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On errors inherent in commonly accepted rate forms of the elastic constitutive law

M. KLEIBER (WARSZAWA)

THE OBJECTIVE of this note is to critically discuss the applicability of a simple hypoelastic constitutive equation in order to describe energy-preserving hyper-elastic systems. The danger of a significant accuracy loss in using the hypo-elastic material model to the analysis of cyclic processes is demonstrated.

W pracy przedyskutowano stosowalność prostego, często wykorzystywanego w literaturze, równania konstytutywnego hiposprężystości do opisu materiałów hyper-sprężystych. Wskazano na niebezpieczeństwo błędów powstających przy stosowaniu tego równania w analizie procesów cyklicznych.

В работе обсуждена применяемость простого, часто используемого в литературе, определяющего уравнения гипоупругости для описания гиперупругих материалов. Указано на опасность ошибок, возникающих при применении этого уравнения в анализе циклических процессов.

1. Introduction

IT IS VERY common both in the theoretical work on foundations of elasto-plasticity as well as in the existing finite element codes, cf. [1—3], for instance, that the rate form of the constitutive law of elasticity is employed as the hypo-elastic relation

$$\overset{\#}{\sigma}_{ij} = \tilde{C}_{ijkl} d_{kl}, \quad i, j, \ldots = 1, 2, 3, \tag{1.1}$$

where $\overset{\#}{\sigma}_{ij}$ is a Cauchy stress flux, d_{ij} is the strain rate equal to the symmetric part of the spatial velocity gradient $v_{i,j}$ and $\tilde{C}_{ijkl}$ is a tensor of elastic moduli assumed constant. This effectively implies that $\tilde{C}_{ijkl}$ is usually taken equal to the moduli C_{ijkl} of the linear Hooke's elasticity

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad C_{ijkl} = C_{klij}. \tag{1.2}$$

Only the Cartesian components of tensors with respect to a fixed coordinate system are considered in Eqs. (1.1) and (1.2) and throughout the rest of the paper.

Even if most of the authors are clearly aware of the theoretical inconsistency introduced by postulating Eq. (1.1) with $\tilde{C}_{ijkl}$ = const. to describe classical hyper-elastic solid behaviour(¹), the popularity of this approximation has prompted the author to evaluate on the basis of a simple example the error inherent in such an approach. It is believed that the results obtained are instructive enough to be presented in some detail. The objective

(¹) The reader is referred to the so-called Bernstein's theorem as described in [8], for instance, which yields the sufficient conditions for a hypo-elastic solid to be hyper-elastic.

of this note is thus to critically discuss the applicability of Eq. (1.1) with $\tilde{C}_{ijkl} = \text{const}$ to simple elastic processes. However, since the elastic range is assumed to exist in almost all theories of inelasticity, the conclusions will have a much more general character.

2. Analysis

For the given initial stress distribution

$$\sigma_{ij}^0 = \sigma_{ij}|_{\tau=t_0}, \tag{2.1}$$

we consider a linear in time τ field of the infinitesimal displacement gradient

$$u_{i,j}(\tau) = a_{ij}\tau, \qquad \tau \in [t_0, t]. \tag{2.2}$$

The constant in time fields of the velocity gradient, strain rate tensor and spin tensor are given by

$$v_{ij} = a_{ij}, \tag{2.3}$$

$$d_{ij} = \operatorname{sym} a_{ij} = a_{(ij)}, \tag{2.4}$$

$$\omega_{ij} = \operatorname{antisym} a_{ij} = a_{[ij]}. \tag{2.5}$$

By moving the corotational terms over to the right-hand side, Eq. (1.1) can generally be presented for any stress flux as

$$\dot{\sigma}_{ij}(\tau) = A_{ijkl}(v_{m,n})\sigma_{kl}(\tau) + \tilde{C}_{ijkl}d_{kl} \tag{2.6}$$

or, in a convenient matrix notation, as

$$\dot{\boldsymbol{\sigma}}(\tau) = \mathbf{A}\boldsymbol{\sigma}(\tau) + \mathbf{F}, \qquad \boldsymbol{\sigma}, \mathbf{F} \in \mathrm{R}^6 \tag{2.7}$$

