Arch. Mech., 38, 1-2, pp. 191-207, 1986

Viscoelastic flows with dominating extensions: application to squeezing flows

S. ZAHORSKI (WARSZAWA)

PLANE viscoelastic flows with dominating extensions are defined as thin-layer flows described by the constitutive equations valid for steady or unsteady extensions and perturbed with respect to the strain-rate invariants. Certain approximate solutions and applications to continuous squeezing flows are discussed in greater detail.

Płaskie przepływy lepkosprężyste z dominującym rozciąganiem zdefiniowano jako przepływy w cienkich warstwach opisywane równaniami konstytutywnymi dla ustalonego lub nieustalonego rozciągania, perturbowane względem niezmienników szybkości odkształcenia. Bardziej szczegółowo przedyskutowano niektóre przybliżone rozwiązania oraz zastosowania do zagadnień wyciskania cieczy.

Плоские вязкоупругие течения с доминирующим растяжением определены как течения в тонких слоях, описыванные определяющими уравнениями для установившегося или неустановившегося растяжения, пертурбированными по отношению к инвариантам скорости деформации. Более детально обсуждены некоторые приближенные решения и применения к задачам выдавливания жидкости.

1. Introduction

It is well known that in many practically important situations the character of flows considered is neither viscometric nor purely extensional. There appears also a variety of important velocity fields and viscoelastic phenomena which are not representable as small perturbations of slow steady flows [1]. On the other hand, there exist numerous flows of particular geometries, e.g., squeezing flows, converging and diverging flows, rolling, calendering and milling flows, etc., in which the extensional parts of deformation are dominating and more essential as compared with shearing deformations, usually prevailing in the regions close to solid boundaries.

For such flows, especially those characterised by high Deborah numbers and low vorticity components, A. B. METZNER [1, 2] introduced and developed the notion of "extensional primary fields" or EPF approximations. These are the flows in which the diagonal components of the stress matrix are much greater than the shearing ones at a given deformation rate level. The EPF approximations are precisely the reverse of those usually made in the boundary layer theory or in the slow flows of lubricating fluids.

In the present paper we adopt the Metzner's idea and discuss a class of plane viscoelastic flows called the "flows with dominating extensions" or briefly FDE. These flows can be defined as thin-layer flows (one of the characteristic dimension is much greater than the other) in which the general constitutive equation of an incompressible simple fluid for steady and unsteady extensional flows (cf. [1, 3]) may be used in a form linearly perturbed

http://rcin.org.pl

with respect to the relevant strain-rate invariants. Such an idea seems to be similar, in a reverse sense, to that used for quasi-viscometric approximations of viscoelastic boundary layers with small Weissenberg numbers (cf. [1, 4]). The FDE considerations lead to relatively simple, analytical or semi-numerical solutions satisfying more or less exact boundary conditions. In the present paper the case of plane squeezing flows is analysed in greater detail, and the effects of an increasing or decreasing extensional viscosity on the velocity profiles and the load-bearing forces are widely discussed.

It is noteworthy that the analysis presented in the paper can be extended, with only slight modifications, to other cases of lubricating, rolling, etc., flows.

2. Flows with dominating extensions (FDE)

Consider a plane flow in which the Cartesian velocity components can be presented in the following form:

$$\begin{aligned} u^* &= q(t)x+u(x,y,t), \\ v^* &= -q(t)y+v(x,y,t), \end{aligned} \tag{2.1}$$

where $q(t)$ may depend at most on time t, and u and v denote the additional velocity components along the axes x and y, respectively([1]).

If, moreover, the above flow is realised in a thin layer of fluid, in which one of the characteristic dimensions L is much greater than the dimension h describing the layer thickness, we can use all the kinematic simplifications resulting from the so-called lubrication (or thin-layer) approximation.

Denoting

$$x = \bar{x}L, \quad y = \bar{y}h, \quad u = \bar{u}U, \quad v = \varepsilon\bar{v}U, \quad \varepsilon = \frac{h}{L} \ll 1, \tag{2.2}$$

where $U = qh$ is a characteristic velocity, and overbars denote dimensionless quantities, we obtain

$$\begin{aligned} &\frac{\partial u^*}{\partial x} = q\left(1+\varepsilon\frac{\partial\bar{u}}{\partial\bar{x}}\right), \qquad \frac{\partial u^*}{\partial y} = q\frac{\partial\bar{u}}{\partial\bar{y}}, \\ &\frac{\partial v^*}{\partial y} = q\left(-1+\varepsilon\frac{\partial\bar{v}}{\partial\bar{y}}\right), \qquad \frac{\partial v^*}{\partial x} = q\varepsilon^2\frac{\partial\bar{v}}{\partial\bar{x}}, \\ &\omega^* = \frac{1}{2}\left(\frac{\partial u^*}{\partial y}-\frac{\partial v^*}{\partial x}\right) = \frac{1}{2}q\left(\frac{\partial\bar{u}}{\partial\bar{y}}-\varepsilon^2\frac{\partial\bar{v}}{\partial\bar{x}}\right). \end{aligned} \tag{2.3}$$

It is seen from the above relations that, under the assumption of dimensionless velocity gradients being of order $O(1)$, the first terms of diagonal components may be more meaningful than the remaining terms, if only the vorticity ω^* (or $\partial u/\partial y$) is sufficiently small. In the case of pure extension ($\omega^* \equiv 0$), the diagonal components are proportional to the extension rate q.

