{(r)}$ evaluated at the lower face of the r^{th} layer, $x_1 = H_{r-1}$ say. The stress functions $T_{ij}^{(r)}(x_1)$ and displacement functions $U_i^{(r)}(x_1)$ throughout the r^{th} layer may then be expressed as linear combinations of the components of the six-vector $\mathbf{X}^{(r)}(H_{r-1})$ where $\mathbf{X}^{(r)}(x_1)$ is defined as $(T_{11}^{(r)}(x_1),\ T_{12}^{(r)}(x_1),\ T_{13}^{(r)}(x_1),\ U_1^{(r)}(x_1),\ U_2^{(r)}(x_1),\ U_3^{(r)}(x_1))^T$ and T denotes the transpose. In particular, the six-vector $\mathbf{X}^{(r)}(H_r)$ evaluated at the upper surface $x_1 = H_{r-1}+h_r = H_r$, is related to $\mathbf{X}^{(r)}(H_{r-1})$ by means of the 6×6 propagator matrix $\mathbf{M}^{(r)}$ through the equation

$$\mathbf{X}^{(r)}(H_r) = \mathbf{M}^{(r)}\mathbf{X}^{(r)}(H_{r-1}). \tag{2.1}$$

The matrix $\mathbf{M}^{(r)}$ is a function of the elastic constants of the material of the r^{th} layer, the wave numbers k_2 and k_3, the angular frequency ω and the layer thickness h_r.

The interface conditions between adjoining layers serve to express the traction components T_{1i} ($i = 1, 2, 3$) and the displacement components U_j ($j = 1, 2, 3$) at the bottom surface of one layer in terms of the same quantities at the upper surface of the layer immediately below. Thus, for an n-layered plate with perfect bonding between each layer, we have

$$\mathbf{X}^{(r)}(H_{r-1}) = \mathbf{X}^{(r-1)}(H_{r-1}) \quad (r = 2, 3, \ldots, n). \tag{2.2}$$

Using Eq. (2.1) in each layer together with Eq. (2.2) then gives

$$\mathbf{X}^{(n)}(H_n) = \mathbf{M}^{(n)}\mathbf{M}^{(n-1)} \ldots \mathbf{M}^{(1)}\mathbf{X}^{(1)}(H_0) = \mathbf{M}\mathbf{X}^{(1)}(H_0), \tag{2.3}$$

where $\mathbf{M}$ is the overall propagator matrix for the composite plate whose lower surface is at $x_1 = H_0$ and whose upper surface is at $x_1 = H_n = H_0+h_1+h_2+ \ldots h_n$.

To examine wave propagation in the plate under traction free conditions at the upper and lower surfaces, we set the three traction components at the lower surface to zero

$$T_{11}^{(1)}(H_0) = T_{12}^{(1)}(H_0) = T_{13}^{(1)}(H_0) = 0. \tag{2.4}$$

Equation (2.3) then relates the components of $\mathbf{X}^{(n)}(H_n)$ to the three displacement components at the lower face $U_i^{(1)}(H_0)$ ($i = 1, 2, 3$). In particular the three traction components $T_{1j}^{(n)}(H_n)$ ($j = 1, 2, 3$) at the upper surface are given as linear combinations of these three displacements. The requirement that these traction components be zero leads to a system of three homogeneous equations for the $U_i^{(1)}(H_0)$ ($i = 1, 2, 3$). These have nontrivial solutions provided the determinant of the coefficients vanishes;

$$\begin{vmatrix} m_{14} & m_{15} & m_{16} \\ m_{24} & m_{25} & m_{26} \\ m_{34} & m_{35} & m_{36} \end{vmatrix} = 0, \tag{2.5}$$

where m_{ij} are the elements of the overall propagator matrix $\mathbf{M}$. Equation (2.5) is the dispersion equation relating the angular frequency ω to the wave numbers k_2, k_3, or relating the phase velocity $v = \omega/k$ to the wave number $k = (k_2^2+k_3^2)^{1/2}$ and the propagation angle γ where $k_2 = k\sin\gamma$, $k_3 = k\cos\gamma$. The dispersion equation involves the elastic constants and density of each layer as well as the layer thicknesses.

Whilst this method of deriving the dispersion equation is straightforward in theory, the practical difficulty arises in forming the matrix product $\mathbf{M}$ of the n individual propagator matrices. This problem is considerably simplified when the laminate is constructed from a periodic arrangement of two or more layers of material with different properties. Thus, if the laminate consists of n identical units with each unit comprised of a layer of material with the propagator matrix $\mathbf{M}_1$ bonded on top of a layer of material with propagator matrix $\mathbf{M}_2$, we have that

$$\mathbf{M} = (\mathbf{M}_1\mathbf{M}_2)^n = (\hat{\mathbf{M}})^n, \tag{2.6}$$

where $\hat{\mathbf{M}} = \mathbf{M}_1\mathbf{M}_2$. By repeated application of the Cayley–Hamilton theorem, it is possible to reduce $(\hat{\mathbf{M}})^n$ to a matrix polynomial of degree 5 in $\hat{\mathbf{M}}$ with known coefficients depending on n and on the eigenvalues of $\hat{\mathbf{M}}$. Even with this simplification, the derivation of the dispersion equation is still a formidable problem.

A further simplification results when the two materials are isotropic. The motion then degenerates into two separate disturbances. One is a horizontally polarized shear (*S.H.*) disturbance with one nonzero displacement component and one associated nonzero stress component. The second disturbance involves two displacement components (coupled *S.V.* and *P*) and their associated stress components. For the *S.H.* waves the propagator matrices $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}$, reduce to 2×2 matrices and $(\hat{M})^n$ is expressible in terms of $\mathbf{I}$ and $\hat{\mathbf{M}}$ only with coefficients depending on n and the eigenvalues of $\hat{\mathbf{M}}$. The dispersion equation then reduces to $m_{12} = 0$ and is independent of n, as shown by HERRMANN *et al.* [15]. In the coupled *S.V.* and *P* wave, the matrices $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}$ are all of order 4 and $(\hat{\mathbf{M}})^n$ is expressible as a polynomial of degree 3 in $\hat{\mathbf{M}}$. The dispersion equation then reduces to a 2×2 determinantal condition.

