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Fundamental equations of continuous structural media
I. General model

S. ZIELIŃSKI (ŁÓDŹ)

THE PAPER is aimed at deriving the complete set of equations of a continuous structural medium in which, in addition to the force stresses p^{ij}, generalized couple stresses m^{ijk} and higher order stresses b^{ijkl} appear. Displacements of such a medium is characterized by the linear displacement vector u^i and by two additional tensors φ^{ij} and ψ^{ijk}. The fundamental concept of the paper is the model of structural particle. In the classical continuous medium the particle is identified with a material point. In the couple stress theory the particle is a rigid body subject to displacements and independent rotations. The structural medium particle is a system consisting of its geometric center and a set of planes passing through that center, the planes representing the corresponding cross-sections of the particle considered. It is convenient to assume the planes to be perpendicular to the directions of the mutual interactions of the neighbouring particles. It is assumed that, in general, the planes may be displaced together with the geometric center and, in addition, are subject to mutually independent displacements and deformations; however, they are assumed to rotate about the lines of their intersections, but they are not allowed to displace along these lines. The structural model particle has 39 degrees of freedom. In order to derive the equations governing the medium, Hamilton's principle is suitably generalized. The resulting 273 equations and the corresponding boundary conditions allow for unique determination of all 273 geometric and static unknowns of the model. In the case of a homogeneous, centrosymmetric structural model, the corresponding set of equations of motion expressed in terms of displacements is derived.

Celem pracy jest otrzymanie kompletu równań ciągłego ośrodka strukturalnego, w którym oprócz napięć siłowych p^{ij} występują uogólnione napięcia momentowe m^{ijk} oraz napięcia wyższego rzędu b^{ijkl}. Przemieszczenia tego ośrodka określane są przez wektor przemieszczeń liniowych u^i oraz dwa dodatkowe tensory φ^{ij} i ψ^{ijk}. Podstawową koncepcją pracy jest zbudowanie modelu cząstki strukturalnej. W klasycznym ośrodku ciągłym cząstkę ośrodka utożsamia się z punktem materialnym. W ośrodku momentowym cząstką ośrodka jest bryła sztywna doznająca przemieszczeń oraz niezależnych obrotów. Modelem cząstki strukturalnej jest układ zbudowany z jej ośrodka geometrycznego oraz z płaszczyzn przechodzących przez ten środek przy czym płaszczyzny reprezentują odpowiednie przekroje poprzeczne rozważanej cząstki. Najwygodniej jest przyjąć, że płaszczyzny te są prostopadłe do kierunków oddziaływań cząstki z sąsiadami. Zakładamy, że w ogólnym przypadku rozważane płaszczyzny oprócz przemieszczeń wraz ze środkiem geometrycznym doznają przemieszczeń (i odkształceń) niezależnie jedna od drugiej. Dalej ograniczamy się do założenia, że płaszczyzny wzdłuż krawędzi przecięcia nie mogą się wzajemnie przemieszczać, a jedynie obracać względem siebie. Przedstawiona za pomocą powyższego modelu cząstka strukturalna jest układem o 39 stopniach swobody. Do otrzymania równań ośrodka ciągłego zbudowanego z cząstek strukturalnych, wykorzystana została odpowiednio uogólniona zasada Hamiltona. Dla omawianego modelu otrzymano 273 równania, które wraz z odpowiednimi warunkami brzegowymi pozwalają wyznaczyć 273 niewiadome statyczne i geometryczne modelu. W przypadku jednorodnego centrosymetrycznego ośrodka strukturalnego otrzymano odpowiedni układ równań przemieszczeniowych.

Целью работы является получение системы уравнений сплошной структурной среды, в которой, кроме силовых напряжений p^{ij}, выступают обобщенные моментные напряжения m^{ijk} и напряжения высшего порядка b^{ijkl}. Перемещения этой среды определяются вектором линейных перемещений u^i и двумя дополнительными тензорами φ^{ij} и ψ^{ijk}. Основной концепцией работы является построение модели структурной частицы. В классической сплошной среде частица среды отождествляется с материальной точкой. В моментной среде частица среды является жестким телом испытывающим перемещения и независимые вращения. Моделью структурной частицы является система построенная из ее геометрического центра и из плоскостей, проходящих через этот центр, причем плоскости представляют соответствующие поперечные сечения рассматриваемой частицы. Наиболее выгодно принять, что эти плоскости перпендикулярны направлениям взаимодействий частицы с соседними частицами. Предполагаем, что в общем случае рассматриваемые плоскости, кроме перемещений совместно с геометрическим центром, испытывают перемещения (и деформации) независимо одна от другой. Далее ограничимся предположением, что плоскости вдоль грани сечения не могут взаимно перемещаться, а только вращаться относительно себя.

Представленная при помощи вышеупомянутой модели структурная частица является системой с 39 степенями свободы. Для получения уравнений сплошной среды, построенной из структурных частиц, использован, соответственно обобщенный, принцип Гамильтона. Для обсуждаемой модели получены 273 уравнения, которые совместно с соответствующими граничными условиями позволяют определить 273 статические и геометрические неизвестные модели. В случае однородной центросимметрической структурной среды получена соответствующая система уравнений в перемещениях.

1. Introduction

Mechanics of continua is usually constructed on the basis of three groups of axioms:

1) dynamical axioms describing the forces acting in the medium,

2) kinematical axioms describing the motion of the medium,

3) constitutive equations describing the properties of the material, independent of the forces and its motion.

The existing theory of continua are based on a number of assumptions and make use of numerous notions defined, for instance, in the monographs by C. Truesdell and R. A. Toupin [1] or C. Truesdell and W. Noll [2]. Some of them will be discussed in this paper (for the sake of clarity), the others will be considered as known.

Following C. Truesdell and R. A. Toupin [1], the continuous medium is defined as a three-dimensional differentiable manifold $\mathscr{B}$ of particles **X**. It is assumed that there exists a one-to-one mapping of the set $\mathscr{B}$ onto a fixed domain V of the three-dimensional reference space E_3. If the domain V is parametrized, a radius-vector with components X^K $(K = 1, 2, 3)$ may be ascribed to each particle $X \in \mathscr{B}$. Numbers X^K are Cartesian coordinates of a point in V; they are called material coordinates of a particle, or Lagrangean coordinates.

A point in a three-dimensional Euclidean space E_3 is denoted by symbol **x**. Position of **x** is determined by curvilinear coordinates x^k $(k = 1, 2, 3)$; they are called the spatial or Eulerian coordinates. The coordinate systems introduced here are discussed in detail in the book by C. Truesdell and R. A. Toupin [1].

Time T is a one-dimensional space each point of which is called an instant. Introducing a coordinate system in T (in our case it is sufficient to use an orthonormal system, orthogonal to x^k), we denote by τ the coordinate of the instant. The corresponding measure unit must be settled and a certain instant must be assumed as original, denoted by the coordinate $\tau = 0$.

The primitive notion of dynamics of continua is the force. It is assumed that at a fixed time instant τ a portion π of body $\mathscr{B}$ bounded by a smooth surface $\partial\pi$ is acted on by a vector field called stress, defined at the surface $\partial\pi$ and referred to a unit of that surface. In the classical continuous medium the state of stress is described by means of a symmetric stress tensor (see, e.g., A. C. Eringen [3]).

The primitive notion of kinematics is the particle. A point in the three-dimensional Euclidean space is ascribed to each particle at each time instant τ. In the classical continuous medium each particle possesses three degrees of freedom, and a change in its position in space (motion) is uniquely described by the displacement vector field. This field is a function of the original position of the particle and time; it is assumed to be continuous and twice differentiable.

The third group of axioms has the form of constitutive equations and must comply with the dynamical and kinematical axioms assumed.

The attempted generalization of the foundation of modern mechanics of continua have resulted in the construction of a theory of media with couple stresses. The generalizations affected all the three groups of axioms. As far as the dynamics is concerned, external body couples and couple stresses have been introduced in addition to the classical notions of external body forces and force stresses. Generalization of kinematics consisted in introducing the assumption that each particle has more than three degrees of freedom. The corresponding generalizations were also introduced into the constitutive equations

The generalizations mentioned above lead to the theory of continuous media with stresses described by a non-symmetric force stress tensor and a couple stress tensor. Depending on the approach assumed, different theories were obtained: oriented media, media with microstructure, moment or multipolar media. All these theories will be called couple stress theories. This paper does not contain the detailed review of the existing literature dealing with the couple stress theory, except for the most fundamental papers on the subject.

General foundations of the theory of media with couple stresses are due to E. and F. COSSERAT [4]. According to their kinematical assumptions, each particle has six degrees of freedom. The particle is treated as a rigid body which, in the process of deformation, is subject not only to displacements but also to rotations. Each particle is then characterized by a position in the space, like in the classical theory, and by the additional orientation of a set of three orthonormal vectors.

Under such kinematic assumptions, the state of stress in the medium is described by means of a non-symmetric force stress tensor and a couple stress tensor. Deformation of the medium is described by the displacement vector field and by an independent field of rotation vectors.

The original concept of E. F. Cosserat was further developed and presented in a modern form by J. L. ERICKSEN and C. TRUESDELL [5]; the theory was also discussed in the monograph by C. TRUSDELL and R. A. TOUPIN [1]. Since then a rapid development of the theory may be observed. However, neither the Cosserat brothers nor Ericksen and Truesdell dealt with the corresponding constitutive equations.

Foundations of the modern theory of media with couple stresses were laid by R. A. TOUPIN [6, 7]. In addition to the position in space, a certain internal structure characterized by a set of Q directional vectors (directors) was ascribed to each material particle of the medium. In the case when $Q = 3$ and the vectors form an orthonormal basis, the equations derived in the papers reduce to those given by E. and F. Cosserat.

Another concept is due to R. P. MINDLIN [8], who attributed a certain "micromedium" to each material particle. Homogeneous deformation of the "micromedium" reduces the model to the oriented medium with deformable directors introduced by J. L. ERICKSEN and C. TRUESDELL [5]. It follows from the fact that a homogeneous deformation is uniquely determined by the motion of three linearly independent direction vectors.

By assuming in the body, in addition to the displacement vector, the so-called multipolar displacement fields, we obtain also the theory of bodies with additional degrees of freedom. Such an approach was proposed in the papers by A. E. GREEN and R. S. RIVLIN [9, 10]. A similar approach was also used by A. C. ERINGEN and E. S. SUHUBI [11, 12] and by some other authors.

