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Mixed boundary-initial value problem for the equations of thermodiffusion in solid body

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THE PROOFS of the existence and uniqueness of the weak solution of the initial-boundary value problem for the equations of linear thermodiffusion in a solid body with a mixed boundary condition for temperature, displacement and stresses and the Dirichlet boundary condition for chemical potential have been presented. These proofs have been obtained using the Faedo–Galerkin method in suitable chosen Sobolev spaces.

Przedstawiono dowody istnienia i jednoznaczności słabego rozwiązania mieszanego zagadnienia brzegowo-początkowego w termodyfuzji ciał stałych z mieszanym warunkiem brzegowym dla temperatury, przemieszczeń i naprężeń oraz warunkiem Dirichleta dla potencjału chemicznego. Dowody te przeprowadzono, stosując metodę Faedo–Galerkina w odpowiednich przestrzeniach Sobolewa.

Представлены доказательства существования и единственности слабого решения краево-начальной задачи в термодиффузии твердых тел со смешанным граничным условием для температуры, перемещений и напряжений, а также с условием Дирихле для химического потенциала. Эти доказательства проведены, применяя метод Фаедо-Галеркина в соответствующих пространствах Соболева.

1. Introduction

UNDER the influence exerted by the action of external loads, heating of the body and diffusion of the matter into the solid will arise in this body a displacement field $u(x, t)$ the temperature $\theta_1(x,t)$, and chemical potential $\theta_2(x, t)$. The relations between these fields, called the equations of thermodiffusion in a solid body, have been investigated by W. NOWACKI (cf. [18, 19, 20, 21]), J. S. PODSTRIGAČ (cf. [22, 23] and other authors. In the paper [19] W. Nowacki derived a form of the equations of thermodiffusion other than in [23] by taking a displacement field $u(x, t)$, the temperature $\theta_1(x, t)$, and the chemical potential $\theta_2(x, t)$ as independent functions. These fields are functions of point $x = (x_1, \ldots, \ldots, x_r)$ $(r = 1, 2, 3)$ and time t. The phenomenon of thermodiffusion (cf. [19, 20, 21]) is described by the coupled system of five second order partial differential equations:

(1.1) $$\varrho\partial_t^2 u = \mu\Delta u+(\lambda+\mu)\nabla(\nabla\cdot u)-\gamma_1\nabla\theta_1-\gamma_2\nabla\theta_2+X, \quad (^1)$$

(1.2) $$c\partial_t\theta_1 = k\Delta\theta_1-\gamma_1\partial_t\nabla\cdot u-d\partial_t\theta_2+Q_1,$$

(1.3) $$n\partial_t\theta_2 = D\Delta\theta_2-\gamma_2\partial_t\nabla\cdot u-d\partial_t\theta_1+Q_2.$$

In these equations by $u = (u_1, \ldots, u_r)$ we denote the displacement vector field of the body, by $\theta_1(x, t)$ — the temperature of the body, by $\theta_2(x, t)$ — the chemical potential, by

(1) $\Delta = \partial_j\partial_j$, $j = 1, \ldots, r$, $\nabla(\nabla\cdot u) = \operatorname{grad}\operatorname{div} u$, $\nabla\theta_i = \operatorname{grad}\theta_i$, $i = 1, 2$.

$X = (X_1, \ldots, X_r)$ the yector of body forces, by Q_1 — the intensity of the heat source, by Q_2 — the intensity of the source of diffusing mass: λ, μ — are Lamé's constants, ϱ — density and $\gamma_1 = 3K\bar{a}_1$, $\gamma_2 = 3K\bar{a}_2$, where $K = \lambda + (2/3)\mu$, while $\bar{a}_1$, $\bar{a}_2$ stand for coefficients of linear thermal and diffusion dilatation.

Quantity k is the coefficient of thermal conductivity, while D — the coefficient of diffusion. Quantities n, c, d are the coefficients of thermodiffusion. These quantities satisfy the (cf. [21]) following relations:

$$(1.4) \quad \mu > 0, \quad \lambda+(2/3)\mu > 0, \quad k > 0, \quad D > 0, \quad c > 0, \quad n > 0, \quad nc > d^2 \quad (^2).$$

The concentration field $\theta_3(x, t)$ is related to the displacement vector $u(x, t)$, the temperature $\theta_1(x, t)$, and chemical potential $\theta_2(x, t)$ as follows:

$$(1.5) \qquad \theta_3 = \gamma_2 \operatorname{div} u + d\theta_1 + n\theta_2 .$$

The system of Eqs. (1.1)—(1.3) is hyperbolic with respect to some of the unknown functions and parabolic with respect to others. This system contains as particular case (assuming some of the coefficients to be equal to zero) the well-known system of partial differential equations of coupled thermoelasticity (cf. [17]). The existence problem in the case of thermoelasticity is studied in [4, 13, 12, 17].

J. S. PODSTRIGAČ (cf. [22, 25]) solved many particular, mostly one-dimensional, problems of thermodiffusion in a solid solution. In the papers [18, 19] W. NOWACKI derived the fundamental theorems for the dynamic problems of diffusion in a solid body such as the theorem of virtual work of variation of displacement and rotations, fundamental energy theorem as well as the theorem of the reciprocity of works. In the paper [21] he reduced the system of thermodiffusion equations to wave equations of a comparatively simple form owing to the introduction of elastic potentials and a Galerkin-type representation.

In [27] the existence of the solution of an initial value problem for Eqs. (1.1)—(1.3) has been proved in the class of smooth functions vanishing at infinity using the method of successive approximations.

The existence and uniqueness of the solution of the first boundary-initial value problem for Eqs. (1.1)—(1.3) was proved by G. FICHERA (cf. [8]) in the class of functions $C^1\{\bar{A} \times [0, +\infty)\} \cap C^2\{A \times [0, +\infty)\}$ using the Laplace transformation (A — a bounded domain (open set) of the three-dimensional space with a piecewise smooth boundary).

T. V. BURCULADZE in the paper [3] using the Laplace transformation reduced the considered initial-boundary value problems for Eqs. (1.1)—(1.3) to the system of integral equations and proved an existence theorem for this system of integral equations.

In this paper, using the Faedo–Galerkin method, the existence and uniqueness of a weak solution of the initial-boundary value problem for Eqs. (1.1)–(1.3) with a mixed boundary condition for temperature, displacement and stresses, and the Dirichlet boundary condition for chemical potential have been investigated in suitably chosen Sobolev spaces.

(2) The inequality $nc > d^2$ and other inequalities for the constitutive constants have been obtained by W. NOWACKI in [21]. The inequality $nc > d^2$ plays a very important role in the proof of the existence and the uniqueness theorem of the weak solution of the problem considered in this paper (cf. formula (4.16)).

The method used in this paper should work even with more general or different boundary conditions. It must also be mentioned that the method developed in this paper applies to nonisotropic, inhomogeneous bodies as well. However, this extension will not be discussed here.

2. Sobolev spaces

By r we denote the dimension of the Euclidean space E^r in which the configuration of a thermo-diffusive-elastic medium is embedded. The analysis will be carried out for general r though the model is physically meaningful only for $r = 1, 2, 3$. By x we denote the typical point of E^r and by $x_1, \ldots, x_r$ the coordinates of x with respect to a fixed Cartesian coordinate system. By $\alpha = (\alpha_1, \ldots, \alpha_r)$ we denote multi-index and by $|\alpha| = \alpha_1 + \ldots + \alpha_r$ its length. We introduce the following notation for derivatives with respect to the space variables $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_r^{\alpha_r}$ where $\partial_j = \dfrac{\partial}{\partial x_j}$ for $j = 1, \ldots, r$. Time derivatives are denoted by

$$\partial_t^s = \frac{\partial^s}{\partial t^s} \quad \text{where} \quad s = 1,2 \quad \left(\partial_t = \frac{\partial}{\partial t}\right).$$

Let G be an open bounded set in E^r (cf [7] p. 13) with regular boundary ∂G.

$L^p(G)$ is the space of(³) (equivalence classes of) measurable functions u such that (p being given with $1 \leqslant p \leqslant \infty$)

$$\|u\|_{L^p(G)} = \Big(\int_G |u(x)|^p dx\Big)^{1/p} < \infty, \quad 1 \leqslant p < \infty, \tag{2.1}$$

$$\|u\|_{L^\infty(G)} = \operatorname*{ess\,sup}_{x \in G} |u(x)|, \quad p = \infty \tag{2.2}$$

taken with the norm (2.1) or (2.2), $L^p(G)$ is a Banach space; if $p = 2$, $L^2(G)$ is a Hilbert space, where the scalar product corresponding to the norm (2.1) (where $p = 2$) is given by

$$(u, v)_{L^2(G)} = \int_G u(x)v(x)\,dx. \tag{2.3}$$

The Sobolev space $W_p^m(G)$ (cf. [2] p. 29—38, [28] p. 53—64), $1 \leqslant p \leqslant \infty$, consists of those functions u belonging to $L^p(G)$ with weak derivatives $\partial^\alpha u$ ($|\alpha| \leqslant m$) belonging to $L^p(G)$

$$W_p^m(G) = \{u : u \in L^p(G) : \partial^\alpha u \in L^p(G)\,;\ |\alpha| \leqslant m\}. \tag{2.4}$$

With the norm

$$\|u\|_{W_p^m(G)} = \Big(\sum_{|\alpha| \leqslant m} \|\partial^\alpha u\|_{L^p(G)}^p\Big)^{1/p} \tag{2.5}$$

it is a Banach space.

