Arch. Mech., 38, 1–2, pp. 117–126, 1986

A remark on kinematic hardening
II. The consequences of introducing yield conditions(*)

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IN A PRECEDING paper [3], a time-dependent theory of plastic multi-component materials was developed. In the present paper yield conditions are introduced for each of the plastic subelements. Investigating quasi-static loading processes with yielding and hardening plastic subelements, we arrive at flow laws of the plastic potential type, and from identifying the limit of hardening with the thermodynamical limit of vanishing dissipation it is concluded that workhardening is necessarily of Ziegler's type. The limit of hardening amounts to a second kind of yield condition similar to the loading surfaces in Phillips' theory. A simple model describing this kind of failure is presented for illustration. Finally an invariant representation of unsymmetrically deforming yield surfaces is given.

W poprzedniej pracy [3] przedstawiono zależną od czasu teorię wieloskładnikowych materiałów. W tej pracy wprowadza się warunki plastyczności dla każdego subelementu plastycznego. Badając procesy quasi-statycznego obciążenia z płynięciem i wzmocnieniem plastycznych subelementów, dochodzimy do praw płynięcia typu potencjału plastycznego. Identyfikując granicę wzmocnienia z termodynamicznym ograniczeniem zanikającej dysypacji wnioskuje się, że wzmocnienie musi być typu Zieglera. Granica wzmocnienia jest równoznaczna drugiemu typowi warunku plastyczności, zbliżonemu do powierzchni obciążenia w teorii Phillipsa. Dla ilustracji przedstawiono prosty model opisujący ten rodzaj zniszczenia. Na zakończenie podano reprezentację niezmiennikową niesymetrycznie deformujących się powierzchni plastyczności.

В предыдущей работе [3] была разработана зависящая от времени теория многокомпонентных материалов. В этой работе вводятся условия пластичности для каждого пластического субэлемента. Исследуя процессы квазистатического нагружения с течением и пластическим упрочнением субэлементов приводит к законам течения типа пластического потенциала. Идентифицируя границы упрочнения с термодинамическим ограничением исчезающей диссипации, приходим к выводу, что упрочнение должно быть типа Циглера. Граница упрочнения эквивалентна второму типу условия пластичности, близкому к поверхности нагружания в теории Филлипса. С целью иллюстрации приведены примеры инвариант несимметрически деформирующихся поверхностей пластичности.

1. Introduction

FOR ALMOST sixty years it is a well-established idea that plastic behaviour of metals under general loading conditions, and especially in the case of cyclic loading, must be interpreted as an interplay of a multitude of differently yielding plastic subelements (cf. Z. MRÓZ [1], or O. ALMROTH [2], for instance).

In a preceding paper [3] a rather general thermo-mechanically consistent time-dependent theory describing plastic multi-component materials was developed, without introducing

(*) The paper has been presented at 3th German-Poljsh Symposium on Mechanics of Inelastic Solids and Structures, Bad Honnef, September 1984.

the concept of yield conditions. Thus the independence of the thermodynamical arguments of supposing yield conditions existing or not existing was demontrated. Two questions were left open there:

1) how to construct laws of plastic flow in accordance with physical reality, and

2) how to decide, whether Prager-hardening or Ziegler-hardening or a combination of both, is the correct one.

In the present paper we shall answer these questions by introducing the most characteristic feature of plastic flow, namely the existence of yield conditions for the different plastic subelements.

2. The main results of [3]

In order to base the theory on a solid kinematical foundation we have started in [3] from the multiplicative decomposition of the deformation gradient **F**,

$$\mathbf{F} = \mathbf{F}_{(e)}\mathbf{F}_{(i)}, \tag{2.1}$$

according to the Eckart–Kondo concept of an intermediate configuration. Equation (2.1) amounts to an additive decomposition of the velocity gradient

$$\mathbf{G} :={}^{t}(\nabla\otimes\mathbf{v}) \tag{2.2}$$

according to

$$\mathbf{G} = \mathbf{G}_{(e)} + \frac{1}{3}e_{(i)}\mathbf{1} + \mathbf{\Phi} \tag{2.3}$$