In the monograph [13] C. WOŹNIAK applied the action functional to systematize the various models of media with couple stresses; his considerations were concerned with dynamic equations of motion of such media.

S. KALISKI [14] applied the theory of media with couple stresses to the analysis of spatial rod systems. He proved that the transition from a discrete rod system to a continuous medium yields the equations of the couple stress theory. He was the first to present the constitutive equations in an explicit form. The same concept was applied in paper [15] by A. ASKAR and A. S. CAKMAK.

A number of interesting applications of the couple stress theory to the analysis of dense grids consisting of solid bars are presented in the book [16] by C. WOŹNIAK.

The first papers dealing with thermoelasticity of the media with couple stresses were written by W. NOWACKI [17] and C. WOŹNIAK [18]. The fundamental results of their research were published in the monographs [19] and [16]. Another trend in the development of the couple stress theory, connected with the problem of stress concentrations, is represented by M. SOKOŁOWSKI [20]. An extensive review of the corresponding literature may be found in the review papers by W. BARAŃSKI, K. WILMAŃSKI and C. WOŹNIAK [21], H. SCHAEFER [22] and M. SOKOŁOWSKI [20], and also in both monographs [19] and [16] written by W. NOWACKI and C. WOŹNIAK.

The models of the media with couple stresses mentioned above correspond to a body with a very simple internal structure and cannot be used to describe the behaviour of more complicated structures such like, for instance, thin-walled structures (Fig. 1). The basic element of such structures is a beam (rod) with thin-walled cross-section. Thin-walled (plane and spatial) gridworks are characterized by considerable load carrying capacity at relatively low dead load; they are frequently applied in the civil and industrial engineering practice and in the aircraft and ship-building industry. Such structures are used as bearing elements in rockets, airplanes and ships, industrial and bridge floors.

Elements of such kind are subject not only to the usual shearing and normal forces, bending and twisting moments, but also to bimoments [23]. Structures of this type cannot be described by the simple model of bodies with couple stresses.

The aim of the present paper is to present a more general model of a continuous medium, which will be called the structural medium, and to derive the fundamental equations of such a medium. We are going to deal with a general, linear theory of this medium. The generalized displacements and deformations willl be assumed to be small, and the material will be linear elastic, homogeneous and isotropic.

The indices appearing in the paper, lower or upper case Latin letters $a, b, c, \ldots, k, l, m$ and $A, B, C, \ldots, K, L, M$, assume the values 1, 2, 3. Covariant differentiation with respect

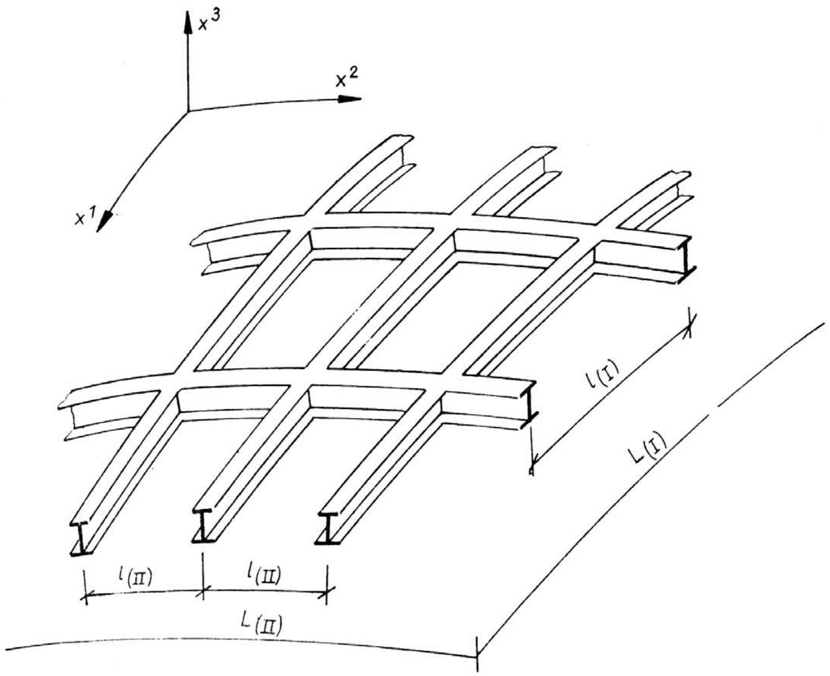

FIG. 1.

to space variables is denoted by a comma, $(\ )_{,i} = \dfrac{\partial(\ )}{\partial x^i}$, a dot denotes the time derivative. Symmetric and skew-symmetric components with respect to a pair of indices are denoted by the parantheses and brackets, respectively,

$$\varphi^{(kl)} \equiv \frac{1}{2}(\varphi^{kl}+\varphi^{lk}),$$

$$\varphi^{[kl]} \equiv \frac{1}{2}(\varphi^{kl}-\varphi^{lk}).$$

The remaining notations will be explained in the text.

2. Structural medium

In defining the structural medium use will be made of the notion of continuous media introduced by C. TRUESDELL and R. A. TOUPIN [1] quoted in the Introduction.

A structural medium is the continuous medium with particles $\mathbf{X}$ to which a certain special internal structure is ascribed, independent of the manifold $\mathscr{B}$. Each particle $\mathbf{X} \in \mathscr{B}$ is assumed to contain a set of microelements $\mathbf{Y}$ forming a differentiable manifold $\mathscr{D}$. Let us map the set $\mathscr{D}$ onto a fixed domain Ω of a three-dimensional Euclidean space. In such a way, each microelement $\mathbf{Y}$ of $\mathscr{D}$ corresponds to three numbers Y^A; they are the Cartesian coordinates from domain Ω and they will be called material micro-coordinates. The structural medium is then the Cartesian product $\mathscr{B} \times \mathscr{D}$, or an ordered set of pairs $(\mathbf{Y}, \mathbf{X})$ with material coordinates (Y^A, X^K).

2.1. Kinematics of a structural medium

Let us assume the material macro-coordinates X^K of particle **X** to coincide with the spatial coordinates of the reference configuration at the initial time instant $\tau = \tau_1$

$$X^K = \delta^K_k X^k(\mathbf{X}, \tau = \tau_1). \tag{2.1}$$

In order to present the following relations in the simplest possible form, let us assume that the material micro-coordinates Y^A satisfy also the relation

$$Y^A = \delta^A_k Y^k(\mathbf{Y}, \mathbf{X} = \mathbf{X}_0, \tau = \tau_1). \tag{2.2}$$

Components of the displacement vector of the material particle **X** are the differences of the final and initial position of the particle (Fig. 2),

$$u^k(X^l, \tau) = x^k(X^l, \tau) - \delta^k_K X^K, \quad k, K = 1, 2, 3. \tag{2.3}$$

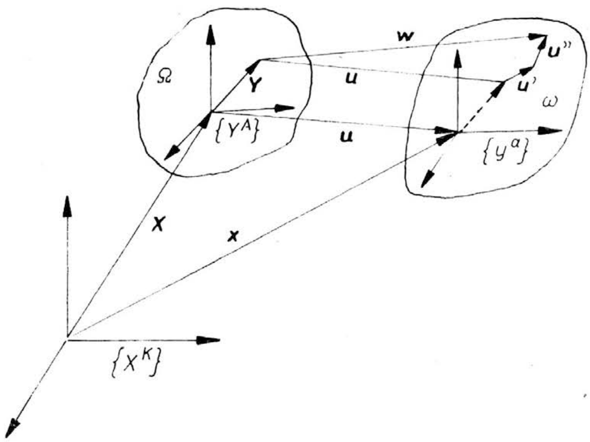

Fig. 2.

For all micro-elements $(\mathbf{Y}, \mathbf{X})$ of particle **X** the displacement u^k is constant and independent of the position of the micro-element **Y**, and hence independent of Y^A. It represents the rigid body displacement of the particle. Moreover, the origin of the micro-coordinate system y^a is attached to the particle **X** with spatial coordinates x^k and is subject to displacement **u** defined by Eq. (2.3). The micro-displacement vector **u** of an arbitrary element $(\mathbf{Y}, \mathbf{X})$ is given by the difference

$$\bar{u}^k(Y^l, X^l, \tau) = y^k(Y^l, X^l, \tau) - Y^k. \tag{2.4}$$

It describes the relative displacement of particle **Y** measured with respect to the micro-coordinate $\{y^k\}$ origin attached to particle **X** with spatial coordinates x^k.

The structural medium considered in this paper consists of such particles which interact with the neighbouring particles in certain definite directions; it is demonstrated in Fig. 3. The interaction takes place in the directions I, II, III only. In this sense the particles are called oriented particles. In addition, the interaction may take place not across the entire cross-section of the particle, but through the shaded areas shown in the figure. The internal structure of the particle is another fundamental problem.

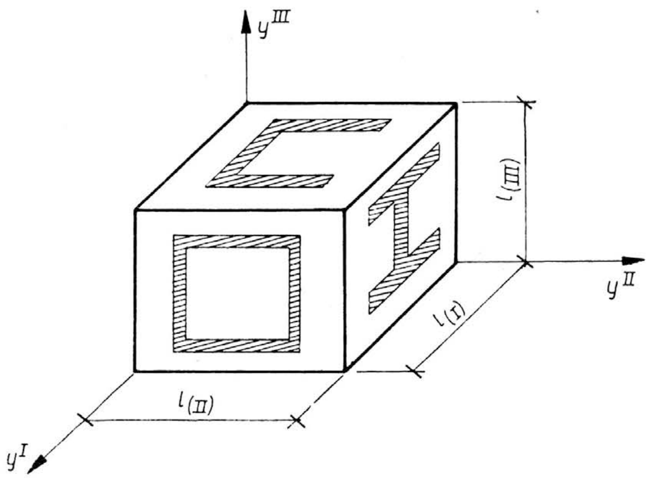

FIG. 3.

If particle **X** is treated as a set of micro-elements **Y** and no additional assumptions are made, determination of the relative displacements $\bar{\mathbf{u}}$ of each particle requires the solution of a corresponding three-dimensional boundary value problem of elasticity. This task may be, even in the case of a small number of particles, extremely difficult. Each particle of such a medium represents (dynamically) a system with an infinite number of degrees of freedom.

The structural medium is defined so that each particle constituting a set of micro-elements **Y** should represent a system with a corresponding, finite number of degrees of freedom.