The case $p = 2$ is fundamental. To simplify the writing, we will put

$$W_2^m(G) = H^m(G)$$

(³) All functions considered here are real-valued.

with the scalar product

$$(u, v)_{H^m(G)} = \sum_{|\alpha| \leqslant m} (\partial^\alpha u, \partial^\alpha v)_{L^2(G)}, \tag{2.6}$$

it is a Hilbert space. The norm in this space is given by

$$||u||_{H^m(G)} = \Big(\sum_{|\alpha| \leqslant m} ||\partial^\alpha u||^2_{L^2(G)} \Big)^{1/2}, \tag{2.7}$$

$C^\infty(G)$ denotes the space of infinitely differentiable real-valued functions defined on G. $C_0^\infty(G)$ consists of those elements of $C^\infty(G)$ with compact support contained in G. By $H_0^m(G)$ we denote the Hilbert space obtained as the completion of $C_0^\infty(G)$ by means of the norm $||\cdot||_{H^m(G)}$ given by the relation (2.7). $H_0^m(G)$ is the subspace of the space $H^m(G)$.

By $\mathsf{L}^2(G)$, $\mathsf{H}^m(G)$ we denote the r-fold Cartesian product of $L^2(G)$, $H^m(G)$, respectively. We denote the scalar product and norms in the space $L^2(G)$, $\mathsf{L}^2(G)$ $(H^m(G), \mathsf{H}^m(G))$ by $(\cdot, \cdot)_{L^2}$, $(\cdot, \cdot)_{\mathsf{L}^2}$, $((\cdot, \cdot)_{H^m}, (\cdot, \cdot)_{\mathsf{H}^m})$ and $||\cdot||_{L^2}$, $||\cdot||_{\mathsf{L}^2}$ $(||\cdot||_{H^m}, ||\cdot||_{\mathsf{H}^m})$, respectively.

In this paper we will investigate the solvability of evolution problems using the Faedo–Galerkin method in the space $L^2(I, X)$ where $I = (0, \vartheta) \subset R$ $(0 < \vartheta < \infty)$ — the time interval, X — the Banach space with its norm denoted by $||\cdot||_X$ (cf. [6]).

By $L^p(I, X)$ we denote the space of (classes of) functions $t \to f(t)$ measurable from $(0, \vartheta) \to X$ (for the measure dt) such that

$$||u||_{L^p(I,X)} = \Big(\int_0^\vartheta ||u(t)||_X^p \, dt \Big)^{1/p}, \quad 1 \leqslant p < \infty, \tag{2.8}$$

$$||u||_{L^\infty(I,X)} = \operatorname*{ess\,sup}_{t \in X} ||u(t)||_X, \quad p = \infty. \tag{2.9}$$

This is a Banach space.

$W_2^k(I, X)$, $k \in \mathsf{N}$ denotes the space of the measurable functions $u: I \to X$, with $d^n u/dt^n \in L^2(I, X)$ for $0 \leqslant n \leqslant k$ (derivatives in the weak sense). The norm in $W_2^k(I, X)$ is given by

$$||u||^2_{W_2^k(I,X)} = \sum_{n=0}^{k} \int_0^\vartheta ||d^n u(t)/dt^n||_X^2 \, dt. \tag{2.10}$$

The space $W_2^k(I, X)$ is the Hilbert space (cf. [29] p. 168).

Let V and H be two Hilbert spaces over R with norms $||\cdot||_V$, $||\cdot||_H$, respectively, their scalar product in H being written $(\cdot, \cdot)_H$; we assume that

$$V \subset H, \quad V \text{ dense in } H \ (^4).$$

Identifying H with its dual $(H = H^*)$ (⁵), H is then identified with a subspace of the dual V^* of V, whence

$$V \subset H \subset V^*. \tag{2.11}$$

The space V, H, V^* which have the property (2.11) form the Gelfand triples (cf. [6, 29]).

(4) Therefore there exists a constant c such that

$$||v||_H \leqslant c||v||_V \quad \forall v \in V.$$

(5) By V^* we denote the dual space to the space V.

In this paper we will use the following inequalities:

1. The Poincaré inequality (cf. [7] p. 347—389).

$$\|u\|_{H^m}^2 \leqslant C \sum_{|\alpha| \leqslant m} \int_G |\partial^\alpha u|^2 dx, \quad u \in H_0^m(G), \tag{2.12}$$

where $C = C(G, m)$.

2. The second Korns inequality (cf. [6] p. 110)

$$\int_G \varepsilon_{ij}(u)\,\varepsilon_{ij}(u)\,dx + \int_G u_i u_i\,dx \geqslant C\|u\|_{H^1}^2, \quad \forall u \in H^1(G), \tag{2.13}$$

where $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ and $C = C(G)$, $C > 0$.

3. Gronwall's inequality (cf. [14] p. 298)

Let g, ϱ be functions with properties $g, \varrho \in C([0, \vartheta])$, $g, \varrho \geqslant 0$ and g is a non-decreasing function.

If ϱ satisfies the inequality

$$\varrho(t) \leqslant g(t) + C_0 \int_0^t \varrho(\sigma)\,d\sigma, \quad 0 \leqslant t \leqslant \vartheta, \quad C_0 \text{ — const} \tag{2.14}$$

then there exists a constant $C_1 = C_1(C_0, \vartheta)$ such that

$$\varrho(t) \leqslant C_1 g(t) \quad \forall t \in [0, \vartheta]. \tag{2.15}$$

REMARK 2.1. The spaces used in our consideration form the Gelfand triples. In the case of the boundary-initial value problem considered in this paper we use the spaces

$$\mathsf{V}_0, \mathsf{L}^2(G), \mathsf{V}_0^*; \quad V_1, L^2(G), V_1^* \quad \text{and} \quad V_2, L^2(G), V_2^*$$

(cf. Definition 3.1), which form the Gelfand triples.

3. Statement of the problem

For Eqs. (1.1)—(1.3) we consider the mixed boundary-initial value problem in the region $I \times G$ with the following initial conditions:

$$u(+0) = \varphi_1, \quad (\partial_t u)(+0) = \varphi_2, \quad \theta_l(+0) = \vartheta_l \quad (l = 1, 2), \quad (^6) \tag{3.1}$$

and boundary conditions:

$$u|_{I \times \partial G_1} = \Phi, \quad \sigma \cdot v|_{I \times \partial G_2} = \Psi, \quad \theta_1|_{I \times \partial G_2} = p, \tag{3.2}$$

$$\left.\frac{d\theta_1}{d\nu}\right|_{I \times \partial G_1} = q, \quad \theta_2|_{I \times \partial G_2} = 0, \quad (^7)$$

where $\varphi_1, \varphi_2, \vartheta_1, \vartheta_2, \Phi, \Psi, p, q$ are given;

$$\sigma\nu = (\sigma_{ji}\nu_j)_{i=1,\dots,r} = \left[\mu \frac{du_i}{d\nu} + \mu\nu_j \frac{\partial u_j}{\partial x_i} + (\lambda\nabla \cdot u - \gamma_1\theta_1 - \gamma_2\theta_2)\nu_i\right]_{i=1,\dots,r},$$

(6) We use the notation $f(t) = f(\cdot, t)$, where $\cdot$ denotes the nondeclared variable (cf. [6] p.32).

(7) The assumption of such a decomposition of the boundary ∂G as in the condition (3.2) does not lead to the loss of generality (cf. [17] p. 66)

G — denotes the bounded domain in r-dimensional Euclidean space $E^r (r = 1, 2, 3)$ with smooth (cf. [1]) boundary ∂G, $I = (0, T)$ the bounded time interval $(T < \infty)$; $\partial G = \partial G_1 \cup \partial G_2$, $\partial G_1 \cap \partial G_2 = \phi$. $I \times \partial G$ — the Cartesian product of I and ∂G, ν — the unit external normal to ∂G.

We will seek a weak solution of the boundary-initial value problem for Eqs. (1.1)—(1.3) with the conditions (3.1), (3.2). In order to do it, we start with the definition of the weak solution of this problem.

DEFINITION 3.1. (a weak solution)
The system of functions

$$(u, \theta_1, \theta_2) \in L^2(I, V_0) \times L^2(I, V_1) \times L^2(I, V_2)$$

will be called a weak solution of the problem (1.1)—(1.3), (3.1), (3.2) *if* (u, θ_1, θ_2) *satisfies the following identities*:

$$\varrho\big(\partial_t^2 u(t), \omega\big) + a_1\big(u(t), \omega\big) = \gamma_1\big(\theta_1(t), \nabla\cdot\omega\big) + \gamma_2\big(\theta_2(t), \nabla\cdot\omega\big) + \big(\Omega_1(t), \omega\big) + \gamma_1\big(\Phi_2(t), \nabla\cdot\omega\big) \quad \forall \omega \in V_0,$$

$$(3.3)\quad c\big(\partial_t\theta_1(t), \beta\big) + a_2\big(\theta_1(t), \beta\big) = -d\big(\partial_t\theta_2(t), \beta\big) - \gamma_1(\partial_t\nabla\cdot u, \beta) + \big(\Omega_2(t), \beta\big) \quad \forall \beta \in V_1,$$

$$n\big(\partial_t\theta_2(t), v\big) = D\big(\Delta\theta_2(t), v\big) - \gamma_2\big(\partial_t\nabla\cdot u(t), v\big) - d\big(\partial_t\theta_1(t), v\big) + \big(\Omega_3(t), v\big) \quad \forall v \in V_2,$$

with the initial conditions

$$(3.4)\qquad u(0) = \tilde{\varphi}_1, \quad (\partial_t u)(0) = \tilde{\varphi}_2, \quad \theta_1(0) = \tilde{\vartheta}_1, \quad \theta_2(0) = \vartheta_2,$$