We are concerned with a multilayered plate made up of a single transversely isotropic elastic material which is inextensible in the direction of transverse isotropy. The plate is constructed of alternate layers of this material of thickness $2h$ and $2d$ respectively, arranged so that the directions of transverse isotropy are at right angles to each other in adjacent layers. In this paper we restrict attention to waves propagating parallel to the axis of transverse isotropy in the layers of thickness $2h$ (and theorefore at right angles to the axis in the layers of thickness $2d$). The motion degenerates in the same way as for isotropic materials into two separate disturbances. The first of these is an *S.H.* wave with displacement polarized at right angles to the direction of propagation and parallel to the plane of the plate, for which the results are identical to those derived by HERRMANN *et al.* [15]. The second disturbance involves coupled *S.V.* and *P* waves, for which the displacement is polarized in the plane containing the direction of propagation and the normal to the plate. Whereas for isotropic materials the propagator matrices for this second disturbance are of order 4, in this problem the constraint of inextensibility associated with one of the layers in each pair allows the propagator matrices to be reduced to order 2. The dispersion equation then reduces to $m_{12} = 0$ where m_{12} is the upper right hand corner element of the overall propagator matrix $\mathbf{M}$. The detailed derivation of these propagator matrices is considered in the following sections.

3. Solutions of the equations of motion

We consider waves propagating in the x_3-direction and with displacements polarized in the x_1x_3-plane so that in each layer the displacements have the form

$$\begin{aligned} u_1(x_k, t) &= U(x_1)\cos(kx_3-\omega t), \qquad u_2(x_k, t) = 0, \\ u_3(x_k, t) &= W(x_1)\sin(kx_3-\omega t) \qquad (k = 1, 2, 3). \end{aligned} \tag{3.1}$$

The layers are arranged with the direction of inextensibility parallel to Ox_3 and Ox_2 alternately, the former being of thickness $2h$ and the latter of thickness $2d$. We shall refer to these as the inextensible layer and the isotropic layer, respectively, since the latter appears isotropic relative to the disturbance under consideration. The governing equations in each layer have been derived by BAYLIS and GREEN [9] and are quoted here without derivation.

Inextensible layer

The inextensibility constraint gives $W(x_1) = 0$ throughout the layer and the equations of motion reduce to

$$c_1^2 \frac{d^2U}{dx_1^2} + k^2(v^2 - c_3^2)U = 0, \tag{3.2}$$

where

$$c_1^2 = \frac{\lambda + 2\mu_T}{\varrho}, \qquad c_3^2 = \frac{\mu_L}{\varrho}, \tag{3.3}$$

ϱ is the density and λ, μ_T and μ_L are elastic moduli. Because of the inextensibility constraint, the longitudinal stress component t_{33} is not determined by the constitutive equations but is obtained directly as a reaction stress from the equation of motion in the x_3-direction. BAYLIS and GREEN [9] have pointed out that this reaction stress can be singular at the interface between two layers, allowing a discontinuity in the shear stress t_{13} across the singularity. The normal stress component t_{11} must be continuous and this is given by the expression

$$t_{11} = T_{11}(x_1)\cos(kx_3 - \omega t) = \varrho c_1^2 \frac{dU}{dx_1}\cos(kx_3 - \omega t). \tag{3.4}$$

Writing $T(x_1) = T_{11}(x_1)/\varrho c_1^2 k$ and defining the vector $\mathbf{X}(x_1)$ by $\mathbf{X}(x_1) = (T(x_1)\,U(x_1))^T$, the solution of Eq. (3.2) is given by

$$\mathbf{X}(x_1) = \mathbf{M}_1(x_1)\mathbf{X}(0). \tag{3.5}$$

Here, $\mathbf{M}_1(x_1)$ is the propagator matrix for the inextensible layer and is defined by

$$\mathbf{M}_1(x_1) = \begin{bmatrix} C & pS \\ \dfrac{S}{p} & C \end{bmatrix}, \tag{3.6}$$

where

$$p^2 = (c_3^2 - v^2)/c_1^2 \tag{3.7}$$

and

$$C = \cosh pkx_1, \qquad S = \sinh pkx_1. \tag{3.8}$$

It is easy to show directly from the definition (3.6) that

$$\mathbf{M}_1(a+b) = \mathbf{M}_1(a)\mathbf{M}_1(b) \tag{3.9}$$

and it follows that the solution for $\mathbf{X}(x_1)$ can equally be expressed in terms of $\mathbf{X}$ at any other point, $x_1 = l$ say. In particular, the value of $\mathbf{X}^u$ at the upper surface of a layer of depth h is given in terms of the value $\mathbf{X}^L$ at the lower surface in the form

$$\mathbf{X}^u = \mathbf{M}_1\mathbf{X}^L, \tag{3.10}$$

where

$$\mathbf{M}_1 = \mathbf{M}_1(h). \tag{3.11}$$

Isotropic layer

The equations of motion in this layer are ([9] Eqs. (3.2))

$$(3.12)\qquad \begin{aligned} &c_1^2\frac{d^2U}{dx_1^2}+k^2(v^2-c_2^2)U+(c_1^2-c_2^2)k\frac{dW}{dx_1}=0,\\ &-(c_1^2-c_2^2)k\frac{dU}{dx_1}+c_2^2\frac{d^2W}{dx_1^2}+k^2(v^2-c_1^2)W=0,\end{aligned}$$

where $c_2^2=\mu_T/\varrho$.

