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Fundamental equations of continuous structural media
II. A simplified structural medium

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THE SUBJECT of considerations is the continuous medium in which, besides the force and couple stresses, p^{ij} and m^{ij}, also higher order stresses b^{ijk} occur. Displacement of such media are described by the linear displacement vector u_i, the vector of unconstrained rotations φ_i and the generalized displacement tensor ψ_{ij}. The starting point of our considerations are the results obtained in the paper [1] dealing with a general case. The model of a particle of the simplified medium represents a system consisting of its geometric center and a set of planes passing through it. The planes may move together with the geometric center and, in addition, may be subject to independent displacements and rotations. The planes are not allowed to move with respect to each other along the lines of intersection, as in the general case; in the simplified model the planes are also prevented to rotate about each other. The particle of such a simplified structural medium is a system with 15 degrees of freedom; the medium consisting of such particles is governed by 105 equations with the corresponding boundary conditions; 15 of them are the equations of motion, 45 geometric equations and 45 physical relations. From such a system it is possible to determine the 105 statical and geometrical unknowns appearing in the model of the simplified structural medium.

Przedmiotem rozważań jest ośrodek ciągły, w którym oprócz napięć siłowych p^{ij}, napięć momentowych m^{ij} występują napięcia wyższego rzędu b^{ijk}. W ośrodku tym przemieszczenia opisane są przez wektor przemieszczeń liniowych u_i, wektor niezależnych obrotów φ_i oraz tensor uogólnionych przemieszczeń ψ_{ij}. Punkt wyjścia stanowią wyniki otrzymane w pracy [1] dla przypadku ogólnego. Model cząstki uproszczonego ośrodka stanowi nadal układ złożony ze środka geometrycznego i odpowiednich płaszczyzn przechodzących przez ten środek. Płaszczyzny te oprócz przemieszczeń wraz ze środkiem geometrycznym doznają niezależnych przemieszczeń i odkształceń. Nadal zakładamy, że wzdłuż krawędzi przecięcia nie mogą się one nawzajem przemieszczać względem siebie. Dla omawianego modelu uproszczonego zakładamy dodatkowo, że płaszczyzny te nie mogą się względem siebie obracać. Reprezentowana przez powyższy model cząstka uproszczonego ośrodka strukturalnego jest układem o 15 stopniach swobody. Dla ośrodka ciągłego złożonego z takich cząstek otrzymano 105 równań z odpowiednimi warunkami brzegowymi. W skład tego układu wchodzi 15 równań ruchu, 45 równań geometrycznych oraz 45 związków fizycznych. Z powyższego układu równań możemy w sposób jednoznaczny wyznaczyć 105 niewiadomych statycznych i geometrycznych występujących w modelu uproszczonego ośrodka strukturalnego.

Предметом рассуждений является сплошная среда, в которой кроме силовых напряжений p^{ij}, моментных напряжений m^{ij}, выступают напряжения высшего порядка b^{ijk}. В этой среде перемещения описываются вектором линейных перемещений u_i, вектором независимых вращений φ_i и тензором обобщенных перемещений φ_{ij}. Исходную точку составляют результаты полученные в работе [1] для общего случая. Модель частицы упрощенной среды составляет в дальнейшем система состоящая из геометрического центра и соответствующих плоскостей, проходящих через этот центр. Эти плоскости, кроме перемещений совместно с геометрическим центром, испытывают независимые перемещения и деформации. В дальнейшем предпологаем, что вдоль грани пересечения они не могут взаимно перемещаться относительно себя. Для обсуждаемой упрощенной модели предполагаем дополнительно, что эти плоскости не могут относительно себя вращаться. Представленная вышеупомянутой моделью частица упрощенной структурной среды является системой с 15 степенями свободы. Для сплошной среды, состоящей из таких частиц, получены 105 уравнений с соответствующими граничными условиями. В состав этой системы входит 15 уравнений движения, 45 геометрических уравнений и 45 физических соотношений. Из вышеупомянутой системы уравнений можем единственным образом определить 105 статических и геометрических неизвестных, выступающих в упрощенной модели структурной среды.