([1]) A similar analysis can be developed in any system of orthogonal coordinates. Also, axially symmetric flows of the FDE-type can be considered (cf. [5]).

http://rcin.org.pl

The constitutive equation of an incompressible simple fluid in any extensional (irrotational) flow, which can be treated as a motion with constant or proportional stretch history (cf. [3, 6, 7]), has the following form:

$$\mathbf{T} = -p\mathbf{1}+\beta_1(t; I_2, I_3)\mathbf{A}_1+\beta_2(t; I_2, I_3)\mathbf{A}_1^2, \tag{2.4}$$

where p is a hydrostatic pressure, $\mathbf{A}_1$ denotes the first Rivlin–Ericksen kinematic tensor (cf. [3]), and

$$I_2 = \mathrm{tr}\mathbf{A}_1^2, \quad I_3 = \mathrm{tr}\mathbf{A}_1^3, \tag{2.5}$$

are the corresponding invariants. It is noteworthy that the material functions β_i $(i = 1, 2)$ may explicitly depend on time t for unsteady flows. Moreover, for plane flows, the term $\beta_2 \mathbf{A}^2$ can be included into the pressure one.

In the case of plane flows described by Eq. (2.1), we formally write

$$[\mathbf{A}_1^*] = [\mathbf{A}_1]+[\mathbf{A}_1]' = \begin{bmatrix} 2q & 0 \\ 0 & -2q \end{bmatrix} + \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} \end{bmatrix}, \tag{2.6}$$

$$[\mathbf{A}_1^{*2}] = [\mathbf{A}_1^2]+[\mathbf{A}_1^2]' = \begin{bmatrix} 4q^2 & 0 \\ 0 & 4q^2 \end{bmatrix} + \begin{bmatrix} 8q\frac{\partial u}{\partial x}+4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 & 0 \\ 0 & 8q\frac{\partial u}{\partial x}+4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 \end{bmatrix}. \tag{2.7}$$

For invariants, we also have

$$\begin{gathered} \mathrm{tr}\mathbf{A}_1^* = 0, \quad \mathrm{tr}\mathbf{A}_1^{*3} = 0, \\ \mathrm{tr}\mathbf{A}_1^{*2} = \mathrm{tr}\mathbf{A}_1^2+(\mathrm{tr}\mathbf{A}_1^2)' = 8q^2+\left[16q\frac{\partial u}{\partial x}+8\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]. \end{gathered} \tag{2.8}$$

Taking into account the order arguments discussed after Eq. (2.3), we can define the plane "flows with dominating extensions" (FDE) as such thin-layer flows in which the constitutive equations (2.4), exact for purely extensional flows of an incompressible simple fluid, may be used in a form linearly perturbed with respect to the invariants (2.5), depending on the extension rate q.

This assumption means that

$$\mathbf{T}^* = -p\mathbf{1}+\beta\mathbf{A}_1+\beta\mathbf{A}_1'+\frac{d\beta}{dq}q'\mathbf{A}_1, \tag{2.9}$$

where for plane flows: $-p = T^{*33}+4\beta_2 q^2$, and the linear increment of q, denoted by q', is

$$q' = \frac{\partial u}{\partial x}+\frac{1}{2q}\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{8q}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2. \tag{2.10}$$

http://rcin.org.pl

Thus we have in a component form

$$
(2.11)\quad
\begin{aligned}
T^{*11} &= -p+2\beta q+2\beta\frac{\partial u}{\partial x}+\frac{1}{4}\frac{d\beta}{aq}\left[8q\frac{\partial u}{\partial x}+4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right],\\
T^{*22} &= -p-2\beta q-2\beta\frac{\partial u}{\partial x}-\frac{1}{4}\frac{d\beta}{dq}\left[8q\frac{\partial u}{\partial x}+4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right],\\
T^{*33} &= -p-4\beta_2 q^2,\\
T^{*12} &= \beta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),
\end{aligned}
$$

where, without any loss of generality, we have omitted the subscript 1 at β.

The terms involved in Eqs. (2.11) are of different orders of magnitude with respect to the parameter $\varepsilon = h/L$. An answer to the question, which of them are really essential for further considerations, results from the dynamic equations of equilibrium written in a dimensionless form. To this end, we introduce Eqs. (2.11) into

$$
(2.12)\quad
\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{\partial T_E^{*11}}{\partial x}+\frac{\partial T^{*12}}{\partial y},\\
\frac{\partial p}{\partial y} &= \frac{\partial T_E^{*22}}{\partial y}+\frac{\partial T^{*12}}{\partial x},
\end{aligned}
$$

where the subscript E denotes the extra-stress components ($T_E^{ii} = T^{ii}+p$, $i = 1, 2$), and the inertia terms have been disregarded([2]).

Expressing the resulting equations in a dimensionless form, by means of Eqs. (2.2) and

$$
(2.13)\quad p = \bar{p}\frac{U\eta L}{h^2},\qquad q = \bar{q}\frac{U}{L},\qquad \beta = \bar{\beta}\eta,
$$

where overbars denote dimensionless quantities and η is a constant with dimension of viscosity, we may retain only terms of the highest order of magnitude with respect to $\varepsilon = h/L$. Such a procedure leads to the following, simplified equations of dynamic equilibrium:

$$
(2.14)\quad
\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{1}{2}\frac{d\beta}{dq}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\beta\frac{\partial^2 u}{\partial y^2},\\
\frac{\partial p}{\partial y} &= -\frac{1}{2}\frac{d\beta}{dq}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}.
\end{aligned}
$$

Eliminating the pressure terms by consecutive differentiation with respect to y and x, we obtain

$$
(2.15)\quad \frac{\partial}{\partial y}\left[\frac{1}{2}\frac{d\beta}{dq}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2+\beta\frac{\partial^2 u}{\partial y^2}\right] = 0.
$$

([2]) Retaining these terms seriously complicates further considerations; the inertia effects, however, can be taken into account in an approximate way (cf. Sect. 4).

http://rcin.org.pl

Introducing the notation:

$$p^* = p - T_E^{*22} = p + \frac{1}{4}\frac{d\beta}{dq}\left(\frac{\partial u}{\partial y}\right)^2 + 2\beta q, \tag{2.16}$$

we can arrive at the alternative system of Eqs. (2.14), viz.

$$\frac{dp^*}{dx} = \frac{1}{2}\frac{d\beta}{dq}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 + \beta\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial p^*}{\partial y} = 0, \tag{2.17}$$

where p^* is a function of x only.