where

$$a_1\big(u(t), \omega\big) = \int_G \left[\mu\left(\frac{\partial u_i(t)}{\partial x_j} + \frac{\partial u_j(t)}{\partial x_i}\right) + \lambda\big(\nabla\cdot u(t)\big)\delta_{ij}\right]\frac{\partial\omega_i}{\partial x_j}\,dx,$$

$$a_2\big(\theta_1(t), \beta\big) = k\int_G \nabla\theta_1(t)\nabla\beta\,dx,$$

$$\big(\Omega_1(t), \omega\big) = \big(X(t), \omega\big) + \int_{\partial G_2}\Psi(t)\omega\,ds - \varrho\big(\partial_t^2\Phi_1(t), \omega\big) - a_1\big(\Phi_1(t), \omega\big),$$

$$\big(\Omega_2(t), \beta\big) = \big(Q_1(t), \beta\big) - a_2\big(\Phi_2(t), \beta\big) - c\big(\partial_t\Phi_2(t), \beta\big) - \gamma_1\big(\partial_t\nabla\cdot\Phi_1(t), \beta\big) + k\int_{\partial G_1}\beta q(t)\,ds,$$

$$\big(\Omega_3(t), v\big) = \big(Q_2(t), v\big) - d\big(\partial_t\Phi_2(t), v\big) - \gamma_2\big(\partial_t\nabla\cdot\Phi_1(t), v\big),$$

$$\tilde{\varphi}_1 = \varphi_1 - \Phi_1(0), \quad \tilde{\varphi}_2 = \varphi_2 - (\partial_t\Phi_1)(0), \quad \tilde{\vartheta}_1 = \vartheta_1 - \Phi_2(0),$$

$\Phi_1(t) \in H^1(G)$ and satisfies the condition $\Phi_1(t)|_{\partial G_1} = \Phi(t)$,

$\Phi_2(t) \in H^1(G)$ and satisfies the condition $\Phi_2(t)|_{\partial G_2} = p(t)$, [8]

$V_0 = \{\omega : \omega \in H^1(G) \wedge \omega|_{\partial G_1} = 0\}$, $V_1 = \{\beta : \beta \in H^1(G) \wedge \beta|_{\partial G_2} = 0\}$,

$V_2 = \{v : v \in H_0^1(G)\}$. [9]

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[8] We use the same notations as in [6] p. 122.

[9] The spaces V_0, V_1, V_2 are separable. ([2, 14, 29]).

Let us notice that the spaces V_0, $\mathsf{L}^2(G)$, V_0^*, V_1, $L^2(G)$, V_1^* and V_2, $L^2(G)$, V_2^* (where the spaces V_0^*, V_1^*, V_2^* denote the dual spaces to the spaces V_0, V_1, V_2 respectively cf. [29]) form the Gelfand triples. The symbol $(\cdot\,,\,\cdot)$ denotes the forms of duality on $(\mathsf{V}_0, \mathsf{V}_0^*)$; (V_1, V_1^*) and (V_2, V_2^*) respectively, which on the Cartesian product $\mathsf{L}^2(G)\times \mathsf{L}^2(G)$ or on the product $L^2(G)\times L^2(G)$ becomes the scalar product in the spaces $\mathsf{L}^2(G)$ or $L^2(G)$, respectively.

4. Existence theorem

THEOREM 4.1. *If the following supplementary conditions are satisfied*:

$$\begin{aligned}
&X\in W_2^1\big(I,\mathsf{L}^2(G)\big), \quad Q_i\in L^2(I,V_i^*), \quad \tilde{\varphi}_1\in \mathsf{V}_0,\\
(4.1)\quad &\tilde{\varphi}_2\in \mathsf{L}^2(G), \quad \tilde{\vartheta}_1\in V_1, \quad \vartheta_2\in V_2, \quad \tilde{q}\in L^2\big(I,L^2(\partial G)\big),\\
&\tilde{\Phi}_1\in W_2^1\big(I,\mathsf{H}^{1/2}(\partial G)\big), \quad \partial_t^2\tilde{\Phi}_1\in W_2^1\big(I,\mathsf{H}^{-1/2}(\partial G)\big)\ (^{10}),\\
&\tilde{\Phi}_2\in W_2^1\big(I,\mathsf{H}^{1/2}(\partial G)\big), \quad \tilde{\Psi}\in W_2^1\big(I,L^2(G)\big) \quad (i=1,2)
\end{aligned}$$

then there exists a weak solution of the problem (1.1)—(1.3), (3.1), (3.2) *with the properties*

$$(4.2)\quad \partial_t u\in L^2\big(I,\mathsf{L}^2(G)\big), \quad \partial_t^2 u\in L^2(I,\mathsf{V}_0^*), \quad \partial_t\theta_1\in L^2(I,V_1^*), \quad \partial_t\theta_2\in L^2(I,V_2^*).$$

REMARK 4.1. By $\tilde{q}$, $\tilde{\Phi}_1$, $\tilde{\Phi}_2$, $\tilde{\Psi}$ we denote the extension of the functions q, Φ_1, Φ_2, Ψ to $I\times\partial G$ (cf. [6] with the properties

$$\tilde{q}(t)|_{I\times\partial G_1} = q(t), \quad \tilde{\Phi}_1(t)|_{I\times\partial G_1} = \Phi(t), \quad \tilde{\Phi}_2(t)|_{I\times\partial G_2} = p(t), \quad \tilde{\Psi}(t)|_{I\times\partial G_2} = \Psi(t).$$

Proof. We prove the theorem 4.1 using the Faedo–Galerkin method. The proof is divided into three steps:

1. The approximation of the solution by a sequence $(u^m, \theta_1^m, \theta_2^m)_{m\in N}$, i.e. the so-called sequence of Galerkin approximations.

2. The estimations of the Galerkin approximations.

3. The convergence of the sequence $(u^m, \theta_1^m, \theta_2^m)_{m\in N}$ to the weak solution of the problem (1.1)–(1.3), (3.1), (3.2).

Ad. 1. Let $\{\omega^m\}$, $\{\beta_m\}$ and $\{v_m\}$ be linear and complete systems in V_0, V_1, V_2, respectively.

We define the Galerkin approximations of the solution (u, θ_1, θ_2) by

$$(4.3)\qquad u^m(t) = \sum_{j=1}^{m} g_j^m(t)\omega^j, \quad \theta_1^m(t) = \sum_{j=1}^{m} h_{mj}(t)\beta_j, \quad \theta_2^m(t) = \sum_{j=1}^{m} k_{mj}(t)v_j,$$

where the functions $g_j^m(\cdot)$, $h_{mj}(\cdot)$, $k_{mj}(\cdot)$ are chosen in such a way that the following system of equations is satisfied:

$$\varrho\big(\partial_t^2 u^m(t),\omega^l\big)_{\mathsf{L}^2}+a_1\big(u^m(t),\omega^l\big) = \gamma_1\big(\theta_1^m(t),\nabla\cdot\omega^l\big)_{L^2} +\gamma_2\big(\theta_2^m(t),\nabla\cdot\omega^l\big)_{L^2}+\gamma_1\big(\Phi_2(t),\nabla\cdot\omega^l\big)+\big(\Omega_1(t),\omega^l\big),$$

(^{10}) The definition of the space $H^s(\partial G)$ for $s\in\mathsf{R}$ may be found in [14].

$$c\big(\partial_t\theta_1^m(t),\beta_l\big)_{L^2}+a_2\big(\theta_1^m(t),\beta_l\big) = -d\big(\partial_t\theta_2^m(t),\beta_l\big)_{L^2}-\gamma_1\big(\partial_t\nabla\cdot u^m(t),\beta_l\big)_{L^2}+\big(\Omega_2(t),\beta_l\big),$$
$$n\big(\partial_t\theta_2^m(t),v_l\big)_{L^2} = D\big(\Delta\theta_2^m(t),v_l\big)_{L^2}-\gamma_2\big(\partial_t\nabla\cdot u^m(t),v_l\big)_{L^2}+\big(\Omega_3(t),v_l\big)-d\big(\partial_t\theta_1^m(t),v_l\big)_{L^2} \tag{4.4}$$

with the initial conditions:

$$u^m(0) = \tilde{\varphi}_1^m = \sum_{j=1}^m a_j^m\omega^j, \qquad (\partial_t u^m)(0) = \tilde{\varphi}_2^m = \sum_{j=1}^m b_j^m\omega^j,$$
$$\theta_1^m(0) = \tilde{\vartheta}_1^m = \sum_{j=1}^m c_{mj}\beta_j, \qquad \theta_2^m(0) = \vartheta_2^m = \sum_{j=1}^m d_{mj}v_j, \tag{4.5}$$

where

$$\tilde{\varphi}_1^m\to\tilde{\varphi}_1 \quad \text{in } \mathsf{V}_0, \qquad \tilde{\varphi}_2^m\to\tilde{\varphi}_2 \quad \text{in} \quad \mathsf{L}^2(G),$$
$$\tilde{\vartheta}_1^m\to\tilde{\vartheta}_1 \quad \text{in } V_1, \qquad \vartheta_2^m\to\vartheta_2 \quad \text{in} \quad V_2$$

if $m\to\infty$.