1. Generalized displacement tensors φ_{ij}, ψ_{ijk}

BEFORE passing to the presentation of simpler models of the medium, let us return to the displacement tensors φ_{ij}, ψ_{ijk} introduced in the paper [1]; all the notations and definitions introduced in that paper remain valid. Let us start with tensor φ_{ij} by decomposing it int the symmetric and antisymmetric parts. According to the definition,

$$\varphi_{(ij)} = \frac{\varphi_{ij}+\varphi_{ji}}{2}, \qquad \varphi_{[ij]} = \frac{\varphi_{ij}-\varphi_{ji}}{2}. \tag{1.1}$$

All components of the tensor are expressed as sums of their symmetric and antisymmetric parts, $\varphi_{ij} = \varphi_{(ij)}+\varphi_{[ij]}$. In the matrix form

$$\varphi_{(ij)} = \begin{bmatrix} \varphi_{11}, & \frac{\varphi_{12}+\varphi_{21}}{2}, & \frac{\varphi_{13}+\varphi_{31}}{2} \\ \frac{\varphi_{21}+\varphi_{12}}{2}, & \varphi_{22}, & \frac{\varphi_{23}+\varphi_{32}}{2} \\ \frac{\varphi_{31}+\varphi_{13}}{2}, & \frac{\varphi_{32}+\varphi_{23}}{2}, & \varphi_{33} \end{bmatrix},$$
$$\varphi_{[ij]} = \begin{bmatrix} 0, & \frac{\varphi_{12}-\varphi_{21}}{2}, & \frac{\varphi_{13}-\varphi_{31}}{2} \\ \frac{\varphi_{21}-\varphi_{12}}{2}, & 0, & \frac{\varphi_{23}-\varphi_{32}}{2} \\ \frac{\varphi_{31}-\varphi_{13}}{2}, & \frac{\varphi_{32}-\varphi_{23}}{2}, & 0 \end{bmatrix}. \tag{1.2}$$

The antisymmetric part of tensor $\varphi_{[ij]}$ is a pseudo-tensor and may be represented in terms of the vector φ^k:

$$\varphi_{[ij]} = \varepsilon_{ij}^{\cdot\cdot k}\varphi_k, \tag{1.3}$$

$\varepsilon_{ij}^{\cdot\cdot k}$ denoting the Ricci tensor. In a matrix notation

$$\varphi_{[ij]} = \begin{bmatrix} 0, & \varphi_3, & -\varphi_2 \\ -\varphi_3, & 0, & \varphi_1 \\ \varphi_2, & -\varphi_1, & 0 \end{bmatrix}. \tag{1.4}$$

Multiplying the expression (1.3) by the Ricci tensor and contraction of the indices yields the formula inverse to (1.3),

$$\varphi_k = \frac{1}{2}\varepsilon_k^{\cdot ij}\varphi_{ij}. \tag{1.5}$$

This relation may also be derived directly by comparing the corresponding terms of matrices (1.2) and (1.4).

Passing to the physical interpretation of φ_{ij} let us observe that its antisymmetric part expressed in terms of the vector φ_k represents the independent rotation of the particle treated as a rigid body (Fig. 1). The individual components $\varphi_1, \varphi_2, \varphi_3$ of $\boldsymbol{\varphi}$ describe the rotations about the corresponding axes $\{y^i\}$.

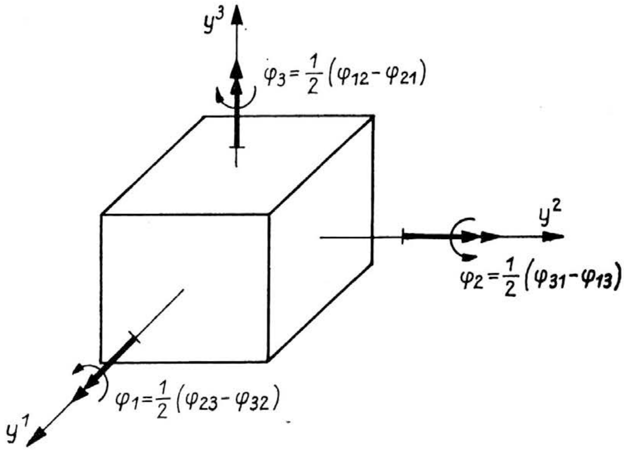

FIG. 1.