To this end let us make an additional kinematic assumption. Namely, the relative displacement of a structural particle consists of two mutually independent components

$$\bar{u}^k = u'^k + u''^k. \tag{2.5}$$

Assume, as it is done in the usual couple stress theories, that the first component u'^k describes the homogeneous displacement of the particle considered, occupying at the actual instant the domain ω. It follows that the gradient

$$\frac{\partial u'^k}{\partial y^l} = \varphi^{lk} \tag{2.6}$$

is constant for the particle considered, that is inside the domain ω. Displacement u'^k may then be represented by the relation

$$u'^k(Y^l, X^l, \tau) = \frac{\partial u'^k}{\partial y^l} y_l \tag{2.7}$$

and, making use of Eq. (2.6), we obtain

$$u'^k(Y^l, X^l, \tau) = \varphi^{lk}(X^L, \tau) y_l(Y). \tag{2.8}$$

Displacement vector u'^k is then a linear function of microcoordinates y_l, and tensor φ^{lk} — a function of macrocoordinates X^K and time τ, but independent of the microcoordinates Y^A.

In order to illustrate the relations derived, a typical displacement connected with the various tensor components φ^{lk} is shown in Fig. 4. Their variation inside the particle is linear, what follows from Eq. (2.8).

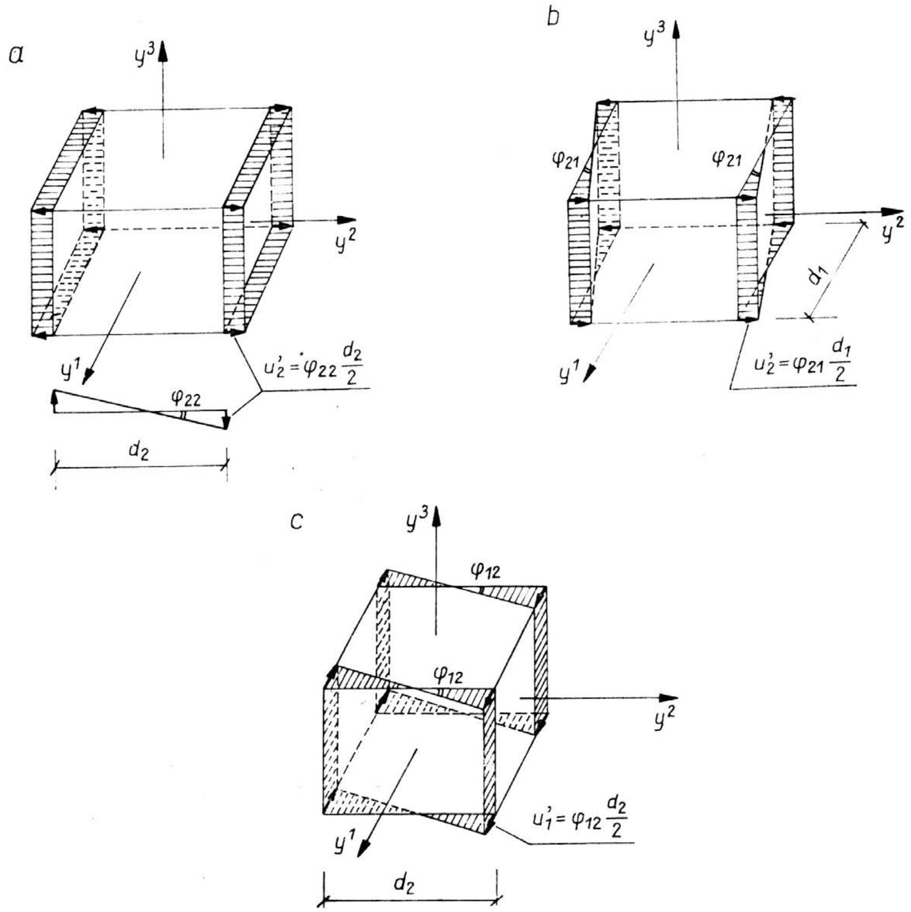

FIG. 4.

The second independent component u''^k of the relative displacement (micro-displacement) (2.5) describes the nonhomogeneous displacement of the microstructure $(\mathbf{Y}, \mathbf{X})$. We assume that it may be represented by the following tensor formula:

$$u_k''(Y^A, X^K, \tau) = \psi_{lmk}(X^K, \tau)\Omega^{lm}(Y^A). \tag{2.9}$$

The third rank tensor ψ_{lmk} is independent of the micro-coordinates Y^A, while Ω^{lm} is a known, prescribed function of Y^A. It should be stressed that the form of tensor function Ω^{lm} depends on the internal structure of particle **X**. On the other hand, knowledge of the function Ω^{lm} makes it possible to describe the displacement of particles of a definite internal structure.

In order to illustrate the assumption made let us consider the elementary example.

Without going into the internal structure of the particle, assume the tensor Ω^{lm} in the simplest form of a tensor product of two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$,

$$\Omega^{lm} = \alpha^l \beta^m. \tag{2.10}$$

Such a tensor is called a dyad. In the matrix notation it assumes the form

$$[\Omega^{lm}] = \begin{bmatrix} \alpha^1\beta^1, & \alpha^1\beta^1, & \alpha^1\beta^3 \\ \alpha^2\beta^1, & \alpha^2\beta^2, & \alpha^2\beta^3 \\ \alpha^2\beta^1, & \alpha^2\beta^2, & \alpha^2\beta^3 \end{bmatrix}. \tag{2.11}$$

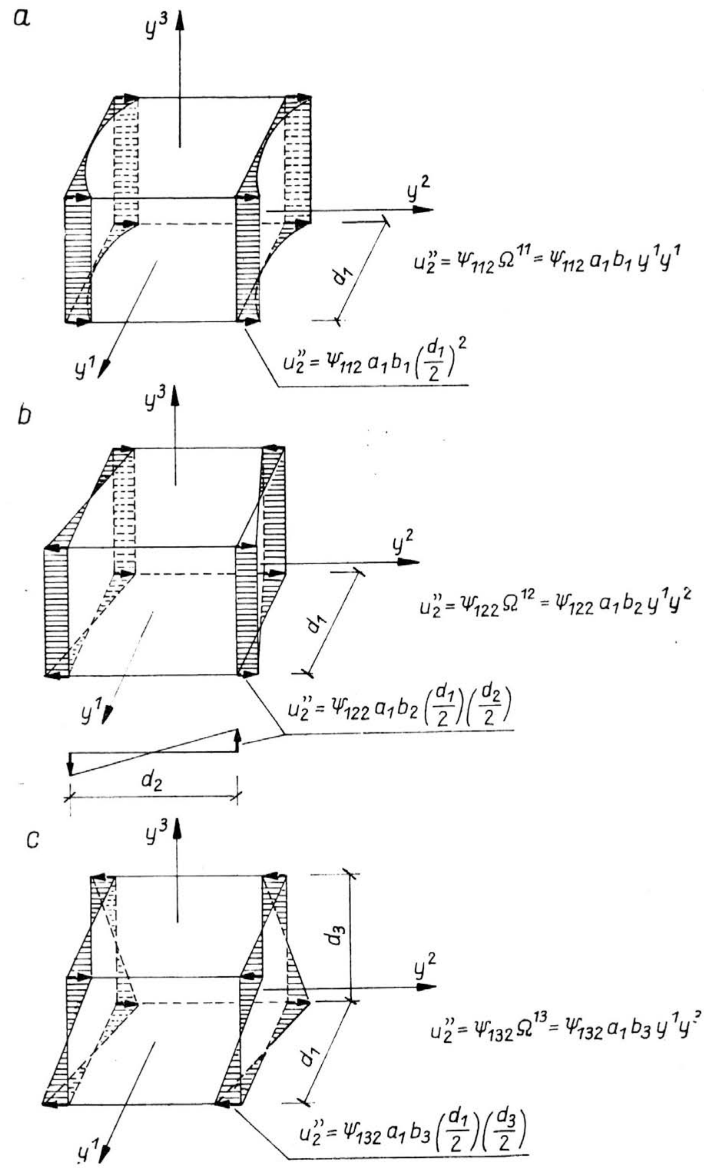

FIG. 5.

The rows of the matrix are seen to be proportional to each other and, hence, it is a tensor of the simplest type.

Assume the corresponding vector components of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to be linear functions of the corresponding micro-coordinates y^i. We obtain the relations

$$\alpha^i = a_i y^i, \quad \beta^k = b_k y^k \text{ (no summation)} \tag{2.12}$$

where a_i and b_k $(i, k = 1, 2, 3)$ are constants. Substitution of the corresponding components of (2.12) into (2.11) yields the matrix of components of $\boldsymbol{\Omega}$ in the final form

$$[\Omega^{lm}] = \begin{bmatrix} a_1 b_1 y^1 y^1, & a_1 b_2 y^1 y^2, & a_1 b_3 y^1 y^3 \\ a_2 b_1 y^1 y^2, & a_2 b_2 y^2 y^2, & a_2 b_3 y^2 y^3 \\ a_3 b_1 y^1 y^3, & a_3 b_2 y^2 y^3, & a_3 b_3 y^3 y^3 \end{bmatrix}. \tag{2.13}$$

Let us return to the formulae (2.9) for the nonhomogeneous displacements u_k''. Some typical displacements corresponding to various components of tensor components ψ_{lmk} and tensor $\boldsymbol{\Omega}$ given in Eq. (2.13) are shown in Fig. 5.

The vector of total displacement $\mathbf{w}$ of micro-element $\mathbf{Y}$ is a sum of macro-displacement $\mathbf{u}$ and the relative displacement (micro-displacement) $\bar{\mathbf{u}}$,

$$w_k(Y^A, X^K, \tau) = u_k(X^K, \tau) + \bar{u}_k(Y^A, X^K, \tau), \tag{2.14}$$

as shown in Fig. 2.

