Arch. Mech., 39, 6, pp. 693–698, Warszawa 1987

BRIEF NOTES

Properties of the sensitivity functions
Part I. Differentiable functions

V. KOMKOV (DAYTON, OHIO)

Assuming differentiable properties for both the state of the system and for the sensitivity, we show a close relation between the state and the sensitivity of the state to design changes.

1. The state sensitivity function

Let the state equations be of the form $\Phi_i(\mathbf{x}, t, u_x, u_t) = 0$, $i = 1, 2, \ldots, m$, $x \in \mathsf{R}_n$, $t \in \mathsf{R}_+$, u: $\{(\Omega \subset \mathsf{R}^n) \times \mathsf{R}_+)\} \to \mathsf{R}$. u_x, u_t are defined in the weak (Sobolev sense). u is called the state, t will be identified with the time, $\mathbf{x}$ with generalized "position" coordinates. The state $u(t, \mathbf{x})$ is a vector in a normed space H_1. Some holonomic constraints are assigned to the system $\psi_j(t, \mathbf{x}, u) \leqslant 0$, or $\psi_k(t, \mathbf{x}, u) = 0$. ψ_α: $\{\mathsf{R}_+ \times \Omega \times H_1\} \to \mathsf{R}$. The spatial domain $\Omega \subset \mathsf{R}^n$ is a manifold, called the (admissible) kinematic manifold. The state function $\mathbf{u}$ depends on a number of physical parameters $h_i(\mathbf{x})$, $i = 1, 2, \ldots k$. The vector $\mathbf{h}(\mathbf{x})$ is an element of a Banach space B, that could be $\mathscr{L}_1$ or $\mathscr{L}_\infty$. $\mathbf{h}(\mathbf{x})$ belongs to $\varkappa \subset B$ called the admissible design. To define the sensitivity function u_h, we first defined the Gateaux difference in the direction of the vector $\boldsymbol{\eta} \in \mathsf{B}$: $\Delta_{\varepsilon\eta} u(\mathbf{h}_0) = u(\mathbf{x}, t, \mathbf{h}_0(\mathbf{x}) + \varepsilon\boldsymbol{\eta}(\mathbf{x})) - u(\mathbf{x}, t, \mathbf{h}_0(\mathbf{x}))$ where ε is a real number.

Let ϕ be a mapping from B into H_1, let $\langle , \rangle$ denote a bilinear product $\langle f, g\rangle \in H_1$, $f \in \mathsf{B}$.

If $\Delta_{\varepsilon\eta} u(\mathbf{h}_0)$ can be represented in the form

$$\Delta_{\varepsilon\eta} u(\mathbf{h}_0) = \varepsilon < \phi(t, \mathbf{x}, \mathbf{h}), \eta > + \zeta(\varepsilon\boldsymbol{\eta})$$

such that $\dfrac{||\zeta(\varepsilon\boldsymbol{\eta})||}{||\varepsilon\boldsymbol{\eta}||} \to 0$ as $\varepsilon \to 0$ and $\phi(t, \mathbf{x}, \mathbf{h})$ is independent of η, then we shall call $\phi(t, \mathbf{x}, \mathbf{h})$ the Frèchet derivative of u with respect to $\mathbf{h}$ and denote $\phi = u_h$. (See [1] and [2].) For small values of ε we designate the function $\mathbf{u_h}$ the "sensitivity" function for the state of the system.

2. The sensitivity equation

We introduce some elementary lemmas permitting us to manipulate the properties of the sensitivity function u_h:$B \to H_1$. Strong or weak continuity and differentiability of u_h with respect to the variables $\mathbf{x}$, t is defined in the usual manner.

LEMMA 1. If continuous differentiability is assumed for $\mathbf{u}$ and for $h(x)$ and u_h exists, then $\partial u_h/\partial x = (u_x)_h$.