The symmetric part $\varphi_{(ij)}$ represents certain deformations of the particle. The components $\varphi_{(11)} = \varphi_{11}, \varphi_{(22)} = \varphi_{22}, \varphi_{(33)} = \varphi_{33}$ lying at the principal diagonal of the matrix $(1.2)_1$ are the elongations measured along the axes y^1, y^2, y^3. For instance, the elongation in the direction of y^2 connected with the component φ_{22} is shown in Fig. 4 of paper [1]. The remaining terms of the matrix $(1.2)_1$: $\varphi_{(12)} = \varphi_{(21)} = \frac{1}{2}(\varphi_{12}+\varphi_{21})$, $\varphi_{(13)} = \varphi_{(31)} = \frac{1}{2}(\varphi_{13}+\varphi_{31})$, $\varphi_{(23)} = \varphi_{(32)} = \frac{1}{2}(\varphi_{23}+\varphi_{32})$ represent the corresponding distortions. The distortion connected with the components $\varphi_{(23)} = \varphi_{(32)}$ is shown in Fig. 2.

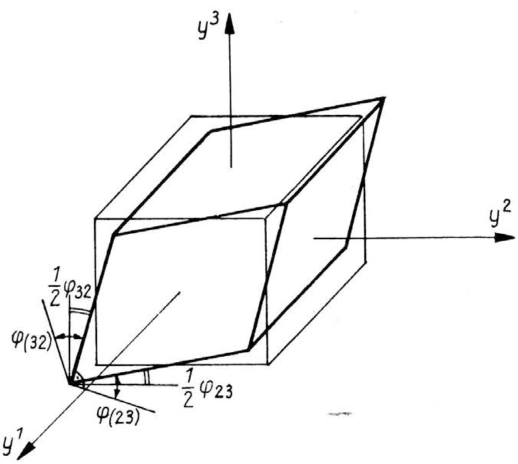

FIG. 2.

Let us return to the decomposition (1.1) of tensor φ_{ij} into the symmetric $\varphi_{(ij)}$ and antisymmetric $\varphi_{[ij]}$ parts. The homogeneous part of the relative displacement $\mathbf{u}'$, Eq. (2.8) in [1], may be represented in the form

$$u'_k = [\varphi_{[lk]} + \varphi_{(lk)}]y^l. \tag{1.6}$$

The antisymmetric component in (1.6) is now replaced with φ_m according to Eq. (1.3). We obtain

$$u'_k = \varepsilon_{lk}{}^{\cdot m}\varphi_m y^l + \varphi_{(lk)}y^l. \tag{1.7}$$

First term of this expression may be considered as a rigid body rotation of the whole particle, and the second one — as the strains shown in Fig. 4a in [1] and in Fig. 2.

Passing to the displacement tensor ψ_{ijk} let us consider the tensor function $\Omega^{ij}(Y^A)$ defined in [1]; decompose it into the symmetric $\Omega^{(ij)}$ and antisymmetric $\Omega^{[ij]}$ parts,

$$\Omega^{ij} = \Omega^{(ij)} + \Omega^{[ij]}. \tag{1.8}$$

Here

$$\Omega^{(ij)} = \frac{\Omega^{ij} + \Omega^{ji}}{2},$$

$$\Omega^{[ij]} = \frac{\Omega^{ij} - \Omega^{ji}}{2}$$

are the respective symmetric and antisymmetric parts of tensor Ω^{ij}. In matrix notation

$$\Omega^{(ij)} = \begin{bmatrix} \Omega^{11}, & \dfrac{\Omega^{12}+\Omega^{21}}{2}, & \dfrac{\Omega^{13}+\Omega^{31}}{2} \\ \dfrac{\Omega^{21}+\Omega^{12}}{2}, & \Omega^{22}, & \dfrac{\Omega^{23}+\Omega^{32}}{2} \\ \dfrac{\Omega^{31}+\Omega^{13}}{2}, & \dfrac{\Omega^{32}+\Omega^{23}}{2}, & \Omega^{33} \end{bmatrix}$$

$$\Omega^{[ij]} = \begin{bmatrix} 0, & \dfrac{\Omega^{12}-\Omega^{21}}{2}, & \dfrac{\Omega^{13}-\Omega^{31}}{2} \\ \dfrac{\Omega^{21}-\Omega^{12}}{2}, & 0, & \dfrac{\Omega^{23}-\Omega^{32}}{2} \\ \dfrac{\Omega^{31}-\Omega^{13}}{2}, & \dfrac{\Omega^{32}-\Omega^{23}}{2}. & 0 \end{bmatrix}. \tag{1.9}$$