with the initial condition

$$\boldsymbol{\sigma}(t_0) = \boldsymbol{\sigma}^0 \tag{2.8}$$

and the definitions of particular terms in Eq. (2.7) following directly by comparing Eqs. (2.6) and (2.7). The solution to Eq. (2.7) with the initial condition (2.8) has the form

$$\boldsymbol{\sigma}(\tau) = e^{\mathbf{A}(\tau - t_0)}\boldsymbol{\sigma}^0 + \int_{t_0}^{\tau} e^{\mathbf{A}(\tau - s)}\mathbf{F}(s)\,ds, \tag{2.9}$$

where

$$e^{\mathbf{A}\tau} = \mathbf{I} + \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} + \ldots + \frac{\mathbf{A}^n\tau^n}{n!} + \ldots \tag{2.10}$$

which converges for all τ. There are several classes of matrices $\mathbf{A}$ for which the infinite series (2.10) can be summed exactly. In general, though, it is not possible to express $e^{\mathbf{A}\tau}$ in closed form. Yet, the remarkable fact is, cf. [4], that if $\mathbf{X}(\tau)$ is a fundamental matrix solution of the differential equation

$$\dot{\boldsymbol{\sigma}}(\tau) = \mathbf{A}\boldsymbol{\sigma}(\tau), \tag{2.11}$$

i.e. columns of $\mathbf{X}(\tau)$ form a set of 6 linearly independent solutions of Eq. (2.11), then

$$e^{\mathbf{A}\tau} = \mathbf{X}(\tau)\mathbf{X}^{-1}(0). \tag{2.12}$$

In other words, the product of any fundamental matrix solution of Eq. (2.11) with its inverse at $\tau = 0$ must yield $e^{\mathbf{A}\tau}$. Thus, for a given initial-value problem (2.7) and (2.8), we can find the solution $\boldsymbol{\sigma}(\tau)$ given by Eqs. (2.9) and (2.12) provided we can find 6 linearly independent solutions of Eq. (2.11).

The most common choices of the objective stress rate $\overset{\#}{\sigma}_{ij}$ in Eq. (1.1) are the Jaumann derivative of the Cauchy stress, Truesdell derivative of the Cauchy stress (which corresponds naturally to the Green strain on the current configuration, [5, 6]) or the Jaumann derivative of the Kirchhoff stress (which corresponds naturally to the logarithmic strain, [5, 6]). To be specific we shall now consider in Eq. (2.1) the stress flux in the form of the Jaumann rate of the Cauchy stress; other rates can be treated similarly. We have

$$\overset{\#}{\sigma}_{ij} = \overset{\circ}{\sigma}_{ij} = \dot{\sigma}_{ij} - \omega_{im}\sigma_{mj} + \sigma_{im}\omega_{mj} \tag{2.13}$$

so that

$$\overset{\circ}{\sigma}_{ij} = \tilde{C}_{ijkl} d_{kl} = \tilde{C}_{ijkl} v_{(k,l)} = \tilde{C}_{ijkl} v_{k,l} \tag{2.14}$$

or

$$\begin{aligned}\sigma_{ij} &= (\omega_{ik}\delta_{jl} - \omega_{kj}\delta_{il})\sigma_{kl} + \tilde{C}_{ijkl} v_{k,l} = \\ &= \frac{1}{2}(\delta_{ik}\sigma_{lj} - \delta_{lj}\sigma_{ik} + \delta_{jk}\sigma_{il} - \delta_{il}\sigma_{kj}) v_{k,l} + \tilde{C}_{ijkl} v_{k,l} = A_{ijkl}(v_{m,n})\sigma_{kl} + F_{ij}.\end{aligned} \tag{2.15}$$

3. Example

For the plane strain case the only nonvanishing strain-rate and spin components are $d_{11}, d_{22}, d_{12}, \omega_{12} = \omega$ so that Eq. (2.15) becomes

$$\underbrace{\begin{bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{12} \end{bmatrix}}_{\dot{\boldsymbol{\sigma}}} = \omega \underbrace{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}}_{\boldsymbol{\sigma}} + \underbrace{\begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & 0 \\ \tilde{C}_{12} & \tilde{C}_{22} & 0 \\ 0 & 0 & \tilde{C}_{33} \end{bmatrix} \begin{bmatrix} d_{11} \\ d_{22} \\ d_{12} \end{bmatrix}}_{\mathbf{F}}. \tag{3.1}$$