In view of relations (2.5), (2.7), (2.9), the total displacement (2.14) is expressed by the formula

$$w_k(Y^A, X^K, \tau) = u_k(X^K, \tau) + \varphi_{lk}(X^K, \tau) y^l(Y^A) + \psi_{lmk}(X^K, \tau)\, \Omega^{lm}(Y^A). \tag{2.15}$$

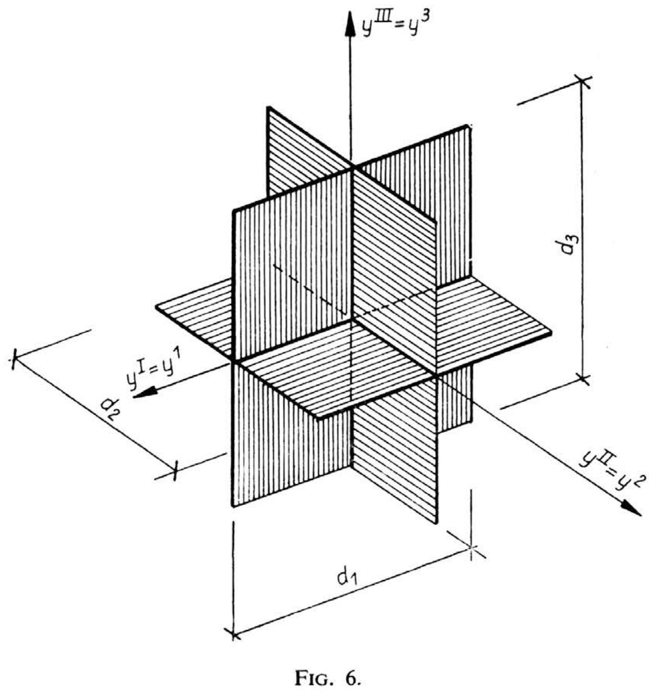

FIG. 6.

This displacement consists of three components: displacement u_k, constant for all micro-elements **Y** of particle **X**; homogeneous displacement expressed by tensor φ_{lk}; and non-homogeneous displacement expressed by tensor ψ_{lmk}. The set of micro-elements **Y** the displacements of which satisfy the relation (2.15) will be called a structural particle. The continuous medium constructed from such particles will be called a structural medium. If all components of ψ_{lmk} vanish, relation (2.15) represents the displacement of a medium with couple stresses.

A model of a structural particle is shown in Fig. 6. It consists of the geometric center of the particle in which the origin of the coordinate system $\{y^k\}$ is located, and of non-parallel planes passing through the geometric center and attached to it. It is convenient to assume the planes to be perpendicular to the directions of interaction with the neighbouring particles. In general, the planes are not interconnected along the lines of their intersection and are free to displace and rotate with respect to each other.

In order to simplify the diagram, cross-sections of the particles by the planes perpendicular to the interaction directions are assumed to be rectangular.

Each plane, in addition to the rigid body displacements, may also be displaced in tangential and normal directions; displacements of the planes are mutually independent.

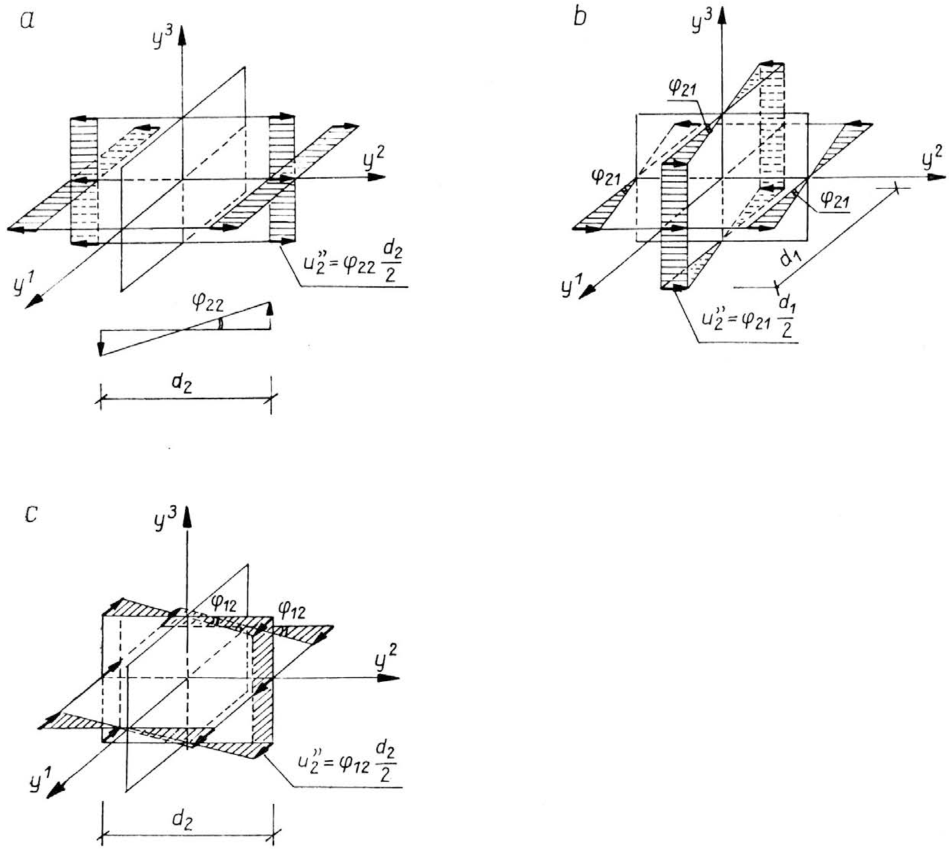

FIG. 7.

Additional kinematic assumptions may constrain the independent motions of individual planes. For instance, from the assumption of plane cross-sections of a particle it follows that the intersecting planes cannot be displaced along their common lines but are allowed to rotate about the lines of intersection. It is easily verified that displacements of the particle shown in Figs. 4 and 5 may be replaced by displacements of the planes passing through the origin and perpendicular to the individual axes $\{y^k\}$; it is shown in Figs. 7 and 8. In

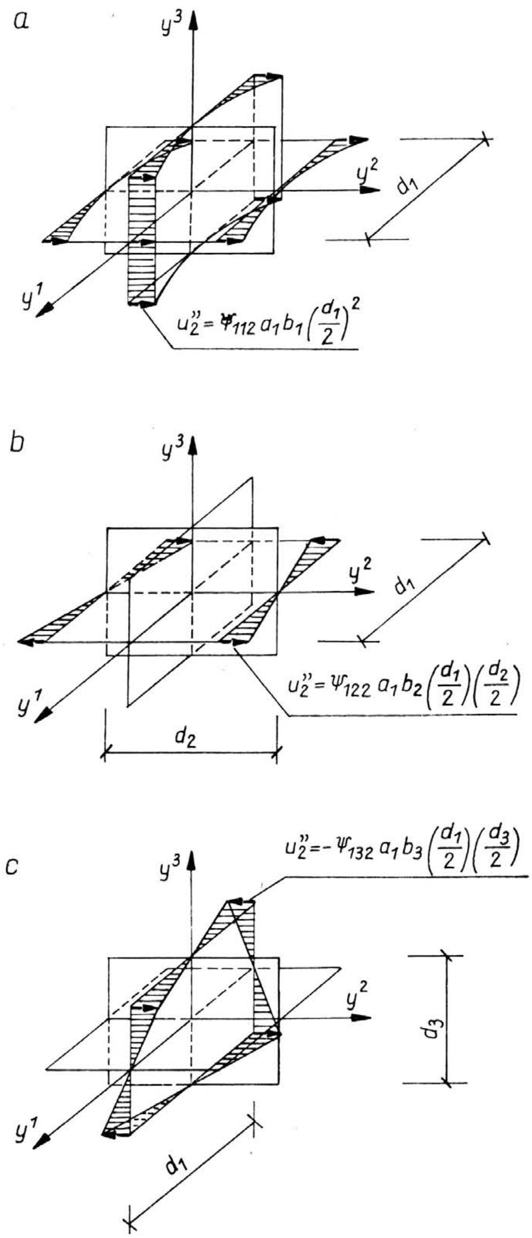

FIG. 8.

Fig. 7 are shown the displacements of planes corresponding to the displacement of the entire particle presented in Fig. 4. Figure 8 illustrates the displacement of planes resulting from the particle displacements shown in Fig. 5.

The structural medium will be considered in the time interval $[\tau_1, \tau_2]$, τ_1 and τ_2 denoting the initial and actual time instants, respectively. All possible motions are taken into account, that is the kinematically admissible motions which carry the system from its initial position occupied at time τ_1 to the final position occupied at time τ_2. The motion of the structural system is described by two independent systems of equations. Motion of particles **X**, that is the macro-motion, is governed by the equations

$$x^k = x^k(X^K, \tau). \tag{2.16}$$

The relative motion of elements **Y** localized at particle **X**, that is the micro-motion, is expressed by the formulae

$$y^a = y^a(Y^A, X^K, \tau). \tag{2.17}$$

The motion may also be described by means of the displacement vectors. From the relations (2.3), (2.4) it follows that

$$\begin{aligned} x^k(X^K, \tau) &= \delta^k_K X^K + u^k(X^K, \tau), \\ y^a(Y^A, X^K, \tau) &= \delta^a_A Y_A + \bar{u}^a(Y^A, X^K, \tau). \end{aligned} \tag{2.18}$$

Here, as before, u^k denotes the macro-displacement of particle **X**, and $\bar{u}^k$ is the relative displacement of element **Y** localized at **X**.

Substitution of the corresponding expressions from Eq. (2.5), (2.8), (2.9) into (2.18) yields

$$\begin{aligned} x^k(X^K, \tau) &= \delta^k_K X^K + u^k(X^K, \tau), \\ y^a(Y^A, X^K, \tau) &= \delta^a_A Y^A + \delta^a_k \varphi^{lk}(X^K, \tau) y_l(Y^A) + \delta^a_k \psi^{lmk}(X^K, \tau) \Omega_{lm}(Y^A). \end{aligned} \tag{2.19}$$

whence it follows that the motion of structural medium is described by the tensor functions

$$\begin{aligned} u^k &= u^k(X^K, \tau), \\ \varphi^{lk} &= \varphi^{lk}(X^K, \tau), \\ \psi^{lmk} &= \psi^{lmk}(X^K, \tau), \end{aligned} \tag{2.20}$$

independent of the micro-coordinates Y^A. The knowledge of all 39 components of these functions makes it possible to determine the position of each particle of the structural medium at each time instant τ; from the dynamical point of view, each particle of the medium represents a system with 39 degrees of freedom.