(**1** means the unit tensor) with

$$\mathbf{G}_{(e)} = \dot{\mathbf{F}}_{(e)}\mathbf{F}_{(o)}^{-1} \tag{2.4}$$

representing the *elastic velocity gradient* and with

$$e_{(i)} = (\ln\det\mathbf{F}_{(i)}) \tag{2.5}$$

meaning the *inelastic expansion* whereas

$$\mathbf{\Phi} = {}^{t}\mathbf{\Phi}, \quad \mathrm{tr}\mathbf{\Phi} = 0, \tag{2.6}$$

indicate the volume conserving symmetric *plastic flow tensor*. From Eqs. (2.3) and (2.4) we have obtained the evolution equation for the elastic deformation gradient $\mathbf{F}_{(e)}$,

$$\dot{\mathbf{F}}_{(e)} = \left({}^{t}(\nabla\otimes\mathbf{v}) - \frac{1}{3}e_{(i)}\mathbf{1} - \mathbf{\Phi}\right)\mathbf{F}_{(e)}, \tag{2.7}$$

where $e_{(i)}$ and $\mathbf{\Phi}$ must be understood as constitutive functions of the stresses as well as of other variables describing the material state.

A second fundamental additive decomposition is that of Cauchy's stress tensor **T**,

$$\mathbf{T} = \boldsymbol{\sigma} + \boldsymbol{\zeta} \tag{2.8}$$

with

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}(\mathbf{D}) \tag{2.9}$$

(**D** means the deformation rate tensor) representing the *stress tensor of internal friction*, and with $\boldsymbol{\sigma}$ indicating the *elastic stress tensor*

$$\boldsymbol{\sigma} = 2\varrho \mathbf{F}_{(e)} \frac{\partial \psi}{\partial \mathbf{C}_{(e)}} {}^t\mathbf{F}_{(e)}, \tag{2.10}$$

where

$$\mathbf{C}_{(e)} = {}^t\mathbf{F}_{(e)}\mathbf{F}_{(e)} \tag{2.11}$$

is the *elastic right Cauchy–Green stretch tensor* and ψ represents the specific free energy.

It is noteworthy that the multiplicative structure of Eq. (2.1) manifests itself in the form of the two equations (2.7) and (2.10) only. This has an important consequence for the case of *isotropy* [4], because $\boldsymbol{\sigma}$ becomes a function of the tensor

$$\mathbf{B}_{(e)} := \mathbf{F}_{(e)}{}^t\mathbf{F}_{(e)} \tag{2.12}$$

then, that means

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{B}_{(e)}) \tag{2.13}$$

($\mathbf{B}_{(e)}$ is the *elastic left Cauchy–Green stretch tensor* or *elastic Finger stretch tensor*) so that Eq. (2.7) may be substituted by

$$\dot{\mathbf{B}}_{(e)} = \left({}^t(\nabla \otimes \mathbf{v}) - \frac{1}{3} e_{(i)}\mathbf{1} - \boldsymbol{\Phi}\right)\mathbf{B}_{(e)} + \mathbf{B}_{(e)}\left(\nabla \otimes \mathbf{v} - \frac{1}{3} e_{(i)}\mathbf{1} - \boldsymbol{\Phi}\right). \tag{2.14}$$

Evidently, by reducing the number of variables, the problem of integration is therewith simplified.

For metals, that means in the case of plasticity, the norm of the *elastic deformation tensor*

$$\boldsymbol{\epsilon}_{(e)} := \frac{1}{2}(\mathbf{B}_{(e)} - \mathbf{1}) \tag{2.15}$$

is a small number in any case,

$$\|\boldsymbol{\epsilon}_{(e)}\| \ll \mathbf{1}, \tag{2.16}$$

so that Eq. (2.14) may be approximated by

$$\dot{\boldsymbol{\epsilon}}_{(e)} = \mathbf{D} - \left(\frac{1}{3} e_{(i)}\mathbf{1} + \boldsymbol{\Phi}\right). \tag{2.17}$$

It is essential to note that Eq. (2.17) — though being identical with the corresponding equation of the small deformation theory — is valid for *arbitrary* plastic deformations. Furthermore it should be noted that the factorial structure of Eq. (2.1) does not any more appear explicitly if we consider $e_{(i)}$ and $\boldsymbol{\Phi}$ as constitutive functions of the stresses and similar other variables, as has been done in [3].