First, we find $e^{\mathbf{A}\tau}$ where

$$\mathbf{A} = \omega \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix}. \tag{3.2}$$

To this end compute

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} -\lambda & 0 & 2\omega \\ 0 & -\lambda & -2\omega \\ -\omega & \omega & 0 \end{bmatrix} = -\lambda(\lambda^2 + 4\omega^2). \tag{3.3}$$

Thus the eigenvalues of $\mathbf{A}$ are $\lambda = 0$ and $\lambda = \pm 2\omega i$.

(i) $\lambda = 0$: we seek a nonzero vector $\mathbf{v}$ such that

$$(\mathbf{A} - 0 \cdot \mathbf{I})\mathbf{v} = \begin{bmatrix} 0 & 0 & 2\omega \\ 0 & 0 & -2\omega \\ -\omega & \omega & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.4}$$

Hence the solution of Eq. (2.19) can be adopted in the form

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{3.5}$$

and, consequently,

$$\boldsymbol{\sigma}_1(\tau) = e^0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{3.6}$$

is a solution of the homogeneous equation $\dot{\boldsymbol{\sigma}} = \mathbf{A}\boldsymbol{\sigma}$.

(ii) $\lambda = 2\omega i$: we seek a nonzero vector $\mathbf{v}$ such that

$$[\mathbf{A} - (2\omega i)\mathbf{I}]\mathbf{v} = \begin{bmatrix} -2\omega i & 0 & 2\omega \\ 0 & -2\omega i & -2\omega \\ -\omega & \omega & -2\omega i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.7}$$

Hence the following solution of Eq. (2.22) can be obtained:

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ i \end{bmatrix} \tag{3.8}$$

and

$$\boldsymbol{\sigma}_2(\tau) = e^{2\omega i\tau} \begin{bmatrix} 1 \\ -1 \\ i \end{bmatrix} \tag{3.9}$$

is a complex-valued solution of $\dot{\boldsymbol{\sigma}} = \mathbf{A}\boldsymbol{\sigma}$. Now

$$\begin{bmatrix} 1 \\ -1 \\ i \end{bmatrix} e^{2\omega i\tau} = (\cos 2\omega\tau + i \sin 2\omega\tau)\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \tag{3.10}$$

$$= \begin{bmatrix} \cos 2\omega\tau \\ -\cos 2\omega\tau \\ -\sin 2\omega\tau \end{bmatrix} + i \begin{bmatrix} \sin 2\omega\tau \\ -\sin 2\omega\tau \\ \cos 2\omega\tau \end{bmatrix}$$

and, consequently,

$$\boldsymbol{\sigma}_2(\tau) = \begin{bmatrix} \cos 2\omega\tau \\ -\cos 2\omega\tau \\ -\sin 2\omega\tau \end{bmatrix}, \quad \boldsymbol{\sigma}_3(\tau) = \begin{bmatrix} \sin 2\omega\tau \\ -\sin 2\omega\tau \\ \cos 2\omega\tau \end{bmatrix} \tag{3.11}$$

are real-valued solutions of $\dot{\boldsymbol{\sigma}} = \mathbf{A}\boldsymbol{\sigma}$. The solutions $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ are linearly independent since their values at $\tau = 0$ are clearly linearly independent vectors of R^3. Therefore

$$\mathbf{X}(\tau) = \begin{bmatrix} 1 & \cos 2\omega\tau & \sin 2\omega\tau \\ 1 & -\cos 2\omega\tau & -\sin 2\omega\tau \\ 0 & -\sin 2\omega\tau & \cos 2\omega\tau \end{bmatrix} \tag{3.12}$$

is a fundamental matrix solution of $\dot{\boldsymbol{\sigma}} = \mathbf{A}\boldsymbol{\sigma}$.