Let us define the strain tensors of the structural medium. They will be expressed in terms of the tensor functions u^k, φ^{lk}, ψ^{lmk} and their derivatives. Assume the following relations to hold true in the structural medium:

$$\begin{aligned} \gamma_{ij}(X^K, \tau) &= u_{j,i} - \varphi_{i,j}, \\ \varkappa_{kij}(X^K, \tau) &= \varphi_{ij,k} - \psi_{kij}, \\ \eta_{lkij}(X^K, \tau) &= \psi_{kij,l}. \end{aligned} \tag{2.21}$$

The non-symmetric strain tensors γ_{ij}, $\varkappa_{kij}$, η_{lkij} are independent of micro-coordinates Y^A. The geometric relations shown above constitute one of the basic groups of equations characterizing the structural medium. In the case when $\psi_{kij} = \psi_{kij,l} \equiv 0$, they define the strain tensors of media with couple stresses [8, 19].

2.2. Generalized Hamilton's principle

Dynamic equations of motion of a structural medium may be derived from the principle of stationary action functional [24]. One has to assume a suitable number of dynamic variables in the expression for the action density and give their physical interpretation. This approach was extensively discussed in Chapter 1 of the paper [25] and will not be dealt with here.

As it was shown in the preceding section of this paper, the motion of structural media is described by three tensors $u^k(x^i, \tau)$, $\varphi^{kl}(x^i, \tau)$ and $\psi^{klm}(x^i, \tau)$. Let us compare them with the corresponding expressions $u^k+\delta u^k$, $\varphi^{kl}+\delta\varphi^{kl}$ and $\psi^{klm}+\delta\psi^{klm}$. Here (and later) symbol $\delta(\cdot)$ denotes the principal, linear component of the variation and, when speaking of variations, we shall have in mind their linear components only. The independent variations δu^k, $\delta\varphi^{kl}$, $\delta\psi^{klm}$ vanish at the ends of the time interval considered $[\tau_1, \tau_2]$. Hence, it is a variational problem with a fixed boundary [26, 27], and

$$
\begin{aligned}
\delta u^k(x^i, \tau_1) &= \delta u^k(x^i, \tau_2) \equiv 0,\\
\delta\varphi^{kl}(x^i, \tau_1) &= \delta\varphi^{kl}(x^i, \tau_2) \equiv 0,\\
\delta\psi^{klm}(x^i, \tau) &= \delta\psi^{klm}(x^i, \tau_2) \equiv 0.
\end{aligned} \tag{2.22}
$$

Our considerations will be based on Hamilton's principle [28] subject to a suitable generalization. The principle may be written in the general form

$$
\delta \int_{\tau_1}^{\tau_2} (\mathscr{K}-\mathscr{E})d\tau + \int_{\tau_1}^{\tau_2} \mathscr{A} d\tau = 0. \tag{2.23}
$$

Here $\mathscr{K}$ denotes the kinetic energy of the system, and symbols $\delta\mathscr{E}$ and $\mathscr{A}$ are the virtual work done by internal and external forces, respectively. The equation holds true in the case when the forces have no potential what happens, for instance, when the external loads depend on the tensors $\mathbf{u}$, $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ and their variation in time. If the external forces possess a potential (what is possible only if they are independent of $\mathbf{u}$, $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ and their derivatives), then

$$
\mathscr{A} = -\left(\frac{\partial\mathscr{P}}{\partial u^k}\delta u^k + \frac{\partial\mathscr{P}}{\partial\varphi^{kl}}\delta\varphi^{kl} + \frac{\partial\mathscr{P}}{\partial\psi^{klm}}\delta\psi^{klm}\right) = -\delta\mathscr{P}. \tag{2.24}
$$

Here $\mathscr{P}$ is the potential of external loads. On substituting this relation into Eq. (2.23) we obtain the following form of Hamilton's principle

$$
\delta \int_{\tau_1}^{\tau_2} (\mathscr{K}-\mathscr{E}-\mathscr{P})d\tau = 0, \tag{2.25}
$$

where $\mathscr{E}+\mathscr{P}$ denotes the total potential energy. The equation may also be written in the known form

$$\delta W = \delta \int_{\tau_1}^{\tau_2} L d\tau = 0, \tag{2.26}$$

where $L = \mathscr{K}-\mathscr{E}-\mathscr{P}$, the difference of the kinetic energy and the total potential energy, is called the Lagrange function. The integral

$$W = \int_{\tau_1}^{\tau_2} L d\tau$$

is called the action integral in the time interval $[\tau_1, \tau_2]$ and $Ld\tau$ is the elementary action [24].

From the suitably generalized Hamilton's principle (2.26) we can obtain the equations of motion and the boundary conditions for structural media in the case of external loads derived from a potential. To this end let us consider the individual terms of Lagrangean function L.

The micro-particle $(\mathbf{x}, \mathbf{y})$ with spatial macro-coordinates x^k and spatial micro-coordinates y^a has all the properties of a classical continuous medium. The kinetic energy density, that is the kinetic energy per unit macro-volume v is expressed by the integral

$$k = \frac{1}{2}\frac{1}{\omega}\int_{\omega} \bar{\varrho}\mathring{w}^k\mathring{w}_k d\omega. \tag{2.27}$$

Here $\bar{\varrho}$ is the micro-material density, w^k are the components of the total displacement vector of the micro-particle, and ω — the microvolume. Small circles denote differentiation with respect to time.

Let us calculate the material derivative of displacement w_i. From Eqs. (2.14) and (2.5) it follows that

$$\mathring{w}_i = \overset{\circ}{\overline{u_i+\bar{u}_i}} = \overset{\circ}{\overline{u_i+u_i'+u_i''}} = \mathring{u}_i+\mathring{u}_i'+\mathring{u}_i''. \tag{2.28}$$

Using Eq. (2.15) we may write

$$\mathring{w}_i(Y^A, X^K, \tau) = \mathring{u}_i(X^K, \tau)+\mathring{\varphi}_{ji}(X^K, \tau)y^j(Y^A)+\mathring{\psi}_{jki}(X^K, \tau)\Omega^{jk}(Y^A). \tag{2.29}$$

Substitution of the above expression into Eq. (2.27) and transformation of the integrand yields

$$k = \frac{1}{2\omega}\int_{\omega} [\mathring{u}^i\mathring{u}_i+2\mathring{u}^i\mathring{\varphi}_{ji}y^j+2\mathring{u}^i\mathring{\psi}_{jki}\Omega^{jk}+\mathring{\varphi}_j^{\cdot i}\mathring{\varphi}_{ki}y^jy^k+2\mathring{\varphi}_j^{\cdot i}\mathring{\psi}_{kli}y^j\Omega^{kl}+\mathring{\psi}_{jk}^{\cdot\cdot i}\mathring{\psi}_{lmi}\Omega^{jk}\Omega^{lm}]\bar{\varrho}d\omega. \tag{2.30}$$

Taking into account the fact that the time derivatives of tensors $\mathbf{u}$, $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ are independent of the micro-coordinates Y^A, we may write the formula (2.30) in the form

$$k = \frac{1}{2}[\mathring{u}^i\mathring{u}_i\varrho+2\mathring{u}^i\mathring{\varphi}_{ji}\varrho^j+\mathring{\varphi}_j^{\cdot i}\mathring{\varphi}_{ki}\varrho^{jk}+2\mathring{u}^i\mathring{\psi}_{jki}\mu^{jk}+2\mathring{\varphi}_j^{\cdot i}\mathring{\psi}_{kli}\mu^{jkl}+\mathring{\psi}_{jk}^{\cdot\cdot i}\mathring{\psi}_{lmi}\mu^{jklm}], \tag{2.31}$$

Here

$$
\varrho = \frac{1}{\omega}\int\limits_{\omega}\bar{\varrho}\,d\omega, \qquad \varrho^{i} = \frac{1}{\omega}\int\limits_{\omega} y^{i}\bar{\varrho}\,d\omega, \qquad \varrho^{ij} = \frac{1}{\omega}\int\limits_{\omega} y^{i}y^{j}\bar{\varrho}\,d\omega,
$$

(2.32)

$$
\mu^{ij} = \frac{1}{\omega}\int\limits_{\omega}\Omega^{ij}\bar{\varrho}\,d\omega, \quad \mu^{ijk} = \frac{1}{\omega}\int\limits_{\omega} y^{i}\Omega^{jk}\bar{\varrho}\,d\omega, \quad \mu^{ijkl} = \frac{1}{\omega}\int\limits_{\omega}\Omega^{ij}\Omega^{kl}\bar{\varrho}\,d\omega
$$

are the generalized micro-densities of the structural material.

The total kinetic energy $\mathcal{K}$ is defined as the integral taken over the macro-volume v,

$$
\mathcal{K} = \int\limits_{v} k\,dv \tag{2.33}
$$

of the kinetic energy density k expressed by the formula (2.31).

Let us calculate the variation of the integral $\int\limits_{\tau_1}^{\tau_1}\mathcal{K}\,d\tau$. From the definition (2.33) it follows that

$$
\delta\int\limits_{\tau_1}^{\tau_2}\mathcal{K}\,d\tau = \delta\int\limits_{\tau_1}^{\tau_2}d\tau\int\limits_{v}k\,dv. \tag{2.34}
$$

Assume the variations of generalized masses to vanish; then

$$
\delta(\varrho\,dv) = \delta(\varrho^{i}dv) = \delta(\varrho^{ij}dv) = \delta(\mu^{ij}dv) = \delta(\mu^{ijk}dv) = \delta(\mu^{ijkl}dv) = 0.
$$

The variation symbol may be written under the integration sign so that Eq. (2.34) may be replaced with

$$
\delta\int\limits_{\tau_1}^{\tau_2}\mathcal{K}\,d\tau = \int\limits_{\tau_1}^{\tau_2}d\tau\int\limits_{v}\delta k\,dv. \tag{2.35}
$$

From Eq. (2.31) it is seen that the kinetic energy density $k = k(\mathring{u}_i, \mathring{\varphi}_{ij}, \mathring{\psi}_{ijk})$ is a function of the time derivatives of tensors $\mathbf{u}$, $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ what means that the kinetic energy density variation may be written in the form

$$
\delta k = \frac{\partial k}{\partial\mathring{u}_i}\delta(\mathring{u}_i) + \frac{\partial k}{\partial\mathring{\varphi}_{ij}}\delta(\mathring{\varphi}_{ij}) + \frac{\partial k}{\partial\mathring{\psi}_{ijk}}\delta(\mathring{\psi}_{ijk}). \tag{2.36}
$$