Next, assuming a multitude of plastic subelements (1), ..., (*n*) we were lead to

$$\boldsymbol{\Phi} = \sum_{j=1}^{n} \boldsymbol{\Phi}_{(j)} \tag{2.18}$$

with $\boldsymbol{\Phi}_{(j)}$ representing the plastic flow tensor of the *j*-th plastic subelement, and with each $\boldsymbol{\Phi}_{(j)}$ being symmetric and deviatoric like $\boldsymbol{\Phi}$.

Then, introducing a linear relation connecting the internal state rates $\dot{q}_1, \dot{q}_2, \ldots$ and the flow tensors $\mathbf{\Phi}_{(j)}$, we obtained a very transparent bilinear representation of the local dissipation function δ_{loc}. Discarding internal friction, inelastic expansion, and anelasticity, δ_{loc} reduced to a purely *plastic dissipation function*

$$\delta_p = \sum_{j=1}^{n} \delta_{(j)},$$

$$\delta_{(j)} := \mathbf{\Phi}_{(j)} \cdot \boldsymbol{\tau}_{(j)}, \tag{2.19}$$

$$\boldsymbol{\tau}_{(j)} := \boldsymbol{\tau} - \boldsymbol{\alpha}_{(j)},$$

with $\boldsymbol{\tau}$ meaning the deviatoric elastic shear-stress tensor, and with $\boldsymbol{\alpha}_{(j)}$ representing the internal back-stress tensor of the j-th subelement. The effective shear-stresses $\boldsymbol{\tau}_{(j)}$ must be interpreted as the thermodynamic forces causing the irreversible fluxes $\mathbf{\Phi}_{(j)}$ so that these should be looked at as constitutive tensor-valued functions

$$\mathbf{\Phi}_{(j)} = \mathbf{\Phi}_{(j)}(\boldsymbol{\tau}_{(1)}, \ldots, \boldsymbol{\tau}_{(n)}; \vartheta, \boldsymbol{\alpha}_{(1)}, \ldots, \boldsymbol{\alpha}_{(n)}), \quad j = 1, \ldots, n, \tag{2.20}$$

fulfilling the thermodynamic conditions

$$\delta_{(j)} := \mathbf{\Phi}_{(j)} \cdot \boldsymbol{\tau}_{(j)} \geqslant 0, \tag{2.21}$$

in accordance with Onsager's general philosophy of thermodynamics of irreversible processes. The functions (2.20) will be called the *laws of plastic flow*.

Finally, as to the time-dependence of the tensors $\boldsymbol{\alpha}_{(j)}$ we assumed independence of the plastic subelements, thus establishing *laws of workhardening including recovery* which most naturally read

$$\overset{\nabla}{\boldsymbol{\alpha}}_{(j)} = \boldsymbol{\varkappa}_{(j)} \cdot \mathbf{\Phi}_{(j)} - \gamma_{(j)} e^{-\frac{Q_{(j)}}{R\vartheta}} \boldsymbol{\alpha}_{(j)}, \tag{2.22}$$

with $\overset{\nabla}{\boldsymbol{\alpha}}_{(j)}$ indicating the corotational (or Jaumann) derivative of $\boldsymbol{\alpha}_{(j)}$. $\gamma_{(j)}$ and $Q_{(j)}$ meaning the coefficient and the activation energy of recovery of the j-th plastic subelement, respectively, and $\boldsymbol{\varkappa}_{(j)}$ represents a fourth-order projection tensor of the following form:

$$\boldsymbol{\varkappa}_{(j)} = \varkappa'_{(j)}\left(\mathbf{S} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right) + \varkappa''_{(j)} \frac{\boldsymbol{\tau}_{(j)}\otimes\boldsymbol{\tau}_{(j)}}{\boldsymbol{\tau}_{(j)} \cdot \boldsymbol{\tau}_{(j)}}, \tag{2.23}$$

where $\mathbf{S}$ is the fourth-order unit tensor on the space of symmetric second-order tensors, whereas $\varkappa'_{(j)}$ and $\varkappa''_{(j)}$ indicate *moduli of Prager-hardening* and *Ziegler-hardening*, respectively.