The traction components $t_{11}=T_{11}(x_1)\cos(kx_3-\omega t)$, and $t_{13}=T_{13}(x_1)\sin(kx_3-\omega t)$ on any plane $x_1=$ constant, are given by the expressions

$$(3.13)\qquad T_{11}=\varrho c_1^2\frac{dU}{dx_1}+\varrho(c_1^2-2c_2^2)kW,\qquad T_{13}=\varrho c_2^2\left(\frac{dW}{dx_1}-kU\right).$$

Writing

$$(3.14)\qquad T(x_1)=T_{11}/\varrho c_1^2k,\qquad S(x_1)=T_{13}/\varrho c_2^2k$$

the solution of Eqs. (3.12) give expressions for $T(x_1)$, $S(x_1)$, $U(x_1)$, $W(x_1)$ in terms of their values at $x_1=0$ say, in the form

$$(3.15)\qquad \mathbf{Y}(x_1)=\mathbf{P}(x_1)\mathbf{Y}(0),$$

where

$$(3.16)\qquad \mathbf{Y}(x_1)=\begin{bmatrix}T(x_1)\\ S(x_1)\\ U(x_1)\\ W(x_1)\end{bmatrix},$$

and

(3.17)

$$\mathbf{P}(x_1)=\begin{bmatrix}\alpha C_1+(1-\alpha)C_2 & \dfrac{\alpha S_1}{q_1}+(1-\alpha)q_2S_2 & -\dfrac{2\gamma\alpha^2}{(1-\alpha)}\dfrac{S_1}{q_1}+2\gamma(1-\alpha)q_2S_2 & 2\gamma\alpha(C_1-C_2)\\ (1-\alpha)q_1S_1+\dfrac{\alpha S_2}{q_2} & (1-\alpha)C_1+\alpha C_2 & -2\gamma\alpha(C_1-C_2) & 2\gamma(1-\alpha)q_1S_1-\dfrac{2\gamma\alpha^2}{(1-\alpha)}\dfrac{S_2}{q_2}\\ -\dfrac{(1-\alpha)}{2\gamma}\left(q_1S_1-\dfrac{S_2}{q_2}\right) & -\dfrac{(1-\alpha)}{2\gamma}(C_1-C_2) & \alpha C_1+(1-\alpha)C_2 & -(1-\alpha)q_1S_1-\alpha\dfrac{S_2}{q_2}\\ \dfrac{(1-\alpha)}{2\gamma}(C_1-C_2) & \dfrac{(1-\alpha)}{2\gamma}\left(\dfrac{S_1}{q_1}-q_2S_2\right) & -\alpha\dfrac{S_1}{q_1}-(1-\alpha)q_2S_2 & (1-\alpha)C_1+\alpha C_2\end{bmatrix},$$

is the propagator matrix for the isotropic layer.

The terms appearing in the matrix in Eq. (3.17) are defined by

$$S_1 = \sinh kq_1 x_1, \quad S_2 = \sinh kq_2 x_1, \quad C_1 = \cosh kq_1 x_1, \quad C_2 = \cosh kq_2 x_1, \tag{3.18}$$

where

$$\alpha = 1 - \frac{2c_2^2}{v^2}, \quad \gamma = \frac{c_2^2}{c_1^2}, \quad q_1 = \left(1 - \frac{v^2}{c_1^2}\right)^{1/2}, \quad q_2 = \left(1 - \frac{v^2}{c_2^2}\right)^{1/2}. \tag{3.19}$$

The matrix $\mathbf{P}(x_1)$ possesses the properties of propagator matrices detailed by GILBERT and BACKUS [1], in particular $\mathbf{P}(x_1)$ has determinant unity and

$$\mathbf{P}(a+b) = \mathbf{P}(a) \cdot \mathbf{P}(b). \tag{3.20}$$

It follows from Eq. (3.20) that

$$\mathbf{Y}(x_1) = \mathbf{P}(x_1 - l)\mathbf{Y}(l), \tag{3.21}$$

for any l.

In the applications being considered here we are concerned with relating the value of the two quantities T and U at the upper surface of the layer to their values at the lower surface, when these two surfaces are themselves subject to some contraint conditions. There are three separate cases to be considered. In the first of these we have a layer of thickness $2d$ with each surface subject to the constraint $W = 0$, corresponding to the layer being bonded at each surface to an inextensible layer. Using these conditions gives an expression for $\mathbf{X}$ at the upper surface $\mathbf{X}^u$ in terms of its value at the lower surface $\mathbf{X}^L$. This may be expressed in terms of elements of the propagator matrix $\mathbf{P}(2d)$ for the complete element, but it is convenient to write this as the product $\mathbf{P}(d)\mathbf{P}(d)$ and the resulting expression has the form

$$\mathbf{X}^u = \mathbf{M}_2 \cdot \mathbf{X}^L = \frac{\mathbf{R} \cdot \hat{\mathbf{R}}}{|\mathbf{R}|} \cdot \mathbf{X}^L, \tag{3.22}$$

where

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \quad \hat{\mathbf{R}} = \begin{pmatrix} r_{22} & r_{12} \\ r_{21} & r_{11} \end{pmatrix}, \tag{3.23}$$

and the elements r_{ij} are related to the elements p_{ij} of the propagator $\mathbf{P}(d)$ by the expressions

$$\begin{aligned} r_{11} &= p_{11} - \frac{p_{14}p_{41}}{p_{44}}, \quad & r_{12} &= p_{13} - \frac{p_{12}p_{43}}{p_{42}}, \\ r_{21} &= p_{31} - \frac{p_{41}p_{34}}{p_{44}}, \quad & r_{22} &= p_{33} - \frac{p_{32}p_{43}}{p_{42}}. \end{aligned} \tag{3.24}$$

The second case we consider is that of a layer of thickness d, subject to the constraint $S = 0$ at the upper surface and $W = 0$ at the lower surface. This will correspond to the top layer of the composite plate. These constraints may be employed to express all the quantities at the upper surface in terms of the two quantities T and U at the lower surface. In particular we have for T and U at the upper surface

$$\mathbf{X}^u = \mathbf{S}\mathbf{X}^L, \tag{3.25}$$

where

$$\mathbf{S} = \begin{pmatrix} p_{11} - \dfrac{p_{12}p_{21}}{p_{22}} & p_{13} - \dfrac{p_{12}p_{23}}{p_{22}} \\ p_{31} - \dfrac{p_{32}p_{21}}{p_{22}} & p_{33} - \dfrac{p_{32}p_{23}}{p_{22}} \end{pmatrix}. \tag{3.26}$$

