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BRIEF NOTES

An improved bound on the error in Reissner's theory of plates

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THIS REPORT shows that the energy-norm error for solutions in Reissner's theory of plates is under special types of boundary conditions, proportional to the thickness cubed; normally the error is known [1] to be proportional to the thickness squared.

1. Introduction

NORDGREN [1] has established the energy-norm error for solutions in Reissner's theory of plate bending to be proportional to the thickness squared. To compare the plate theory (approximate) solutions with the corresponding three-dimensional elasticity theory (exact) solutions, he employed the hypercircle theorem due to PRAGER and SYNGE [2] with specially constructed, from the plate theory displacements and moments, thickness distributions of the stresses and displacements. The boundary conditions of the elasticity problem were assumed to conform to these distributions, thus being "regular" in the sense defined by KOITER [3]. Later BERDICHEVSKI [4] suggested that if the plate carries no face loads and no body forces, being under certain displacement boundary conditions, the error for Reissner's theory may be a quantity proportional to the thickness cubed. In this report the latter observation is proved to be valid not only for the regular displacement boundary conditions of [1] and [4] but also for certain regular stress boundary conditions resulting from a more elaborate than in [1] distribution of the stresses through the thickness. More precisely, we are able to prove that the relative energy-norm error is, under the indicated circumstances, a quantity of order $O(h^3/L^3)$, where $2h$ is the thickness and L a characteristic wavelength of the deformation pattern. The analysis is restricted to isotropic homogeneous plates of constant thickness.

2. Preliminary relations

Let the three-dimensional stress fields $\boldsymbol{\sigma}$, $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ denote, respectively, the exact solution, a statically admissible solution and a kinematically admissible solution to a given three-dimensional problem in the linear theory of elasticity, and let $C[\cdot]$ be a positive definite homogeneous quadratic functional representing the stress energy. The hypercircle theorem asserts that [2]

$$C\left[\boldsymbol{\sigma}-\frac{1}{2}(\tilde{\boldsymbol{\sigma}}+\hat{\boldsymbol{\sigma}})\right]=C\left[\frac{1}{2}(\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}})\right], \tag{2.1}$$

thus enabling quantitative estimates of the difference between the exact (usually unknown) solution $\boldsymbol{\sigma}$ and the sum of the two supposedly known solutions $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$. In view of Eq. (2.1), the relative error ε for the approximate solution $(\tilde{\boldsymbol{\sigma}}+\hat{\boldsymbol{\sigma}})/2$ may be computed from (see [5])

$$\varepsilon^2=C\left[\frac{1}{2}(\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}})\right]\Big/C[\hat{\boldsymbol{\sigma}}]. \tag{2.2}$$

The space occupied by the plate is parametrized here by cylindrical coordinates (x^α, $x^3 = z$), where x^α ($\alpha = 1, 2$) is an arbitrary set of curvilinear coordinates on the middle plane and z denotes the distance from that plane.

For isotropic materials, the stress energy of an arbitrary stress field $\boldsymbol{\sigma}$ reads

$$C[\boldsymbol{\sigma}]=\frac{1}{2E}\iint\limits_A\int\limits_{-h}^{h}[(1+\nu)\sigma^i_j\sigma^j_i-\nu\sigma^i_i\sigma^j_j]\,dA\,dz \qquad (i,j=1,2,3), \tag{2.3}$$

where E is Young's moduls, ν is Poisson's ratio, A denotes the region occupied by the midplane and $2h$ is the constant thickness of the plate. The corresponding constitutive relations are

$$\begin{aligned}
\sigma_{\alpha\beta}&=\frac{E}{1-\nu^2}[(1-\nu)u_{(\alpha|\beta)}+\nu a_{\alpha\beta}u^\lambda|_\lambda]+\frac{\nu}{1-\nu}a_{\alpha\beta}\sigma_{33},\\
\sigma_{\alpha3}&=\frac{E}{2(1+\nu)}(u_{\alpha,3}+u_{3,\alpha}),\\
\sigma_{33}&=\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)u_{3,3}+\nu u^\lambda|_\lambda],
\end{aligned} \tag{2.4}$$

$\mathbf{u}$ being the displacement vector, $a_{\alpha\beta}$ the covariant components of the midplane metric tensor, the subscripts preceded by a comma and a vertical stroke denote, respectively, partial and surface covariant differentiation, and the indices placed in parentheses mean symmetrization.