It is also noteworthy that the simplified constitutive equations, leading immediately to Eqs. (2.15) or (2.17), can be written as follows:

$$\begin{aligned} T^{*11} &= -p + 2\beta q + \frac{1}{4}\frac{d\beta}{dq}\left(\frac{\partial u}{\partial y}\right)^2, \\ T^{*22} &= -p - 2\beta q - \frac{1}{4}\frac{d\beta}{dq}\left(\frac{\partial u}{\partial y}\right)^2, \\ T^{*12} &= \beta\frac{\partial u}{\partial y}, \end{aligned} \tag{2.18}$$

where the function β may also depend on time t for unsteady flows. For steady flows, this function can be related to the corresponding elongational viscosity $\eta^*(q)$. Thus, e.g., for a pure two-dimensional extension, we have

$$\eta^*(q) = \frac{1}{q}(T^{*11} - T^{*22}) = 4\beta(q), \tag{2.19}$$

since then $\partial u/\partial y = 0$. In what follows, we shall call $\beta(q)$ the extensional viscosity function.

Eq. (2.15) is a third order nonlinear partial differential equation, a solution of which, even for simple boundary conditions imposed on u or its derivatives, is not known at all. On the other hand, the solution of the simplified equation

$$\frac{dp^*}{dx} = \beta_0\frac{\partial^2 u}{\partial y^2} = C_0(x), \tag{2.20}$$

valid for a Newtonian fluid, can be presented as

$$u = \frac{1}{\beta_0}\left[\frac{1}{2}C_0(x)y^2 + C_1(x)y + C_2(x)\right], \tag{2.21}$$

where $C_1(x)$ and $C_2(x)$ are functions depending on the boundary conditions.

In more general cases of viscoelastic fluids, we shall seek an approximate solution of Eq. (2.15) in the following form:

$$u = (x+a)\big(w(y)+b\big), \tag{2.22}$$

where a and b are constants and $w(y)$ is a function of y only. After introducing Eq. (2.22) into Eq. (2.15), we arrive at

$$(x+a)\left(\beta w''(y) + \frac{d\beta}{dq}w'^2(y)\right) = C_0(x), \tag{2.23}$$

http://rcin.org.pl

where primes denote differentiation with respect to y. In particular, any solution of the equation

$$\beta w''(y)+\frac{d\beta}{dq}w'^2(y) = C = \text{const} \tag{2.24}$$

also satisfies Eq. (2.23) when $C_0(x) = C'(x+a)$, i.e. for a parabolic dependence of p^* on x. More general distributions of the thrusts $p^*(x)$ can be taken into account under the assumption that $C = C(x)$, if x is treated as some parameter, on which the solutions of Eq. (2.24) may depend.

The simplified Eq. (2.24) represents a special Riccati equation for $w'(y)$; its general solution satisfying the boundary condition: $w'(0) = 0$, is

$$\begin{aligned} w'(y) &= \frac{B}{\sqrt{-AB}}\,\mathrm{tg}\left(\sqrt{-AB}y\right) \quad \text{for} \quad AB < 0,\\ w'(y) &= \frac{B}{\sqrt{AB}}\,\mathrm{th}\left(\sqrt{AB}y\right) \quad \text{for} \quad AB > 0, \end{aligned} \tag{2.25}$$

where

$$A = \frac{1}{\beta}\frac{d\beta}{dq}, \quad B = \frac{C}{\beta} = \frac{1}{(x+a)\beta}\frac{dp^*}{dx}. \tag{2.26}$$

This gives

$$\begin{aligned} w(y) &= \frac{1}{A}\ln\cos\left(\sqrt{-AB}y\right)+C_1 \quad \text{for} \quad AB < 0,\\ w(y) &= \frac{1}{A}\ln\mathrm{ch}\left(\sqrt{AB}y\right)+C_1 \quad \text{for} \quad AB > 0, \end{aligned} \tag{2.27}$$

where C_1 denotes an integration constant. It is easy to see that $A > 0$ corresponds to an increasing function $\beta(q)$, while $A < 0$ — to a decreasing $\beta(q)$. The sign of B depends exclusively on the sign of $C(\beta > 0)$, i.e. on whether the thrust $p^*(x)$ is an increasing or decreasing function of x.

For Newtonian fluids ($\beta = \beta_0 = \text{const}$), we obtain instead of Eqs. (2.25) and (2.27):

$$w'(y) = \frac{C}{\beta_0}y, \quad w(y) = \frac{C}{2\beta_0}y^2+C_1. \tag{2.28}$$

The above considerations can be applied, in principle, to any plane FDE in a sufficiently thin layer of viscoelastic fluid. To this end, however, a knowledge of the appropriate boundary conditions is necessary. In every case we must verify that the parameter $\varepsilon = h/L$ is sufficiently small in the range of x considered.

3. Plane squeezing flows of viscoelastic fluids

Various problems of plane and axially symmetric squezing flows were widely discussed elsewhere (for references cf. [8, 9, 10, 11]). In what follows, we shall be interested mainly

http://rcin.org.pl

in the case of continuous squeezing flows with some simulated approach velocity as described by D. R. OLIVER and M. SHAHIDULLAH [12, 13].