Finally we may derive the corresponding expression for a plate of thickness d subject to the constraint $W = 0$ at the upper surface and $S = 0$ at the lower surface, in the form

$$\mathbf{X}^u = \hat{\mathbf{S}}\mathbf{X}^L. \tag{3.27}$$

Here $\hat{\mathbf{S}}$ is given in terms of the components s_{ij} of the matrix S defined in Eq. (3.26) by

$$\hat{\mathbf{S}} = \begin{pmatrix} s_{22} & s_{12} \\ s_{21} & s_{11} \end{pmatrix}. \tag{3.28}$$

In deriving the results (3.28) we have made use of the relations

$$p_{33} = p_{11}, \quad p_{44} = p_{22}, \quad p_{23} = -p_{14}, \quad p_{41} = -p_{32}, \quad p_{43} = -p_{12}, \quad p_{34} = -p_{21}, \tag{3.29}$$

which follow from the expression for $\mathbf{P}(x_1)$ given in Eq. (3.17).

We note that the transfer matrices $\mathbf{M}_2$, S and $\hat{\mathbf{S}}$ defined by Eqs. (3.22), (3.26) and (3.28) are not true propagator matrices in that they do not possess all the properties detailed by GILBERT and BACKUS [1]. They do, however, each possess the property of having a unit determinant.

In the next section we shall make use of these transfer matrices to derive the overall transfer matrices and hence the dispersion equations for multilayered plates.

4. Dispersion equations

We consider as a basic unit a layer of thickness $2d$ of the apparently isotropic material, sandwiched between two layers, each of thickness h, of the inextensible material, as shown in Fig. 1(a). From this we form two symmetric multi-ply plates, each of thickness $2n(d+h)$.

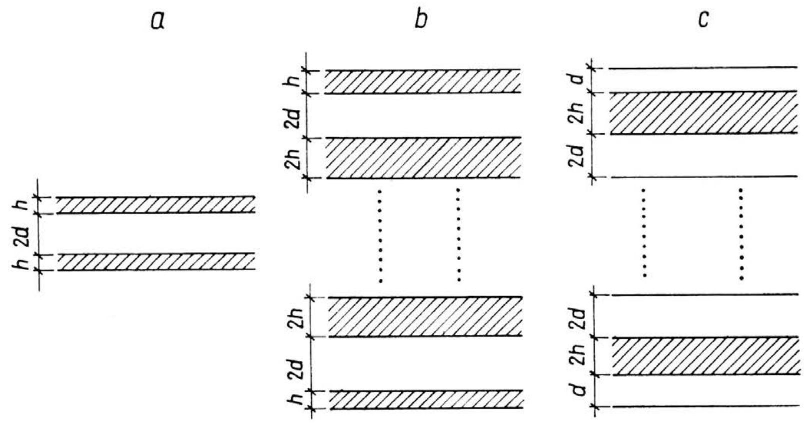

FIG. 1a. Basic unit. b. Type I n-ply plate. c. Type II n-ply plate.

The first of these consists of n of the basic units bonded together as shown in Fig. 1(b), with outer layers being each of an inextensible material. The second configuration consists of $(n-1)$ basic units bonded together and bounded at the top and bottom by a half unit, so that the outer layers are each of an isotropic material, as shown in Fig. 1(c).

In order to obtain the transfer matrix for the basic unit, we must make use of the interface conditions between the isotropic layers and the two inextensible layers bonded to it. These are that U and T must be continuous across the interfaces and that $W = 0$ at the interfaces. The value of S can be discontinuous in view of the possible singularity in the reaction stress at the boundary in the inextensible layers.

The vector $\mathbf{X}^T$ at the top of the basic unit is then given in terms of the vector $\mathbf{X}^B$ at the bottom by the expression

$$\mathbf{X}^T = \mathbf{D}\mathbf{X}^B = \mathbf{M}_1\mathbf{M}_2\mathbf{M}_1\mathbf{X}^B = \frac{\mathbf{Q}\hat{\mathbf{Q}}}{|\mathbf{Q}|}\mathbf{X}^B, \tag{4.1}$$

where

$$\mathbf{Q} = \mathbf{M}_1\mathbf{R}, \qquad \hat{\mathbf{Q}} = \hat{\mathbf{R}}\mathbf{M}_1, \tag{4.2}$$

and

$$|\mathbf{Q}| = |\mathbf{M}_1\mathbf{R}| = |\mathbf{R}|. \tag{4.3}$$

The overall transfer matrix $\mathbf{M}$ for the symmetric configuration of Fig. 1(b) is then given by

$$\mathbf{M} = \mathbf{D}^n \tag{4.4}$$

and on using the Cayley–Hamilton theorem this reduces to

$$\mathbf{M} = \frac{(\lambda_1^n - \lambda_2^n)}{(\lambda_1 - \lambda_2)}\mathbf{D} - \lambda_1\lambda_2\frac{(\lambda_1^{n-1} - \lambda_2^{n-1})}{(\lambda_1 - \lambda_2)}\mathbf{I}, \tag{4.5}$$

where λ_1 and λ_2 are the eigenvalues of $\mathbf{D}$.