Computing

$$\mathbf{X}^{-1}(0) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.13}$$

we see that

$$e^{\mathbf{A}\tau} = \begin{bmatrix} 1 & \cos 2\omega\tau & \sin 2\omega\tau \\ 1 & -\cos 2\omega\tau & -\sin 2\omega\tau \\ 0 & -\sin 2\omega\tau & \cos 2\omega\tau \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

$$= \begin{bmatrix} \frac{1}{2}(1+\cos 2\omega\tau) & \frac{1}{2}(1-\cos 2\omega\tau) & \sin 2\omega\tau \\ \frac{1}{2}(1-\cos 2\omega\tau) & \frac{1}{2}(1+\cos 2\omega\tau) & -\sin 2\omega\tau \\ -\frac{1}{2}\sin 2\omega\tau & \frac{1}{2}\sin 2\omega\tau & \cos 2\omega\tau \end{bmatrix}.$$

Using Eq. (2.9) we arrive at

$$\begin{bmatrix} \sigma_{11}(\tau) \\ \sigma_{22}(\tau) \\ \sigma_{12}(\tau) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[1+\cos 2\omega(\tau-t_0)] & \frac{1}{2}[1-\cos 2\omega(\tau-t_0)] & \sin 2\omega(\tau-t_0) \\ \frac{1}{2}[1-\cos 2\omega(\tau-t_0)] & \frac{1}{2}[1+\cos 2\omega(\tau-t_0)] & -\sin 2\omega(\tau-t_0) \\ -\frac{1}{2}\sin 2\omega(\tau-t_0) & \frac{1}{2}\sin 2\omega(\tau-t_0) & \cos 2\omega(\tau-t_0) \end{bmatrix} \tag{3.15}$$

$$\times \begin{bmatrix} \sigma_{11}^0 \\ \sigma_{22}^0 \\ \sigma_{12}^0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\tau-t_0)+\frac{1}{4\omega}\sin 2\omega(\tau-t_0) & \frac{1}{2}(\tau-t_0)-\frac{1}{4\omega}\sin 2\omega(\tau-t_0) & \frac{1}{2\omega}[1-\cos 2\omega(\tau-t_0)] \\ \frac{1}{2}(\tau-t_0)-\frac{1}{4\omega}\sin 2\omega(\tau-t_0) & \frac{1}{2}(\tau-t_0)+\frac{1}{4\omega}\sin 2\omega(\tau-t_0) & -\frac{1}{2\omega}[1-\cos 2\omega(\tau-t_0)] \\ -\frac{1}{4\omega}[1-\cos 2\omega(\tau-t_0)] & \frac{1}{4\omega}[1-\cos 2\omega(\tau-t_0)] & \frac{1}{2\omega}[\sin 2\omega(\tau-t_0)] \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}.$$

In the limiting case of the „zero-order" theory (terms up to zero order in ω are retained) we correctly have from Eq. (3.15)

$$\begin{bmatrix}\sigma_{11}(\tau)\\ \sigma_{22}(\tau)\\ \sigma_{12}(\tau)\end{bmatrix} = \begin{bmatrix}\sigma^0_{11}\\ \sigma^0_{22}\\ \sigma^0_{12}\end{bmatrix} + (\tau-t_0)\begin{bmatrix}F_1\\ F_2\\ F_3\end{bmatrix}. \tag{3.16}$$

For the first order theory we have

$$\begin{bmatrix}\sigma_{11}(\tau)\\ \sigma_{22}(\tau)\\ \sigma_{12}(\tau)\end{bmatrix} = [\mathbf{I}+\mathbf{B}\cdot\omega(\tau-t_0)]\begin{bmatrix}\sigma^0_{11}\\ \sigma^0_{22}\\ \sigma^0_{12}\end{bmatrix} + \left[\mathbf{I}+\frac{1}{2}\mathbf{B}\cdot\omega(\tau-t_0)\right](\tau-t_0)\begin{bmatrix}F_1\\ F_2\\ F_3\end{bmatrix}, \tag{3.16}$$

where

$$\mathbf{B} = \begin{bmatrix}0 & 0 & 2\\ 0 & 0 & -2\\ -1 & 1 & 0\end{bmatrix}. \tag{3.17}$$