The antisymmetric part $\Omega^{[ij]}$ may be replaced with the vector $\bar{\Omega}^k$ in the following manner:

$$\Omega^{[ij]} = \begin{bmatrix} 0, & \Omega^3, & -\Omega^2 \\ -\bar{\Omega}^3, & 0, & \Omega^1 \\ \bar{\Omega}^2, & -\Omega^1, & 0 \end{bmatrix} \tag{1.10}$$

or, in tensorial notation, with

$$\Omega^{[ij]} = \varepsilon^{ij}{}_{\cdot k}\bar{\Omega}^k. \tag{1.11}$$

The inverse formula takes the form

$$\bar{\Omega}^k = \frac{1}{2}\,\varepsilon^{k}_{.ij}\,\Omega^{ij} \tag{1.12}$$

Let us assume the tensor $\mathbf{\Omega}$ to be of the form (2.10) of [1]. In such a case its symmetric and antisymmetric parts shown in formula (1.9) are transformed and take the form

$$\Omega^{(ij)} = \begin{bmatrix} \alpha^1\beta^1 & \dfrac{\alpha^1\beta^2+\alpha^2\beta^1}{2}, & \dfrac{\alpha^1\beta^3+\alpha^3\beta^1}{2} \\ \dfrac{\alpha^2\beta^1+\alpha^1\beta^2}{2}, & \alpha^2\beta^2 \quad , & \dfrac{\alpha^2\beta^3+\alpha^3\beta^2}{2} \\ \dfrac{\alpha^3\beta^1+\alpha^1\beta^3}{2}, & \dfrac{\alpha^3\beta^2+\alpha^2\beta^3}{2} & \alpha^3\beta^3 \end{bmatrix},$$

$$\Omega^{[ij]} = \begin{bmatrix} 0 \quad , & \dfrac{\alpha^1\beta^2-\alpha^2\beta^1}{2}, & \dfrac{\alpha^1\beta^3-\alpha^3\beta^1}{2} \\ \dfrac{\alpha^2\beta^1-\alpha^1\beta^2}{2}, & 0 \quad , & \dfrac{\alpha^2\beta^3-\alpha^3\beta^2}{2} \\ \dfrac{\alpha^2\beta^1-\alpha^1\beta^3}{2}, & \dfrac{\alpha^3\beta^2-\alpha^2\beta^3}{2}, & 0 \end{bmatrix}, \tag{1.13}$$

Vector $\bar{\Omega}^k$ defined by Eq. (1.12) assumes now the form

$$\bar{\Omega}^k = \frac{1}{2}\,\varepsilon^{k}_{.ij}\,\alpha^i\beta^j \tag{1.14}$$

what follows also from a direct comparison of the corresponding terms of matrices (1.10) and $(1.13)_2$.

The non-homogeneous part of the relative displacement $\mathbf{u}''$ given by Eq. (2.9) of paper [1] may, in view of relation (1.8), be represented in the form

$$u''_k(Y^A, X^K, \tau) = \psi_{ijk}(\Omega^{[ij]}+\Omega^{(ij)}). \tag{1.15}$$

The first right-hand term is antisymmetric in the dummy indices i, j, and the second one is symmetric. It follows that

$$u''_k(Y^A, X^K, \tau) = \psi_{[ij]k}\Omega^{[ij]}+\psi_{(ij)k}\Omega^{(ij)}. \tag{1.16}$$

Pseudotensor $\psi_{[ij]k}$ may be expressed by means of a tensor of rank 2,

$$\psi_{[ij]k} = \varepsilon_{ij}^{\cdot l}\,\psi_{lk}. \tag{1.17}$$

The inverse formula holds also true,

$$\psi_{lk} = \frac{1}{2}\,\varepsilon_{l}^{\cdot ij}\,\psi_{ijk}. \tag{1.18}$$

Substitution of the right-hand sides of Eqs. (1.11) and (1.17) into the formula (1.16) yields

$$u''_k(Y^A, X^K, \tau) = \varepsilon_{ij}^{\cdot l}\,\psi_{lk}\,\varepsilon^{ij}_{\cdot\cdot m}\,\bar{\Omega}^m+\psi_{(ij)k}\Omega^{(ij)}. \tag{1.19}$$