Consider a test fluid contained between two horizontal plates of width $2a$, at the distance $2h$ one from the other, and loaded with the force P per unit length (Fig. 1). In a traditional

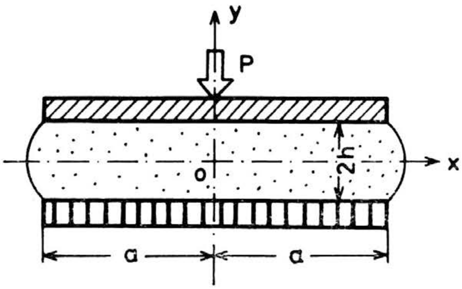

FIG. 1.

flow the top plate is released at some instant $t = 0$ and falls down freely under the constant load; the distance h is measured as a function of current time. The resulting flow is unsteady one, in which inertia effects may be of great importance. In a continuous flow, the fluid moves through a stationary lower plate, being extruded from numerous holes uniformly distributed over the lower surface, with neither plate moving (cf. [12]). The force P corresponding to the simulated approach velocity $\dot{h}$ is measured. Such a flow is steady, since the ratio of the approach velocity to the fluid-layer thickness is kept constant.

For the case of plane continuous squeezing flows, we assume that

$$\varepsilon = \frac{h}{a} \ll 1, \quad q = -\frac{\dot{h}}{h} = \text{const}, \quad v^*(h) = \dot{h} = \text{const}, \tag{3.1}$$

and the velocity field is of the form:

$$u^* = qx + w(y)x, \quad v^* = -qy + v(x, y). \tag{3.2}$$

The kinematic boundary conditions, expressing adhesion of the fluid to the walls and symmetry of the flow considered, lead to

$$u^*(\pm h) = 0, \quad \frac{\partial u^*}{\partial y}(0) = 0, \tag{3.3}$$

or

$$w'(\pm h) = -q, \quad w'(0) = 0. \tag{3.4}$$

On the basis of the above conditions, we arrive at Eqs. (2.25) and

$$\begin{aligned} w(y) &= \frac{1}{A} \ln \frac{\cos(\sqrt{-AB}y)}{\cos(\sqrt{-AB}h)} - q \quad \text{for} \quad AB < 0, \\ w(y) &= \frac{1}{A} \ln \frac{\text{ch}(\sqrt{AB}y)}{\text{ch}(\sqrt{AB}h)} - q \quad \text{for} \quad AB > 0, \end{aligned} \tag{3.5}$$

where A and B have been defined in Eqs. (2.26).

http://rcin.org.pl

For Newtonian fluid ($\beta = \beta_0 = \text{const}$), we obtain Eq. $(2.28)_1$ and

$$w(y) = \frac{C_N}{2\beta_0}(y^2-h^2)-q. \tag{3.6}$$

The continuity condition, or the equivalent requirement that the volume discharge Q of fluid during the flow is conserved, leads to the condition:

$$\frac{1}{2}Q = -ah = 2\int_0^h u^*|_{x=a}\,dy = 2qah+2\int_0^h w(y)a\,dy. \tag{3.7}$$

Since for Newtonian fluids, after integrating Eq. (3.6), we have

$$C_N = \frac{3}{2}\frac{\beta_0\dot{h}}{h^3} < 0 \quad \text{or} \quad \frac{dp^*}{dx} = C_N x, \tag{3.8}$$

it can be deduced that also for viscoelastic fluids

$$C = -H\frac{\beta q}{h^2} < 0, \tag{3.9}$$

where H is a dimensionless number determined later in this Section.

Introducing the results (3.5) into Eq. (3.7) and performing integration, we arrive at

$$\begin{aligned} \frac{\dot{h}}{2} &= \frac{1}{A}\left[\frac{1}{\sqrt{-AB}}L\left(\sqrt{-AB}h\right)+h\ln\cos\left(\sqrt{-AB}h\right)\right] \quad &\text{for} \quad \frac{d\beta}{dq} > 0,\\ \frac{\dot{h}}{2} &= -\frac{1}{A}\left[\frac{1}{\sqrt{AB}}\overline{L}\left(\sqrt{AB}h\right)-h\ln\text{ch}\left(\sqrt{AB}h\right)\right] \quad &\text{for} \quad \frac{d\beta}{dq} < 0, \end{aligned} \tag{3.10}$$

where

$$L(x) = -\int_0^x \ln\cos z\,dz \quad \text{and} \quad \overline{L}(x) = \int_0^x \ln\text{ch}\,z\,dz, \tag{3.11}$$

denote the Lobachevsky and the modified Lobachevsky functions, respectively. Only the first of them is tabelarised (cf. [14]); the second one can easily be calculated.

For simplicity of further calculations, the notations used in Eqs. (3.10) require some slight alterations. To this end, we introduce the new parameter:

$$\gamma = AHq = \frac{1}{\beta}\frac{d\beta}{dq}Hq, \tag{3.12}$$

where H is related to C by Eq. (3.9). Then, instead of Eqs. (3.10), we can write

$$\begin{aligned} \frac{|\gamma|}{2H} &= -\left[\ln\cos\sqrt{|\gamma|}+\frac{1}{\sqrt{|\gamma|}}L\left(\sqrt{|\gamma|}\right)\right] \quad &\text{for} \quad \gamma > 0,\\ \frac{|\gamma|}{2H} &= \left[\ln\text{ch}\sqrt{|\gamma|}-\frac{1}{\sqrt{|\gamma|}}\overline{L}\left(\sqrt{|\gamma|}\right)\right] \quad &\text{for} \quad \gamma < 0. \end{aligned} \tag{3.13}$$

The above relations connect H (or C) with other kinematic and material parameters involved in γ.

http://rcin.org.pl

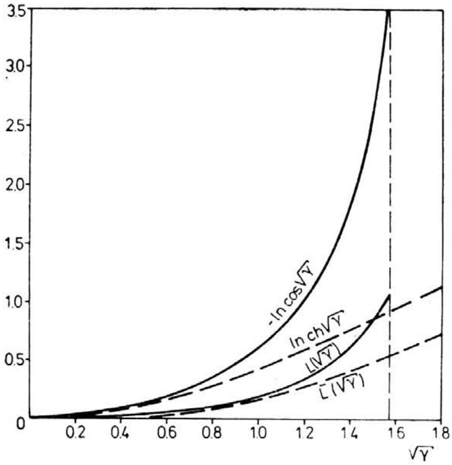

FIG. 2.