Setting the traction component T_{11} to zero at the lower surface of the composite, the condition that T_{11} should vanish at the top of the plate then becomes

$$m_{12} = \frac{(\lambda_1^n - \lambda_2^n)}{(\lambda_1 - \lambda_2)}d_{12} = \frac{(\lambda_1^n - \lambda_2^n)}{(\lambda_1 - \lambda_2)}2q_{11}q_{12} = 0. \tag{4.6}$$

Equation (4.6) is satisfied by either of the equations

$$q_{11} = \frac{q_2v^2CC_1C_2 + c_1^2pS(C_1S_2 - q_1q_2S_1C_2)}{q_2\{2c_2^2C_1 + (v^2 - 2c_2^2)C_2\}} = 0$$

(4.7) or

$$q_{12} = \frac{q_2v^2CS_1S_2 + c_1^2pS(S_1C_2 - q_1q_2C_1S_2)}{c_1^2(S_1 - q_1q_2S_2)} = 0,$$

and these are the dispersion equations for the composite plate. These equations are independent of n and were previously derived for the plate for which $n = 1$ by BAYLIS and GREEN [9].

The configuration 1(c) consists of $(n-1)$ basic units bounded at the top and tne bottom by one half a basic unit so that the outer layer at both surfaces is an isotropic layer. The traction free conditions require that both T and S should vanish at the two surfaces and it is therefore necessary to employ the transfer matrix $\mathbf{S}$ defined in Eq. (3.26) for the upper ayer and the matrix $\hat{\mathbf{S}}$ defined in Eq. (3.28) for the lower layer. The overall transfer matrix $\mathbf{M}^*$ is then given by

$$\begin{aligned} \mathbf{M}^* &= \mathbf{S}\mathbf{M}_1\mathbf{D}^{n-1}\mathbf{M}_1\hat{\mathbf{S}} \\ &= \mathbf{V}\mathbf{D}^{n-1}\hat{\mathbf{V}}, \end{aligned} \tag{4.8}$$

where

$$\mathbf{V} = \mathbf{S}\mathbf{M}_1, \quad \hat{\mathbf{V}} = \mathbf{M}_1\hat{\mathbf{S}}. \tag{4.9}$$

Using the Cayley–Hamilton theorem Eq. (4.8) reduces to

$$\mathbf{M}^* = \frac{(\lambda_1^{n-1}-\lambda_2^{n-1})}{(\lambda_1-\lambda_2)}\mathbf{V}\mathbf{D}\hat{\mathbf{V}} - \frac{(\lambda_1^{n-2}-\lambda_2^{n-2})}{(\lambda_1-\lambda_2)}\lambda_1\lambda_2\mathbf{V}\hat{\mathbf{V}}. \tag{4.10}$$

Setting the traction T at the lower boundary equal to zero, the condition that T should vanish at the upper boundary is then that

$$m_{12}^* = \frac{2(\lambda_1^{n-1}-\lambda_2^{n-1})}{(\lambda_1-\lambda_2)}\frac{(v_{11}q_{11}+v_{12}q_{21})(v_{11}q_{12}+v_{12}q_{22})}{(q_{11}q_{22}-q_{12}q_{21})} - \frac{2(\lambda_1^{n-2}-\lambda_2^{n-2})}{(\lambda_1-\lambda_2)}\lambda_1\lambda_2 v_{11}v_{12} = 0. \tag{4.11}$$

The eigenvalues λ_1, λ_2 of $\mathbf{D}$ may be written explicitly as

$$\lambda_1 = \frac{(\sqrt{q_{11}q_{22}}+\sqrt{q_{12}q_{21}})^2}{(q_{11}q_{22}-q_{12}q_{21})}, \quad \lambda_2 = \frac{(\sqrt{q_{11}q_{22}}-\sqrt{q_{12}q_{21}})^2}{(q_{11}q_{22}-q_{12}q_{21})}, \tag{4.12}$$

and these when substituted into Eq. (4.11) give the dispersion equation in the form

$$\frac{\lambda_1^{n-1}(v_{11}\sqrt{q_{11}q_{12}}+v_{12}\sqrt{q_{21}q_{22}})^2-\lambda_2^{n-1}(v_{11}\sqrt{q_{11}q_{12}}-v_{12}\sqrt{q_{21}q_{22}})^2}{\sqrt{q_{11}q_{12}q_{21}q_{22}}} = 0. \tag{4.13}$$

When $n = 1$, Eq. (4.13) reduces to $v_{11}v_{12} = 0$ which agrees with the equation derived by BAYLIS and GREEN [9] for the triple plate consisting of a single inextensible layer of thickness $2h$ bounded by isotropic layers of thickness d at top and bottom.

5. Dispersion curves

The dispersion equations (4.7) and (4.13) are readily solved using a micro-computer, and the resulting dispersion curves are shown in Fig. 2—4. These results are based on the elastic constants measured by MARKHAM [16] for a carbon-fibre epoxy resin composite and for which $c_1^2/c_2^2 = 4.297$ and $c_3^2/c_2^2 = 2.301$.

The curves shown in Fig. 2 relate to the plate configuration of Fig. 1(b) and are obtained by solving Eqs. (4.7). The motion is symmetric relative to some plane $x_1 = H$ say, if the displacement U is an odd function of the variable $X = x_1 - H$ (i.e. if $U(-X) = -U(X)$