Contraction with the Ricci tensors in the above formula leads to the result

$$u_k''(Y^A, X^K, \tau) = \psi_{lk}(2\bar{\Omega}^l) + \psi_{(ij)k}\Omega^{(ij)}. \tag{1.20}$$

Let us introduce the vector $\mathbf{\Omega}$ and define it as the vector product of vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, $\mathbf{\Omega} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$. In indicial notation

$$\Omega^l = \varepsilon^l_{.ij}\alpha^i\beta^j. \tag{1.21}$$

On comparing the formulae (1.12) with (1.21) we conclude that $\Omega^k = 2\bar{\Omega}^k$. Substituting this result into the Eq. (1.20) we obtain

$$u_k''(Y^A, X^K, \tau) = \psi_{lk}(X^K, \tau)\Omega^l(Y^A) + \psi_{(ij)k}(X^K, \tau)\Omega^{(ij)}(Y^A). \tag{1.22}$$

The total displacement defined by Eq. (2.15) of paper [1] assumes, in view of Eqs. (1.7) and (1.22), the form

$$\begin{aligned} w_k(Y^A, X^K, \tau) = u_k(X^K, \tau) + \varepsilon_{lk}^{\cdot m}\varphi_m(X^K, \tau)y^l(Y^A) + \varphi_{(lk)}(X^K, \tau)y^l(Y^A) \\ + \psi_{lk}(X^K, \tau)\Omega^l(Y^A) + \psi_{(ij)k}(X^K, \tau)\Omega^{(ij)}(Y^A). \end{aligned} \tag{1.23}$$

Let us now pass to the simplified model of a structural medium. In such a model it will be assumed that the symmetric part $\varphi_{(ij)}$ of φ_{ij} and the symmetric (in indices i, j) part of tensor $\psi_{(ij)k}$ vanish, $\varphi_{(ij)} = \psi_{(ij)k} = 0$. Total displacement of the micro-element $\mathbf{Y}$ of particle $\mathbf{X}$ expressed by Eq. (1.23) is simplified to the form

$$w_k(Y^A, X^K, \tau) = u_k(X^K, \tau) + \varepsilon_{lk}^{\cdot m}\varphi_m(X^K, \tau)y^l(Y^A) + \psi_{lk}(X^K, \tau)\Omega^l(Y^A). \tag{1.24}$$

Vector u_k describes the rigid body displacement of the whole particle $\mathbf{X}$; the rotation vector φ_m corresponds to the particle rotation about the axes passing through its geometric center, and tensor ψ_{lk} represents its nonhomogeneous displacement. The first two right-hand terms of the expression (1.24) describe the behaviour of the particle treated as a rigid body; its strains are connected with tensor ψ_{lk} only. To illustrate the character of these strains let us assume the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ defining the vector function Ω^l of (1.21) to be linear in the coordinates $\{y^i\}$ and to satisfy the relations (2.12) of paper [1].

Substitution of relations (2.12) [1] into the Eq. (1.21) makes it possible to calculate the components of Ω^l. We obtain

$$\Omega^l = \begin{bmatrix} c_1 y^2 y^3 \\ c_2 y^3 y^1 \\ c_3 y^1 y^2 \end{bmatrix}, \tag{1.25}$$

where

$$c_i = \varepsilon_i^{\cdot jk} a_j b_k. \tag{1.26}$$

The non-homogeneous relative displacement $\mathbf{u}''$ is described by the third term of expression (1.24),

$$u_k''(Y^A, X^K, \tau) = \psi_{lk}(X^K, \tau)\Omega^l(Y^A). \tag{1.27}$$

The physical character of displacements u_2'' connected with the components ψ_{12}, ψ_{22}, ψ_{32} of tensor $\boldsymbol{\psi}$, under the assumption that the vector function Ω^l has the form (1.25), is illustrated by Fig. 3. Displacements u_1'' and u_2'' connected with the remaining components of $\boldsymbol{\psi}$ are of a similar character.