In Fig. 2, the diagrams of the Lobachevsky functions L and $\bar{L}$ are compared with the functions $-\ln\cos\sqrt{\gamma}$ and $\ln\mathrm{ch}\sqrt{\gamma}$ occurring in Eqs. (3.13). It is obvious that $L(x)$ is determined for $0 < x < \pi/2$ only.

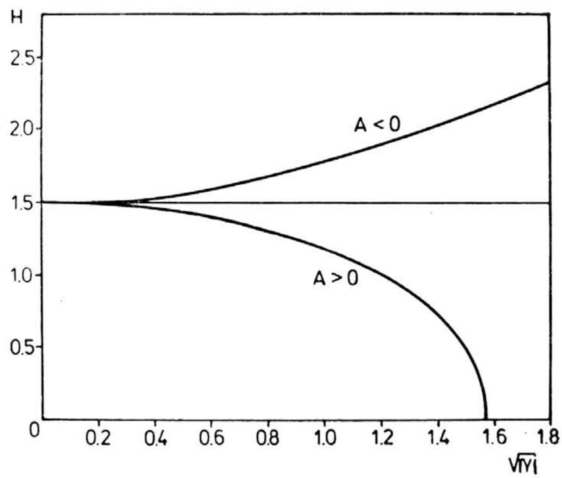

FIG. 3.

The relations between $\sqrt{|\gamma|}$ and H resulting from Eqs. (3.13) are shown in Fig. 3. Since γ is defined by Eq. (3.12), it is easy to observe that H does not differ significantly from $H = 1.5$ (for Newtonian fluids) for sufficiently small values of the extension rate $q = -\dot{h}/h$. The upper curve refers to a decreasing extensional viscosity $\beta(q)$, while the lower one — to an increasing $\beta(q)$.

The illustrative velocity profiles at the ends of plates for $\gamma = 2.25$ and $\gamma = -2.47$ are drawn in Fig. 4. The Newtonian profile is marked with a broken line. It is seen, that for an increasing extensional viscosity ($A > 0$), the profile may be much more flattened

http://rcin.org.pl

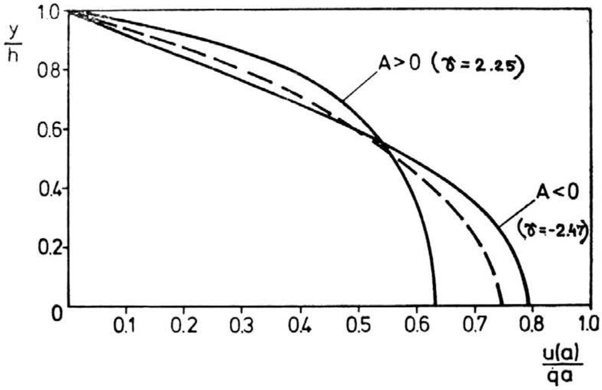

FIG. 4.

as compared with the Newtonian one. It is also interesting that the limit case of plug flow (with entirely flattened velocity profile) corresponds to (cf. (2.25))

$$\sqrt{-AB}h = \frac{\pi}{2} \quad \text{or} \quad \frac{d\beta}{dq} q \simeq 1.63\beta, \tag{3.14}$$

i.e., for sufficiently high rates of increase of the extensional viscosity $\beta(q)$. For a constant rate of increase of β, the above case is impossible at all.

4. Load-bearing forces for various boundary conditions at the adge

The velocity profiles could be determined on the basis of the boundary conditions satisfied on surfaces of the upper and lower plates. For satisfactory determination of the total thrust on the top plate or the load-bearing force, one has to know something more about the boundary conditions at the edges. The whole problem may become very complex, if we want to take into account all the possible edge effects (cf. [15, 16]). For sake of simplicity we restrict ourselves to certain approximate boundary conditions valid for the cases of free and drowned edges of the plates.

At this moment, we also want to comment briefly possible influences of inertia effects, which may be essential for viscoelastic fluids (cf. [8, 12]). For the case of continuous squeezing flows both plates are stationary. If the top plate moves down under the total force P (involving its weight), we have

$$F = P - m\ddot{h}, \tag{4.1}$$

where $m\ddot{h}$ represents the load inertia, and F is the force exerted by the fluid on the upper plate. The fluid inertia effects have been disregarded in Eqs. (2.12); they can be taken into account in an approximate way, considering the total mass balance (cf. [17]). This gives the following values of fluid inertia forces:

$$F_I = 0{,}031 \div 0{,}048 \frac{\varrho \dot{h}^2 a^3}{h^2}, \tag{4.2}$$

which should be added to the force F exerted by the fluid itself (cf.e.g. (4.9)).

http://rcin.org.pl

4.1. The case of free edge

If the outer surfaces are free from any loading, like in the Mooney plastometer (cf. Fig. 1), we may assume either

$$T^{*11}|_{x=\pm a} = 0 \quad (= -p_a), \tag{4.3}$$

where p_a denotes the atmospheric pressure, or

$$T^{*11}|_{x=\pm a} = -p_a - HS\frac{\dot{h}a}{h^3}, \tag{4.4}$$

where S denotes a constant surface-tension coefficient. The above conditions are approximately valid, if the wetting angle at the edge, characterising relation among the fluid, the solid surface and the atmosphere or vapor in a surrounding region, is close to 90° (cf. [16]).

In some other cases, Eq. (4.3) can be replaced by the condition (cf. [8]):

$$\int_0^h T^{*11}|_{x=\pm a} dy = \int_0^h (-p+T_E^{*11})|_{x=\pm a} dy = 0, \tag{4.5}$$

expressing the fact that the resulting force on the free surface is equal to zero.

