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Nonlinear torsional vibrations of rigid bodies

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WE DISCUSS nonlinear dynamical models of a rigid body working under oscillatory conditions with damping, and fastened at a fixed point, e.g. centre of mass. Special stress is laid on isotropic models, invariant under so-called hyperrotations, i.e. rotations of the rotation vector. For the damping-free (purely oscillatory) case we determine completely degenerate potentials and discuss general solutions of equations of motion. These results can be helpful in designing instruments sensitive to inertial forces and their moments, e.g. sensors of angular acceleration in automatic control systems and navigation.

Omawiamy nieliniowe modele dynamiczne ciała sztywnego pracującego w reżimie oscylacyjnym z ewentualnym tłumieniem i zamocowanego w ustalonym punkcie, np. środku masy. Szczególny nacisk położony został na modele izotropowe, niezmiennicze względem tzw. hiperobrotów, tzn. obrotów wektora obrotu. Dla zagadnień bez tłumienia znalezione zostały całkowicie zdegenerowane (periodyczne) modele potencjalne. Przedyskutowano też ogólne rozwiązanie równań ruchu. Wyniki te mogą być przydatne przy projektowaniu urządzeń reagujących na siły bezwładności i ich momenty, np. czujników przyspieszenia kątowego w układach autoregulacji i przyrządach nawigacyjnych.

Обсуждаем нелинейные динамические модели жесткого тела, работающего в осцилляционном режиме с возможным затуханием и закрепленного в установленной точке, например в центре масс. Особенное внимание обращено на изотропные модели, инвариантные по отношению к т. наз. гипервращениям, т.е. вращениям вектора вращения. Для задач без затухания найдены полностью вырожденные (периодические) потенциальные модели. Обсуждено тоже общее решение уравнений движения. Эти результаты могут быть пригодны при проектировании устройств, реагирующих на силы инецрии и их моменты, например датчиков углового ускорения в системах авторегулировки и в навигационных приборах.

1. Generalized coordinates

IF A REFERENCE configuration is fixed, the configuration space of a rigid body can be identified with $SO[3, \mathsf{R}] \times \mathsf{R}^3$ — the semi-direct product of the three-dimensional group of real rotations and the additive group R^3. In this paper we deal exclusively with a rigid body without translational degrees of freedom (fastened at a fixed point); its configuration space is $SO[3, \mathsf{R}]$. All symbols of vector calculus used below, e.g. scalar and vector products, $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \times \mathbf{B}$ are understood in the R^3-sense.

The choice of generalized coordinates is usually motivated by practical purposes and by peculiar dynamical properties of the object. From the purely geometric point of view, canonical coordinates of the first kind provide the most intuitive parametrization of Lie groups. In the special case of $SO[3, \mathsf{R}]$, these coordinates are identical with components of the so-called *rotation vector* $\mathbf{k} \in \mathsf{R}^3$. The direction of this vector, i.e. $\mathbf{n} = \frac{1}{k}\mathbf{k}$, fixes the oriented rotation axis (right-handed screw rule), and the length k equals the rotation angle. Thus $0 \leqslant k \leqslant \pi$, and the antipodal points on the sphere $k = \pi$ are identified. The rotation operator $R(\mathbf{k})$ is given by

$$R(\mathbf{k})\cdot\mathbf{u} = \cos k\mathbf{u} + k^{-2}(1-\cos k)(\mathbf{k}\cdot\mathbf{u})\mathbf{k} + k^{-1}\sin k\,\mathbf{k}\times\mathbf{u}, \tag{1.1}$$

thus, for small rotations,

$$R(\mathbf{k})\cdot\mathbf{u} \approx \mathbf{u}+\mathbf{k}\times\mathbf{u}, \quad k \approx 0.$$

The character of $\mathbf{k}$ as first-kind canonical coordinates is expressed by the equations

$$R(\mathbf{k}) = \exp(k_j A_j) = \sum_{n=0}^{\infty} \frac{1}{n!}(k_j A_j)^n \quad \text{(summation convention)}, \tag{1.2}$$

where the matirces A_a have components $(A_a)_{bc} = -\varepsilon_{abc}$, ε is totally antisymmetric and $\varepsilon_{123} = 1$. These matrices satisfy the standard commutation rules for $SO[3, \mathrm{R}]$:

$$[A_a, A_b] = \varepsilon_{abc} A_c. \tag{1.3}$$

Another convenient parametrization of $SO[3, \mathrm{R}]$ consists in using the so-called *vector of finite rotation*

$$\boldsymbol{\theta} = \frac{2}{k}\,\mathrm{tg}\,\frac{k}{2}\,\mathbf{k}. \tag{1.4}$$

For small values of k vectors $\boldsymbol{\theta}$ and $\mathbf{k}$ are asymptotically equivalent. The main advantage of $\boldsymbol{\theta}$ is the suggestive and simple form of the composition rule, $R[\boldsymbol{\theta}_1]R[\boldsymbol{\theta}_2] = R[\boldsymbol{\theta}]$, where

$$\boldsymbol{\theta} = \left(1-\frac{1}{4}\boldsymbol{\theta}_1\cdot\boldsymbol{\theta}_2\right)^{-1}\left(\boldsymbol{\theta}_1+\boldsymbol{\theta}_2+\frac{1}{2}\boldsymbol{\theta}_1\times\boldsymbol{\theta}_2\right); \tag{1.5}$$

the formula (1.1) is equivalent to

$$R[\boldsymbol{\theta}]\cdot\mathbf{u} = \mathbf{u}+\left(1+\frac{1}{4}\boldsymbol{\theta}^2\right)^{-1}\boldsymbol{\theta}\times\left(\mathbf{u}+\frac{1}{2}\boldsymbol{\theta}\times\mathbf{u}\right). \tag{1.6}$$

The mapping $\mathbf{k}\mapsto\boldsymbol{\theta}(\mathbf{k})$ identifies $SO[3, \mathrm{R}]$ with the three-dimensional projective space $P\mathrm{R}^3$. Rotations by π, i.e. nontrivial square roots of identity are then represented by points at infinity.