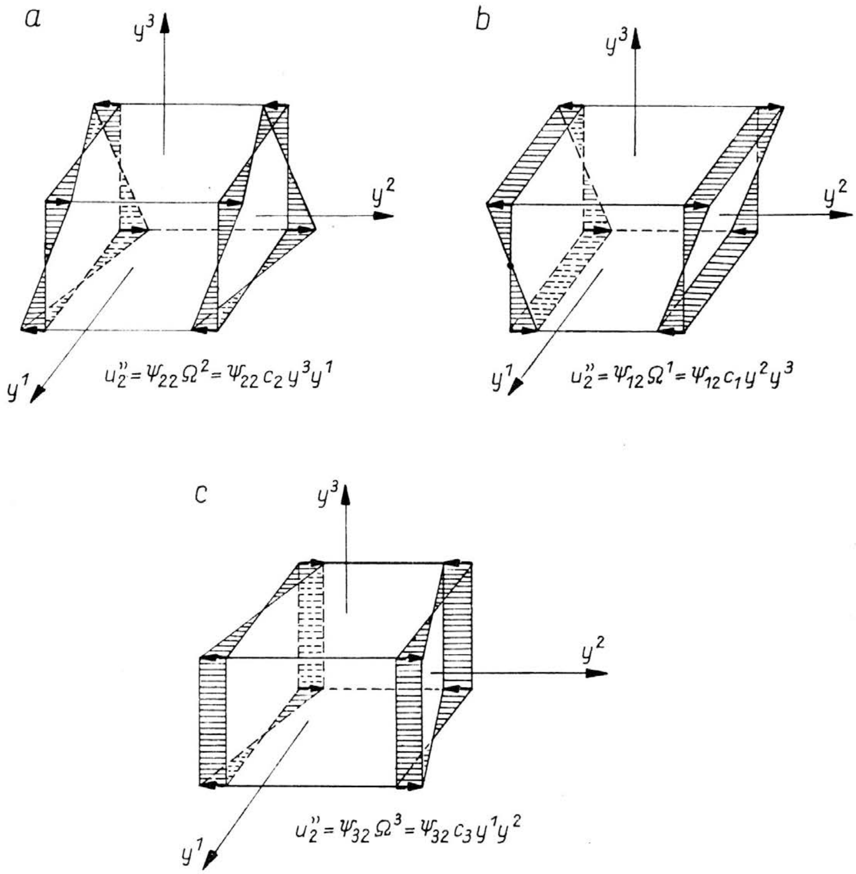

FIG. 3.

The model of a particle of the simplified structural medium remains the same as in the general case discussed in [1]. The simplification consists in the assumption that the planes passing through the geometric center are not allowed to rotate with respect to each other. However, they may be subject to independent warping. Warping of one of the planes is shown in Fig. 8c of paper [1].

In addition to warping, each plane is subject to independent tangential displacements (and deformations) as shown in Fig. 8b of paper [1].

Motion of the simplified structural medium is described by the tensor functions

$$\begin{aligned} u_k &= u_k(X^K, \tau), \\ \varphi_k &= \varphi_k(X^K, \tau), \qquad (1.28) \\ \psi_{lk} &= \psi_{lk}(X^K, \tau). \end{aligned}$$

Determination of all the 15 components of these functions determines the position of each particle of the medium considered. From the dynamical point of view, each particle represents a system with 15 degrees of freedom.

2. Equations of the simplified structural medium

The individual groups of the simplified medium result from the corresponding equations of the general model presented in [1], in which suitable simplifications discussed in the preceding section must be introduced.

The geometric relations (2.20) of paper [1] are reduced to the form

$$
\begin{aligned}
\gamma_{ij} &= u_{j,i} - \varepsilon_{ij}^{\cdot\cdot k}\varphi_k,\\
\varkappa_{ij} &= \varphi_{j,i} - \psi_{ij},\\
\eta_{kij} &= \psi_{ij,k}.
\end{aligned}
\tag{2.1}
$$

Tensor $\varkappa_{ij}$ is derived from Eqs. $(2.20)_2$ [1] as the antisymmetric (in indices k, l) part of tensor $\varkappa_{ikl}$, use being made of the definition $\varkappa_{i[kl]} = \varepsilon_{kl}^{\cdot\cdot j}\varkappa_{ij}$ and the relations (1.3) and (1.18). By introducing the definition $\eta_{ki[mn]} = \varepsilon_{mn}^{\cdot\cdot j}\eta_{kij}$ and the relation (1.18) into the equation $(2.20)_3$ of paper [1], relation $(2.1)_3$ is obtained.

In order to derive the equations of motion, the boundary conditions and the constitutive relations, the generalized Hamilton principle formulated in [1] (Eq. (2.25)) must be used. The procedure is analogous to that applied in the general case. It was extensively discussed in the previous paper [1] and there is no need to return to the problem here.