The internal energy from t_0 to τ is given by $W_{(t_0,\tau)} = \int\limits_{t_0}^{\tau}\sigma_{ij}(s)a_{ij}ds$ and can be computed by noting that the integral $\int\limits_{t_0}^{\tau}\sigma_{ij}(s)ds$ can be specified to yield

$$\begin{bmatrix}
\frac{1}{2}(\tau-t_0)+\frac{1}{4\omega}\sin 2\omega(\tau-t_0) & \frac{1}{2}(\tau-t_0)-\frac{1}{4\omega}\sin 2\omega(\tau-t_0) & \frac{1}{2\omega}[1-\cos 2\omega(\tau-t_0)]\\
\frac{1}{2}(\tau-t_0)-\frac{1}{4\omega}\sin 2\omega(\tau-t_0) & \frac{1}{2}(\tau-t_0)+\frac{1}{4\omega}\sin 2\omega(\tau-t_0) & -\frac{1}{2\omega}[1-\cos 2\omega(\tau-t_0)]\\
-\frac{1}{4\omega}[1-\cos 2\omega(\tau-t_0)] & \frac{1}{4\omega}[1-\cos 2\omega(\tau-t_0)] & \frac{1}{2\omega}\sin 2\omega(\tau-t_0)
\end{bmatrix}\begin{bmatrix}\sigma^0_{11}\\ \sigma^0_{22}\\ \sigma^0_{12}\end{bmatrix} + \tag{3.18}$$

$$+\begin{bmatrix}
\frac{1}{4}(\tau-t_0)^2-\frac{1}{8\omega^2}[\cos 2\omega(\tau-t_0)-1] & \frac{1}{4}(\tau-t_0)^2+\frac{1}{8\omega^2}[\cos 2\omega(\tau-t_0)-1] & \frac{1}{2\omega}(\tau-t_0)-\frac{1}{4\omega^2}\sin 2\omega(\tau-t_0)\\
\frac{1}{4}(\tau-t_0)^2+\frac{1}{8\omega^2}[\cos 2\omega(\tau-t_0)-1] & \frac{1}{4}(\tau-t_0)-\frac{1}{8\omega^2}[\cos 2\omega(\tau-t_0)-1] & -\frac{1}{2\omega}(\tau-t_0)+\frac{1}{4\omega^2}\sin 2\omega(\tau-t_0)\\
-\frac{1}{4\omega}(\tau-t_0)+\frac{1}{8\omega^2}\sin 2\omega(\tau-t_0) & \frac{1}{4\omega}(\tau-t_0)^2-\frac{1}{8\omega^2}\sin 2\omega(\tau-t_0) & -\frac{1}{4\omega^2}[\cos 2\omega(\tau-t_0)-1]
\end{bmatrix}\begin{bmatrix}F_1\\ F_2\\ F_3\end{bmatrix}$$

The „zero-order" approximation to the energy expression is

$$W^{(0)}_{(t_0,\tau)} = \mathbf{d}^T\boldsymbol{\sigma}^0(\tau-t_0)+\frac{1}{2}\mathbf{d}^T\tilde{\mathbf{C}}\mathbf{d}(\tau-t_0)^2 \tag{3.19}$$

whereas the "first order" approximation can be shown to be

$$W^{(I)}_{(t_0,\tau)} = \mathbf{d}^T\left[\mathbf{I}+\frac{1}{2}\mathbf{B}\omega(\tau-t_0)\right]\boldsymbol{\sigma}^0(\tau-t_0)+\frac{1}{2}\mathbf{d}^T\left[\mathbf{I}+\frac{1}{3}\mathbf{B}\omega(\tau-t_0)\right]\tilde{\mathbf{C}}\mathbf{d}(\tau-t_0)^2. \tag{3.20}$$

It can easily be proved that the work given by Eq. (3.20) vanishes for the displacement gradient path $0-a-0$, Fig. 1, as it should for any closed path in the classical elastic material. For we have

$$W^{(I)}_{(0,1)} = \mathbf{d}^T\left[\mathbf{I}+\frac{1}{2}\mathbf{B}\omega\right]\boldsymbol{\sigma}^0+\frac{1}{2}\mathbf{d}^T\left[\mathbf{I}+\frac{1}{3}\mathbf{B}\omega\right]\tilde{\mathbf{C}}\mathbf{d} \tag{3.21}$$

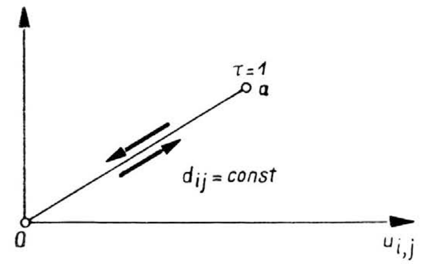

FIG. 1.