We now turn to the question how to specify the constitutive Equations (2.20) and (2.23) if we introduce yield conditions for the plastic subelements.

3. The consequences of introducing yield conditions

The primary problem of a time-dependent theory of plasticity with a yield condition is the proof that both phenomena are compatible, namely the evolutionary character of the theory on the one hand and, on the other hand, the demand that every incremental loading which leads into the plastic range readjusts itself to the yield surface. A secondary problem is the question how to interpret the limit case of perfectly plastic behaviour from the viewpoint of irreversible thermodynamics.

Now we introduce n yield functions, $f_{(1)}, \ldots, f_{(n)}$, determining the *n yield conditions* of the n subelements

(3.1) $$\mathbf{\Phi}_{(j)} \neq \mathbf{0} \Leftrightarrow f_{(j)} > 0, \quad j = 1, \ldots, n.$$

(Note that in a time-dependent theory the elastic range is a closed set in stress space). According to the *hypothesis of independent plastic subelements*, a change of a yield function $f_{(j)}$ by hardening may only be effected by a change of the respective back-stress tensor $\boldsymbol{\alpha}_{(j)}$ alone. We thus assume

(3.2) $$\left.\begin{array}{l} f_{(1)} = f_{(1)}(\boldsymbol{\tau}, \boldsymbol{\alpha}_{(1)}; \vartheta, \ldots), \\ \cdots\cdots\cdots\cdots \\ f_{(n)} = f_{(n)}(\boldsymbol{\tau}, \boldsymbol{\alpha}_{(n)}; \vartheta, \ldots), \end{array}\right\}$$

and naturally

(3.3) $$\left.\begin{array}{l} \mathbf{\Phi}_{(1)} = \mathbf{\Phi}_{(1)}(\boldsymbol{\tau}, \boldsymbol{\alpha}_{(1)}; \vartheta, \ldots), \\ \cdots\cdots\cdots\cdots \\ \mathbf{\Phi}_{(n)} = \mathbf{\Phi}_{(n)}(\boldsymbol{\tau}, \boldsymbol{\alpha}_{(n)}; \vartheta, \ldots), \end{array}\right\}$$

too, with the points indicating other variables. In this paper, however, we shall neglect other variables, as well as temperature effects and recovery.

Next, we consider a quasi-static loading process. After having obtained a static state with all flow tensors $\mathbf{\Phi}_{(j)}$ vanishing, we add a further stress increment $\Delta\boldsymbol{\tau}$ leading into the plastic range of one, or more, of the yield conditions by shifting the values of the respective yield functions $f_{(y)}$ from $f_{(y)} = 0$ to positive values:

(3.4) $$\Delta f_{(y)} = \frac{\partial f_{(y)}}{\partial \boldsymbol{\tau}} \cdot \Delta\boldsymbol{\tau} > 0.$$

We thus obtain plastic flows $\mathbf{\Phi}_{(y)}$ which, in turn, entail workhardening according to

(3.5) $$\overset{\nabla}{\boldsymbol{\alpha}}_{(y)} = \boldsymbol{\varkappa}_{(y)} \cdot \mathbf{\Phi}_{(y)},$$

and therewith induce rates of change of the $f_{(y)}$ because of hardening:

(3.6) $$\dot{f}_{(y)}\big|_{\text{hard}} = \frac{\partial f_{(y)}}{\partial \boldsymbol{\alpha}_{(y)}} \cdot (\boldsymbol{\varkappa}_{(y)} \cdot \mathbf{\Phi}_{(y)}),$$

with $\boldsymbol{\varkappa}_{(y)}$ representing the Prager–Ziegler tensor of hardening moduli of the y-th plastic subelement, according to Eq. (2.23).