THEOREM 1. *If suffcient smoothness assumptions are made, then the sensitivity function for the system Φ and the constraints χ*

$$(1)\qquad \Phi(u, u_x, u_t, \mathbf{x}, t, h(x)) = 0, \quad \chi_i(t, \mathbf{x}, u) \leqslant 0,$$

obeys a quasi-linear equation

$$(2)\qquad a_2 \cdot (u_h)_t + \sum_{i=l}^{n} a_{l_i} \cdot (u_h)_{x_i} + a_0 \cdot u_h + c = 0.$$

Proof. Assuming that all the derivatives that are displayed here exist, we use the chain rule

$$\frac{d\Phi}{dh} = \frac{\partial\Phi}{\partial h} + \frac{\partial\Phi}{\partial u} u_h + \frac{\partial\Phi}{\partial u} (u_h)_x + \frac{\partial\Phi}{\partial u_t} (u_h)_t + \sum_i \left(\lambda_i \frac{\partial\chi_i}{\partial u}\right) u_h = 0,$$

or

$$(2')\qquad C_0(\boldsymbol{\lambda}, t, x, u, u_x, u_t h) + a_0(\ldots) u_h + \mathbf{a}_1(\ldots)(u_h)_x + a_2(\ldots)(u_h)_t = 0.$$

We note that $\Phi(x, t, u, u_x, u_t, \mathbf{h}) = 0$ for any $\mathbf{h} \in \varkappa \subset B$, that is for any admissible value of the design variable $\mathbf{h}(\mathbf{x}) \cdot \lambda_i \geqslant 0$ are Lagrangian multipliers.

LEMMA 3. If $\mathscr{L}$ is a linear differential operator $\mathscr{L} = \mathscr{L}(\mathbf{h})$ and $\mathscr{L}_h$ is defined in the neighborhood of $h_0 \in R^m$ then $(\mathscr{L}u)_h = \mathscr{L}_h u + \mathscr{L} u_h$.

THEOREM 2. *If $\Phi(\ldots) = 0$ is a linear partial differential equation, then the sensitivity equation (2′) is again a partial differential equation of the same form.*

Outline of the proof:

Basically the proof follows from the formula (2′), by putting $\Phi(u, u_x, u_t, x, t, h) = \mathscr{L}(h)$, $\mathbf{u} = q(\mathbf{x}, t)$, where we may consider $\mathbf{u}$ to be a vector rather than scalar element of an appropriate space H. For example, if second derivatives $u_{x_i x_j}$ occur in the state equation, then we replace u by the vector

$$(u, v_1, v_2 \ldots v_n) = \{u, v\} = u_{x_1}, \ldots, v_n = u_{x_n}, \quad \text{or by} \begin{bmatrix} u \\ v_i \end{bmatrix} = \mathbf{u}.$$

Thus $u_{x_i x_j} = u_{i_{x_j}}$.

This is a standard method of conversion of a high order system to a system of first order equations. Thus $\Phi = 0$ is converted to a system of linear differential first order equations

$$\mathscr{L}(\mathbf{h}) u_i = q_i(\mathbf{x}, t), \qquad \mathbf{h} \in \varkappa,$$

which is of the form

$$(3)\qquad \sum_{i,j} a_{ij} U_{i,_{kj}} + \sum_{k} a_k U_{k,_t} + \sum_i b_i U_i + C = q.$$

Thus the sensitivity equation is given by

$$(3')\qquad \sum_{i,j} (a_{ij_\mathbf{h}} U_{i_{x_j}} + a_{ij} U_{i_{\mathbf{h},x}}) + \sum a_k U_{k\mathbf{h},_t} + b_\mathbf{h} U + b U_\mathbf{h} + C_\mathbf{h} = 0,$$

or

$$\sum_{i,j} \alpha_{ij} U_{i_{\mathbf{h},x_j}} + \sum_{k} \alpha_k U_{k_{\mathbf{h},t}} + \sum_{i} \beta_i U_{h_i} + C_{\mathbf{h}} = 0.$$

The constraints have been ignored. It should be clear that if the Lagrangian multipliers are not functions of **h**, then the general term $\sum_i \lambda_i \chi_{h_i}$ will be included in Eq. (3′) modifing only the terms $C_{\mathbf{h}}$ and $\beta_i U_{h_i}$. This completes the outline of the proof.