The manifold $SO[3, R]$ is doubly-connected. The curves connecting antipodal points $\mathbf{k} = \pm\pi\mathbf{n}[\mathbf{n}\cdot\mathbf{n} = 1]$ on the sphere $k = \pi$ in the space of rotation vectors, are closed due to the antipodal identification. At the same time they are topologically nontrivial, i.e. non-contractible to points via continuous deformation process. Even without any sophisticated topology this fact can be easily discovered when one solves differential equations on $SO[3, \mathrm{R}]$. In quite a natural way there appear two-valued functions. To make them one-valued, one introduces the universal covering group of $SO[3, R]$ which, as is well-known, is isomorphic with $SU[2]$ — the group of complex 2×2 unimodular unitary matrices ($u \in SU[2]$ satisfies $u^+u = I$, $\det u = 1$, u^+ being the Hermitean conjugate of u). $SU[2]$ can be also parametrized with the help of the rotation vector. The difference is that the range of k is doubled, $0 \leqslant k \leqslant 2\pi$, there is no antipodal identification on the sphere

$k = \pi$, and all points on the sphere $k = 2\pi$ are identified; they represent the minus identity element of $SU[2]$. Explicitly,

$$u(\mathbf{k}) = \exp\left(k_a \frac{\sigma_a}{2\mathrm{i}}\right) = \cos\frac{k}{2} I - k_a \,\mathrm{i}\, \sigma_a, \ \frac{1}{k} \sin\frac{k}{2} \tag{1.7}$$

where σ are Pauli matrices and I is the identity matrix,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The matrices $a_j := \frac{1}{2\mathrm{i}} \sigma_j$ exponentiated in Eq. (1.7) satisfy the commutation rules

$$[a_i, a_j] = \varepsilon_{ijk} a_k. \tag{1.8}$$

The natural projection pr: $SU[2] \to SO[3, R]$ is given by $v \mapsto R$, where

$$vu(\mathbf{k})v^{-1} = u(R\mathbf{k}). \tag{1.9}$$

If $k \leqslant \pi$, then $pr(u(\mathbf{k})) = R(\mathbf{k})$; if $k > \pi$, then $pr(u(\mathbf{k})) = R\left(-(2\pi - k)\frac{1}{k}\mathbf{k}\right)$.

Thus

$$pr^{-1}\left(R(\mathbf{k})\right) = \left\{u(\mathbf{k}), u\left(-(2\pi-k)\frac{1}{k}\mathbf{k}\right)\right\} = \left\{u(\mathbf{k}), -u(\mathbf{k})\right\}. \tag{1.10}$$

The $SU[2]$ — representation of gyroscopic degrees of freedom is in many respects computationally simpler than the natural $SO[3, R]$ — representation just because of using "smaller" matrices. That is why it is commonly used in various practical problems including navigation [6, 7, 9] and theoretical design of navigation instruments. The same can be said about the vector of finite rotation $\boldsymbol{\theta}$.

There are formal transitions independent of whether we use $SO[3, R]$ or $SU[2]$ as a mathematical description of gyroscopic degrees of freedom. In all these problems we shall use the common symbol G for $SO[3, R]$ and $SU[2]$, and the common symbol $Pr: G \to SO[3, R]$ which is to be understood as the above-introduced projection pr if $G = SU[2]$ and as identity transformation if $G = SO[3, R]$. Similarly, we shall use the same symbol g to denote Lie algebras $so[3, R]$, $su[2]$, consisting, respectively, of real skew-symmetric ($w^T = -w$) and complex-antihermite'an ($w^+ = -w$) matrices. The base elements A_i, a_i., will be denoted by the common symbol E_i, $i = 1, 2, 3$. The Killing tensor on g (and its manifold extension on G), denoted by Γ, is normalized in such a way that $\{E_i\}$ is orthonormal,

$$(E_i|E_j) := \Gamma(E_i, E_j) = \delta_{ij};$$

$(a|b) := \Gamma(a, b)$ denotes the Γ-scalar product of elements a, b. Thus $\Gamma(a, b) = -\frac{1}{2} Tr(ab)$ if $G = SO[3, R]$, and $\Gamma(a, b) = -2Tr(ab)$ if $G = SU[2]$. The standard base $\{E_i\}$ identifies g (i.e. both Lie algebras) with R^3 endowed with the vector product Lie bracket.

Killing tensor on G can be expressed in terms of canonical coordinates as follows:

$$\Gamma_{ij} = 4k^{-2}\sin^2\frac{k}{2}\,\delta_{ij} + k^{-2}\left(1 - 4k^{-2}\sin^2\frac{k}{2}\right) k_i k_j, \tag{1.11}$$

thus the corresponding arc element on G is given by

$$ds^2 = \Gamma_{ij} dk^i dk^j = dk^2 + 4\sin^2\frac{k}{2}\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{1.12}$$

where (k, ϑ, φ) are polar variables in the $\mathbf{k}$-space.