The equations (4.4) with the initial conditions (4.5) are a system of ordinary linear differential equations for unknowns g_j^m, h_{mj}, k_{mj} and can be written in the form

$$\varrho G_{1m}\frac{d^2G_m}{dt^2}+A_{1m}G_m = \gamma_1 M_{1m}H_m+\gamma_2 M_{2m}K_m+F_{1m},$$
$$cG_{2m}\frac{dH_m}{dt}+A_{2m}H_m = -dD_{1m}\frac{dK_m}{dt}-\gamma_1 U_{1m}\frac{dG_m}{dt}+F_{2m}, \tag{4.6}$$
$$nG_{3m}\frac{dK_m}{dt} = M_{2m}K_m-\gamma_2 U_{2m}\frac{dG_m}{dt}-dD_{2m}\frac{dH_m}{dt}+F_{3m}$$

and the inital conditions

$$G_m(0) = \big(g_j^m(0)\big)_{j=1,\ldots,m} = (a_{01}^m, a_{02}^m, \ldots, a_{0m}^m),$$
$$(\partial_t G_m)(0) = \big(\partial_t g_j^m(0)\big)_{j=1,\ldots,m} = (b_{11}^m, b_{12}^m, \ldots, b_{mm}^m),$$
$$H_m(0) = \big(h_{mj}(0)\big)_{j=1,\ldots,m} = (c_{01}^m, c_{02}^m, \ldots, c_{mm}^m), \tag{4.7}$$
$$K_m(0) = \big(k_{mj}(0)\big)_{j=1,\ldots,m} = (d_{01}^m, \ldots, d_{mm}^m),$$

where

$$G_{1m} = \big((w^j, w^l)_{L^2}\big)_{j,l=1,\ldots,m},$$
$$G_m(t) = \big(g_j^m(t)\big)_{j=1,\ldots,m},$$
$$A_{1m} = \big(a_1(w^j, w^l)\big)_{j,l=1,\ldots,m},$$
$$H_m(t) = \big((h_{mj}(t)\big)_{j=1,\ldots,m},$$
$$M_{1m} = \big((\beta_j, \nabla\cdot w^l)_{L^2}\big)_{j,l=1,\ldots,m},$$
$$K_m(t) = \big(k_{mj}(t)\big)_{j=1,\ldots,m},$$
$$M_{2m} = \big((v_j, \nabla\cdot\omega^l)_{L^2}\big)_{j,l=1,\ldots,m},$$
$$F_{1m} = \big((\Omega_1(t),\omega^l)+\gamma_1\big(\Phi_2(t),\nabla\cdot\omega^l\big)\big)_{l=1,\ldots,m},$$

$$
\begin{aligned}
G_{2m} &= \big((\beta_j, \beta_l)_{L^2}\big)_{j,l=1,\dots,m},\\
A_{2m} &= \big(a_2(\beta_j, \beta_l)\big)_{j,l=1,\dots,m},\\
D_{1m} &= \big((v_j, \beta_l)_{L^2}\big)_{j,l=1,\dots,m},\\
U_{1m} &= \big((\nabla \cdot \omega^j, \beta_l)_{L^2}\big)_{j,l=1,\dots,m},\\
F_{2m} &= \big((\Omega_2(t), \beta_l)\big)_{l=1,\dots,m};\\
G_{3m} &= \big((v_j, v_l)_{L^2}\big)_{j,l=1,\dots,m},\\
M_{2m} &= \big((\nabla v_j, v_l)\big)_{j,l=1,\dots,m},\\
U_{2m} &= \big((\nabla \cdot \omega^j, v_l)_{L^2}\big)_{j,l=1,\dots,m},\\
D_{2m} &= \big((\beta_j, v_l)_{L^2}\big)_{j,l=1,\dots,m},\\
F_{3m} &= \big((\Omega_3(t), v_l)\big)_{l=1,\dots,m}.
\end{aligned}
$$

Equations (4.6) with the initial conditions (4.7) have a unique solution in the interval $I = (0, T)$ $(T < \infty)$ (cf. [15] p. 327—328). This follows from the general theory of ordinary differential equations (cf. [26] p. 157—187). Thus the Galerkin approximation sequences $(u^m)_{m\in N}$, $(\theta_1^m)_{m\in N}$, $(\theta_2^m)_{m\in N}$ are uniquely determined by the system (4.4)—(4.5).

Ad. 2. By multiplying the relations $(4.4)_1$ by $(\partial_t g_l^m)(t)$, $(4.4)_2$ and $(4.4)_3$ by $h_{ml}(t)$, $k_{ml}(t)$ respectively, and taking the sum over l for $(1 \leqslant l \leqslant m)$, we obtain

$$
\begin{aligned}
&\varrho\big(\partial_t^2 u^m(t), \partial_t u^m(t)\big)_{L^2} + a_1\big(u^m(t), \partial_t u^m(t)\big) = \gamma_1\big(\theta_1^m(t), \nabla\cdot\partial_t u^m(t)\big)_{L^2}\\
&\qquad + \gamma_2\big(\theta_2^m(t), \nabla\cdot\partial_t u^m(t)\big)_{L^2} + \big(\Omega_1(t), \partial_t u^m(t)\big) + \gamma_1\big(\Phi_2(t), \nabla\cdot\partial_t u^m(t)\big),\\
&c\big(\partial_t\theta_1^m(t), \theta_1^m(t)\big)_{L^2} + a_2\big(\theta_1^m(t), \theta_1^m(t)\big) = -d\big(\partial_t\theta_2^m(t), \theta_1^m(t)\big)_{L^2}\\
&\qquad - \gamma_1\big(\partial_t\nabla\cdot u^m(t), \theta_1^m(t)\big)_{L^2} + \big(\Omega_2(t), \theta_1^m(t)\big),\\
&n\big(\partial_t\theta_2^m(t), \theta_2^m(t)\big)_{L^2} = D\big(\Delta\theta_2^m(t), \theta_2^m(t)\big) - \gamma_2\big((\partial_t\nabla\cdot u^m(t), \theta_2^m(t)\big)_{L^2}\\
&\qquad - d\big(\partial_t\theta_1^m(t), \theta_2^m(t)\big)_{L^2} + \big(\Omega_3(t), \theta_2^m(t)\big).
\end{aligned}
\tag{4.8}
$$

It is easy to see that the following identities are true:

$$
\begin{aligned}
\varrho\big(\partial_t^2 u^m(t), \partial_t u^m(t)\big)_{L^2} &= \frac{1}{2}\varrho\frac{d}{dt}\|\partial_t u^m(t)\|_{L^2}^2,\\
a_1\big(u^m(t), \partial_t u^m(t)\big) &= \frac{1}{2}\frac{d}{dt}a_1\big(u^m(t), u^m(t)\big),\\
c\big(\partial_t\theta_1^m(t), \theta_1^m(t)\big)_{L^2} &= \frac{1}{2}c\frac{d}{dt}\|\theta_1^m(t)\|_{L^2}^2,\\
n\big(\partial_t\theta_2^m(t), \theta_2^m(t)\big)_{L^2} &= \frac{1}{2}n\frac{d}{dt}\|\theta_2^m(t)\|_{L^2}^2,\\
D\big(\Delta\theta_2^m(t), \theta_2^m(t)\big)_{L^2} &= -D\|\nabla\theta_2^m(t)\|_{L^2}^2.
\end{aligned}
\tag{4.9}
$$

Substituting the relations (4.9) into Eqs. (4.8), adding Eqs. (4.8) by sides and integrating the obtained result over the interval $(0, t)$, we get

$$\varrho\|\partial_t u^m(t)\|^2_{L^2}+a_1\big(u^m(t),u^m(t)\big)+c\|\theta_1^m(t)\|^2_{L^2}+n\|\theta_2^m(t)\|^2_{L^2}$$
$$+2D\int_0^t\|\nabla\theta_2^m(\tau)\|^2_{L^2}d\tau+2\int_0^t a_2\big(\theta_1^m(\tau),\theta_1^m(\tau)\big)d\tau=-2d\big(\theta_1^m(t),\theta_2^m(t)\big)_{L^2}$$
$$+2\int_0^t\big(\Omega_1(\tau),\partial_\tau u^m(\tau)\big)d\tau+2\int_0^t\big(\Omega_2(\tau),\theta_1^m(\tau)\big)d\tau+2\int_0^t\big(\Omega_3(\tau),\theta_2^m(\tau)\big)d\tau$$
$$+\varrho\|\tilde\varphi_2^m\|^2_{L^2}+a_1(\tilde\varphi_1^m,\tilde\varphi_1^m)+c\|\vartheta_1^m\|^2_{L^2}+n\|\vartheta_2^m\|^2_{L^2}+2d(\tilde\vartheta_1^m,\vartheta_2^m)+2\gamma_1\int_0^t\big(\Phi_2(\tau),\nabla\cdot\partial_\tau u^m(\tau)\big)d\tau. \tag{4.10}$$