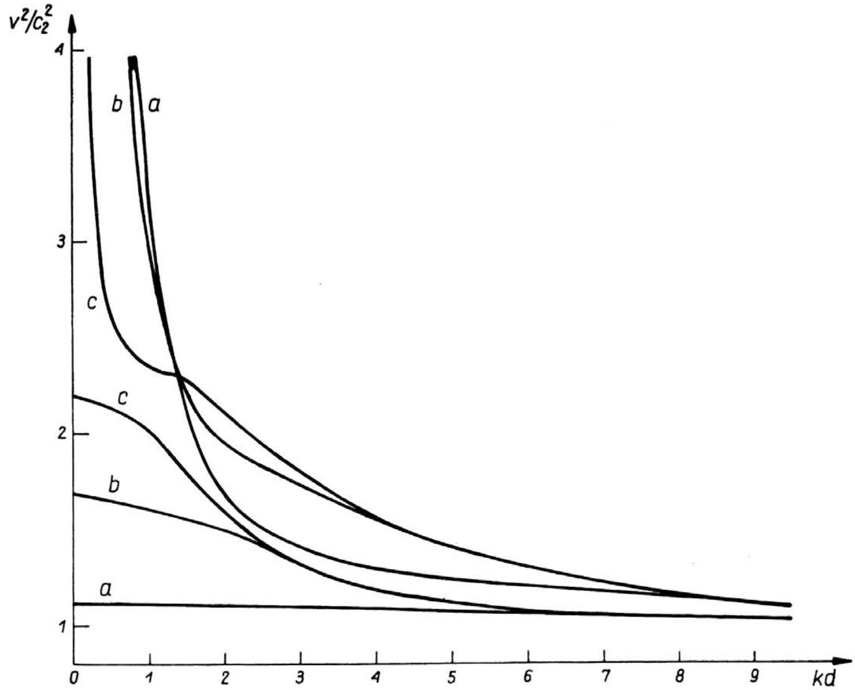

FIG. 2. Variation of phase velocity with wavelength for Type I plate and a — $\nu = 0.1$, b — $\nu = 1$, c — $\nu = 10$.

and the traction T is an even function of X ($T(-X) = T(X)$). The motion is antisymmetric relative to $x_1 = H$ if U is an even function of X and T an odd function of X. It may be shown that the first of Eqs. (4.7) ($q_{11} = 0$) corresponds to a motion which is symmetric with respect to the middle plane of each of the isotropic layers and antisymmetric with respect to the middle plane of each inextensible layer. Following HERRMANN *et al.* [15], we will designate this motion as SA. The second of Eqs. (4.7) ($q_{12} = 0$) corresponds to a motion which is antisymmetric relative to the middle planes of each of the isotropic and the inextensible layers and this will be designated AA. For a plate with $n = 1$, the SA motion corresponds to a longitudinal wave whilst the AA motion corresponds to a flexural wave.

Figure 2 shows graphs of the phase velocity versus the reduced wave number (kd) for the fundamental modes of both the SA wave and the AA wave for three different values of $\nu = d/h$. Note that the SA curves have a common point of intersection for all values of ν. This is the point $v = c_3$, $kd = \dfrac{\pi}{2}\left(\dfrac{c_3^2}{c_2^2} - 1\right)^{-\frac{1}{2}} = 1.377$, corresponding to $p = 0$, $C_2 = 0$. Since the wave number k is related to the wave length Λ by the equation $k = 2\pi/\Lambda$, the limiting behaviour as $kd \to 0$ (and therefore $kh \to 0$) corresponds to long wave disturbances in the composite plate. It is easy to show that the limiting velocity of the long wave AA motion is given by

$$v^2 = \frac{c_3^2 + \nu c_2^2}{1 + \nu},$$

whereas the limiting velocity for the long wave SA motion tends to infinity. The latter corresponds to a finite cut-off frequency $\overline{\omega}$, in the long wave limit, given by the equation

(5.1) $$\cos\frac{\overline{\omega}d}{c_2}\cos\frac{\overline{\omega}(d+h)}{c_1}=0.$$

Equation (5.1) has the solutions

(5.2) $$\frac{\overline{\omega}d}{c_2}=\frac{\pi}{2}\quad\text{or}\quad\frac{\overline{\omega}(d+h)}{c_1}=\frac{\pi}{2},$$

which correspond to a variation of velocity with kd of the form

(5.3) $$\frac{v}{c_2}=\frac{\pi}{2kd}\quad\text{or}\quad\frac{v}{c_2}=\frac{c_1}{c_2}\frac{\nu}{(1+\nu)}\frac{\pi}{2kd},$$

in the limit as $kd\to 0$.

The cut-off frequency for the fundamental SA mode is given by the first of Eqs. (5.2) for values of $\nu\geqslant c_2/(c_1-c_2)(=0.932)$. The limiting form of the dispersion curve is then given by the first of Eqs. (5.3) and is independent of ν. For values of $\nu<c_2/(c_1-c_2)$, the fundamental mode cut-off frequency is given by the second of Eqs. (5.2) and the limiting form of the dispersion curve is dependent on ν, being given by the second of Eqs. (5.3). These conclusions are borne out by the data presented in Table 1 where it is seen that the dispersion curves for $\nu=1.0$ and $\nu=10.0$ agree with each other and with the first of Eqs. (5.3) in the limit as $kd\to 0$ but they differ from the curve for $\nu=0.1$ which is in close agreement with the second of Eqs. (5.3).

Table 1. Long wavelength solutions for Type I plate

$$R_1=\left[\frac{c_1}{c_2}\frac{\nu}{(1+\nu)}\frac{\pi}{2kd}\right]^2\Big/\left[\frac{v^2}{c_2^2}\right],\quad R_2=\left[\frac{\pi}{2kd}\right]^2\Big/\left[\frac{v^2}{c_2^2}\right].$$

$\frac{v^2}{c_2^2}$	$\nu=0.1$		$\nu=1$		$\nu=10$	
	kd	R_1	kd	R_2	kd	R_2
10	0.107	0.77	0.462	1.16	0.519	0.91
20	0.070	0.89	0.327	1.15	0.359	0.96
30	0.056	0.93	0.269	1.13	0.291	0.97
40	0.048	0.94	0.234	1.12	0.251	0.98
50	0.043	0.95	0.211	1.11	0.224	0.984
100	0.030	0.977	0.151	1.08	0.158	0.992
200	0.021	0.989	0.108	1.05	0.111	0.996
300	0.017	0.992	0.089	1.04	0.091	0.997
400	0.015	0.994	0.077	1.03	0.079	0.998
500	0.013	0.995	0.069	1.03	0.070	0.998