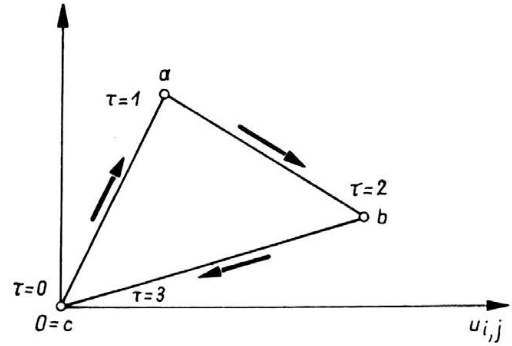

FIG. 2.

and

$$W^{(I)}_{(1,0)} = -\mathbf{d}^T\left[\mathbf{I}-\frac{1}{2}\mathbf{B}\omega\right]\boldsymbol{\sigma}^a+\frac{1}{2}\mathbf{d}^T\left[\mathbf{I}-\frac{1}{3}\mathbf{B}\omega\right]\tilde{\mathbf{C}}\mathbf{d} = -W^{(I)}_{(0,1)}. \tag{3.22}$$

Let us see now whether the same can be said of the closed path $0-a-b-0$ shown in Fig. 2. We parametrize the deformation gradient as

$$\begin{aligned} u_{i,j}(\tau) &= a_{ij}\tau, && \tau\in[0,1],\\ u_{i,j}(\tau) &= (2-\tau)a_{ij}+(\tau-1)b_{ij}, && \tau\in[1,2],\\ u_{i,j}(\tau) &= (3-\tau)b_{ij}, && \tau\in[2,3] \end{aligned} \tag{3.23}$$

so that

$$\begin{aligned} v_{i,j} &= a_{ij}, && \tau\in[0,1],\\ v_{i,j} &= -a_{ij}+b_{ij}, && \tau\in[1,2],\\ v_{i,j} &= -b_{ij}, && \tau\in[2,3]. \end{aligned} \tag{3.24}$$

and with the additional notation for the plane strain case

$$\begin{gathered} \omega^a = a_{[12]}, \quad \omega^b = -a_{[12]}+b_{[12]}, \quad \boldsymbol{\sigma}^a = \boldsymbol{\sigma}|_{\tau=1}, \quad \boldsymbol{\sigma}^c = \boldsymbol{\sigma}|_{\tau=3},\\ \mathbf{d}^{aT} = [a_{(11)}a_{(22)}a_{(12)}], \ldots, \quad \boldsymbol{\sigma}^b = \boldsymbol{\sigma}|_{\tau=2}, \end{gathered} \tag{3.25}$$

we arrive at

$$\boldsymbol{\sigma}^a = [\mathbf{I}+\mathbf{B}\omega^a]\boldsymbol{\sigma}^0+\left[\mathbf{I}+\frac{1}{2}\mathbf{B}\omega^a\right]\tilde{\mathbf{C}}\mathbf{d}^a,$$

$$\sigma^b = [\mathbf{I}+\mathbf{B}\omega^b]\sigma^0 + \frac{1}{2}\mathbf{B}\omega^b\tilde{\mathbf{C}}\mathbf{d}^a + \left[\mathbf{I}+\frac{1}{2}\mathbf{B}(-\omega^a+\omega^b)\right]\tilde{\mathbf{C}}\mathbf{d}^b, \tag{3.26}$$

$$\sigma^c = [\mathbf{I}-\mathbf{B}\omega^b]\sigma^b - \left[\mathbf{I}-\frac{1}{2}\mathbf{B}\omega^b\right]\tilde{\mathbf{C}}\mathbf{d}^b = \sigma^0 + \frac{1}{2}\mathbf{B}(\omega^b\tilde{\mathbf{C}}\mathbf{d}^a - \omega^a\tilde{\mathbf{C}}\mathbf{d}^b);$$

$$W^{(I)}_{(0,1)} = \mathbf{d}^{aT}\left[\mathbf{I}+\frac{1}{2}\mathbf{B}\omega^a\right]\sigma^0 + \frac{1}{2}\mathbf{d}^{aT}\left[\mathbf{I}+\frac{1}{3}\mathbf{B}\omega^a\right]\tilde{\mathbf{C}}\mathbf{d}^a,$$