Using the Schwarz inequality (cf. [15], [28]) and taking into account the inequality

$$2|ab|\leqslant\varepsilon a^2+\frac{1}{\varepsilon}b^2\qquad\forall\varepsilon>0,\ (^{11}) \tag{4.11}$$

we get the following estimates:

$$\left|2\int_0^t\big(\Omega_1(\tau),\partial_\tau u^m(\tau)\big)d\tau\right|\leqslant c_1\varepsilon_1\|u^m(t)\|^2_{\mathring V_0}+c_2\varepsilon_2\int_0^t\|u^m(\tau)\|^2_{\mathring V_0}d\tau$$
$$+\frac{c_1}{\varepsilon_1}\|\Omega_1(t)\|^2_{\mathring V_0^*}+\frac{c_2}{\varepsilon_2}\int_0^t\|\partial_\tau\Omega_1(\tau)\|^2_{\mathring V_0^*}d\tau+\frac{c_3}{\varepsilon_3}\|\Omega_1(0)\|^2_{\mathring V_0^*}+c_3\varepsilon_3\|\tilde\varphi_1^m\|^2_{\mathring V_0}\ (^{12}), \tag{4.12}$$

where (cf. [6] p. 122)

$$\|\Omega_1(t)\|_{\mathring V_0^*}=\sup_{\|u^m(t)\|\leqslant1}|(\Omega_1(t),u^m(t))|,\ \left|2\int_0^t\big(\Omega_2(\tau),\theta_1^m(\tau)\big)d\tau\right|\leqslant c_4\varepsilon_4\int_0^t\|\theta_1^m(\tau)\|^2_{V_1}d\tau$$
$$+\frac{c_4}{\varepsilon_4}\int_0^t\big(\|Q_1(\tau)\|^2_{V_1^*}+\|\partial_\tau\Phi_1(\tau)\|^2_{V_0}+\|\Phi_2(\tau)\|^2_{V_1}+\|\partial_\tau\Phi_2(\tau)\|^2_{V_1}+\|\tilde q(\tau)\|^2_{L^2(\partial G)}\big)d\tau,$$
$$\left|2\int_0^t\big(\Omega_3(\tau),\theta_2^m(\tau)\big)d\tau\right|\leqslant c_5\varepsilon_5\int_0^t\|\theta_2^m(\tau)\|^2_{V_2}d\tau+\frac{c_5}{\varepsilon_5}\int_0^t\big(\|Q_2(\tau)\|^2_{V_2^*}$$
$$+\|\partial_\tau\Phi_1(\tau)\|^2_{\mathring V_0}+\|\partial_\tau\Phi_2(\tau)\|^2_{V_1}\big)d\tau,$$
$$\left|2\gamma_1\int_0^t\big(\Phi_2(\tau),\nabla\partial_\tau u^m(\tau)\big)d\tau\right|\leqslant c_6\varepsilon_6\Big(\|u^m(t)\|^2_{\mathring V_0}+\|\Phi_2(0)\|^2_{V_1}+\int_0^t\|u^m(\tau)\|^2_{\mathring V_0}d\tau\Big)$$
$$+\frac{c_6}{\varepsilon_6}\Big(\|\Phi_2(\tau)\|^2_{V_1}+\|\tilde\varphi_1^m\|^2_{\mathring V_0}+\int_0^t\|\partial_\tau\Phi_2(\tau)\|^2_{V_1}d\tau\Big),$$

(^{11}) It is easy to see that $2|ab|=2|\varepsilon^{1/2}a\varepsilon^{-1/2}b|\leqslant(\varepsilon^{1/2}a)^2+\left(\frac{1}{\varepsilon^{1/2}}b\right)^2$ for $\forall\varepsilon>0$.

(^{12}) We take into account that $(\Omega_1,\partial_\tau u^m(\tau))=\partial_\tau(\Omega_1,u^m(\tau))-(\partial_\tau\Omega_1,u^m(\tau))$ and we use the Schwarz inequality and the inequality (4.11).

$$|-2d(\theta_1^m(t), \theta_2^m(t))_{L^2}| \leqslant d\varepsilon||\theta_1^m(t)||^2_{L^2} + \frac{1}{\varepsilon} d||\theta_2^m(t)||^2_{L^2},$$

$$|2d(\tilde{\vartheta}_1^m, \vartheta_2^m)| \leqslant d||\tilde{\vartheta}_1^m||^2_{L^2} + \frac{d}{\varepsilon} ||\vartheta_2^m||^2_{L^2},$$

$$a_1(\tilde{\varphi}_1^m, \tilde{\varphi}_1^m) \leqslant c_7||\tilde{\varphi}_1^m||^2_{\mathrm{V}_0}.$$

Moreover, in view of Poincaré's inequality (cf. [7]) and Korn's second inequality (cf. [6], we get

$$(4.13)\qquad \begin{aligned} &||\nabla\theta_2^m(t)||^2_{L^2} \geqslant \delta_1||\theta_2^m(t)||^2_{V_2} \quad \text{where} \quad \delta_1 = \delta_1(G),\\ &a_1(u^m(t), u^m(t)) \geqslant \mu_1||u^m(t)||^2_{\mathrm{V}_0} - \lambda_1||u^m(t)||^2_{L^2},\\ &a_2(\theta_1^m(t), \theta_1^m(t)) \geqslant \mu_2||\theta_1^m(t)||^2_{V_1} - \lambda_2||\theta_1^m(t)||^2_{L^2}.\end{aligned}$$

The constants $c_1, c_2, \ldots, c_7$ follow from Schwarz's inequality μ_1, μ_2, λ_1, λ_2 — from Korn's second inequality and δ_1 from Poincaré's inequality and are independent of m. However, the constants $\varepsilon, \varepsilon_1, \ldots, \varepsilon_6$ follow from the inequality (4.11) and are arbitrary positive constants. Taking into account the above estimates and using the inequality[13]

$$(4.14)\qquad ||u^m(t)||^2_{L^2} \leqslant 2||u^m(0)||^2_{L^2} + 2T\int_0^t ||\partial_\tau u^m(\tau)||^2_{L^2}d\tau$$

and the imbedding theorem (cf. [28, 29]), we have

$$(4.15)\qquad \varrho||\partial_t u^m(t)||^2_{L^2} + (\mu_1 - c_1\varepsilon_1 - c_6\varepsilon_6)||u^m(t)||^2_{\mathrm{V}_0} + (c - d\varepsilon)||\theta_1^m(t)||^2_{L^2}$$

$$+\left(n - \frac{d}{\varepsilon}\right)||\theta_2^m(t)||^2_{L^2} + (2\mu_2 - c_4\varepsilon_4)\int_0^t ||\theta_1^m(\tau)||^2_{V_1}d\tau + (2D\delta_1 - c_5\varepsilon_5)\int_0^t ||\theta_2^m(\tau)||^2_{V_2}d\tau$$

$$\leqslant 2\lambda_1 T\int_0^t ||\partial_\tau u^m(\tau)||^2_{L^2}d\tau + (c_2\varepsilon_2 + c_6\varepsilon_6)\int_0^t ||u^m(\tau)||^2_{\mathrm{V}_0}d\tau + 2\lambda_2\int_0^t ||\theta_1^m(\tau)||^2_{L^2}d\tau$$

$$+\left(\tilde{\lambda}_1 + c_3\varepsilon_3 + \frac{c_6}{\varepsilon_6} + c_7\right)||\tilde{\varphi}_1^m||^2_{\mathrm{V}_0} + \varrho||\tilde{\varphi}_2^m||^2_{L^2} + (c + d\varepsilon)||\tilde{\vartheta}_1^m||^2_{L^2} + (c_6\varepsilon_6 + 2)||\Phi_2(0)||^2_{V_1}$$

$$+\left(n + \frac{d}{\varepsilon}\right)||\vartheta_2^m||^2_{L^2} + \frac{c_4}{\varepsilon_4}\int_0^t ||Q_1(\tau)||^2_{V_1^*}d\tau + \frac{c_5}{\varepsilon_5}\int_0^t ||Q_2(\tau)||^2_{V_2^*}d\tau$$

$$+\left(\frac{c_4}{\varepsilon_4} + \frac{c_5}{\varepsilon_5}\right)\int_0^t ||\partial_\tau\Phi_1(\tau)||^2_{\mathrm{V}_0}d\tau + \left(\frac{c_4}{\varepsilon_4} + \frac{c_5}{\varepsilon_5} + \frac{c_6}{\varepsilon_6} + 2T\right)\int_0^t ||\partial_\tau\Phi_2(\tau)||^2_{V_1}d\tau$$

$$+\frac{c_4}{\varepsilon_4}\int_0^t ||\phi_2(\tau)||^2_{V_1}d\tau + \frac{c_4}{\varepsilon_4}\int_0^t ||\tilde{q}(\tau)||^2_{L^2(\partial G)}d\tau + \frac{c_2}{\varepsilon_2}\int_0^t ||\partial_\tau\Omega_1(\tau)||^2_{\mathrm{V}_0^*}d\tau$$

$$+\frac{c_1}{\varepsilon_1}||\Omega_1(t)||^2_{\mathrm{V}_0^*} + \frac{c_3}{\varepsilon_3}||\Omega_1(0)||^2_{\mathrm{V}_0^*}.$$

---

[13] It is easy to see that $||u^m(t)||_{L^2} \leqslant ||u^m(0)||_{L^2} + \int_0^t ||\partial_\tau u^m(\tau)||_{L^2}d\tau$. After simple transformations we get the inequality (4.14)

We take the constants $\varepsilon, \varepsilon_1, \varepsilon_4, \varepsilon_5, \varepsilon_6$ so that the following conditions will be satisfied:

$$(4.16)\qquad c-d\varepsilon > 0, \quad n-\frac{d}{\varepsilon} > 0, \quad \mu_1-c_1\varepsilon_1-c_6\varepsilon_6 > 0,$$
$$2D\delta_1-c_5\varepsilon_5 > 0, \quad 2\mu_2-c_4\varepsilon_4 > 0.$$

It is easy to see that the constant $\varepsilon = \sqrt{c/n}$ satisfies the inequality (4.16) and the inequality $cn > d^2$.