In order to examine the high frequency (short wavelength) behaviour of the dispersion curves, it is necessary to determine whether v is less than or greater than c_2 as $kd\to\infty$. Setting $v=c_2$ in the two equations (4.7), it is possible to show that neither equation is satisfied for any value of kd and none of the dispersion curves can therefore cross the line $v=c_2$. Since the fundamental modes for both the SA motion and the AA motion lie above $v = c_2$ as $kd \to 0$, it follows that they will do so for all kd. It is then straightforward to deduce from Eqs. (4.7) that the limiting velocity as $kd \to \infty$ is given for the SA mode ($q_{11} = 0$) by the conditions

$$S_2 = 0 \quad \text{and} \quad q_2 C_2 = 0, \tag{5.4}$$

and for the AA mode ($q_{12} = 0$) by the conditions

$$C_2 = 0 \quad \text{and} \quad q_2 S_2 = 0. \tag{5.5}$$

Equations (5.4) are satisfied by

$$\lim_{kd\to\infty} q_2 kd = m\pi \quad (m = 1, 2, \ldots) \tag{5.6}$$

which gives for the fundamental SA mode ($m = 1$), that v varies with kd in the limit as $kd \to \infty$, according to the equation

$$\frac{v^2}{c_2^2} = 1 + \frac{\pi^2}{k^2d^2}. \tag{5.7}$$

The solutions of Eq. (5.5) are

$$\lim_{kd\to\infty} q_2 kd = \left(m + \frac{1}{2}\right)\pi \quad (m = 0, 1, 2, \ldots) \tag{5.8}$$

and the variation of v with kd (as $kd \to \infty$) for the fundamental AA mode ($m = 0$) is

$$\frac{v^2}{c_2^2} = 1 + \frac{\pi^2}{4k^2d^2}. \tag{5.9}$$

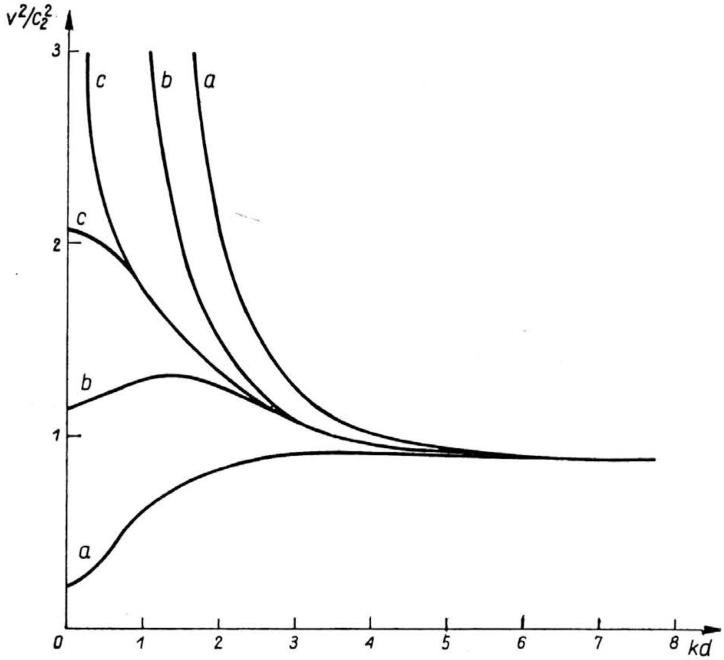

FIG. 3. Variation of phase velocity with wavelength for Type II plate for $n = 1$ and a — $\nu = 0.1$, b — $\nu = 1$ c — $\nu = 10$.

Both solutions (5.7) and (5.9) are independent of ν and this is evident from the graphs in Fig. 2.

For the plate configuration shown in Fig. 1(c), the dispersion curves are dependent on the value of n as well as on the parameter ν. Figure 3 shows the curves for the case $n = 1$, for which the dispersion equation (4.13) factorises into the two equations

$$v_{11} = 0 \quad \text{and} \quad v_{12} = 0. \tag{5.10}$$

The first of Eqs. (5.10) corresponds to the longitudinal motion of the plate whilst the second gives the dispersion curve for flexural waves. The fundamental modes for each of these are plotted in Fig. 3 for 3 values of ν. In the long wave limit ($kd \to 0$) the flexural wave velocity tends to the value given by

$$v^2 = \frac{c_3^2}{(1+\nu)}, \tag{5.11}$$

whereas the velocity of longitudinal waves tends to infinity, corresponding to a finite cut-off frequency. The cut-off frequency is again determined by Eq. (5.1) and the same considerations apply as for the plate in Fig. 1(b). Detailed results for $kd \to 0$ are presented in Table 2 which also contains the solutions (5.3) for comparison. It may be seen from Fig. 3 that in the short wave limit ($kd \to \infty$) each of the dispersion curves asymptotes the Rayleigh wave velocity in the isotropic layer, all the curves running together from $kd = 5$ onwards.

Table 2. Long wavelength solutions for Type II plate for $n = 1$

$$R_1 = \left[\frac{c_1}{c_2}\frac{\nu}{(1+\nu)}\frac{\pi}{2kd}\right]^2 \Big/ \left[\frac{v^2}{c_2^2}\right], \quad R_2 = \left[\frac{\pi}{2kd}\right]^2 \Big/ \left[\frac{v^2}{c_2^2}\right].$$

$\frac{v^2}{c_2^2}$	$\nu = 0.1$ kd	R_1	$\nu = 1$ kd	R_2	$\nu = 10$ kd	R_2
10	0.104	0.81	0.515	0.93	0.599	0.68
20	0.070	0.91	0.356	0.97	0.383	0.84
30	0.056	0.94	0.289	0.987	0.303	0.89
40	0.048	0.95	0.249	0.992	0.259	0.92
50	0.043	0.96	0.223	0.995	0.230	0.94
100	0.030	0.98	0.157	0.999	0.160	0.97
200	0.021	0.99	0.111	1.00	0.112	0.984
300	0.017	0.994	0.091	1.00	0.091	0.989
400	0.015	0.995	0.079	1.00	0.079	0.992
500	0.013	0.996	0.070	1.00	0.070	0.993