By way of illustration, Eq. (4.3) together with Eq. (2.18) leads to

$$T^{*22}|_{x=\pm a} = -4\beta q - \frac{1}{2}\frac{d\beta}{dq} w'^2(y)a^2. \tag{4.6}$$

The load-bearing force (without inertia effects) is calculated from

$$F = -2\int_0^a T^{*22}|_{y=\pm h} dx = -2(T^{*22}|_{y=\pm h}x)|_0^a + 2\int_0^a \frac{\partial T^{*22}}{\partial x} x|_{y=\pm h} dx, \tag{4.7}$$

where integration by parts is used. Bearing in mind Eqs. (4.6) and (2.12), we obtain

$$F = 8\beta q a + \frac{d\beta}{dq} w'^2(h)a^3 + 2\int_0^a \left(-\frac{\partial T^{*12}}{\partial y} - \frac{\partial}{\partial x}(T^{*11} - T^{*22})\right) x|_{y=\pm h} dx \tag{4.8}$$

and after introducing Eqs. (3.9), (2.18)

$$F = -\frac{2}{3} C a^3 \left(1 + \frac{3}{2}\mathrm{tg}^2 \sqrt{\frac{1}{\beta}\frac{d\beta}{dq}} qH + 12\frac{h^2}{Ha^2}\right) \tag{4.9}$$

if $d\beta/dq > 0$. Since under the assumption that $\varepsilon = h/a \ll 1$ the last term can be neglected, we finally have

$$F = -\frac{2}{3} H \frac{\beta \dot{h} a^3}{h^3}\left(1 + \frac{3}{2}\mathrm{tg}^2 \sqrt{\frac{1}{\beta}\frac{d\beta}{dq}} qH\right) \quad \text{for} \quad \frac{d\beta}{dq} > 0, \tag{4.10}$$

and

$$F = -\frac{2}{3} H \frac{\beta \dot{h} a^3}{h^3}\left(1 - \frac{3}{2}\mathrm{th}^2 \sqrt{-\frac{1}{\beta}\frac{d\beta}{dq}} qH\right) \quad \text{for} \quad \frac{d\beta}{dq} < 0. \tag{4.11}$$

http://rcin.org.pl

In the case of Newtonian fluids, similar consideration lead to the well-known expression:

$$F_N = -\frac{\beta_0 \dot{h} a^3}{h^3}. \tag{4.12}$$

It should be emphasized, however, that Eqs. (4.10), (4.11) essentially depend on boundary conditions imposed on T^{*11} as well as on any additional hypothesis concerned with the normal-stress differences. By way of illustration, we can consider the case of Weissenberg hypothesis ($T_E^{*22} = T_E^{*33}$) and that of biaxial extension ($T_E^{*11} = T_E^{*33}$ at the edge) (cf. [17]). The number coefficients appearing in the parantheses on the right hand sides of Eqs. (4.10), (4.11), i.e. 3/2, change into 9/4 for the first case, and into 9/8 in the second.

A graphical presentation of the ratio F/F_N (multiplied by $\beta_0/\beta(q)$) in function of $\sqrt{\gamma}$ ($A > 0$) or $\sqrt{-\gamma}$ ($A < 0$) is shown in Fig. 5 (solid line). In the same Fig. 5, we plot the

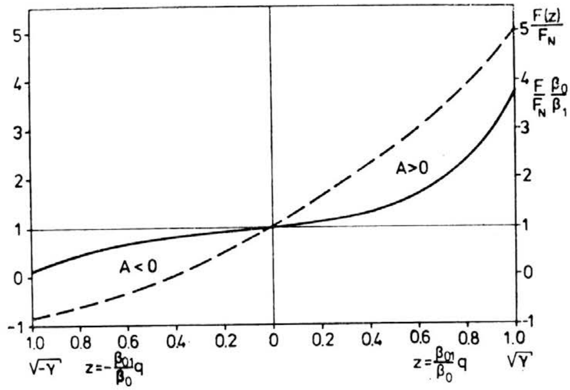

FIG. 5.

dependence of F/F_N on $z = \pm\beta_{01}/\beta_0 q$, under the assumption that the extensional viscosity $\beta(q)$ is a linear function of q, viz.

$$\beta(q) = \beta_0 + \beta_{01} q, \quad \beta_{01} = \text{const}. \tag{4.13}$$

The above diagrams are qualitatively in agreement with available experimental results (cf. [12, 13]); they show definitive load-enhancement effects, if the extension viscosity is an increasing function of the extension rate. For more precise quantitative comparisons, further accurate numerical data would be desirable.

It is doubtful, however, whether Eqs. (4.10), (4.11) can be used for determination of the extensional viscosity function $\beta(q)$, since the ratio F/F_N depends on $\beta(q)$ and $d\beta/dq$ as well. Such a determination is possible, in principle, if the form of $\beta(q)$ can be deduced from other considerations. For instance, if $\beta(q)$ is a linear function of q, like in Eq. (4.13), we arrive at

$$\frac{F}{F_N} = -\frac{2}{3} H\left(1+\frac{\beta_{01}}{\beta_0} q\right)\left(1+\frac{3}{2}\mathrm{tg}^2\sqrt{\frac{qH\beta_{01}}{\beta_0+\beta_{01} q}}\right), \tag{4.14}$$

from which the ratio β_{01}/β_0 can be calculated.

http://rcin.org.pl

If the term under square root in Eq. (4.10) is small enough to be treated as a small parameter, the squares of which can be disregarded as compared with the first powers, we obtain the following approximate relation:

(4.15)
$$\dot{h} \simeq \frac{2}{3} H\dot{h}_N \frac{\beta_0}{\beta(q)} \left(1 - \frac{2}{3} H \frac{d\beta}{dq} \frac{q_N}{\beta}\right),$$

where the subscript N refers to Newtonian quantities. It is seen from Eq. (4.15) that for $d\beta/dq > 0$, the velocity $\dot{h}$ is always less than $\dot{h}_N$.