Due to the commutation law of differentiation and variation we obtain the identities

$$
\frac{\partial k}{\partial\mathring{u}_i}\delta(\mathring{u}_i) = \left(\frac{\partial k}{\partial\mathring{u}_i}\delta u_i\right)^{\circ} - \left(\frac{\partial k}{\partial\mathring{u}_i}\right)^{\circ}\delta u_i,
$$

$$
\frac{\partial k}{\partial\mathring{\varphi}_{ij}}\delta(\mathring{\varphi}_{ij}) = \left(\frac{\partial k}{\partial\mathring{\varphi}_{ij}}\delta\varphi_{ij}\right)^{\circ} - \left(\frac{\partial k}{\partial\mathring{\varphi}_{ij}}\right)^{\circ}\delta\varphi_{ij},
$$

$$
\frac{\partial k}{\partial\mathring{\psi}_{ijk}}\delta(\mathring{\psi}_{ijk}) = \left(\frac{\partial k}{\partial\mathring{\psi}_{ijk}}\delta\psi_{ijk}\right)^{\circ} - \left(\frac{\partial k}{\partial\mathring{\psi}_{ijk}}\right)^{\circ}\delta\psi_{ijk},
$$

which may be substituted into the expression (2.36) for the variation δk of the kinetic energy density,

$$(2.37)\qquad \delta k = -\left(\frac{\partial k}{\partial \mathring{u}_i}\right)^{\circ}\delta u_i - \left(\frac{\partial k}{\partial \mathring{\varphi}_{ij}}\right)^{\circ}\delta\varphi_{ij} - \left(\frac{\partial k}{\partial \mathring{\psi}_{ijk}}\right)^{\circ}\delta\psi_{ijk} + \left(\frac{\partial k}{\partial \mathring{u}_i}\delta u_i\right)^{\circ} + \left(\frac{\partial k}{\partial \mathring{\varphi}_{ij}}\delta\varphi_{ij}\right)^{\circ} + \left(\frac{\partial k}{\partial \mathring{\psi}_{ijk}}\delta\psi_{ijk}\right)^{\circ}.$$

Substitution of this relation into Eq. (2.35) yields in turn

$$(2.38)\qquad \delta\int_{\tau_1}^{\tau_2}\mathcal{K}\,d\tau = -\int_{\tau_1}^{\tau_2}d\tau\int_v\left\{\left(\frac{\partial k}{\partial \mathring{u}_i}\right)^{\circ}\delta u_i + \left(\frac{\partial k}{\partial \mathring{\varphi}_{ij}}\right)^{\circ}\delta\varphi_{ij} + \left(\frac{\partial k}{\partial \mathring{\psi}_{ijk}}\right)^{\circ}\delta\psi_{ijk}\right\}dv + \left[\int_v \frac{\partial k}{\partial \mathring{u}_i}\delta u_i\,dv\right]_{\tau_1}^{\tau_2} + \left[\int_v \frac{\partial k}{\partial \mathring{\varphi}_{ij}}\delta\varphi_{ij}\,dv\right]_{\tau_1}^{\tau_2} + \left[\int_v \frac{\partial k}{\partial \mathring{\psi}_{ijk}}\delta\psi_{ijk}\,dv\right]_{\tau_1}^{\tau_2}.$$

brackets denoting here the difference

$$[---]_{\tau_1}^{\tau_2} = [---]_{\tau-\tau_2} - [---]_{\tau-\tau_2}.$$

According to the assumptions (2.22), variations δu_i, $\delta\varphi_{ij}$, $\delta\psi_{ijk}$ vanish at the ends of the time interval $[\tau_1, \tau_2]$ what means that the expressions in brackets also vanish. Thus the relation is simplified to the form

$$(2.39)\qquad \delta\int_{\tau_1}^{\tau_2}\mathcal{K}\,d\tau = -\int_{\tau_1}^{\tau_2}d\tau\int_v\left[\left(\frac{\partial k}{\partial \mathring{u}_i}\right)^{\circ}\delta u_i + \left(\frac{\partial k}{\partial \mathring{\varphi}_{ij}}\right)^{\circ}\delta\varphi_{ij} + \left(\frac{\partial k}{\partial \mathring{\psi}_{ijk}}\right)^{\circ}\delta\psi_{ijk}\right]dv.$$

Formula (2.31) is used to evaluate the following expressions:

$$\left(\frac{\partial k}{\partial \mathring{u}_i}\right)^{\circ} = \overset{\circ\circ}{u}{}^i\varrho + \overset{\circ\circ}{\varphi}{}_j^{\cdot i}\varrho^j + \overset{\circ\circ}{\psi}{}_{jk}^{\cdot\cdot i}\mu^{jk},$$

$$\left(\frac{\partial k}{\partial \mathring{\varphi}_{ij}}\right)^{\circ} = \overset{\circ\circ}{u}{}^j\varrho^i + \overset{\circ\circ}{\varphi}{}_k^{\cdot j}\varrho^{ik} + \overset{\circ\circ}{\psi}{}_{kl}^{\cdot\cdot j}\mu^{ikl},$$

$$\left(\frac{\partial k}{\partial \mathring{\psi}_{ijk}}\right)^{\circ} = \overset{\circ\circ}{u}{}^k\mu^{ij} + \overset{\circ\circ}{\varphi}{}^{\cdot k}\mu^{lij} + \overset{\circ\circ}{\psi}{}_{lm}^{\cdot\cdot k}\mu^{ijlm}.$$

Right-hand sides of these expressions are now substituted into Eq. (2.39) to yield

$$(2.40)\qquad \delta\int_{\tau_1}^{\tau_2}\mathcal{K}\,d\tau = -\int_{\tau_1}^{\tau_2}d\tau\int_v[(\overset{\circ\circ}{u}{}^i\varrho + \overset{\circ\circ}{\varphi}{}_j^{\cdot i}\varrho^j + \overset{\circ\circ}{\psi}{}_{jk}^{\cdot\cdot i}\mu^{jk})\,\delta u_i + (\overset{\circ\circ}{u}{}^j\varrho^i + \overset{\circ\circ}{\varphi}{}_k^{\cdot j}\varrho^{ik} + \overset{\circ\circ}{\psi}{}_{kl}^{\cdot\cdot j}\mu^{ikl})\,\delta\varphi_{ij} + (\overset{\circ\circ}{u}{}^k\mu^{ij} + \overset{\circ\circ}{\varphi}{}_l^{\cdot k}\mu^{lij} + \overset{\circ\circ}{\psi}{}_{lm}^{\cdot\cdot k}\mu^{ijlm})\,\delta\psi_{ijk}]\,dv.$$

In the particular case of $\varrho^i = \mu^{ij} = \mu^{ijk} = 0$ (what is true, e.g., in microstructures characterized by three axes of symmetry), the above expression is simplified to the form

$$(2.41)\qquad \delta\int_{\tau_1}^{\tau_2}\mathcal{K}\,d\tau = -\,d\tau\int_v[\overset{\circ\circ}{u}{}^i\varrho\,\delta u_i + \overset{\circ\circ}{\varphi}{}_k^{\cdot j}\varrho^{ik}\delta\varphi_{ij} + \overset{\circ\circ}{\psi}{}_{lm}^{\cdot\cdot k}\mu^{ijlm}\delta\psi_{ijk}]\,dv.$$

Let us now pass to the problem of work done by the body and surface forces; it is written as a sum of volume and surface integrals

$$(2.42)\qquad -\mathcal{P} = \int_v F_{(v)}\,dv + \int_s T_{(s)}\,ds,$$

taken over the macro-volume v and surface s bounding v, respectively. $F(v)$ is the work density of the body forces, and $T(s)$ — the work density of surface forces. Consider both the integrands. The density of work done by surface forces is defined by the integral

$$T_{(s)} = \int_{\sigma} \bar{p}^i_{(n)} w_i \, d\sigma \tag{2.43}$$

of the work done by the mean surface stresses $\bar{p}^i(u)$ taken over the surface σ bounding the micro-volume ω. Using the relation (2.15) we obtain

$$T_{(s)} = \int_{\sigma} \bar{p}^i_{(n)}(u_i + \varphi_{ji} y^j + \psi_{jki} \Omega^{jk}) \, d\sigma. \tag{2.44}$$

Tensors u_i, φ_{ij}, ψ_{ijk} are independent of σ, and hence

$$T_{(s)} = u_i p^i_{(n)} + \varphi_{ji} m^{ji}_{(n)} + \psi_{jki} b^{jki}_{(n)} \tag{2.45}$$

with the following notations:

$$\begin{aligned} p^i_{(n)} &= \int_{\sigma} \bar{p}^i_{(n)} \, d\sigma, \\ m^{ji}_{(n)} &= \int_{\sigma} y^j \bar{p}^i_{(n)} \, d\sigma, \\ b^{jki}_{(n)} &= \int_{\sigma} \Omega^{jk} \bar{p}^i_{(n)} \, d\sigma. \end{aligned} \tag{2.46}$$

From Eq. (2.45) it follows that the surface forces work density $T_{(s)} = T_{(s)}(u_i, \varphi_{ij}, \psi_{ijk})$ is a function of tensors u, $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$. Variation $\delta T_{(s)}$ of the density $T_{(s)}$ is given by the formula

$$\delta T_{(s)} = p^i_{(n)} \delta u_i + m^{ij}_{(n)} \delta \varphi_{ij} + b^{ijk}_{(n)} \delta \psi_{ijk}. \tag{2.47}$$

The density of work done by body forces is defined by the integral taken over the micro-volume ω,

$$F_{(v)} = \frac{1}{\varrho \omega} \int_{\omega} \bar{\varrho} \bar{f}^i w_i \, d\omega. \tag{2.48}$$

Here again ω is the micro-volume, $\bar{\varrho}$ — micro-material density, ϱ — macro-density, and w_i — the corresponding component of the micro-particle displacement vector. Symbol $\bar{f}^i$ denotes the body force density of the micro-particle.

Equation (2.15) may be used to write the expression (2.48) in the form

$$F_{(v)} = f^i u_i + h^{ij} \varphi_{ij} + g^{jki} \psi_{ijk}, \tag{2.49}$$

where

$$\begin{aligned} f^i &= \frac{1}{\varrho \omega} \int_{\omega} \bar{f}^i \bar{\varrho} \, d\omega, \\ h^{jk} &= \frac{1}{\varrho \omega} \int_{\omega} y^j \bar{f}^k \bar{\varrho} \, d\omega, \\ g^{jki} &= \frac{1}{\varrho \omega} \int_{\omega} \Omega^{jk} \bar{f}^i \bar{\varrho} \, d\omega \end{aligned} \tag{2.50}$$

are the definitions of the generalized body forces.