In order to restore a static state, we demand that

(3.7) $$\dot{f}_{(y)}\big|_{\text{hard}} < 0$$

so that the yield functions $f_{(y)}$ of the yielding plastic subelements return to zero after elapsing the *periods of plastic flow*

(3.8) $$\Delta t_{(y)} = \left| \frac{\Delta f_{(y)}}{\dot{f}_{(y)}\big|_{\text{hard}}} \right|.$$

The fourth-order tensors $\boldsymbol{\varkappa}_{(J)}$ are nonnegative, according to experimental experience. Therefore we conclude from Eqs. (3.6) and (3.7) that necessarily

$$\begin{aligned} \boldsymbol{\Phi}_{(J)} &= -\varphi_{(J)} \frac{\partial f_{(J)}}{\partial \boldsymbol{\alpha}_{(J)}}, \\ \varphi_{(J)} &= \begin{cases} 0, & \text{if} \quad f_{(J)} \leqslant 0, \\ \lambda_{(J)} > 0, & \text{if} \quad f_{(J)} > 0, \end{cases} \end{aligned} \tag{3.9}$$

thus verifying the yield functions $f_{(J)}$ to represent *plastic potentials*.

Inserting Eqs. (2.23) and (3.9) into Eq. (3.6), we arrive at

$$\dot{f}_{(y)}|_{\text{hard}} = -\frac{1}{\lambda_{(y)}} \left(\varkappa'_{(y)} ||\boldsymbol{\Phi}_{(y)}||^2 + \varkappa''_{(y)} \frac{\delta^2_{(y)}}{||\boldsymbol{\tau}_{(y)}||^2} \right), \tag{3.10}$$

with $\delta_{(y)}$ meaning the dissipation function of the y-th yielding subelement. $\delta_{(y)}$ tending to zero represents an unattainable limit case, according to the second part of the second law of thermodynamics. In order to identify the thermodynamical limit case with the quasi-static limit case of $\Delta t_{(y)}$ tending to infinity, we must assume

$$\varkappa'_{(y)} = 0, \tag{3.11}$$

thus specializing workhardening to pure *Ziegler-hardening*,

$$\begin{aligned} \boldsymbol{\varkappa}_{(J)} &= \varkappa_{(J)} \frac{\boldsymbol{\tau}_{(J)} \otimes \boldsymbol{\tau}_{(J)}}{\boldsymbol{\tau}_{(J)} \cdot \boldsymbol{\tau}_{(J)}}, \\ \varkappa_{(J)} &> 0. \end{aligned} \tag{3.12}$$

NOTE. The functions

$$F_{(J)}(\boldsymbol{\tau}, \boldsymbol{\alpha}_{(J)}) := -\frac{\partial f_{(J)}(\boldsymbol{\tau}, \boldsymbol{\alpha}_{(J)})}{\partial \boldsymbol{\alpha}_{(J)}} \cdot (\boldsymbol{\tau} - \boldsymbol{\alpha}_{(J)}) \tag{3.13}$$

may be considered as a *second kind of yield functions* because

$$F_{(J)} = 0 \tag{3.14}$$

represents the *limit of hardening* beyond which further quasi-static loading becomes impossible so that it might be called the limit case of *ideal plastic behaviour*. The thermodynamic limit conditions $F_{(J)} = 0$ indicate a new *criterion of material failure*, remembering A. PHILLIPS' concept of loading surfaces [5, 6]. In contradiction to Phillips' idea, however, the surfaces $F_{(J)} = 0$ are moving surfaces like the ordinary yield surfaces $f_{(J)} = 0$.

4. A simple model describing material failure

For simplicity we consider one plastic subelement only. Let us assume a Mises-type yield function showing kinematic hardening, as well as isotropic hardening according to

$$\begin{aligned} f(\boldsymbol{\tau}, \boldsymbol{\alpha}) &= \frac{1}{2} ||\boldsymbol{\tau} - \boldsymbol{\alpha}||^2 - k^2(\boldsymbol{\alpha}), \\ k^2(\boldsymbol{\alpha}) &= k_0^2 - \frac{a}{2} ||\boldsymbol{\alpha}||^2. \end{aligned} \tag{4.1}$$

The associated second kind of yield function (Eq. (3.13)) reads

$$F(\boldsymbol{\tau}, \boldsymbol{\alpha}) = \big(\boldsymbol{\tau}-(1+a)\boldsymbol{\alpha}\big)\cdot(\boldsymbol{\tau}-\boldsymbol{\alpha}), \tag{4.2}$$

and the quasi-static incremental hardening equation (3.27) amounts to

$$\Delta\boldsymbol{\alpha} = \frac{(\boldsymbol{\tau}-\boldsymbol{\alpha})\cdot\Delta\boldsymbol{\tau}}{F(\boldsymbol{\tau}, \boldsymbol{\alpha})}(\boldsymbol{\tau}-\boldsymbol{\alpha}). \tag{4.3}$$