The conformal flatness of ds^2 is explicitly seen in other convenient coordinates, $\mathbf{r} = \frac{a}{k}\operatorname{tg}\frac{k}{4}\mathbf{k}$, $a > 0$, namely

$$ds^2 = 16a^2(a^2+r^2)^{-1}[dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)].$$

It is interesting that $r \in [0, a]$ if $G = SO[3, R]$, but if $G = SU[2]$, then $r \in [0, \infty]$, thus $\mathbf{k} \mapsto \mathbf{r}(\mathbf{k})$ describes the conformal mapping of $SU[2]$ onto R^3.

We shall use exclusively the variables $\mathbf{k}$ or spherical coordinates k, ϑ, φ. They are most adequate and convenient in dynamical problems of a rigid body working under oscillatory dynamical conditions. However, they are not very popular in literature and typical applications, because most of the scientific work on rigid body mechanics concentrated on non-oscillatory problems. The free rigid body, the rigid body influenced by the gravitational field, gyroscopes and more complicated gyroscopic-like instruments are usually analyzed in terms of *Euler angles* $(\varphi, \vartheta, \psi)$. The element $g_{(\varphi,\vartheta,\psi)}$ corresponding to the Euler angles $(\varphi, \vartheta, \psi)$ is given by

$$g_{(\varphi,\vartheta,\psi)} := g\big((0,0,\varphi)\big)g\big((0,\vartheta,0)\big)g\big((0,0,\psi)\big) = \exp(\varphi E_3)\exp(\vartheta E_2)\exp(\psi E_3), \tag{1.13}$$

where $g(\mathbf{k})$, as previously, denotes the element indexed by the rotation vector $\mathbf{k} \in R^3$. The range of the Euler angles is as follows:

$$\begin{aligned} SO(3,R)&: \quad 0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant \psi \leqslant 2\pi, \quad 0 \leqslant \vartheta \leqslant \pi,\\ SU(2)&: \quad 0 \leqslant \varphi \leqslant 4\pi, \quad 0 \leqslant \psi \leqslant 4\pi, \quad 0 \leqslant \vartheta \leqslant 2\pi. \end{aligned} \tag{1.14}$$

Thus the range of ϑ is twice as small as that of φ, ψ. Obviously $g_{(\varphi,\vartheta,\psi)}$ is well-defined also for higher ϑ; however, we repeat then the elements of G parametrized by the relations (1.14).

Canonical coordinates of second kind are given by

$$g_{\{\alpha,\beta,\gamma\}} = g\big((\alpha,0,0)\big)g\big((0,\beta,0)\big)g\big((0,0,\gamma)\big) = \exp\alpha E_1 \exp\beta E_2 \exp\gamma E_3.$$

They are not very popular. The coordinates (α, β, γ) may be useful in certain dynamical problems concerning gyroscopes in Cardan suspensions.

2. Transformations and Hamiltonian formalism

There are two natural transformation groups on G, namely, left and right regular translations

$$x \mapsto L_h(x) = hx, \qquad x \mapsto R_h(x) = xh. \tag{2.1}$$

They obey the following representation rules:

$$L_h \circ L_k = L_{hk}, \quad R_h \circ R_k = R_{kh}, \quad L_h \circ R_k = R_k \circ L_h. \tag{2.2}$$

The left regular translations describe usual rotations of the body in the physical space. The right translations are less intuitive material rotations; one can interpret them as rotations of the reference configuration. If $G = SO[3, R]$, these groups intersect trivially, i.e. they have no common elements but identity transformation. If $G = SU[2]$, there are two common elements, $x \mapsto x$, $x \mapsto -x$.

The rotation vector is not a spatial vector in the usual sense, i.e. the left regular translation $R(\mathbf{k}) \mapsto RR(\mathbf{k})$ of $SO[3, R]$ does not transform $\mathbf{k}$ into $R\mathbf{k}$; $RR(\mathbf{k}) \neq R(R\mathbf{k})$. This follows from the formula (1.5). Overlooking this fact has often accounted for mistakes in micropolar elasticity.

There is an important subgroup of the 6-dimensional group $L_H R_G$ of all translations in G, namely, the *adjoint group*, or *automorphism group*, consisting of the transformations:

$$Ad_k,\ k \in G: x \mapsto kxk^{-1}. \tag{2.3}$$

Obviously, $Ad_h \circ Ad_k = Ad_{hk}$.

This group is three-dimensional and isomorphic with $SO[3, R]$; its action consists in rotating the rotation vector,

$$\begin{aligned} Ad_U\big(R(\mathbf{k})\big) &= R(U\mathbf{k}) \quad \text{on} \quad SO[3, R].\\ Ad_v\big(u(k)\big) &= u\big(pr(v)\mathbf{k}\big) \quad \text{on} \quad SU[2]. \end{aligned} \tag{2.4}$$

In some sense, the transformations (2.3) can be interpreted as spatial rotations accompanied by appropriate rotations of the reference state. In fact, any configuration $x \in G$ is generated from the standard configuration I (identity matrix) by the spatial rotation x, $x = xI$. In turn, the spatially rotated configuration hx can be interpreted as generated by the spatial rotation hxh^{-1} from the spatially co-rotated new reference configuration $h = hI$,

$$(hx) = (hxh^{-1})(hI).$$

The quantity hxh^{-1} measures the "rotation exceed" appearing, due to the non-Abelian character of G, when we simultaneously rotate the current and reference configurations. Because of this, we shall also use the term "*hyperrotations*" for mappings Ad . In contrast to the group of spatial rotations, the hyperrotation group is not transitive; its orbits are given by spheres k = const. The identity matrix I is a singular, one-element, orbit. The very definition of hyperrotations presupposes a fixed reference state—the centre of rotation. In this paper we deal with oscillatory motions of a rigid body, when some equilibrium configuration is fixed by dynamical conditions. Thus our choice of the hyperrotation centre is free of nonphysical arbitrariness.