COMMENTS. If the terms $a_{ij} U_{i,x_j}$, $a_k U_{k,t}$ $b_i U_i$ and c are of the same order of magnitude, and the same may be said of all terms in (3′), then no serious problems arise in computing the sensitivity of the system or the sensitivity of a related functional $J(u, u_x, u, \mathbf{x}, t, \mathbf{h}(x))$. In that case the sensitivity equation is a mirror image of the original state equation.

EXAMPLE. Consider the state function $\hat{u}(h(x), x, t) = u(x, t)$ and the linear equation $m(x, t)\ddot{u}(x, t) + \mathbf{k} \cdot \operatorname{grad}(u) = q(t)$, $(\cdot = \partial/\partial t)$.

The sensitivity equation is

$$m(x, t)\ddot{S}(x, t) + (k \cdot \operatorname{grad} S(x, t) = 0.$$

For a nonlinear system

$$m(x, t)\ddot{u}(x, t) + k(u \cdot \operatorname{grad} u) = q(x, t)$$

the sensitivity equation is

$$m\ddot{s} + k(u \cdot \operatorname{grad} s) + k(s \cdot \operatorname{grad} u) = 0.$$

An example. Burger's equation

(4) $$\Phi(u, u_x, u_t, \varepsilon) = u_t + u u_x - \varepsilon u_{xx} = q(x, t), \qquad u = u(\varepsilon)$$

and $4^{(H)}$ obtained by setting $q(x, t) \equiv 0$.

Then

(5) $$\frac{d\Phi}{d\varepsilon} = \frac{\partial \Phi}{\partial \varepsilon} + \frac{\partial \Phi}{\partial u} u_\varepsilon + \frac{\partial \Phi}{\partial u_x} u_{\varepsilon_x} + \frac{\partial \Phi}{\partial u_{xx}} u_{\varepsilon_{xx}} + \frac{\partial \Phi}{\partial u_t} u_{\varepsilon_t} = -u_{xx} + u_x u_\varepsilon$$
$$+ u u_{\varepsilon_x} - \varepsilon u_{\varepsilon_{xx}} + u_{\varepsilon_t} = -u_{xx} + u_x s + u s_x + s_t - \varepsilon s_{xx} = 0,$$

where $s(x, t, \varepsilon) = u_\varepsilon$.

Behavior of the sensitivity equation for large values of x can be deduced from our assumptions:

$$\lim_{X \to \pm\infty} u(x, t) = u_\infty = \text{const},$$
$$\lim_{X \to \pm\infty} u_x = 0,$$
$$\lim_{X \to \pm\infty} u_{xx} = 0.$$

It is convenient to use non-standard symbolism.

Thus, if $x \in {}^*\mathrm{R}_\infty$, then

$$u(x, t) = u_\infty + \xi,$$
$$u_x(x, t) = \eta,$$
$$u_{xx}(x, t) = \xi.$$

and s satisfies the equation

$$(5') \qquad -\xi+\eta\cdot s+(u_{\infty}+\zeta)s_x+s_t-\varepsilon s_{xx} = 0,$$

where $\xi, \eta, \zeta \in \mu(0)$ (monad of zero).

The original equation ($4^{(H)}$) has a conservation law derived from

$$(4') \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\varepsilon u_x - \frac{u^2}{2}\right).$$

If conditions at infinity are

$$\lim_{X\to\pm\infty} u(x,t) = \text{const.}$$

$$\lim_{X\to\pm\infty} u_x = 0$$

then

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)\,dx = [\varepsilon u_x - u^2/2]_{-\infty}^{\infty} = 0$$

and

$$\int_{X=-\infty}^{X=\infty} u(x,t)\,dx$$

is an invariant.