For the case of traditional squeezing flow under a constant load, i.e. when h depends on time t, S. J. LEE *et al.* [8] explained the oscillatory behaviour of $h(t)$ throughout the competition between inertia and elastic forces. It turns out that an oscillatory compressive flow is expected, if only the modified elasticity number is high enough. According to our analysis, even a single "bounce", observed by G. BRINDLEY *et al.* [9] for viscoelastic fluids under more severe loading conditions may appear when $\dot{h} = 0$. Eq. (4.15) leads to the following condition of bouncing:

(4.16)
$$\frac{1}{\beta} \frac{d\beta}{dq} q = \frac{2}{3H},$$

which, for a parabolic function: $\beta = \beta_0 + \beta_2 q^2$, gives

(4.17)
$$q^2 = \frac{\beta_0}{\beta_2(3H-1)}.$$

It can also be seen from Eq. (4.16) that the above described phenomenon of bouncing is impossible at all, if $\beta(q)$ is a linear function.

In general, after introducing Eq. (4.10) into Eq. (4.1), we obtain a complex nonlinear equation involving, apart from material characteristics of a fluid, the distance h and its time derivatives $\dot{h}$ and $\ddot{h}$.

4.2. The case of drowned edge

If the lower plate is identified with the bottom of a large container, while the top plate, being rather a parallelpiped, is drowned in a fluid at the depth l (Fig. 6), we can take adventage of a procedure similar to that used by R. I. TANNER [18] for viscometric flows. To this end, we shift the origin of Cartesian coordinates to the left upper edge (cf. Fig. 6).

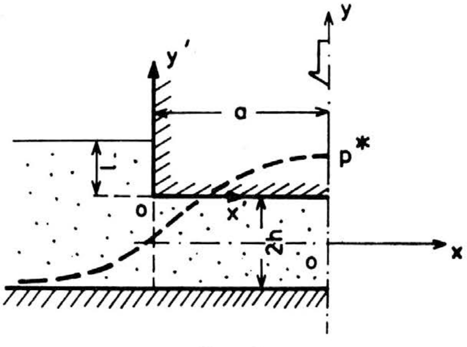

FIG. 6.

http://rcin.org.pl

For these new coordinates, viz.

$$x' = x+a, \quad y' = y-h, \tag{4.18}$$

the force exerted by the fluid on the top part can be expressed as

$$F = -2\int_0^a T^{*22}|_{y'=0}dx' - 2\int_0^l T^{*'2}|_{x'=0}dy'. \tag{4.19}$$

Taking into account that

$$\frac{dp^*}{dx'} = -\frac{\partial T^{*22}}{\partial x'} = C(x'-a), \quad T^{*12} = \beta w'(y)(x'-a), \tag{4.20}$$

we have

$$p^* = \frac{1}{2}C(x'-a)^2 + p_m^* \quad \text{if} \quad p^*(a) = p_m^*, \tag{4.21}$$

where p_m^* denotes the maximum thrust in the centre of the upper plate. After integrating Eq. (4.19), we arrive at

$$F = 2p_m^* a + \frac{Ca^3}{3} + 2\frac{Cal}{\sqrt{-AB}}\operatorname{tg}\left(\sqrt{-AB}h\right) \quad \text{for} \quad \frac{d\beta}{dq} > 0, \tag{4.22}$$

and the similar expression with tg replaced by th, for $d\beta/dq < 0$. Since for $l \approx h$ or $l \approx a$, the last terms on the right-hand side of Eq. (4.22) can be disregarded as being of order ε^2 or ε, respectively, we finally obtain

$$F^+ = 2p_m^* a + \frac{Ca^3}{3}, \quad C = H\frac{\beta\dot{h}}{h^3}, \tag{4.23}$$

independently of the character of function $\beta(q)$.

The above result has been obtained for C = const, i.e. for a parabolic distribution of thrusts on the top plate. In more realistic situations, we may have (cf. (2.20))

$$p^* = p_m^* f(x'), \tag{4.25}$$

where

$$f(-\infty) = f'(-\infty) = 0, \quad f(a) = 1, \quad f'(a) = 0. \tag{4.26}$$

The total force acting on the bottom of the container can be written as

$$F^- = 2\int_{-\infty}^a p^* dx' = 2p_m^* \int_{-\infty}^a f(x')dx', \tag{4.27}$$

while that acting on the upper plate as

$$F^+ = 2\int_0^a p^* dx' = 2p_m^* \int_0^a f(x')dx'. \tag{4.28}$$

If we assume, moreover, that the forces acting on both plates must be mutually equilibrated, and as a consequence, that the force (4.28) is equal to that described by Eq. (4.27), we obtain

http://rcin.org.pl

$$(4.29) \qquad p_m^* = -\frac{Ca^3}{6} \frac{1}{\left(1 - \int\limits_{-\infty}^{a} f(x)\,dx\right)}.$$

The above expression introduced into Eq. (4.28) finally gives

$$(4.30) \qquad F^+ = -H \frac{\beta(q)\dot{h}a^3}{3h^3} \frac{\int\limits_0^a f(x)\,dx}{1 - \int\limits_{-\infty}^{a} f(x)\,dx}.$$

By way of illustration, assume that the real distribution of thrusts on the bottom may be described by the following exponential function:

$$(4.31) \qquad p^* = p_m^* e^{(x'-a)^2}, \quad p^*(-\infty) = 0, \quad p^*(a) = p_m^*,$$

and

$$(4.32) \qquad \frac{dp^*}{dx'} = -2p_m^*(x'-a)e^{-(x'-a)^2}, \quad \frac{dp^*}{dx'}(a) = 0.$$

These relations used in the above outlined procedure lead to

$$(4.33) \qquad F^+ = -H \frac{\beta(q)\dot{h}a^3}{3h^3} \frac{\mathrm{Erf}(a)}{a - \left(\frac{\sqrt{\pi}}{2} + \mathrm{Erf}\left(\frac{a}{\sqrt{2}}\right)\right)},$$

where

$$(4.34) \qquad \mathrm{Erf}(x) = \int\limits_0^x e^{-t^2} dt,$$

denotes the error function (cf. [14]). It is seen from Eq. (4.21) that the approximation applied is valid for

$$(4.35) \qquad p_m^*(1 - e^{-a^6}) > -\frac{1}{2} Ca^2.$$

This inequality can be satisfied if

$$(4.36) \qquad a - \mathrm{Erf}\left(\frac{a}{\sqrt{2}}\right) > \frac{\sqrt{\pi}}{2} \simeq 0{,}886,$$

i.e. approximately for $a > 1.8$.