We denote the following constants by C_8, C_9 and A:

$$C_8 = \min\left[\varrho, \mu_1-c_1\varepsilon_1-c_6\varepsilon_6, c-d\varepsilon, n-\frac{d}{\varepsilon}, 2\mu_2-c_4\varepsilon_4, 2D\delta_1-c_5\varepsilon_5\right],$$

$$(4.17)\qquad C_9 = \max\left[2\lambda_1 T, c_2\varepsilon_2+c_6\varepsilon_6, 2\lambda_2, \tilde{\lambda}_1+c_3\varepsilon_3+\frac{c_6}{\varepsilon_6}+c_7, c+d\varepsilon, c_6\varepsilon_6+2\varrho,\right.$$
$$\left. n+\frac{d}{\varepsilon}, \frac{c_4}{\varepsilon_4}, \frac{c_5}{\varepsilon_5}, \frac{c_4}{\varepsilon_4}+\frac{c_5}{\varepsilon_5}, \frac{c_4}{\varepsilon_4}+\frac{c_5}{\varepsilon_5}+\frac{c_6}{\varepsilon_6}+2T, \frac{c_2}{\varepsilon_2}, \frac{c_1}{\varepsilon_1}, \frac{c_3}{\varepsilon_3}\right],$$

$$A = \frac{C_9}{C_8}.$$

Using the above symbols we can write the inequality (4.15) as

$$(4.18)\qquad ||\partial_t u^m(t)||^2_{L^2}+||u^m(t)||^2_{V_0}+\sum_{i=1}^{2}||\theta_i^m(t)||^2_{L^2}+\sum_{i=1}^{2}\int_0^t||\theta_i^m(\tau)||^2_{V_i}d\tau$$
$$\leqslant A\int_0^t\Big(||\partial_\tau u^m(\tau)||^2_{L^2}+||u^m(\tau)||^2_{V_0}+\sum_{i=1}^{2}||Q_i^m(\tau)||^2_{L^2}\Big)d\tau+A\Big[||\tilde{\varphi}_1^m||^2_{V_0}+||\tilde{\varphi}_2^m||^2_{L^2}$$
$$+||\tilde{\vartheta}_1^m||^2_{L^2}+||\vartheta_2^m||^2_{L^2}+||\Phi_2(0)||^2_{V_1}+\int_0^t\Big(\sum_{i=1}^{2}||Q_i(\tau)||^2_{V_i^*}+||\partial_\tau\Phi_1(\tau)||^2_{V_0^*}+||\partial_\tau\Phi_2(\tau)||^2_{V_1}$$
$$+||\Phi_2(\tau)||^2_{V_1}\Big)d\tau+\int_0^t\big(||\tilde{q}(\tau)||^2_{L^2(\partial G)}+||\partial_\tau\Omega_1(\tau)||^2_{V_0^*}\big)d\tau+||\Omega_1(\tau)||^2_{V_0^*}+||\Omega_1(0)||^2_{V_0^*}\Big].\ (^{14})$$

It is easy to notice that the following relations are true (cf. [29])

$$(4.19)\qquad ||\tilde{\varphi}_1^m||^2_{V_0} \leqslant C_{10}||\tilde{\varphi}_1||^2_{V_0}, \quad ||\tilde{\varphi}_2^m||^2_{L^2} \leqslant C_{11}||\tilde{\varphi}_2||^2_{L^2},$$
$$||\tilde{\vartheta}_1^m||^2_{L^2} \leqslant C_{12}||\tilde{\vartheta}_1||^2_{V_1}, \quad ||\vartheta_2^m||^2_{L^2} \leqslant C_{13}||\vartheta_2||^2_{V_2},$$

where the constants $C_{10}, C_{11}, C_{12}, C_{13}$ are positive constants and are independent of m,

$$\int_0^t\sum_{i=1}^{2}||Q_i(\tau)||^2_{V_i^*}d\tau \leqslant \sum_{i=1}^{2}||Q_i||^2_{L^2(I,V_i^*)},$$

$$\int_0^t\big(||\partial_\tau\Phi_2(\tau)||^2_{V_1}+||\Phi_2(\tau)||^2_{V_1}\big)d\tau \leqslant ||\Phi_2||^2_{W_2^1(I,V_1)},$$

---

(14) The inequalities (4.19) follow from the relations (4.5) (cf. [29]).

$$(4.20)\qquad \int_0^t ||\partial_\tau \Phi_1(\tau)||^2_{\mathsf{V}_0} d\tau \leqslant ||\Phi_1||^2_{W_2^1(I,\mathsf{V}_0)},$$

$$\int_0^t ||q(\tau)||^2_{L^2(\partial G)} d\tau \leqslant ||\tilde{q}||^2_{L^2(I,L^2(\partial G))},$$

$$\int_0^t ||\partial_t \Omega_1(\tau)||^2_{\mathsf{V}_0^*} d\tau \leqslant ||\Omega_1||^2_{W_2^1(I,\mathsf{V}_0^*)}.$$

Substituting the inequalities (4.19) and (4.20) into the inequality (4.18) and denoting the following constants by B,

$$B = A\Big(C_{10}||\tilde{\varphi}_1||^2_{\mathsf{V}_0} + C_{11}||\tilde{\varphi}_2||^2_{\mathsf{L}^2} + C_{12}||\tilde{\vartheta}_1||^2_{V_1} + C_{13}||\vartheta_2||^2_{V_2} + ||\Phi_2(0)||^2_{V_1} + ||\Omega_1(0)||^2_{\mathsf{V}_0^*}$$
$$+ \max_{t\in[0,T]} ||\Omega_1(t)||^2_{\mathsf{V}_0^*} + \sum_{i=1}^{2} ||Q_i||^2_{L^2(I,V_1^*)} + ||\Phi_2||^2_{W_2^1(I,V_1)} + ||\Phi_1||^2_{W_2^1(I,\mathsf{V}_0^*)}$$
$$+ ||\tilde{q}||^2_{L^2(I,L^2(\partial G))} + ||\Omega_1||^2_{W_2^1(I,\mathsf{V}_0^*)}\Big),$$

we get

$$(4.21)\qquad ||\partial_t u^m(t)||^2_{\mathsf{L}^2} + ||u^m(t)||^2_{\mathsf{V}_0} + \sum_{i=1}^{2} ||\theta_i^m(t)||^2_{L^2} + \sum_{i=1}^{2} \int_0^t ||\theta_i^m(\tau)||^2_{V_i} d\tau$$
$$\leqslant B + A\int_0^t \Big(||\partial_\tau u^m(\tau)||^2_{\mathsf{L}^2} + ||u^m(\tau)||^2_{\mathsf{V}_0} + \sum_{i=1}^{2} ||\theta_i^m(\tau)||^2_{L^2}\Big) d\tau.$$

From the inequality (4.21) we get directly

$$(4.22)\qquad ||\partial_t u^m(t)||^2_{\mathsf{L}^2} + ||u^m(t)||^2_{\mathsf{V}_0} + \sum_{i=1}^{2} ||\theta_i^m(t)||^2_{L^2} \leqslant B + A\int_0^t \Big(||\partial_\tau u^m(\tau)||^2_{\mathsf{L}^2}$$
$$+ ||u^m(\tau)||^2_{\mathsf{V}_0} + \sum_{i=1}^{2} ||\theta_i^m(\tau)||^2_{L^2}\Big) d\tau.$$

Applying Gronwall's inequality (cf. [14, 16]) to the relation (4.22), we have

$$(4.23)\qquad ||\partial_t u^m(t)||^2_{\mathsf{L}^2} + ||u^m(t)||^2_{\mathsf{V}_0} + \sum_{i=1}^{2} ||\theta_i^m(t)||^2_{L^2} \leqslant C(T,A)B \quad \forall t \in I,$$

where $C(T, A) = e^{AT}$ (cf. [14], p. 298, [16], p. 46).

The estimates (4.23)—(4.22) imply that the sequences $(u^m)_{m\in\mathsf{N}}$, $(\partial_t u^m)_{m\in\mathsf{N}}$, $(\theta_i^m)_{m\in\mathsf{N}}$ $(i = 1, 2)$ are bounded in the spaces $L^2(I, \mathsf{V}_0)$, $L^2(I, \mathsf{L}^2(G))$, $L^2(I, L^2(G)) \cap L^2(I, V_i)$ $(i = 1, 2)$ for any $m \in \mathsf{N}$ and $t \in (0, T)$, respectively.

Ad. 3. Consequently, there exist the weakly convergent subsequences (u^{m_n}), $(\partial_t u^{m_n})$, $(\theta_i^{m_n})$ of the sequences $(u^m)_{m\in\mathsf{N}}$, $(\partial_t u)^m_{m\in\mathsf{N}}$, $(\theta_i^m)_{m\in\mathsf{N}}$ they will be denoted by the same symbols as Galerkin sequences, i.e. $(u^\nu)_{\nu\in\mathsf{N}}$, $(\partial_t u^\nu)_{\nu\in\mathsf{N}}$, $(\theta_i^\nu)_{\nu\in\mathsf{N}}$ $(i = 1, 2)$. Without loss of generality we may assume that

$$
\begin{aligned}
&u^\nu \to z \text{ (weakly) in } L^2(I, \mathsf{V}_0),\\
(4.24)\qquad &\partial_t u^\nu \to z' \text{ (weakly) in } L^2\big(I, \mathsf{L}^2(G)\big)\\
&\theta_i^\nu \to y_i \text{ (weakly) in } L^2\big(I, L^2(G)\big)\cap L^2(I, V_1).
\end{aligned}
$$