We confine the discussion to the *uniaxial tension test* starting from a virgin material ($\boldsymbol{\alpha} = \mathbf{0}$) so that

$$\begin{gathered} \boldsymbol{\tau} = \tau\mathbf{e}_{\text{tension}}, \\ \tau = \sqrt{\frac{2}{3}}\sigma, \end{gathered} \tag{4.4}$$

(σ: uniaxial tensional stress; $\mathbf{e}_{\text{tension}}$: unit vector of uniaxial tension test in deviatoric stress space), and

$$\boldsymbol{\alpha} = \alpha\mathbf{e}_{\text{tension}} \tag{4.5}$$

with

$$\alpha = 0, \quad \text{if} \quad \sigma \leqslant \sigma_e = \sqrt{3}k_0, \tag{4.6}$$

where σ_e indicates the elastic limit. If σ becomes greater than σ_e, we may either integrate the differential equation

$$\frac{d\alpha}{d\tau} = \frac{\tau-\alpha}{\tau-(1+a)\alpha} \tag{4.7}$$

taking into consideration the initial condition Eq. (4.6), or we may deduce α directly from solving the quadratic equation $f = 0$, so that

$$\begin{gathered} \alpha = \frac{\tau}{1+a} - \sqrt{\left(\frac{\tau}{1+a}\right)^2 - \frac{\tau^2-2k_0^2}{1+a}}, \\ \tau \geqslant \sqrt{2}k_0, \end{gathered} \tag{4.8}$$

where the sign of the root has been determined from the relation (4.6).

Material failure $F = 0$ (indicating ideal plasticity) is reached if the denominator of Eq. (4.7) vanishes, or, which is the same, if the radicand of Eq. (4.8) vanishes. Thus the *stress of failure* σ_f amounts to

$$\sigma_f = \sqrt{1+\frac{1}{a}}\,\sigma_e \tag{4.9}$$

so that material failure occurs if, and only if, $a > 0$, and the hardening parameter α assumes the finite value of failure

$$\alpha_f = \sqrt{\frac{2}{3}}\frac{\sigma_f}{1+a}, \tag{4.10}$$

then.

5. Invariant description of unsymmetrically deforming yield sufaces

Because of workhardening, yield surfaces may undergo considerable changes of shape during plastic deformation (cf. A. PHILLIPS [7], or E. SHIRATORI *et al.* [8], for instance). Assuming only one single yield condition, at first, we are thus confronted with the question how to construct yield surfaces

$$f(\boldsymbol{\tau}, \boldsymbol{\alpha}) = 0 \tag{5.1}$$

in an invariant manner so that all essential features of the strange behaviour are included which we observe in combined loading experiments.

First we draw attention to the observation of G. I. TAYLOR and H. QUINNEY [9] that Mises' yield function

$$f_M := J_2 - k^2, \tag{5.2}$$

$$J_2 := \frac{1}{2}\operatorname{tr}\boldsymbol{\tau}^2, \tag{5.3}$$

describes virgin materials very well. In the σ, τ-plane of combined tension and torsion this function reads

$$f_M = \frac{1}{3}\sigma^2 + \tau^2 - k^2. \tag{5.4}$$

Next we state the following invariant distinction of pure tension and pure torsion:

(a) pure tension:

$$\begin{aligned} \det\boldsymbol{\tau} &\neq 0, \\ \operatorname{discr}\boldsymbol{\tau} &= 0; \end{aligned} \tag{5.5}$$