Variation $\delta F_{(v)}$ of the body forces are calculated by means of the relation (2.49):

$$\delta F_{(v)} = f^i \delta u_i + h^{ij} \delta\psi_{ij} + g^{ijk} \delta\psi_{ijk} \tag{2.51}$$

and variation $\delta\mathscr{P}$ of the work done by body and surface forces — from Eq. (2.42),

$$-\delta\mathscr{P} = \delta \int_v F_{(v)} dv + \delta \int_s T_{(s)} ds. \tag{2.52}$$

Writing the variation symbol under the integration sign we obtain

$$-\delta\mathscr{P} = \int_v \delta F_{(v)} dv + \int_s \delta T_{(s)} ds. \tag{2.53}$$

Substitution of the right-hand sides of Eqs. (2.47) and (2.51) for the variations $\delta F_{(v)}$ and $\delta T_{(s)}$ in (2.53) yields

$$-\delta\mathscr{P} = \int_v (f^i \delta u_i + h^{ij} \delta\varphi_{ij} + g^{ijk} \delta\psi_{ijk}) dv + \int_s (p^i_{(n)} \delta u_i + m^{ij}_{(n)} \delta\varphi_{ij} + b^{ijk}_{(n)} \delta\psi_{ijk}) ds. \tag{2.54}$$

Let us finally define the internal energy of a structural medium. Assume the internal energy density U per unit macro-volume v to be a function of the strain tensors γ_{ij}, $\varkappa_{ijk}$, η_{ijkl}. It may be written that

$$U = U(\gamma_{ij}, \varkappa_{ijk}, \eta_{ijkl}). \tag{2.55}$$

Let us evaluate the variation δU of the internal energy density. In view of the above relation, it may be written in the form

$$\delta U = \frac{\partial U}{\partial \gamma_{ij}} \delta\gamma_{ij} + \frac{\partial U}{\partial \varkappa_{ijk}} \delta\varkappa_{ijk} + \frac{\partial U}{\partial \eta_{ijkl}} \delta\eta_{ijkl}. \tag{2.56}$$

Generalized stresses in a structural medium are defined as partial derivatives of the internal energy density U with respect to the corresponding strains,

$$p^{ij} = \frac{\partial U}{\partial \gamma_{ij}}, \qquad m^{ijk} = \frac{\partial U}{\partial \varkappa_{ijk}}, \qquad b^{ijkl} = \frac{\partial U}{\partial \eta_{ijkl}}. \tag{2.57}$$

Substitution of the above definitions into Eq. (2.56) yields

$$\delta U = p^{ij} \delta\gamma_i + m^{ijk} + \delta\varkappa_{ijk} + b^{ijkl} \delta\psi_{ijkl}$$

and using the geometric relations (2.21) we obtain

$$\delta U = p^{ij} \delta(u_{ji} - \varphi_{ij}) + m^{ijk} \delta(\varphi_{jk,i} - \psi_{ijk}) + b^{ijkl} \delta(\psi_{jkl,i}). \tag{2.58}$$

The operations of variation and differentiations are commutative, so that Eq. (2.58), after suitable rearrangements, assumes the form

$$\begin{aligned} \delta U = (p^{ij} \delta u_j)_{,i} - p^{ij}_{,i} \delta u_j - p^{ij} \delta\varphi_{ij} + (m^{ijk} \delta\varphi_{jk})_{,i} \\ - m^{ijk}_{,i} \delta\varphi_{jk} - m^{ijk} \delta\psi_{ijk} + (b^{ijkl} \delta\psi_{ijkl})_{,i} - b^{ijkl}_{,i} \delta\psi_{jkl}. \end{aligned} \tag{2.59}$$

Internal energy $\mathscr{E}$ of the structural medium is the volume integral

$$\mathscr{E} = \int_v U dv \tag{2.60}$$

of the internal energy density U taken over the macro-volume v. Variation $\delta\mathscr{E}$ of the internal energy $\mathscr{E}$ is calculated from the formula

$$\delta\mathscr{E} = \delta\int_v U dv = \int_v \delta U dv. \tag{2.61}$$

Substituting here the expression (2.59) for the variation δU of the energy density we obtain

$$\begin{aligned}\delta\mathscr{E} = &-\int_v p^{ij}_{.i}\,\delta u_j dv - \int_v (m^{ijk}_{.i} + p^{jk})\,\delta\varphi_{jk} dv - \int_v (b^{ijkl}_{.i} + m^{jkl})\,\delta\psi_{jkl} dv\\ &+ \int_v (p^{ij}\delta u_j)_{,i} dv + \int_v (m^{ijk}\delta\varphi_{jk})_{,i} dv + \int_v (b^{ijkl}\delta\psi_{ijkl})_{,i} dv.\end{aligned} \tag{2.62}$$

From the Gauss–Ostrogradskii theorem the following equalities may be derived:

$$\int_v (p^{ij}\delta u_j)_{,i} dv = \int_s p^{ij} n_i\,\delta u_j ds,$$

$$\int_v (m^{ijk}\delta\varphi_{jk})_{,i} dv = \int_s m^{ijk} n_j \delta\varphi_{jk} ds,$$

$$\int_v (b^{ijkl}\delta\psi_{jkl})_{,i} dv = \int_s b^{ijkl} n_i\,\delta\psi_{jkl} ds.$$

Here n_i are components of the unit outer normal vector $\mathbf{n}$ of surface s. These equations enable us to replace the volume integrals in Eq. (2.62) with the surface integrals,

$$\begin{aligned}\delta\mathscr{E} = &-\int_v [p^{ij}_{.i}\,\delta u_j + (m^{ijk}_{.i} + p^{jk})\,\delta\varphi_{jk} + (b^{ijkl}_{.i} + m^{jkl})\,\delta\psi_{jkl}] dv\\ &+ \int_s (p^{ij} n_i\,\delta u_j + m^{ijk} n_i\,\delta\varphi_{jk} + b^{ijkl} n_i\,\delta\psi_{jkl}) ds.\end{aligned} \tag{2.63}$$

2.3. Equations of motion and constitutive relations

The generalized Hamilton principle (2.62) according to which variation δW of the action functional should vanish may, on the basis of preceding section, be represented in the form

$$\int_{\tau_1}^{\tau_2} \delta(\mathscr{K} - \mathscr{E} - \mathscr{P}) d\tau = 0. \tag{2.64}$$

This equation will be used to derive the equations of motion and the constitutive relations of structural media.

Let us substitute here expressions (2.40), (2.54) and (2.63) for the variations $\delta\mathscr{K}$, $\delta\mathscr{P}$ and $\delta\mathscr{E}$. Disregarding the time integrals we obtain

$$\begin{aligned}&\int_v (p^{ij}_{.i} + f^j - \varrho\ddot{u}^j - \varrho^k\ddot{\varphi}^{.j}_k - \mu^{kl}\ddot{\psi}^{.j}_{kl})\,\delta u_j dv + \int_v (m^{ijk}_{.i} + p^{jk} + h^{jk}\\ &- \varrho^j\ddot{u}^k - \varrho^{ji}\ddot{\varphi}^{.k}_i - \mu^{jil}\ddot{\psi}^{.k}_{il})\,\delta\varphi_{jk} dv + \int_v (b^{ijkl}_{.i} + m^{jkl} + g^{jkl} - \mu^{jk}\ddot{u}^l - \mu^{mjk}\ddot{\varphi}^{.l}_m - \mu^{jkmn}\ddot{\psi}^{.\,l}_{mn})\,\delta\psi_{jkl} dv\\ &- \int_s (p^{ij} n_i - p^j_{(n)})\,\delta u_j ds - \int_s (m^{ijk} n_i - m^j_{(n)})\,\delta\varphi_{jk} ds - \int_s (b^{ijkl} n_i - b^{jkl}_{(n)})\,\delta\psi_{jkl} ds = 0.\end{aligned} \tag{2.65}$$

According to the former assumption, variations δu_i, $\delta\varphi_{ij}$, $\delta\psi_{ijk}$ are mutually independent, and the corresponding integrands must vanish separately. From Eq. (2.65) it follows that

$$(2.66)\qquad \begin{aligned} p^{ij}_{.i}+f^j &= \varrho\overset{\circ\circ}{u}{}^j+\varrho^k\overset{\circ\circ}{\varphi}{}^{.j}_k+\mu^{kl}\overset{\circ\circ}{\psi}{}^{.j}_{kl},\\ m^{ijk}_{.i}+p^{jk}+h^{jk} &= \varrho^j\overset{\circ\circ}{u}{}^k+\varrho^{ji}\overset{\circ\circ}{\varphi}{}^{.k}_i+\mu^{jlm}\overset{\circ\circ}{\psi}{}^{.k}_{lm},\\ b^{ijkl}_{.i}+m^{jkl}+g^{jkl} &= \mu^{jk}\overset{\circ\circ}{u}{}^l+\mu^{mjk}\overset{\circ\circ}{\varphi}{}^{.l}_m+\mu^{jkmn}\overset{\circ\circ}{\psi}{}^{.l}_{mn}.\end{aligned}$$

These equations are called the equations of motion of a three-dimensional structural medium.

From Eq. (2.65) it may also be found that

$$(2.67)\qquad \begin{aligned} b^{ij}n_i &= p^j_{(n)},\\ m^{ijk}n_i &= m^{jk}_{(n)},\\ b^{ijkl}n_i &= b^{jkl}_{(n)},\end{aligned}$$

and these relations are the static boundary conditions of a structural medium expressed in terms of generalized stresses.