Equations of motion of the simplified medium have the form

$$
\begin{aligned}
p^{ij}_{\cdot,i} + f^j &= \varrho\ddot{u}^j + \varepsilon^{\cdot j}_{\cdot\,ik}\varrho^i\ddot{\varphi}^k + \mu^i\ddot{\psi}_i^{\cdot j},\\
m^{ij}_{\cdot,i} + \varepsilon^{\cdot j}_{\cdot\,ik}p^{ik} + h^j &= \varepsilon^{\cdot j}_{\cdot\,ik}[\varrho^{i}\ddot{u}^k + \varepsilon^{klm}\varrho^{\cdot i}_{\cdot\,l}\ddot{\varphi}_m + \nu^{il}\ddot{\psi}_l^{\cdot k}],\\
b^{ijk}_{\cdot,i} + m^{jk} + g^{jk} &= \mu^j\ddot{u}^k + \varepsilon^{kil}\nu_i^{\cdot j}\ddot{\varphi}_l + \mu^{ji}\ddot{\psi}_i^{\cdot k}.
\end{aligned}
\tag{2.2}
$$

A number of new notations have been introduced here. The first group,

$$
p^{ij} = \frac{\partial U}{\partial \gamma_{ij}},\qquad m^{ij} = \frac{\partial U}{\partial \varkappa_{ij}},\qquad b^{ijk} = \frac{\partial U}{\partial \eta_{ijk}}
\tag{2.3}
$$

defines the force stress tensors, couple stress tensors and higher order stress tensors, respectively. U is, as before, the internal energy density, and it is assumed to be a function of the strain tensors introduced in Eqs. (2.1), $U = U(\gamma_{ij}, \varkappa_{ij}, \eta_{ijk})$, representing the counterpart of Eqs. (2.25) in paper [1].

The second group contains, in addition to the definitions (2.32) of ϱ, ϱ^i, ϱ^{ij} ([1]), the following notations:

$$
\nu^{ij} = \frac{1}{\omega}\int_\omega y^i\Omega^j\bar{\varrho}\,d\omega,\qquad \mu^i = \frac{1}{\omega}\int_\omega \Omega^i\bar{\varrho}\,d\omega,\qquad \mu^{ij} = \frac{1}{\omega}\int_\omega \Omega^i\Omega^j\bar{\varrho}\,d\omega.
\tag{2.4}
$$

which may be termed the generalized densities of the medium.

The following symbols

$$
\begin{aligned}
f^j &= \frac{1}{\varrho\omega}\int_\omega \bar{f}^j\bar{\varrho}\,d\omega,\\
h^j &= \varepsilon^{\cdot j}_{\cdot\,ik}\frac{1}{\varrho\omega}\int_\omega y^i\bar{f}^k\bar{\varrho}\,d\omega,
\end{aligned}
\tag{2.5}
$$

$$g^{jk} = \frac{1}{\varrho\omega} \int_{\omega} \Omega^j \bar{f}^k \bar{\varrho}\, d\omega$$

denote the densities of the generalized body forces.

The boundary conditions are reduced to the form

$$\begin{aligned} p^{ij} n_i &= p^j_{(n)}, \\ m^{ij} n_i &= m^j_{(n)}, \\ b^{ijk} n_i &= b^{jk}_{(n)} \end{aligned} \tag{2.6}$$

where n_i are the components of the outer normal $\mathbf{n}$ to the boundary bounded by the surface s. The following notations are used in Eq. (2.6):

$$\begin{aligned} p^j_{(n)} &= \int_{\sigma} \bar{p}^j_{(n)}\, d\sigma, \\ m^j_{(n)} &= \varepsilon^j_{.ik} \int_{\sigma} y^i \bar{p}^k_{(n)}\, d\sigma, \\ b^{jk}_{(n)} &= \int_{\sigma} \Omega^j \bar{p}^k_{(n)}\, d\sigma. \end{aligned} \tag{2.7}$$

The remaining notations have been explained in paper [1]. Equations (2.6) represent the static boundary conditions expressed in terms of the force, couple and higher order stress tensors.