(b) pure torsion:

$$\begin{aligned} \det\boldsymbol{\tau} &= 0, \\ \operatorname{discr}\boldsymbol{\tau} &\neq 0, \end{aligned} \tag{5.6}$$

with the determinant and the discriminant being expressed by

$$\det\boldsymbol{\tau} = J_3, \tag{5.7}$$

and

$$\operatorname{discr}\boldsymbol{\tau} = 4J_2^3 - 27J_3^2 \tag{5.8}$$

with J_2 given by Eq. (5.3) and J_3 defined as

$$J_3 := \frac{1}{3}\operatorname{tr}\boldsymbol{\tau}^3. \tag{5.9}$$

Contrary to the determinant, however, the discriminant is positive for nonvanishing τ so that we must consider rather the square-root of the discriminant. We are thus lead to distinguish the two third-degree stress invariants

$$\begin{aligned} S &:= \sqrt{\frac{27}{4}}J_3, \\ T &:= \left(J_2^3 - \frac{27}{4}J_3^2\right)^{1/2} \end{aligned} \tag{5.10}$$

so that

$$(5.11)\qquad -\infty < S < \infty, \quad T = 0,$$

describes *pure tension*, whereas *pure torsion* is represented by

$$(5.12)\qquad S = 0, \quad -\infty < T < \infty.$$

We then obtain

$$(5.13)\qquad J_2^3 = S^2 + T^2,$$

and Mises' yield criterion for virgin materials can equivalently be expressed by help of a yield function of degree six

$$(5.4)\qquad f_M^* := S^2 + T^2 - k^6,$$

because

$$(5.15)\qquad f_M^* \equiv f_M \cdot (J + k^2 J_2 + k^4).$$

After this it is quite evident how to formulate yield functions describing moving and unsymmetrically deforming yield surfaces. Especially, if we are considering different plastic subelements (j) with different back-stress tensors $\boldsymbol{\alpha}_{(j)}$, we shall define corresponding S- and T-invariants

$$(5.16)\qquad \begin{aligned} S_{(j)} &:= \sqrt{\frac{3}{4}} \operatorname{tr} \boldsymbol{\tau}_{(j)}^3, \\ T_{(j)} &:= \left[\frac{1}{8} (\operatorname{tr} \boldsymbol{\tau}_{(j)}^2)^3 - \frac{3}{4} (\operatorname{tr} \boldsymbol{\tau}_{(j)}^3)^2 \right]^{1/2}, \end{aligned}$$

with $\boldsymbol{\tau}_{(j)}$ meaning the effective shear stress

$$(5.17)\qquad \boldsymbol{\tau}_{(j)} := \boldsymbol{\tau} - \boldsymbol{\alpha}_{(j)}.$$

Then a moving and deforming yield surface $f_{(j)}^{(\exp)} = 0$, determined experimentally can be approximated by help of a dimensionless yield function of the kind

$$(5.18)\qquad f_{(j)} := \frac{S_{(j)}^2}{k_{S,j}^6} + \frac{T_{(j)}^2}{k_{T,j}^6} - 1$$

with the denominators representing positive functions

$$(5.18)\qquad \begin{aligned} k_{S,j}^6 &= g_{S,j}(S_{(j)}; T_{(j)}, \boldsymbol{\alpha}_{(j)}), \\ k_{T,j}^6 &= g_{T,j}(T_{(j)}; S_{(j)}, \boldsymbol{\alpha}_{(j)}), \end{aligned}$$

where the first arguments indicate the possibility of unsymmetrical deformation if $g_{s,j}$ not even regards $S_{(j)}$, or if $g_{T,j}$ not even regards $T_{(j)}$, respectively.

Attention should be drawn to the fact that Eqs. (5.18) do not depend on the four simultaneous invariants of the tensors $\boldsymbol{\tau}_{(j)}$ and $\boldsymbol{\alpha}_{(j)}$. This is justified by stating that these equations seem to be sufficiently general for describing experimental reality.

6. Additional remarks

Two important questions have not been touched at all in this paper:

(1) What do the plastic subelements really mean?

(2) What are the consequences if the yield functions are no sharp step functions?

As to the first point it must be stressed that in continuum mechanics we are forced to discard in principle all microscopic details. Nevertheless, the physics of microscopic phenomena plays an important heuristic role. As to multicomponent models, especially, Z. MRÓZ [1] and O. ALMROTH [2] have given literary surveys to which I would like to add three more references, namely L. PRANDTL [10], and J. BURBACH [11, 12].

As to the second point, no drastic changes are afforded if the noncontinuous yield functions become continuous ones. As a consequence of the time-dependent character of our theory, we only need to substitute the scalar functions $\varphi_{(J)}$ in Eq. (3.9) by smooth ones.

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Received March 13, 1985.