Assuming zero body forces, the equilibrium equations read

$$\sigma^{\alpha3}_{,3}+\sigma^{\alpha\beta}|_\beta=0, \qquad \sigma^{33}_{,3}+\sigma^{\alpha3}|_\alpha=0. \tag{2.5}$$

We also assume that the faces of the plate carry no load, i.e.

$$\sigma_{\alpha3}(z=\pm h)=\sigma_{33}(z=\pm h)=0, \tag{2.6}$$

whereas on the cylindrical edge surface S, with the unit vector n_j normal to S, the boundary conditions are "regular" (see [3]), that is have the form

$$\begin{aligned}
\sigma^{ij}n_j&=\tilde{\sigma}^{ij}n_j \quad \text{on} \quad S_\sigma,\\
u_i&=\hat{u}_i \quad \text{on} \quad S_u.
\end{aligned} \tag{2.7}$$

$\hat{u}_i$ being a kinematically admissible displacement field which, by definition, produces via the constitutive equations (2.4) the stress field $\hat{\boldsymbol{\sigma}}$.

The basic relations of Reissner's theory are adopted here in the form corresponding to that of [1]. For future use we record the equilibrium equations

$$M^{\beta}_{\alpha|\beta} - Q_\alpha = 0, \quad Q^\alpha|_\alpha = 0, \tag{2.8}$$

which are homogeneous due to the absence of face and body forces, the constitutive relations

$$\begin{aligned} M_{\alpha\beta} &= \frac{2Eh^3}{3(1-\nu^2)}\left[(1-\nu)\beta_{(\alpha|\beta)} + \nu a_{\alpha\beta}\beta^\lambda|_\lambda\right], \\ Q_\alpha &= \frac{5Eh}{6(1+\nu)}(\beta_\alpha + w_{,\alpha}), \end{aligned} \tag{2.9}$$

and the boundary conditions along the edge curve C of the midplane

$$\begin{aligned} M_{\alpha\beta}n^\beta = M^*_{\alpha\beta}n^\beta, \quad Q_\alpha n^\alpha = Q^*_\alpha n^\alpha \quad &\text{on} \quad C_\sigma, \\ w = w^*, \quad \beta_\alpha = \beta^*_\alpha \quad &\text{on} \quad C_u, \end{aligned} \tag{2.10}$$

the starred quantities being prescribed. In the foregoing β_α, w, $M_{\alpha\beta}$ and Q_α are resultant rotations of normals to the midplane, resultant lateral deflection, moments and transverse shearing forces, related to the three-dimensional displacements and stresses as follows:

$$\begin{aligned} \beta_\alpha &= \frac{3}{2h^3}\int_{-h}^{h} u_\alpha z\,dz, \quad w = \frac{3}{4h}\int_{-h}^{h}\left(1-\frac{z^2}{h^2}\right)u_3\,dz, \\ M_{\alpha\beta} &= \int_{-h}^{h}\sigma_{\alpha\beta}z\,dz, \quad Q_\alpha = \int_{-h}^{h}\sigma_{\alpha 3}\,dz. \end{aligned} \tag{2.11}$$

In view of the relations (2.11), the regular boundary conditions (2.7) of the three-dimensional elasticity problem and the reduced boundary conditions (2.10) of the corresponding two-dimensional plate problem are related by

$$\begin{aligned} &\int_{-h}^{h}\tilde{\sigma}_{\alpha\beta}n^\beta z\,dz = M^*_{\alpha\beta}n^\beta, \quad \int_{-h}^{h}\tilde{\sigma}_{\alpha 3}n^\alpha dz = Q^*_\alpha n^\alpha \quad \text{on } C_\sigma, \\ &\frac{3}{2h^3}\int_{-h}^{h}\hat{u}_\alpha z\,dz = \beta^*_\alpha, \quad \frac{3}{4h}\int_{-h}^{h}\left(1-\frac{z^2}{h^2}\right)\hat{u}_3\,dz = w^* \quad \text{on } C_u. \end{aligned} \tag{2.12}$$

3. Error estimates

Suppose that we know the quantities w, β_α, $M_{\alpha\beta}$ and Q_α satisfying Eqs. (2.8)–(2.10) of Reissner's theory. This solution may be concisely characterized by the absolute maximum of the moments M and a characteristic wavelength L of the deformation pattern as follows:

$$|M_{\alpha\beta}| \leqslant M, \quad |M_{\alpha\beta|\eta}| \leqslant M/L, \quad |M_{\alpha\beta|\eta\lambda}| \leqslant M/L^2. \tag{3.1}$$

From this two-dimensional solution we shall construct three-dimensional fields $\hat{\mathbf{u}}$, $\hat{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\sigma}}$, so that the relative error ε defined by Eq. (2.2) is of the order $O(h^3/L^3)$.