The internal energy density $U = U(\gamma_{ij}, \varkappa_{ij}, \eta_{ijk})$ is assumed to be a function of strain tensors. It may be expanded, as in the general case, into Taylor's series in the neighbourhood of the undeformed (natural) state ($\gamma^0_{ij} = \varkappa^0_{ij} = \eta^0_{ijk} = 0$). Making use of the definitions (2.3) we obtain the following constitutive relations:

$$\begin{aligned} p^{ij} &= A^{ijkl}\gamma_{kl} + D^{ijkl}\varkappa_{kl} + E^{ijklm}\eta_{klm}, \\ m^{ij} &= D^{ijkl}\gamma_{kl} + C^{ijkl}\varkappa_{kl} + F^{ijklm}\eta_{klm}, \\ b^{ijk} &= E^{ijklm}\gamma_{klm} + F^{ijklm}\varkappa_{lm} + B^{ijklm}\eta_{lmn}, \end{aligned} \tag{2.8}$$

with the notations

$$\begin{aligned} A^{ijkl} &= \left(\frac{\partial^2 U}{\partial\gamma_{ij}\partial\gamma_{kl}}\right)_0, & D^{ijkl} &= \left(\frac{\partial^2 U}{\partial\gamma_{ij}\partial\varkappa_{kl}}\right)_0, \\ E^{ijklm} &= \left(\frac{\partial^2 U}{\partial\gamma_{ij}\partial\eta_{klm}}\right)_0, & C^{ijkl} &= \left(\frac{\partial^2 U}{\partial\varkappa_{ij}\partial\varkappa_{kl}}\right)_0, \\ F^{ijklm} &= \left(\frac{\partial^2 U}{\partial\varkappa_{ij}\partial\eta_{klm}}\right)_0, & B^{ijklm} &= \left(\frac{\partial^2 U}{\partial\eta_{ijk}\partial\eta_{lmn}}\right)_0. \end{aligned} \tag{2.9}$$

The constitutive relations express the stress tensors in terms of the strain tensors. The magnitudes **A**, **B**, **C**, **D**, **E**, **F** defined in Eq. (2.9) are the elasticity tensors of the medium.

In a medium in which each particle has a center of symmetry, tensors **D**, **E**, **F** vanish, and the constitutive relations assume the form

$$\begin{aligned} p^{ij} &= A^{ijkl}\gamma_{kl}, \\ m^{ij} &= C^{ijkl}\varkappa_{kl}, \\ b^{ijk} &= B^{ijklmn}\eta_{lmn}. \end{aligned} \tag{2.10}$$

In the equations of motion (2.2) it should be assumed in such a case that $\varrho^i = \mu^i = \nu^{ij} = 0$.

The simplified structural model is described by means of 105 equations: 15 equations of motion (2.2), 45 geometric relations (2.1) and 45 physical relations (2.8) or (2.10). These equations, together with the boundary conditions (2.6), allow for a unique determination of 105 unknown statical and geometrical parameters of the medium considered.

45 statical unknowns consist of 9 force stress components p^{ij}, 9 couple stress components m^{ij} and 27 higher order stress components b^{ijk}. 60 geometrical unknowns are represented by 45 strain tensor components and 15 displacement components. The strain tensor components may be divided into 9 components of γ_{ij}, 9 components $\varkappa_{ij}$ and 27 components of the tensor η_{ijk}. The displacement tensor components consist of three vector components u_i, 3 vector components φ_i, and 9 tensor components ψ_{ij}.

As in the general case discussed in [1], also here the equations of motions may be expressed in terms of displacements. Stresses p^{ij}, m^{ij} and b^{ijk} are eliminated from the equations of motion (2.2) by means of the physical relations (2.8) or (2.10), and then the geometric relations (2.1) are used to eliminate the strain tensors. The resulting 15 equations represent the set of equations of motion containing 15 unknown components of the generalized displacement tensors u_i, φ_i, ψ_{ij}. This problem has been discussed in detail in paper [1].

The models of continuous media presented in this paper are rich in possible applications but rather complicated, since they involve a high number of unknown geometrical and statical parameters. In many practical applications such complicated models are not necessary; that is why a much simpler model of a medium called a bimoment medium is used in some cases. In such a medium, besides the force and couple stresses, also the socalled bimoment stresses are introduced. The geometrical magnitudes are represented by linear displacement, unconstrained rotations, and, additionally, by independent warping functions. The bimoment medium and its possible applications will be dealt with in a separate paper.

References

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