A similar conservation law can be derived for Eq. (4) if, for example,

$$q(x,t) = q(x) = \frac{\partial Q(x)}{\partial x},$$

with

$$\lim_{X\to\pm\infty} Q(x) = \text{const.}$$

The invariants of the sensitivity equation can be derived from the observation that Eq. (5) can be rewritten as

$$\frac{\partial}{\partial x}\left(+u_x-(us)+\varepsilon\cdot s_x\right) = s_t.$$

Thus, if

$$\lim_{X\to\pm\infty} \sup\{|s|+|s_x|\} < \infty,$$

and

$$[u_x-(u\cdot s)+\varepsilon s_x]_{-\infty}^{\infty} = 0,$$

then

$$\int_{-\infty}^{\infty}(s_t)\,dx = \frac{d}{dt}\int_{-\infty}^{\infty} s\,dx = 0,$$

and

$$\int_{-\infty}^{\infty} s(x, t)\,dx \equiv \text{const}.$$

Therefore, redesigning Burgers' flow by changing the parameter ε amounts to redistributing the sensitivity function $s(x, t)$ over the real line $(-\infty < x < \infty)$ while keeping "the total amount of sensitivity" $\left(\int_{-\infty}^{\infty} s\,dx\right)$ constant.

Differentiability of the state

Let $a(u, v)$ be a bilinear form, i.e. $a(u, v) = \langle \mathscr{L}(h)u, v\rangle_{\mathscr{L}^2(\Omega)}$ (according to the Lax-Milgram representation theorem). u is the state (or displacement) of the system. $\langle\ ,\ \rangle$ is the usual $\mathscr{L}^2(\Omega)$ product. The notation $\mathscr{L}(h)$ implies that the operator $\mathscr{L}$ depends on the design vector h.

If $\mathscr{L}(h)$ is a positive definite operator, then $\mathscr{L}(h)$ can be written in the form $\mathscr{L}(h) = A(h)A^*(h)$.

Thus $a(u, v) = \langle Au, Av\rangle$ where A depends on a design vector $\mathbf{h}(x)$. u and v are $\mathscr{L}^2(\Omega)$ functions, but v can be selected from a set of admissible generalized forces where could be a dense subset of $\mathscr{L}^2(\Omega)$.

Differentiability of $a(u, v)$ with respect to the design $\mathbf{h}$ implies that

(6) $$\langle \mathscr{L}(\mathbf{h}+\varepsilon\boldsymbol{\eta})u, v\rangle = \langle \mathscr{L}(\mathbf{h})u, v\rangle + \langle \mathscr{L}u, v\rangle_{\mathbf{h}} + r(u, v, \varepsilon\boldsymbol{\eta}), \qquad \varepsilon \in R, \qquad ||\boldsymbol{\eta}|| \leqslant 1,$$

where

(6') $$\frac{||\zeta(u, v)||}{\varepsilon||\boldsymbol{\eta}||} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

However, whether $a(u, v)$ is Fréchet differentiable or not, we have

$$\langle \mathscr{L}(\mathbf{h}+\varepsilon\boldsymbol{\eta})u, v\rangle = \langle \mathscr{L}(\mathbf{h})u, v\rangle + \varepsilon < \alpha(\boldsymbol{\eta})\mathscr{L}(\mathbf{h})u,\ v > + \varepsilon^2\langle \beta(\boldsymbol{\eta})\mathscr{L}(\mathbf{h})u, v\rangle + \alpha(\varepsilon^3).$$

KATO [5] shows that a perturbation of a linear operator $\mathscr{L}(\mathbf{h})$ results in the following expansion:

(6) $$\langle \mathscr{L}(\mathbf{h}+\varepsilon\boldsymbol{\eta})u, v\rangle = \langle \mathscr{L}(\mathbf{h})u, v\rangle + \langle \mathscr{L}u, v_{\mathbf{h}}\rangle + r(u, v, \varepsilon\boldsymbol{\eta}),$$

where $r(u, v, \varepsilon\eta)$ may not have the property (6'). However, if the remainder term $r(u, v, \varepsilon\eta)$ is bounded by $\varepsilon^2||\eta||\cdot a(u, v)$, then the conclusion (6) and (6') follows.