The group structure of G enables us to identify the Newtonian state space TG, i.e. the space of positions and velocities, with $G \times g$, or simply with $G \times R^3$, because the standard bases (1.3), (1.8) identify Lie algebras $so[3, R]$, $su[2]$ with R^3. This splitting of TG is based on the use of *angular velocities* (nonholonomic quasi-velocities corresponding to the action of G). There are two representations of angular velocities,

$$\omega = \frac{dg}{dt} g^{-1} = \omega_a E_a, \quad \Omega = g^{-1}\frac{dg}{dt} = \Omega_a E_a, \tag{2.5}$$

related to each other through the formula

$$\omega(g, \dot{g}) = g\Omega(g, \dot{g})g^{-1}, \tag{2.6}$$

i.e.

$$\omega_i(R, \dot{R}) = R_{ij}\Omega_j(R, \dot{R}), \quad \omega_i(u, \dot{u}) = (\mathrm{pr}(u))_{ij}\Omega_j(u, \dot{u}), \tag{2.7}$$

respectively on $SO[3, R]$ and $SU[2]$.

The quantities ω_i are components of the angular velocities with respect to the *space-fixed* orthonormal frame (reference configuration). Similarly, Ω_i are components of the angular velocity with respect to the *co-moving*, *body-fixed* frame. Obviously, ω and Ω obey the following transformation rules:

$$\begin{array}{llll} \text{Spatial rotations,} & x' = \mathrm{k}x, & \omega' = k\omega k^{-1}, & \Omega' = \Omega, \\ \text{Material rotations} & x' = xk, & \omega' = \omega, & \Omega' = k^{-1}\Omega k. \end{array} \tag{2.8}$$

If $G = SO[3, R]$, this means that spatial and material R-rotations result in

$$\begin{array}{ll} \omega'_i = R_{ij}\omega_j, & \Omega'_i = \Omega_i, \\ \omega'_i = \omega_i, & \Omega'_i = R^{-1}{}_{ij}\Omega_j = \Omega_j R_{ji}; \end{array} \tag{2.9}$$

if $G = SU[2]$, R is to be replaced by $pr(k)$.

The relationship between angular velocity and holonomic velocity $d\mathbf{k}/dt$ is given by the formulas

$$\begin{aligned} \boldsymbol{\omega} &= k^{-1}\sin k\frac{d\mathbf{k}}{dt} + k^{-2}(1 - k^{-1}\sin k)\left(\frac{d\mathbf{k}}{dt}\cdot\mathbf{k}\right)\mathbf{k} + 2k^{-2}\sin^2\frac{k}{2}\,\mathbf{k}\times\frac{d\mathbf{k}}{dt}, \\ \boldsymbol{\Omega} &= k^{-1}\sin k\frac{d\mathbf{k}}{dt} + k^{-2}(1 - k^{-1}\sin k)\left(\frac{d\mathbf{k}}{dt}\cdot\mathbf{k}\right)\mathbf{k} - 2k^{-2}\sin^2\frac{k}{2}\,\mathbf{k}\times\frac{d\mathbf{k}}{dt}. \end{aligned} \tag{2.10}$$

REMARK. Let us recall, we have identified the physical space, the space of rotation vectors, and Lie algebras of $SO[3, R]$ and $SU[2]$ with R^3. Thus all vectors $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$, $\mathbf{k}$, $d\mathbf{k}/dt$ etc. belong to the same R^3. Without a fixed reference configuration, all these vectors would have to be considered as elements of different linear spaces.

The *kinetic energy* is a quadratic form of Ω with configuration-independent coefficients,

$$T = \frac{1}{2}I_{ab}\Omega_a\Omega_b, \tag{2.11}$$

I_{ab} being *co-moving* components of the *tensor of inertia*. If the co-moving frame coincides with the system of principal directions of I, then

$$T = \frac{1}{2}\sum_a I_a\Omega_a^2. \tag{2.12}$$

T is invariant under the spatial rotations, cf the transformation rules (2.8). Under the group of material rotations R_G it is invariant if the rigid body is spherical, $I_a = I$, $a = 1, 2, 3$, i.e.

$$T_{sp} = \frac{I}{2}\sum_a \Omega_a^2 = \frac{I}{2}\Gamma_{ab}\frac{dk^a}{dt}\frac{dk^b}{dt} = \frac{I}{2}\left[\left(\frac{dk}{dt}\right)^2 + 4\sin^2\frac{k}{2}\left(\left(\frac{d\vartheta}{dt}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{dt}\right)^2\right)\right] \tag{2.13}$$

Kinematical angular momentum with respect to the centre of mass, i.e. kinematical spin, is given by

(2.14) $$l_a = I[g]_{ab}\omega_b,$$

where $I[g]_{ab}$ are components of the inertial tensor in the space-fixed firame, thus

(2.15) $$I[R]_{ab} = R_{ac}R_{bd}I_{cd},$$

if we use $SO[3, R]$-description of the configuration space. The co-moving components of kinematical spin are given by

(2.16) $$L_a = I_{ab}\Omega_b = I_{\mathbf{a}}\Omega_{\mathbf{a}}$$

if we use the co-moving frame diagonalizing I.