A direct inspection of Eqs. (4.29) or (4.30) shows that the dependence of F^+ on $q = -\dot{h}/h$ is generally nonlinear, if the extensional viscosity $\beta(q)$ is not constant. For Newtonian fluids ($\beta = \beta_0 = \mathrm{const}$) the load-bearing force F^+ is a linear function of q, although its slope essentially depends on the approximation used for p^*.

The continuous squeezing flows with drowned edges can be used, in principle, for determination of the extensional viscosity function $\beta(q)$. To this end, however, fluid inertia effects as well as more realistic (measured experimentally) thrust distributions should be taken into account.

http://rcin.org.pl

5. Concluding remarks

The problems discussed in the present paper enable to formulate the following remarks:

1) The concept of flows with dominating extensions (FDE), developed in this paper, can be used in all the cases of thin-layer flows in which the extensional rates of deformations are important (high Deborah numbers and small vorticities), but shearing effects cannot be disregarded, what is done under the extensional primary field (ETF) approximations proposed by A. B. Metzner.

2) Numerous plane steady flows in thin layer of viscoelastic fluids, e.g. squeezing and lubricating flows, rolling, calendering, milling etc. flows, can be dealt with the formalism introduced. To this end, however, appropriate geometrical assumptions, ensuring the validity of thin-layer approximations, should be made.

3) In steady flows of the FDE type, an essential role is played by the extensional viscosity function. Many results depend on whether the extensional viscosity is an increasing or decreasing function in the range of extension rates considered.

4) The FDE approximations can be applied to plane and axially symmetric, traditional unsteady and continuous steady squeezing flows. At least in the last case, the obtained results, i.e., the velocity profiles, the load-bearing forces etc., are in a good qualitative agreement with the behaviour observed experimentally.

5) In certain particular situations, the continuous squeezing flows may be used as flows enabling the extensional viscosity measurements. For such cases, possible effects of inertia should be determined in an accurate way.

References

1. A. B. METZNER, *Extensional primary field approximations for viscoelastic media*, Rheol. Acta, **10**, 434, 1971.
2. A. B. METZNER and A. P. METZNER, *Stress levels in rapid extensional flows of polymeric fluids*, Rheol. Acta, **9**, 174, 1970.
3. S. ZAHORSKI, *Mechanics of viscoelastic fluids*, Chap. 4, M. Nijhoff Publishers, The Hague–Boston–London 1982.
4. S. ZAHORSKI, *Viscoelastic boundary layer: the governing equations and boundary conditions*, Arch. Mech., **37**, 75, 1985.
5. S. ZAHORSKI, *Axially-symmetric squeezed films as viscoelastic flows with dominating extensions*, Engin. Trans., **34**, 181, 1986.
6. S. ZAHORSKI, *Flows with proportional stretch history*, Arch. Mech., **24**, 681, 1972.
7. S. C. CHOW, M. M. CARROLL, *Motions of proportional extensions*, Quart, J. Mech. Appl. Math., **26**, 471, 1973.
8. S. J. LEE, M. M. DENN, M. J. CROCHET, A. B. METZNER and G. J. RIGGINS, *Compressive flow between parallel disks. II. Oscillatory behavior of viscoelastic material under a constant load*, J. Non-Newtonian Fluid Mech., **14**, 301, 1984.
9. G. BRINDLEY, J. M. DAVIES and K. WALTERS, *Elastico-viscous squeeze films*, J. Non-Newtonian Fluid Mech., **1**, 19, 1976.
10. S. ZAHORSKI, *Viscoelastic properties in plane squeeze-film flows*, J. Non-Newtonian Fluid Mech., **4**, 217, 1978.
11. S. ZAHORSKI, *Viscoelastic properties in axially symmetric squeeze-film flows*, Arch. Mech., **31**, 431, 1979.

http://rcin.org.pl

12. D. R. OLIVER and M. SHAHIDULLAH, *Definitive load-enhancement effects by polymer-thickened oils in a squeeze film experiment*, J. Non-Newtonian Fluid Mech., **9**, 257, 1981.
13. D. R. OLIVER, M. SHAHIDULLAH, *Reverse squeeze film flow in a continuous flow system*, J. Non-Newtonian Fluis Mech., **15**, 331, 1984.
14. I. S. GRADSTEYN and J. W. RYZHIK, *Tables of integrals, series and products*, Academic Press, New York 1965.
15. C. M. TAYLOR, *Separation cavitation in lightly loaded fluid film bearings with both surfaces in motions*, J. Mech. Eng. Sci., **16**, 147, 1974.
16. D. DOWSON, E. H. SMITH, C. M. TAYLOR, *An experimental study of hydrodynamic film rupture in a steadily-loaded, non-conformal contact*, J. Mech. Eng. Sci., **22**, 71, 1980.
17. D. R. OLIVER, *The influence of fluid inertia, viscosity and extra stress on the load bearing capacity of a squeeze film of oil*, Appl. Sci. Res., **35**, 217, 1979.
18. R. I. TANNER, *Some methods for estimating the normal stress functions in viscometric flows*, Trans. Soc. Rheol., **14:4**, 483, 1970.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEA CH.

Received May 17, 1985.

http://rcin.org.pl