Obviously (cf. [29]) $z' = \partial_t z(t)$ and since $u^\nu(0) \to z(0)$ in V_0 if $\nu \to \infty$ we get $z(0) = \tilde{\varphi}_1$. Let $\xi \in C^\infty(I)$ and satisfy the condition $\xi(T) = 0$. We put $\xi^l(t) = \xi(t)w^l$, $\xi_l(t) = \xi(t)\beta_l$ and $\zeta_l = \xi(t)v_l$. By multiplying Eqs. (4.4) by $\xi(t)$, taking $m = \nu > l$ and integrating by parts on the interval $[0, T]$ in view of condition $\xi(T) = 0$, we have

$$
\begin{aligned}
&-\varrho\int_0^T \big(\partial_t u^\nu(t), \partial_t \xi^l(t)\big)_{\mathsf{L}^2}dt + \int_0^T a_1\big(u^\nu(t), \xi^l(t)\big)dt = \sum_{i=1}^{2}\int_0^T \big(\theta_i^\nu(t), \nabla\cdot\xi^l(t)\big)_{L^2}dt\\
&\qquad + \int_0^T \big(\Omega_1(t), \xi^l(t)\big)dt + \gamma_2\int_0^T \big(\phi_2(t), \nabla\cdot\xi^l(t)\big)dt + \varrho\big(\tilde{\varphi}_2^\nu, \xi^l(0)\big)_{\mathsf{L}^2},\\
(4.25)\qquad &-c\int_0^T \big(\theta_1^\nu(t), \partial_t\xi_l(t)\big)_{L^2}dt + \int_0^T a_2\big(\theta_1^\nu(t), \xi_l(t)\big)dt = -d\int_0^T \big(\partial_t\theta_2^\nu(t), \xi_l(t)\big)dt\\
&\qquad -\gamma_1\int_0^T \big(\partial_t\nabla\cdot u^\nu(t), \xi_l(t)\big)dt + \int_0^T \big(\Omega_2(t), \xi_l(t)\big)dt + c\big(\tilde{\vartheta}_1^\nu, \xi_l(0)\big)_{L^2},\\
&-n\int_0^T \big(\theta_2^\nu(t), \partial_t\zeta_l(t)\big)_{L^2}dt = D\int_0^T \big(\Delta\theta_2^\nu(t), \zeta_l(t)\big)dt - \gamma_2\int_0^T \big(\partial_t\nabla\cdot u^\nu(t), \zeta_l(t)\big)dt\\
&\qquad -d\int_0^T \big(\partial_t\theta_1^\nu(t), \zeta_l(t)\big)_{L^2}dt + \int_0^T \big(\Omega_3(t), \zeta_l(t)\big)dt + n\big(\vartheta_2^\nu, \zeta_l(0)\big)_{L^2}.
\end{aligned}
$$

In view of the weak convergence (4.24), taking $\nu \to \infty$ in Eq. (4.25), we get

$$
\begin{aligned}
&-\varrho\int_0^T \big(\partial_t z(t), \partial_t\xi^l(t)\big)dt + \int_0^T a_1\big(z(t), \xi^l(t)\big)dt = \sum_{i=1}^{2}\gamma_i\int_0^T \big(y_i(t), \nabla\cdot\xi^l(t)\big)dt\\
&\qquad + \int_0^T \big(\Omega_1(t), \xi^l(t)\big)dt + \gamma_2\int_0^T \big(\phi_2(t), \nabla\cdot\xi^l(t)\big)dt + \varrho\big(\tilde{\varphi}_2, \xi^l(0)\big)_{\mathsf{L}^2},\\
(4.26)\qquad &-c\int_0^T \big(y_1(t), \partial_t\xi_l(t)\big)dt + \int_0^T a_2\big(y_1(t), \xi_l(t)\big)dt = -d\int_0^T \big(\partial_t y_2(t), \xi_l(t)\big)dt\\
&\qquad -\gamma_1\int_0^T \big(\partial_t\nabla\cdot z(t), \xi_l(t)\big)dt + \int_0^T \big(\Omega_2(t), \xi_l(t)\big)dt + c\big(\tilde{\vartheta}_1, \xi_l(0)\big)_{L^2},\\
&-n\int_0^T \big(y_2(t), \partial_t\zeta_l(t)\big)dt = D\int_0^T \big(\Delta y_2(t), \zeta_l(t)\big)dt - \gamma_2\int_0^T \big(\partial_t\nabla\cdot z(t), \zeta_l(t)\big)dt\\
&\qquad -d\int_0^T \big(\partial_t y_1(t), \zeta_l(t)\big)dt + \int_0^T \big(\Omega_3(t), \zeta_l(t)\big)dt + n\big(\vartheta_2, \zeta_l(0)\big)_{L^2}.
\end{aligned}
$$

In particular, Eqs. (4.26) are true for any $\xi \in C_0^\infty(I)$.

Therefore we get

$$(\tilde{\varphi}_2, \xi^l(0))_{L^2} = 0, \quad (\tilde{\vartheta}_1, \xi_l(0))_{L^2} = 0 \quad \text{and} \quad (\vartheta_2, \zeta_l(0))_{L^2} = 0.$$

Taking it into account and using the simple transformation (integration by parts), we have

$$\varrho \int_0^T (\partial_t^2 z(t), \xi^l(t))dt + \int_0^T a_1(z(t), \xi^l(t))dt = \sum_{i=1}^2 \gamma_i \int_0^T (y_i(t), \nabla \cdot \xi^l(t))dt + \int_0^T (\Omega_1(t), \xi^l(t))dt + \gamma_2 \int_0^T (\phi_2(t), \nabla \cdot \xi^l(t))\,dt,$$

$$(4.27) \quad c \int_0^T (\partial_t y_1(t), \xi_l(t))dt + \int_0^T a_2(y_1(t), \xi_l(t))dt = -d \int_0^T (\partial_t y_2(t), \xi_l(t))dt - \gamma_1 \int_0^T (\partial_t \nabla \cdot z(t), \xi_l(t))dt + \int_0^T (\Omega_2(t), \xi_l(t))\,dt,$$

$$n \int_0^T (\partial_t y_2(t), \zeta_l(t))dt = D \int_0^T (\Delta y_2(t), \zeta_l(t))dt - \gamma_2 \int_0^T (\partial_t \nabla \cdot z(t), \zeta_l(t))dt - d \int_0^T (\partial_t y_1(t), \zeta_l(t))dt + \int_0^T (\Omega_3(t), \zeta_l(t))\,dt.$$

Since the functions $\xi^l(t)$, $\xi_l(t)$ and $\zeta_l(t)$ are arbitrary, the following identities follow from the relation (4.27):

$$\varrho(\partial_t^2 z(t), \omega^l) + a_1(z(t), \omega^l) = \gamma_1(y_1, \nabla \cdot \omega^l) + \gamma_2(y_2, \nabla \cdot \omega^l) + (\Omega_1(t), \omega^l) + \gamma_2(\Phi_2(t), \nabla \cdot \omega^l),$$

$$(4.28) \quad c(\partial_t y_1(t), \beta_l) + a_2(y_1(t), \beta_l) = -d(\partial_t y_2(t), \beta_l) - \gamma_1(\partial_t \nabla \cdot z(t), \beta_l) + (\Omega_2(t), \beta_l),$$

$$n(\partial_t y_2(t), v_l) = D(\Delta y_2(t), v_l) - \gamma_2(\partial_t \nabla \cdot z(t), v_l) - d(\partial_t y_1(t), v_l) + (\Omega_3(t), v_l).$$

Thus the system of the functions (z, y_1, y_2) is a weak solution of Eqs. (1.1)—(1.3) in the sense of the definition (3.1). Now we show that this solution satisfies the initial condition (3.4). In order to do it, it is sufficient to perform integration by parts in Eqs. (4.26), and to take into account the relations (4.27). After performing this operation we get

$$(4.29) \quad ((\partial_t z)(0), \omega^l)_{L^2}\xi(0) = (\tilde{\varphi}_2, \omega^l)_{L^2}\xi(0), \text{ [15]}$$

$$(y_1(0), \beta_l)_{L^2}\xi(0) = (\tilde{\vartheta}_1, \beta_l)_{L^2}\xi(0), \quad (y_2(0), v_l)_{L^2}\xi(0) = (\vartheta_2, v_l)_{L^2}\xi(0).$$

As the functions ω^l, β_l, v_l are arbitrary, we have $(\partial_t z)(0) = \tilde{\varphi}_2$, $y_1(0) = \tilde{\vartheta}_1$, $y_2(0) = \vartheta_2$.

There the system of the functions (z, y_1, y_2) is a weak solution of the problem (1.1)—(1.3), (3.1), (3.2) in the meaning of the definition (3.1). From the properties of the sequence

([15]) In Eqs. (4.29) ξ — denotes the function which belongs to the space $C^\infty(I)$, with the property $\xi(T) = 0$, so the symbol $\xi(0)$ is meanigful.

of Galerkin approximations (cf. relation (4.24)) and in view of the definition of the weak solution of the problem (1.1)—(1.3), (3.1), (3.2) we obtain

$$\partial_t z \in L^2(I, \mathsf{L}^2(G)), \quad \partial_t^2 z \in L^2(I, \mathsf{V}_0^*), \quad \partial_t y_i \in L^2(I, V_i^*) \quad (i = 1, 2).$$

This completes the proof of Theorem 4.1.