When dealing with nondissipative dynamical problems, one formulates the theory in Hamiltonian canonical terms. Just as the Newtonian state space TG, the Hamiltonian phase space T^*G (the cotangent bundle over G) can be canonically trivialized:

(2.17) $$T^*G \simeq G\times g \simeq G\times g^* \simeq G\times R^3.$$

There are two natural trivializations representing Hamiltonian states as pairs $(g, \boldsymbol{\sigma}) = (g, \sigma_i E_i)$ or $(g, \boldsymbol{\Sigma}) = (g, \Sigma_i E_i)$, where g represents the configuration of a rigid body and σ_i, Σ_i are components of the canonical spin, respectively, in the space-fixed and body-fixed frame. Let $\mathbf{p}$ denote the canonical momentum conjugate to $\mathbf{k}$. The identification (2.17) is understood in the sense

$$p_i\frac{dk}{dt^i} = \sigma_i\omega_i = \Sigma_i\Omega_i,$$

thus, after some calculations, one obtains for the R^3-vectors $\boldsymbol{\sigma}$, $\boldsymbol{\Sigma}$,

$$\boldsymbol{\sigma} = \frac{k}{2}\operatorname{ctg}\frac{k}{2}\mathbf{p}+k^{-2}\left(1-\frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)(\mathbf{p}\cdot\mathbf{k})\mathbf{k}+\frac{1}{2}\mathbf{k}\times\mathbf{p},$$

$$\boldsymbol{\Sigma} = \frac{k}{2}\operatorname{ctg}\frac{k}{2}\mathbf{p}+k^{-2}\left(1-\frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)(\mathbf{p}\cdot\mathbf{k})\mathbf{k}-\frac{1}{2}\mathbf{k}\times\mathbf{p}.$$

The relationship between $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ and their transformation properties are identical with Eqs. (2.6), (2.8) ($\boldsymbol{\omega}$, $\boldsymbol{\Omega}$ replaced by $\boldsymbol{\sigma}$, $\boldsymbol{\Sigma}$). The quantities $\boldsymbol{\sigma}$, $\boldsymbol{\Sigma}$ are, respectively, Hamiltonian generators of spatial and material rotations. Their *Poisson brackets* have the form

(2.19) $$\{\sigma_i, \sigma_j\} = \varepsilon_{ijk}\sigma_k,\ \{\Sigma_i, \Sigma_j\} = -\varepsilon_{ijk}\Sigma_k, \qquad \{\sigma_i, \Sigma_j\} = 0,$$

(compare with the rules (2.2)). For any function $f(\mathbf{k})$ depending only on generalized coordinates, we have

(2.20) $$\{\sigma_i, f\} = -\mathscr{L}_i f, \qquad \{\Sigma_i, f\} = -\mathscr{R}_i f,$$

where $\mathscr{L}_i$, $\mathscr{R}_i$ are differential operators on G generating, respectively, left and right regular translations (spatial and material rotations), thus

(2.21)
$$\mathscr{L}_i = \frac{k}{2}\operatorname{ctg}\frac{k}{2}\frac{\partial}{\partial k_i}+k^{-2}\left(1-\frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)k_ik_j\frac{\partial}{\partial k_j}+\frac{1}{2}\varepsilon_{ijm}k_j\frac{\partial}{\partial k_m},$$
$$\mathscr{R}_i = \frac{k}{2}\operatorname{ctg}\frac{k}{2}\frac{\partial}{\partial k_i}+k^{-2}\left(1-\frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)k_ik_j\frac{\partial}{\partial k_j}-\frac{1}{2}\varepsilon_{ijm}k_j\frac{\partial}{\partial k_m}.$$

Any other Poisson bracket for the functions $f(\mathbf{k}, \boldsymbol{\sigma})$ or $f(\mathbf{k}, \boldsymbol{\Sigma})$ can be easily obtained from Eqs. (2.19), (2.20) if we use universal relationships

$$\{f, g\} = -\{g, f\}, \quad \{f(h), g\} = f'(h)\{h, g\}.$$

The quantity

$$\mathbf{M} := \boldsymbol{\sigma} - \boldsymbol{\Sigma} = \mathbf{k} \times \mathbf{p} \tag{2.22}$$

generates hyperrotations; thus we shall call it *hyperspin* ("hyper", because it measures the exceed of laboratory components of spin over co-moving ones). M obeys the Poisson bracket rules

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k. \tag{2.23}$$

We shall often use spherical variables in the $\mathbf{k}$-space, and their conjugate momenta $(k, \vartheta, \varphi; p_k, p_\vartheta, p_\varphi)$. Obviously $\mathbf{M}$ depends only on $\vartheta, \varphi, p_\vartheta, p_\varphi$, and

$$\begin{aligned} \boldsymbol{\sigma} &= \frac{1}{k} p_k \mathbf{k} - \frac{1}{2k} \operatorname{ctg} \frac{k}{2} \mathbf{k} \times \mathbf{M} + \frac{1}{2} \mathbf{M} \\ \boldsymbol{\Sigma} &= \frac{1}{k} p_k \mathbf{k} - \frac{1}{2k} \operatorname{ctg} \frac{k}{2} \mathbf{k} \times \mathbf{M} - \frac{1}{2} \mathbf{M}. \end{aligned} \tag{2.24}$$