In Fig. 4 we present dispersion curves for three different values of $n(1, 2, 11)$ all for the case of $d = h(\nu = 1)$. For $n = 1$ we plot the fundamental mode of each of the two equations (5.10), corresponding to longitudinal and flexural waves respectively. For values of $n > 1$ the dispersion equation (4.13) is solved for both the fundamental mode and the first harmonic and each of these is plotted in Fig. 4. It is evident that the fundamental mode corresponds to what becomes the flexural wave for $n = 1$ whilst the first harmonic degenerates into the longitudinal wave for $n = 1$. The noteworthy feature

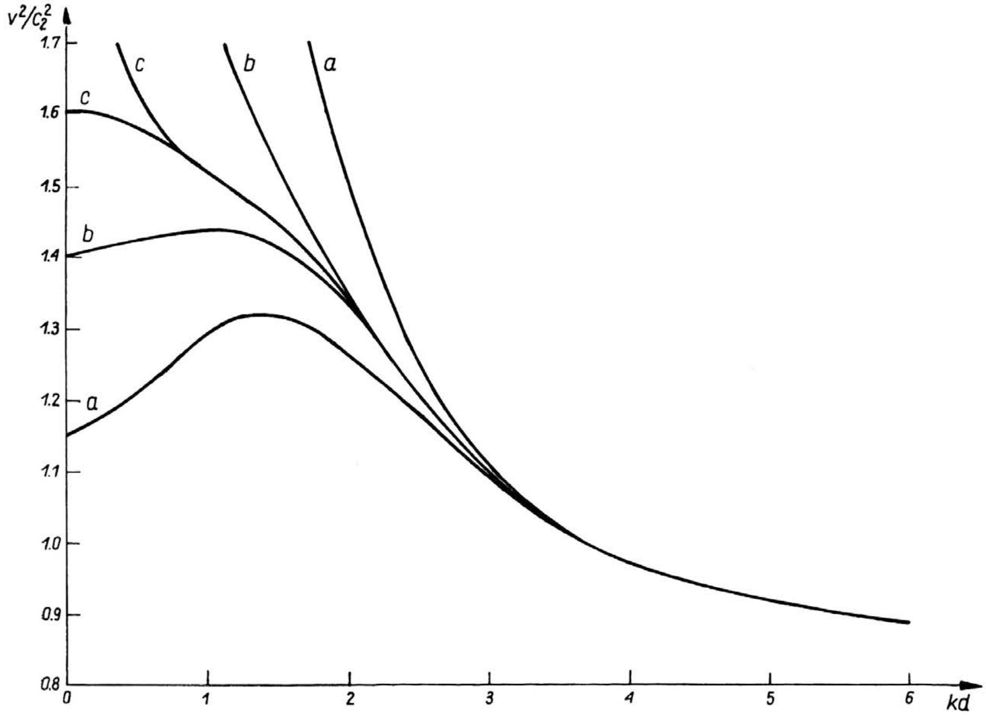

FIG. 4. Variation of phase velocity with wavelength for Type II plate with $\nu = 1$ and a — $n = 1$, b — $n = 2$, c — $n = 11$.

of these curves is that they all come together into a common curve and this occurs for decreasing values of kd as n increases. It is straightforward to show that as $kd \to 0$ the limiting velocity for the fundamental mode of Eq. (4.13) is given by

$$v^2 = \frac{nc_3^2 + (n-1)\nu c_2^2}{n(1+\nu)}, \tag{5.12}$$

whereas the limiting velocity for the first harmonic tends to infinity, corresponding to a cut-off frequency $\hat{\omega}$ given by

$$\cos\frac{\hat{\omega}d}{c_2}\cos\frac{\hat{\omega}n(d+h)}{c_1} = 0. \tag{5.13}$$

Equation (5.13) has the solutions

$$\frac{\hat{\omega}d}{c_2} = \frac{\pi}{2} \quad \text{or} \quad \frac{\hat{\omega}n(d+h)}{c_1} = \frac{\pi}{2}, \tag{5.14}$$

and the variation of velocity with kd as $kd \to 0$ is given by

$$\frac{v}{c_2} = \frac{\pi}{2kd} \quad \text{or} \quad \frac{v}{c_2} = \frac{c_1}{c_2}\frac{\nu}{(1+\nu)}\frac{1}{n}\frac{\pi}{2kd}. \tag{5.15}$$

The limiting behaviour is given by the first of Eqs. (5.15) for values of ν and n satisfying $\nu c_1/[n(1+\nu)c_2] \geqslant 1$ and by the second of Eqs. (5.15) otherwise. The curves in Fig. 4 all relate to $\nu = 1$ and limiting behaviour is given by the first of Eqs. (5.15) for $n \leqslant 1.036$ and by the second of Eqs. (5.15) for $n > 1.036$. These conclusions may be verified from the data given in Table 3.

Table 3. Long wavelength solutions for Type II plate $\nu = 1$

$$R_2 = \left[\frac{\pi}{2kd}\right]^2 \Bigg/ \left[\frac{v^2}{c_2^2}\right], \quad R_3 = \left[\frac{c_1}{c_2}\frac{1}{2n}\frac{\pi}{2kd}\right]^2 \Bigg/ \left[\frac{v^2}{c_2^2}\right].$$

$\frac{v^2}{c_2^2}$	$n = 1$ kd	R_2	$n = 2$ kd	R_3	$n = 11$ kd	R_3
10	0.515	0.93	0.269	0.92	0.051	0.84
20	0.356	0.97	0.186	0.96	0.034	0.92
30	0.289	0.987	0.151	0.97	0.028	0.95
40	0.249	0.992	0.130	0.98	0.024	0.96
50	0.223	0.995	0.116	0.985	0.021	0.97
100	0.157	0.999	0.082	0.992	0.015	0.984
200	0.111	1.00	0.058	0.996	0.010	0.992
300	0.091	1.00	0.047	0.997	0.009	0.995
400	0.079	1.00	0.041	0.998	0.007	0.996
500	0.070	1.00	0.036	0.998	0.007	0.997

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