Similarly to the preceding section it is now assumed that the internal energy density U is a function of strain tensors γ_{ij}, χ_{ijk}, η_{ijkl}. Let us now expand the internal energy density function $U(\gamma_{ij}, \varkappa_{ijk}, \eta_{ijkl})$ into a Taylor series in the neighbourhood of the undeformed state $(\overset{\circ}{\gamma}_{ij} = \overset{\circ}{\varkappa}_{ijk} = \overset{\circ}{\eta}_{ijkl} = 0)$,

$$(2.68)\qquad \begin{aligned} U(\gamma_{ij}, \varkappa_{ijk}, \eta_{ijkl}) &= U_0+\left(\frac{\partial U}{\partial\gamma_{ij}}\right)_0\gamma_{ij}+\left(\frac{\partial U}{\partial\varkappa_{ijk}}\right)_0\varkappa_{ijk}+\left(\frac{\partial U}{\partial\eta_{ijkl}}\right)_0\eta_{ijkl}\\ &+\frac{1}{2}\Bigg[\left(\frac{\partial^2 U}{\partial\gamma_{ij}\partial\gamma_{kl}}\right)_0\gamma_{ij}\gamma_{kl}+2\left(\frac{\partial^2 U}{\partial\gamma_{ij}\partial_{aklm}}\right)_0\gamma_{ij}\varkappa_{klm}+2\left(\frac{\partial^2 U}{\partial\gamma_{ij}\partial\eta_{klmn}}\right)_0\gamma_{ij}\eta_{klmn}\\ &+\left(\frac{\partial^2 U}{\partial\varkappa_{ijk}\partial\varkappa_{lmn}}\right)_0\varkappa_{ijk}\varkappa_{lmn}+2\left(\frac{\partial^2 U}{\partial\varkappa_{ijk}\partial\eta_{lmns}}\right)_0\varkappa_{ijk}\eta_{lmns}+\left(\frac{\partial^2 U}{\partial\eta_{ijkl}\partial\eta_{mnrs}}\right)_0\eta_{ijkl}\eta_{mnrs}\Bigg]\end{aligned}$$

disregarding the terms of order higher than two. In a medium without initial stresses in the undeformed state at $\gamma_{ij} = 0$, $\varkappa_{ijk} = 0$, $\eta_{ijkl} = 0$ the stresses cannot appear and $p^{ij} = 0$, $m^{ijk} = 0$, $b^{ijkl} = 0$. In view of the definitions (2.57) it means that the coefficients multiplying the linear terms in (2.68) must vanish. Vanishing of the internal energy in the undeformed state of the medium implies $U_0 = 0$ and, consequently, the expression for the internal energy density is simplified to the form

$$(2.69)\qquad \begin{aligned} U &= \frac{1}{2}A^{ijkl}\gamma_{ij}\gamma_{kl}+D^{ijklm}\gamma_{ij}\varkappa_{klm}+E^{ijklmn}\gamma_{ij}\eta_{klmn}\\ &+\frac{1}{2}C^{ijklmn}\varkappa_{ijk}\varkappa_{lmn}+F^{ijklmnr}\varkappa_{ijk}\eta_{lmnr}+\frac{1}{2}B^{ijklmnrs}\eta_{ijkl}\eta_{mnrs}.\end{aligned}$$

The following notations have been introduced here:

$$A^{ijkl} = \left(\frac{\partial^2 U}{\partial\gamma_{ij}\partial\gamma_{kl}}\right)_0,\qquad D^{ijklm} = \left(\frac{\partial^2 U}{\partial\gamma_{ij}\partial\varkappa_{klm}}\right)_0,$$

$$E^{ijklmn} = \left(\frac{\partial^2 U}{\partial \gamma_{ij} \partial \eta_{klmn}}\right)_0, \qquad C^{ijklmn} = \left(\frac{\partial^2 U}{\partial \varkappa_{ijk} \partial \varkappa_{lmn}}\right)_0, \tag{2.70}$$

$$F^{ijklmnr} = \left(\frac{\partial^2 U}{\partial \varkappa \eta_{jk} \partial \eta_{lmnr}}\right)_0, \qquad B^{ijklmnrs} = \left(\frac{\partial^2 U}{\partial \eta_{ijkl} \partial \eta_{mnrs}}\right)_0.$$

Let us substitute the right-hand expression of Eq. (2.69) into the definition (2.57). Performing the operations prescribed we obtain

$$\begin{aligned} p^{ij} &= A^{ijkl}\gamma_{kl} + D^{ijklm}\varkappa_{klm} + E^{ijklmn}\eta_{klmn}, \\ m^{ijk} &= D^{ijklm}\gamma_{lm} + C^{ijklmn}\varkappa_{lmn} + F^{ijklmnr}\eta_{lmnr}, \\ b^{ijkl} &= E^{ijklmn}\gamma_{mn} + F^{ijklmnr}\varkappa_{mnr} + B^{ijklmnrs}\eta_{mnrs}. \end{aligned} \tag{2.71}$$

These formulae express the stress tensors in terms of strains and are called the constitutive relations of the structural medium. Symbols **A**, **B**, **C**, **D**, **E**, **F** defined by Eqs. (2.70) are the elasticity tensors of the medium.

For instance, let us define the centrosymmetric structural medium in which no "cross-effects" take place. This condition is satisfied provided

$$\begin{aligned} p^{ij} &= p^{ij}(\gamma_{kl}), \\ m^{ijk} &= m^{ijk}(\varkappa_{lmn}), \\ b^{ijkl} &= b^{ijkl}(\eta_{mnrs}). \end{aligned} \tag{2.74}$$

that is when tensors **D**, **E**, **F** vanish. In a centrosymmetric medium expression (2.69) for the internal energy density U is simplified to the form

$$U = \frac{1}{2}A^{ijkl}\gamma_{ij}\gamma_{kl} + \frac{1}{2}C^{ijklmn}\varkappa_{ijk}\varkappa_{lmn} + \frac{1}{2}B^{ijklmnrs}\eta_{ijkl}\eta_{mnrs}. \tag{2.73}$$

and the constitutive relations assume the form

$$\begin{aligned} p^{ij} &= A^{ijkl}\gamma_{kl}, \\ m^{ijk} &= C^{ijklmn}\varkappa_{lmn} \\ b^{ijkl} &= B^{ijklmnrs}\eta_{mnrs}. \end{aligned} \tag{2.74}$$

The elasticity tensors **A**, **B**, **C** appearing here were defined by Eqs. (2.70). In the case when $\psi_{ijkl} = \psi_{ijkl,m} = 0$, Eqs. (2.71) or (2.74) yield the constitutive relations of the media with couple stresses. In centrosymmetric media the generalized body forces are uncoupled and we should assume $\varrho^k = \mu^{kl} = \mu^{klm} = 0$ in the equations of motion (2.66).

In the contitnuous model of structural media presented above 117 static unknowns appear: 9 components of the stress tensor p^{ij}, 27 components of the tensor m^{ijk} and 81 components of b^{ijkl}. The geometric unknowns are represented by 9 components of the strain tensor γ_{ij}, 27 components of $\varkappa_{ijk}$ and 81 components of the η_{ijkl} tensor, what makes together 117 unknown strain tensor components. Other geometric unknowns are the 39 components of the displacement tensors: 3 components of the vector u_i, 9 components of the tensor φ_{ij} and 27 components of the tensor ψ_{ijk}. Summing up, the number of static and geometric unknowns occurring in the theory of structural media equals 273.

The unknowns are determined from 39 equations of motion (2.66), 117 geometric relations (2.21) and 117 physical relations (2.71) or (2.74). The 273 equations, together with the boundary conditions (2.67), enable a unique determination of the 273 static and geometric unknowns of the model considered.

The most convenient set of equations to deal with, both in the classical elasticity and here, are the equations of motion expressed in terms of displacements. Let us consider this problem in the case of a centrosymmetric medium. To this end let us substitute the right-hand sides of the physical relations (2.74) into the equations of motion (2.66); assuming $\varrho^k = \mu^{kl} = \mu^{klm} = 0$ we obtain

$$(2.75)\quad \begin{aligned} &(A^{ijkl}\gamma_{kl})_{,i} + f^j = \varrho \ddot{u}^j, \\ &(C^{ijklmn}\varkappa_{lkn})_{,i} + (A^{jklm}\gamma_{lm}) + h^{jk} = \varrho^{kl}\ddot{\varphi}^{j}_{\cdot l}, \\ &(B^{ijklmnrs}\eta_{mrs})_{,i} + (C^{jklmnr}\varkappa_{mnr}) + g^{jkl} = \mu^{klmn}\ddot{\psi}^{j}_{\cdot mn}. \end{aligned}$$

Elimination of the strain tensors γ_{ij}, $\varkappa_{ijk}$, η_{ijkl} by means of the geometric relations (2.21) leads to the set of equations

$$(2.76)\quad \begin{aligned} &[A^{ijkl}(u_{l,k} - \varphi_{kl})]_{,i} + f^j = \varrho \ddot{u}^j, \\ &[C^{ijklmn}(\varphi_{mn,l} - \psi_{lmn})]_{,i} + [A^{jklm}(u_{m,l} - \varphi_{lm})] + h^{jk} = \varrho^{kl}\ddot{\varphi}^{j}_{\cdot l}, \\ &[B^{ijklmnrs}(\psi_{nrs,m})]_{,i} + [C^{jklmnr}(\varphi_{nr,m} - \psi_{mnr})] + g^{jk} = \mu^{klmn}\ddot{\psi}^{j}_{\cdot mn}. \end{aligned}$$

Additional assumption of homogeneity of the medium (each particle has the same elastic properties, irrespective of its position) makes all the elasticity tensors independent of the coordinates $\{x^i\}$. Then Eq. (2.76) may be replaced with the system of equations

$$(2.77)\quad \begin{aligned} &A^{ijkl}(u_{l,ki} - \varphi_{kl,i}) + f^j = \varrho \ddot{u}^j, \\ &C^{ijklmn}(\varphi_{mn,li} - \psi_{lmn,i}) + A^{jklm}(u_{m,l} - \varphi_{lm}) + h^{jk} = \varrho^{kl}\ddot{\varphi}^{j}_{\cdot l}, \\ &B^{ijklmnrs}\psi_{nrs,mi} + C^{jklmnr}(\varphi_{nr,m} - \psi_{mnr}) + g^{jkl} = \mu^{klmn}\ddot{\psi}^{j}_{\cdot mn}. \end{aligned}$$

which represents the equations of motion of a homogeneous, centrosymmetric structural medium, expressed in terms of displacements. Solution of this system under the suitable boundary conditions allows for a unique determination of all the generalized displacement vector components u_i, φ_{ij}, ψ_{ijk}. From the geometric relations (2.21) we calculate the strain tensors γ_{ij}, $\varkappa_{ijk}$, η_{ijkl}. Finally, the physical relations (2.74) are used to evaluate the stress tensor components p^{ij}, m^{ijk}, b^{ijkl}. Thus all the static and geometric unknowns of a homogeneous, centrosymmetric structural medium may be considered as determined.

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TECHNICAL UNIVERSITY OF ŁÓDŹ.

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