The Casimir invariant of the rotation group, i.e. squared magnitude of the angular momentum, can be expressed as

$$S^2 = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = \boldsymbol{\Sigma} \cdot \boldsymbol{\Sigma} = p_k^2 + \frac{1}{4} \sin^{-2} \frac{k}{2} M^2, \tag{2.25}$$

where

$$M^2 = \mathbf{M} \cdot \mathbf{M} = p_\vartheta^2 + \sin^{-2}\vartheta p_\varphi^2, \quad M_3 = p_\varphi. \tag{2.26}$$

The kinematical and canonical quantities $\mathbf{p}$ and $d\mathbf{k}/dt$ or, similarly, $\boldsymbol{\sigma}$ and $\mathbf{l}$ are logically independent. They can be related to each other only on the basis of some Lagrangian dynamical model. If the Lagrange function is given by $L = T - V$, where V is a velocity-independent potential function, and T is given by Eq. (2.11), then, the Legendre transformation tells us that

$$\Sigma_a = L_a, \quad \sigma_a = l_a, \tag{2.27}$$

and the Hamiltonian has the form

$$H = \sum_a \frac{1}{2I_a} \Sigma_a^2 + V. \tag{2.28}$$

If we consider a spherical rigid body, then

$$\Sigma_a = I\Omega_a, \quad \sigma_a = I\omega_a, \tag{2.29}$$

$$\frac{1}{I} \mathbf{p} = 4k^{-2} \sin^2 \frac{k}{2} \frac{d}{dt} \mathbf{k} + k^{-2} \left(1 - 4k^{-2} \sin^2 \frac{k}{2}\right) \left(\mathbf{k} \cdot \frac{d\mathbf{k}}{dt}\right) \mathbf{k}, \tag{2.30}$$

$$\mathbf{M} = \mathbf{k} \times \mathbf{p} = 4Ik^{-2} \sin^2 \frac{k}{2} \mathbf{k} \times \frac{d\mathbf{k}}{dt}, \tag{2.31}$$

$$H = \frac{S^2}{2I} + V, \tag{2.32}$$

S^2 given by Eq. (2.25). Therefore

$$\begin{aligned} M^2 &= 16I^2 \sin^4 \frac{k}{2} \left(\left(\frac{d\vartheta}{dt} \right)^2 + \sin^2\vartheta \left(\frac{d\varphi}{dt} \right)^2 \right), \\ M_3 &= 4I \sin^2 \frac{k}{2} \sin^2\vartheta \frac{d\varphi}{dt}. \end{aligned} \tag{2.33}$$

3. Dynamical models

Equations of motion of a rigid body without translational degrees of freedom are equivalent to the balance law of angular momentum,

$$\frac{dl^i}{dt} = n^i, \tag{3.1}$$

where n^i are components of the *moment of forces* with respect to the *space-fixed* frame. Reformulating these equations with the help of co-moving components, we obtain the Euler equations

$$\frac{dL^i}{dt} = -\frac{1}{2} \sum_{jk} \varepsilon_{ijk} \left(\frac{1}{I_j} - \frac{1}{I_k} \right) L_j L_k + N^i, \tag{3.2}$$

i.e.

$$\frac{d\Omega^i}{dt} = \frac{1}{2} \sum_{jk} \varepsilon_{ijk} \frac{I_j - I_k}{I_i} \Omega_j \Omega_k + N^i, \tag{3.3}$$

N^i being *co-moving* components of the *moment of forces*.

If we consider a potential dynamical model $L = T - V$, then

$$l_i = \sigma_i, \quad L_i = \Sigma_i, \quad n_i = \{\sigma_i, V\} = -\mathscr{L}_i V, \quad N_i = \{\Sigma_i, V\} = -\mathscr{R}_i V, \tag{3.4}$$

where $\mathscr{L}_i$, $\mathscr{R}_i$ are differential operators defined by Eq. (2.21).

If a rigid body is spherical, the nondynamical terms on the right-hand side of Eqs. (3.2), (3.3) vanish.

Rigid body mechanics is essentially nonlinear, because there are no coordinates which could reduce the Lagrangian to a quadratic form, even in the interaction-free case. This nonlinearity is implied by the very geometry of degrees of freedom, namely, by the compactness of the configuration space and by the curvature of the metric tensor underlying the kinetic energy form. Thus linear dynamical models of large torsional vibrations are not only physically nonadequate (as they usually are in vibration problems), but also mathematically impossible. The topological structure of G implies that also the potential energy V, if well-defined all over G, is nonquadratic in any sensible coordinates.

The theory of small vibrations is based on potentials of the form

$$V = \frac{1}{2} \varkappa_{ij} k_i k_j, \tag{3.5}$$

where $\varkappa$ is a positive, symmetric and constant matrix. The coordinates k_i are angular variables; thus, if considered as global functions on G, they are either multivalued or non-smooth. They are evidently singular at the half-rotations $k = \pi$. Therefore, realistic models of finite torsional vibrations cannot be obtained from the local quadratic background (3.5) by introducing low-order polynomial corrections. Rather, one has to use from the very beginning essentially nonlinear trigonometric functions suited to the angular character of k_i and asymptotically equivalent to Eq. (3.5) for $k \approx 0$.