$$W^{(I)}_{(1,2)} = (-\mathbf{d}^a+\mathbf{d}^b)^T\left[\mathbf{I}+\frac{1}{2}\mathbf{B}(-\omega^a+\omega^b)\right]\sigma^a + \frac{1}{2}(-\mathbf{d}^a+\mathbf{d}^b)\left[\mathbf{I}+\frac{1}{3}\mathbf{B}(-\omega^a+\omega^b)\right]\tilde{C}(-\mathbf{d}^a+\mathbf{d}^b), \tag{3.27}$$

$$W^{(I)}_{(2,3)} = -\mathbf{d}^{bT}\left(\mathbf{I}-\frac{1}{2}\mathbf{B}\omega^b\right)\sigma^b + \frac{1}{2}\mathbf{d}^{bT}\left[\mathbf{I}-\frac{1}{3}\mathbf{B}\omega^b\right]\tilde{\mathbf{C}}\mathbf{d}^b,$$

which finally yields

$$\Sigma W^{(I)} = W^{(I)}_{(0,1,2,3)} = \frac{1}{2}(\mathbf{d}^{bT}\omega^a - \mathbf{d}^{aT}\omega^b)\mathbf{B}\sigma^0 + \frac{1}{3}(\mathbf{d}^{bT}\mathbf{B}\omega^a\tilde{\mathbf{C}}\mathbf{d}^b - \mathbf{d}^{aT}\mathbf{B}\omega^b\tilde{\mathbf{C}}\mathbf{d}^a) + \frac{1}{6}[\mathbf{d}^{bT}\mathbf{B}(\omega^a-\omega^b)\tilde{\mathbf{C}}\mathbf{d}^a + \mathbf{d}^{aT}\mathbf{B}(\omega^a-\omega^b)\tilde{\mathbf{C}}\mathbf{d}^b]. \tag{3.28}$$

We see that in accordance with what we have predicted the total strain energy does not generally sum up to zero in the deformation cycle considered. For the special case of the rotationless straining we have $\omega^a = \omega^b = 0$ and

$$W^{(I)}_{(0,1,2,3)} = 0 \tag{3.29}$$

e.g. the energy balance is correctly fulfilled.

Let us further specify Eq. (3.28) to the case in which $\mathbf{d}^a = \mathbf{d}^b = \mathbf{d}$ (no strain rate increase from a to b), $\sigma^\circ = 0$ (no initial stress) and $\omega^b = 0$ (no spin from a to b). Denoting for the comparison $E = \frac{1}{2}\mathbf{d}^T\tilde{\mathbf{C}}\mathbf{d}$ we arrive at

$$W^{(I)}_{(0,1,2,3)} = \frac{2}{3}\mathbf{d}^T\mathbf{B}\tilde{\mathbf{C}}\mathbf{d}\omega^a \tag{3.40}$$

which means that energy is produced as

$$\Sigma W^{(I)}_{(0,1,2,3)} \sim E\times\gamma\times(\text{number of cycles}), \tag{3.41}$$

where γ is, say, the maximum value of rotation at any cycle. $\Sigma W^{(I)}_{(0,1,2,3)}$ can clearly be a significant figure — for rotations of the order 0.01 (which are not uncommon) the solution can in several cycles lead to the energy error comparable with the maximum energy attained during the cycle thus rendering the results totally useless.

In view of the result (3.41) it is also doubtful that the use of the constitutive law in the form (1.1) with $\tilde{C}_{ijkl} = C_{ijkl}$ (which is common in many finite element programs) may yield more accurate results than those corresponding to the simple (i.e. non-objective) stress rate on the right-hand side of Eq. (1.1).

4. Conclusions

It has been noted in this paper that the hypo-elastic law (1.1) has often been taken in the literature as the basis for the derivation of the elastic-plastic constitutive law and then applied to the analysis of cyclic processes. The danger of a significant accuracy loss in such an analysis has been demonstrated. The way out is obviously to take the objective rate form of the hyper-elasticity law in a correct, energy-preserving form. This is not a trivial task, though [7].

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