5. Uniqueness theorem

THEOREM 5.1. (Uniqueness). *Let $X, Q_i, \tilde{\Psi}, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\vartheta}_1, \vartheta_2, \tilde{q}, \tilde{\Phi}_1, \tilde{\Phi}_2$ be such that*

$$X \in W_2^2(I, \mathsf{L}^2(G)), \quad Q_i \in W_2^1(I, V_i^*), \quad Q_i(0) \in V_i, \quad \tilde{\Psi} \in W_2^2(I, \mathsf{L}^2(\partial G)),$$

$$(5.1) \quad \tilde{\varphi}_1 \in \mathsf{H}^2(G) \cap \mathsf{V}_0, \quad \tilde{\varphi}_2 \in \mathsf{V}_0, \quad \tilde{\vartheta}_1 \in V_1 \cap H^3(G), \quad \vartheta_2 \in H^3(G),$$

$$\tilde{q} \in W_2^1(I, L^2(\partial G)),$$

$$\tilde{\Phi}_1 \in W_2^2(I, \mathsf{H}^{1/2}(\partial G)), \quad \partial_t^3 \tilde{\Phi}_1 \in W_2^1(I, \mathsf{H}^{-1/2}(\partial G)),$$

$$\tilde{\Phi}_2 \in W_2^2(I, H^{1/2}(\partial G)) \quad (i = 1, 2).$$

Then the problem (1.1)—(1.3), (3.1), (3.2) *has a unique weak solution with the properties*

$$(5.2) \quad \partial_t^2 u \in L^2(I, \mathsf{L}^2(G)), \quad \partial_t^3 u \in L^2(I, \mathsf{V}_0^*), \quad \partial_t^2 \theta_i \in L^2(I, V_i^*) \quad (i = 1, 2).$$

P r o o f. First we consider the following initial-boundary value problem for the equations

$$(5.3) \quad \begin{aligned} \varrho \partial_t^2 P &= \mu \Delta P + (\lambda + \mu) \nabla (\nabla \cdot P) - \sum_{i=1}^{2} \gamma_i \nabla S_i + \partial_t X, \\ c \partial_t S_1 &= k \Delta S_1 - \gamma_1 \partial_t \nabla \cdot P - d \partial_t S_2 + \partial_t Q_1, \\ n \partial_t S_2 &= D \Delta S_2 - \gamma_2 \partial_t \nabla \cdot P - d \partial_t S_1 + \partial_t Q_2 \end{aligned}$$

with the initial conditions

$$P(0) = \varphi_2, \ (\partial_t P)(0) = \frac{\mu}{\varrho} \Delta \varphi_1 + \frac{\lambda + \mu}{\varrho} \nabla (\nabla \varphi_1) - \frac{1}{\varrho} \sum_{i=1}^{2} \gamma_i \nabla \vartheta_i + \frac{1}{\varrho} X(0),$$

$$(5.4) \quad S_1(0) = \frac{1}{\alpha} [nk \Delta \vartheta_1 - dD \Delta \vartheta_2 + (d\gamma_2 - n\gamma_1) \nabla \varphi_2 + nQ_1(0) - d\mathrm{Q}_2(0)],$$

$$S_2(0) = \frac{1}{\alpha} [-dk \Delta \vartheta_1 + cD \Delta \vartheta_2 + (\gamma_1 d - \gamma_2 c) \nabla \varphi_2 - dQ_1(0) + cQ_2(0)],$$

(where $\alpha = cn - d^2$) and the boundary conditions

$$(5.5) \quad P|_{I \times \partial G_1} = \partial_t \Phi, \ \sigma \cdot \nu|_{I \times \partial G_2} = \partial_t \Psi, \ S_1|_{I \times \partial G_2} = \partial_t p, \ \frac{dS_1}{d\nu}\bigg|_{I \times \partial G_1} = \partial_t q,$$

$$S_2|_{I \times \partial G} = 0.$$

It is easy to see that in view of Theorem 4.1 and the conditions (5.1) the system of Eqs. (5.3) with the conditions (5.4) and (5.5) has a weak solution:

$$(5.6) \quad (P, S_1, S_2) \in L^2(I, \mathsf{V}_0) \times L^2(I, V_1) \times L^2(I, V_2)$$

with the properties

$$\partial_t P \in L^2(I, \mathsf{L}^2(G)),\ \partial_t^2 P \in L^2(I, \mathsf{V}_0^*), \qquad \partial_t S_i \in L^2(I, V_i^*) \qquad (i = 1, 2) \tag{5.7}$$

Now we introduce the functions

$$w(t) = \varphi_1 + \int_0^t P(\tau)d\tau, \qquad \eta_i(t) = \vartheta_i + \int_0^t S_i(\tau)d\tau \qquad (i = 1, 2). \tag{5.8}$$

From the functions (5.8) it follows that

$$\begin{gathered} \partial_t w(t) = P(t), \quad \partial_t \eta_i(t) = S_i(t), w(0) = \varphi_1, \quad (\partial_t w)(0) = \varphi_2, \\ \eta_i(0) = \vartheta_i \quad (i = 1, 2). \end{gathered} \tag{5.9}$$

We prove that the functions (5.8) satisfy the system of Eqs. (1.1)—(1.3) with the conditions (3.1), (3.2). Integrating Eqs. (5.3) on the interval $(0, t)$ and taking into account the conditions (5.4), (5.5) and (5.9), we get

$$\begin{aligned} \varrho\partial_t^2 w &= \mu\Delta w + (\lambda+\mu)\nabla(\nabla\cdot w) - \gamma_1\nabla\eta_1 - \gamma_2\nabla\eta_2 + X, \\ c\partial_t\eta_1 &= k\Delta\eta_1 - \gamma_1\partial_t\nabla\cdot w - d\partial_t\eta_2 + Q_1, \\ n\partial_t\eta_2 &= D\Delta\eta_2 - \gamma_2\partial_t\nabla\cdot w - d\partial_t\eta_1 + Q_2, \end{aligned}$$

where $w(0) = \varphi_1$, $(\partial_t w)(0) = \varphi_2$, $\eta_i(0) = \vartheta_i$ $(i = 1, 2)$

$$\begin{gathered} w|_{I\times\partial G_1} = 0, \quad \sigma\cdot\nu|_{I\times\partial G_2} = \Psi, \quad \eta_1|_{I\times\partial G_2} = p, \\ \left.\frac{d\eta_1}{d\nu}\right|_{I\times\partial G_1} = q, \quad \eta_2|_{I\times\partial G} = 0. \end{gathered}$$

So, in view of Theorem (4.1) and the conditions (5.7), (5.8), (5.9) we have the following relations:

$$\begin{gathered} w \in L^2(I, \mathsf{V}_0), \quad \eta_i \in L^2(I, V_i), \quad \partial_t w \in L^2(I, \mathsf{V}_0), \\ \partial_t^2 w \in L^2(I, \mathsf{L}^2(G)), \quad \partial_t^3 w \in L^2(I, \mathsf{V}_0^*), \quad \partial_t \eta_i \in L^2(I, V_i), \\ \partial_t^2 \eta_i \in L^2(I, V_i^*) \quad (i = 1, 2). \end{gathered} \tag{5.10}$$

From the relation (5.10) we conclude that the functions (w, η_1, η_2) satisfy the conditions (5.2).

Now we assume that there exist two different solutions (u, θ_1, θ_2) and w, η_1, η_2 of Eqs. (1.1)—(1.3) with the conditions (3.1), (3.2). The differences $U = u - w$, $\varkappa_i = \theta_i - \eta_i$ satisfy the homogeneous equations

$$\begin{aligned} \varrho\partial_t^2 U &= \mu\Delta U + (\lambda+\mu)\nabla(\nabla\cdot U) - \gamma_1\nabla\varkappa_1 - \gamma_2\nabla\varkappa_2, \\ c\partial_t\varkappa_1 &= k\Delta\varkappa_1 - \gamma_1\partial_t(\nabla\cdot U) - d\partial_t\varkappa_2, \\ n\partial_t\varkappa_2 &= D\Delta\varkappa_2 - \gamma_2\partial_t(\nabla\cdot U) - d\partial_t\varkappa_1 \end{aligned} \tag{5.11}$$

with the homogeneous initial and boundary conditions

$$\begin{gathered} U(0) = 0, \quad (\partial_t U)(0) = 0, \quad U|_{I\times\partial G_1} = 0, \quad \sigma\cdot\nu|_{I\times\partial G_2} = 0, \quad \varkappa_1(0) = 0, \\ \varkappa_1|_{I\times\partial G_2} = 0, \quad \left.\frac{d\varkappa_1}{d\nu}\right|_{I\times\partial G_1} = 0, \quad \varkappa_2(0) = 0, \quad \varkappa_2|_{I\times\partial G} = 0 \end{gathered} \tag{5.12}$$

From Eqs. (5.11), in view of the definition (3.1) (a weak solution), after taking into account the conditions (5.12), we get

$$\varrho\big(\partial_t^2 U(t),\partial_t U(t)\big)+a_1\big(U(t),\partial_t U(t)\big)=\gamma_1\big(\varkappa_1(t),\nabla\cdot\partial_t U(t)\big)+\gamma_2\big(\varkappa_2(t),\nabla\cdot\partial_t U(t)\big),$$

$$c\big(\partial_t\varkappa_1(t),\varkappa_1(t)\big)+a_2\big(\varkappa_1(t),\varkappa_1(t)\big)=-d\big(\partial_t\varkappa_2(t),\varkappa_1(t)\big)-\gamma_1\big(\partial_t\nabla\cdot U(t),\varkappa_1(t)\big), \tag{5.13}$$

$$n\big(\partial_t\varkappa_2(t),\varkappa_2(t)\big)=D\big(\Delta\varkappa_2(t),\varkappa_2(t)\big)-\gamma_2\big(\partial_t\nabla\cdot U(t),\varkappa_2(t)\big)-d\big(\partial_t\varkappa_1(t),\varkappa_2(t)\big).$$

Using the same calculations as in the proof of Theorem 4.1, we obtain the following estimates:

$$||\partial_t U(t)||^2_{L^2}+||U(t)||^2_{\mathring{V}_0}+\sum_{i=1}^{2}||\varkappa_i(t)||^2_{L^2}\leqslant 0 \tag{5.14}$$

from the inequality (5.14) it follows at once

$$||\partial_t U(t)||^2_{L^2}=0,\quad ||U(t)||^2_{\mathring{V}_0}=0,\quad ||\varkappa_i(t)||^2_{L^2}=0,\quad \forall t\in I.$$

Therefore $u=w$, $\theta_i=\eta_i$ $(i=1,2)$.

This completes the proof of Theorem 5.2.

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