For example, we can expand the dependence of V on the variables (ϑ, φ) into spherical functions series or, equivalently, into a series of homogeneous polynomials of the vector $\frac{1}{k}\mathbf{k}$,

$$(3.6)\qquad V = V_0(k)+V_2(k)_{ij}\frac{k^i}{k}\frac{k^j}{k}+V_3(k)_{ijm}\frac{k^i}{k}\frac{k^j}{k}\frac{k^m}{k}+\ldots$$

$$= V_0(k)+\sum_{l=2}^{\infty}\sum_{m=-l}^{l} V_{lm}(k)Y_{lm}(\vartheta, \varphi),$$

where the functions $V_{l,m}$ depend on k in such a way that the total V is well-defined and differentiable at singular points of coordinates, i.e. at the half-rotations sphere $k = \pi$. To attain this, we must assume that either the functions $V_{l,m}$ themselves, or their derivatives $V_{l,m}$, (depending on the parity of l) vanish at $k = \pi$. Obviously, realistic potentials are given by the expressions (3.6) terminating at low values of l.

Let us note there is also another possibility of avoiding problems with the coordinates k and their polynomials at $k = \pi$, namely, just to shift the sphere of half-rotations to infinity by means of the projective transformation (1.4). In other words, one should test phenomenological potentials constructed as polynomials of the finite rotation vector $\boldsymbol{\theta}$. The variables $\boldsymbol{\theta}$ have the advantage of being one-valued in all the region of their finiteness, in contrast to $\mathbf{k}$. Therefore, the quadratic potentials

$$(3.7)\qquad V = \frac{1}{2}\varkappa_{ij}\theta_i\theta_j$$

are admissible. Obviously the problem does not become linear with this potential because kinetic energy depends nontrivially on the variables $\boldsymbol{\theta}$. Nevertheless Eq. (3.7) is, so to speak, "most harmonic" among all (non-harmonic) potentials on G. Nonlinearity of the model based on Eq. (3.7) is contained only in its kinetic energy term. If $\varkappa_{ij} = \varkappa\delta_{ij}$, Eq. (3.7) becomes invariant under the hyperrotations,

$$(3.8)\qquad V = \frac{\varkappa}{2}\theta^2 = 2\varkappa \mathrm{tg}^2\frac{k}{2}.$$

Let us observe that Eq. (3.7) and all other potentials polynomial in $\boldsymbol{\theta}$ are singular at $k = \pi$. This means that they describe *pure torsional vibrations* without admixture of total rotations (librations). The positive infinity of Eq. (3.7) at $k = \pi$ implies that angular deflections from the equilibrium configuration are bounded by the straight angle.

There is also another reasonable representation of phenomenological potentials, based on the Cayley transform concept. Let us recall that the Cayley transform of a matrix A is defined as

$$C(A) = (A-I)(A+I)^{-1} = (A+I)^{-1}(A-I), \tag{3.9}$$

I denoting the identity matrix. It is well-known that for typical matrix groups, the Cayley transform is a local diffeomorphism between the group and its Lie algebra. Thus $C(R)^T = -C(R)$ if $R \in SO[3, R]$, and $C(u)^+ = -C(u)$ if $u \in SU[2]$. If $k \approx 0$, $C(g(\mathbf{k}))$ asymptotically becomes $k_i E_i$, just as $\ln(g(\mathbf{k}))$. However, operations with Cayley transforms are evidently less embarrassing than those with logarithms of matrices. The parameters ξ_i defined by $C(g) = \xi_i(g)E_i$ have properties analogous to θ_i, and probably there exists a simple relationship between $\boldsymbol{\xi}_i$ and $\boldsymbol{\theta}$ because on $SO[3, R]$ ξ_i are also infinite at $k = \pi$ (straight-angle rotations R satisfy the condition $\det(I+R) = 0$). We can represent phenomenological potentials as polynomials or other simple functions of ξ_i; for example, the natural analogue of (3.7) is

$$V = \frac{1}{2}\varkappa_{ij}\xi_i\xi_j,$$

or, in matrix terms,

$$V(g) = -\frac{1}{4}\operatorname{Tr}\big(C(g)C(g)K\big) = -\frac{1}{4}\operatorname{Tr}\big(C(g)KC(g)\big),$$

K being a constant, symmetric and positively-definite matrix. Higher-order polynomials of $\boldsymbol{\xi}$ can be represented as

$$V = \sum_{i=1}^{N}\operatorname{Tr}\big(C(g)\overset{i}{K}_1 C(g)\overset{i}{K}_2 C(g) \ldots C(g)\overset{i}{K}_i C(g)\big),$$

$\overset{i}{K}_p$ being constant matrices, $p = 1, \ldots, i$.

Parametrization of configurations by means of three-dimensional vectors (in various versions, $\mathbf{k}$, $\boldsymbol{\theta}$, $\boldsymbol{\xi}$) suggests us to follow certain methods and models used in mechanics of a material point. Thus one can expect that "central" potentials, depending only on absolute values of those vectors, will be mathematically convenient and physically useful. Using Cayley transforms we can represent them as

$$V(g) = f\big(\operatorname{Tr}(C^2(g))\big).$$

Such potentials have the remarkable property that they are invariant under hyperrotations, thus we shall call them *isotropic*. Let us stress, however, that they are not isotropic in the sense of spatial rotations; as a matter of fact, every non-trivial rigid-body potential is non-isotropic in the spatial sense. Indeed, regular group translations act not only transitively but also freely on G; thus non-constant functions on G are never invariant under all of them.

Nevertheless, potentials invariant under hyperrotations are intuitively isotropic in the sense that the potential energy increase depends only on the angular deflection from the equilibrium, but not on the rotation axis. There is no increase of energy if we simultaneously rotate in space both the rigid body in question and mechanical agents responsible for its equilibrium configuration.