Moreover, there exist operators $\alpha(\boldsymbol{\eta})$, $\beta(\boldsymbol{\eta})$ such that

$$\langle \mathscr{L}(h+\varepsilon\eta)u, v\rangle = a(u, v)_h + \langle \mathscr{L}(h)u, v\rangle + r = < \{A(h) + \varepsilon[(\alpha\cdot A(h)) + \varepsilon\beta A(h)]\}u, A^*(h)v\rangle.$$

The operator $A(\mathbf{h})\cdot(I+\varepsilon\alpha+\varepsilon^2\beta) = A\cdot I_\varepsilon$, replaces $A(\mathbf{h})$.

Here α, β are continuous operators from $\mathscr{L}^2(\Omega)$ to $\mathscr{L}^2(\Omega)$. Thus, if $||\alpha||$ and $||\beta||$ are bounded by some constant and ε is chosen sufficiently small, the operator I_ε is invertible.

It follows after some manipulation that the operator $[\mathscr{L}(\mathbf{h}+\varepsilon\boldsymbol{\eta})]^{-1}$ is defined, and is given by $A(\mathbf{h})^{-1}I_\varepsilon^{-1}A(\mathbf{h})^{-1}$. (A similar line of argumet can be found in' the HAUG–CHOI–KOMKOV monograph [3] and earlier, in the paper of HAUG and ROUSSELET [4]). This means that the Fréchet differentiability of the bilinear functional $a(u, v)$ implies the Fréchet dfferentiability of the state u with respect to the design $\mathbf{h}(\mathbf{x})$, provided the load is either independent of the design, or is a smooth function of the design.

In the static case we have the following corrolaries.

CORROLARY 1. If the basic potential energy functional $a(u, v)$ is a Fréchet differentiable function of the design $\mathbf{h}(x)$, then the sensitivity of the state function $u_{\mathbf{h}}$ is a continuous function of the design.

CORROLARY 2. A discontinuity in the sensitivity function $u_{\mathbf{h}}(x)$ at the point $\mathbf{h} = \mathbf{h}_0(x)$ implies that the potential functional has at best only directional derivatives with respect to the design variable at the point $\mathbf{h}(x) = \mathbf{h}_0(x)$ in the admissible design space.

Comments

Since continuous differentiability of boththe state and the sensitivity function have been assumed, bifurcation phenomena have been automatically excluded from this discussion.

In fact, the discontinuities in derivatives of the sensitivity function accompanying a bifurcation phenomena is the topic of part II of this work.

References

1. R. GATEAUX, *Sur les fonctionelles continues et les fonctionelles analytiques*, Comptes Rendues, **157**, 325–327, 1913.
2. M. FRÉCHET, *Sur la notion differentielle*, J. de Math., **16**, 233–250, 1937.
3. E. J. HAUG, K. K. CHOI and V. KOMKOV, *Design sensitivity analysis of structural systems*, Monographs in Science and Engineering, vol. 177, Academic Press, New York 1986.
4. E. J. HAUG and B. ROUSSELET, *Design sensitivity in structural mechanics, Parts I and II*, J. Struct. Mech., **8**, p. 161–186, 1980.
5. T. KATO, *Perturbation theory for linear operators*, Springer Verlag, Berlin, Heidelberg and New York 1976.

DEPARTMENT OF MATHEMATICS
WINTROP COLLEGE, ROCK HILL, SOUTH CAROLINA, USA.

Received July 14, 1986.