We start with the displacement field of [1]

$$\hat{u}_\alpha = z\beta_\alpha + \left(z^3 - \frac{3}{5}h^2 z\right) f_\alpha, \qquad \hat{u}_3 = w + (z^2 - h^2/5)g, \tag{3.2}$$

which clearly conforms to Eqs. $(2.12)_{3,4}$. The unknown functions $f_\alpha(x^\lambda)$ and $g(x^\lambda)$ wil be chosen so that

$$\hat{\sigma}_{\alpha 3} = \frac{3}{4h}\left(1 - \frac{z^2}{h^2}\right) Q_\alpha, \qquad \hat{\sigma}_{33} = 0, \tag{3.3}$$

these stresses being related to the displacements (3.2) through the constitutive equations (2.4). This is achieved by setting

$$f_\alpha = -\frac{1}{3} g_{,\alpha} - \frac{1+\nu}{2Eh^3} Q_\alpha, \qquad g = -\frac{\nu}{2(1-\nu)} \beta^\alpha|_\alpha. \tag{3.4}$$

Indeed, introducing Eqs. (3.2) into Eq. $(2.4)_2$ and then using Eqs. (3.4) and $(2.9)_2$, we obtain the equality $(3.3)_1$. Similarly, substitution of the relations (3.2) into Eq. $(2.4)_3$ indicates that Eq. $(3.3)_2$ is true provided that

$$f^\alpha|_\alpha = 0. \tag{3.5}$$

From Eqs. (3.4) we have

$$f^\alpha|_\alpha = -\frac{1+\nu}{2Eh^3} Q^\alpha|_\alpha + \frac{\nu}{6(1-\nu)} \beta^\lambda|^\alpha_{\lambda\alpha}, \tag{3.6}$$

the first term in the right side being zero by Eq. $(2.8)_2$; the second term is seen to vanish after introduction of Eq. $(2.8)_1$ into Eq. $(2.8)_2$ and substitution of Eq. $(2.9)_1$ for $M_{\alpha\beta}$.

With Eqs. $(3.2)_1$, $(3.3)_2$, $(2.9)_1$ and (3.5), Eq. $(2.4)_1$ gives

$$\hat{\sigma}_{\alpha\beta} = \frac{3z}{2h^3} M_{\alpha\beta} + \frac{E}{1+\nu}\left(z^3 - \frac{3}{5}h^2 z\right) f_{(\alpha|\beta)} \tag{3.7}$$

and so we know all the components of $\hat{\boldsymbol{\sigma}}$.

A sufficiently close to $\hat{\boldsymbol{\sigma}}$ statically admissible stress field $\tilde{\boldsymbol{\sigma}}$ is

$$\tilde{\sigma}_{\alpha\beta} = \hat{\sigma}_{\alpha\beta},$$
$$\tilde{\sigma}_{\alpha 3} = \frac{3}{4h}\left(1 - \frac{z^2}{h^2}\right) Q_\alpha - \frac{E}{1+\nu}\left(\frac{z^4}{4} - \frac{3}{10}h^2 z^2 + \frac{h^4}{20}\right) f_{(\alpha}|^{\beta}_{\beta)}, \tag{3.8}$$
$$\tilde{\sigma}_{33} = 0.$$

It is obtained by starting from Eq. $(3.8)_1$ and integrating successively the equations of equilibrium (2.5) with Eqs. (3.7), (2.8) and (3.5), under the boundary conditions (2.6). Evidently, the expressions $(3.8)_{1,2}$ conform to the boundary relations $(2.12)_{1,2}$.

Comparison between Eqs. (3.8) and (3.3) gives

$$\tilde{\sigma}_{\alpha\beta} - \hat{\sigma}_{\alpha\beta} = \tilde{\sigma}_{33} - \hat{\sigma}_{33} = 0, \qquad \tilde{\sigma}_{\alpha 3} - \hat{\sigma}_{\alpha 3} = O(Mh/L^3). \tag{3.9}$$

Now from Eq. (2.2) we derive with Eqs. (2.3), (3.9) and (3.7) our final error estimate for the solutions in Reissner's theory:

$$\varepsilon = O(h^3/L^3). \tag{3.10}$$

Recall that this result is valid under no body forces and face loads. If a lateral load on the faces is present, the corresponding error is of the relative order $O(h^2/L^2)$ [1]. For irregular boundary conditions, not conforming to the relations (2.7), the error may be still greater on account of the boundary effects.

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