Let us quote a simple mechanical model. Within the spherical cavity in a material block, and concentrically with it, we place the homogeneous ball and connect its surface with the cavity boundary by means of strained springs. The centre of mass is immobilized, e.g. by means of a few thin rods soldered to the ball and skimming over the cavity surface. All springs are identical and densely distributed in a homogeneous way. Obviously the ball will work as a torsional oscillator without translational degrees of freedom; in equilibrium configuration all springs are perpendicular to the boundary. When "the number of springs tends to infinity", the resulting torsional potential becomes isotropic, i.e. hyperrotations-invariant. The more springs, the better approximation to isotropy. Instead of a system of springs we can use an elastic medium filling the empty region between the ball and cavity surface and rigidly adhering (glued) to both surfaces. Instruments constructed according to such a scheme, and endowed with appropriately suited damping, can work as spatial (three-dimensional) sensors of angular acceleration in navigation and control systems. The advantage of such devices is that they indicate the complete three-dimensional vector of angular acceleration, in contrast to the usual disk-like instruments which are only sensitive to the component orthogonal to the disk plane. If the potential is isotropic, i.e. depends on k through its magnitude (rotation angle), then the moment of forces has a characteristic central structure in the $\mathbf{k}$-space,

$$\mathbf{n} = \mathbf{N} = -\frac{V'(k)}{k}\mathbf{k}, \tag{3.10}$$

cf. the expressions (3.4). Therefore, there is an obvious analogy to the central motion of material points. However, in rigid body mechanics, the isotropy of V does not imply the system to be dynamically invariant under hyperrotations because in general T is not materially isotropic. Equations of motion resulting from a Lagrangian $L = T-V$ are hyperrotations-invariant if: (i) the inertial tensor is spherical, $I_{ab} = I\delta_{ab}$, (ii) V is central (independent of the rotation axis). The torsional oscillator of this form will be called *isotropic.*

Let us notice that in a complete analogy to central problems in material point mechanics, the motion of an isotropic torsional oscillator is flat in the $\mathbf{k}$-space ($\boldsymbol{\theta}$-space). In fact, invariance under hyperrotations implies that their Hamiltonian generator $\mathbf{M} = \mathbf{k}\times\mathbf{p}$ is a constant of motion, $\{M_i, H\} = 0$. But Eq. (2.31) tells us that for such systems

$$\mathbf{M} = \mathbf{k}\times\mathbf{p} = 4Ik^{-2}\sin^2\frac{k}{2}\mathbf{k}\times\frac{d\mathbf{k}}{dt},$$

thus the direction of $\mathbf{k}\times\dfrac{d\mathbf{k}}{dt}$ is a constant of motion (because that of $\mathbf{M}$ is). Therefore $\mathbf{k}$ undergoes a flat motion in a fixed plane depending on initial conditions. Thus, when analysing isotropic torsional vibrations, we can apply the effective mathematical techniques elaborated in mechanics of material points.

Let us now discuss phenomenological models of damping forces. The simplest, and usually sufficient, model of viscous friction is based on forces linear in velocities. Thus it is natural to postulate damping moments in the form

$$\underset{(d)}{n_i} = -\nu_{ij}\omega_j, \tag{3.11}$$

ν being velocity-independent, symmetric and positively semi-definite. The moment (3.11) is explicitly invariant under material rotations. It will be invariant under spatial rotations if and only if $\nu_{ij} = \nu\delta_{ij}$, i.e. for the spherical friction tensor. This invariance is meant in the sense: $\underset{(d)}{\mathbf{n}}(R\boldsymbol{\omega}) = R\underset{(d)}{\mathbf{n}}(\boldsymbol{\omega})$ for any rotation R. Such moments are automatically invariant under hyperrotations.

Let us notice that damping moments partially destroy analogy with the central motion of material points. In fact, if a material point is subject to the central elastic force $\mathbf{F}_{el} = \frac{V'(r)}{r}\mathbf{r}$ and the isotropic friction force $\mathbf{F}_d = -\nu\mathbf{v}$ ($\mathbf{r}$ denoting the radius vector, and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ the translational velocity), then the total force $\mathbf{F} = \mathbf{F}_{el}+\mathbf{F}_d$ is a linear combination of vectors $\mathbf{r}$, $\mathbf{v}$ (with coefficients depending on state variables); thus, in the presence of friction forces, motion is still flat. The angular momentum vector decays exponentially, $L = L_0\exp\left(-\frac{\nu}{m}t\right)$; thus it is not a constant of motion, but its direction is still constant in time. In contrast to this, according to the formulas (2.10), the moment of isotropic damping $-\nu\boldsymbol{\omega}$ is a linear combination of $\mathbf{k}, \frac{d\mathbf{k}}{dt}$ and $\mathbf{k}\times\frac{d\mathbf{k}}{dt}$; thus it has a component perpendicular to the instantaneous plane of motion. Therefore this plane changes and motion is non-flat. The structure of this moment resembles the Magnus effect. Coefficients in the expression of $\boldsymbol{\omega}$ through $\mathbf{k}, \frac{d\mathbf{k}}{dt}$ and $\mathbf{k}\times\frac{d\mathbf{k}}{dt}$, and coefficients of the kinetic energy expressed through $\frac{d\mathbf{k}}{dt}$, are functions of the same kind; hence the mentioned deviation of trajectories from the plane form may be neglected only for geometrically small motions.

Let us notice that besides the elastic potential term $n_{el^i} = -\mathscr{L}_i V$, and linear damping term $n_{d^i} = -\nu_{ij}\omega_j$, practically used moments contain also gyroscopic terms of the form $n_{g^i} = -F_{ij}\omega_j$, where F is skew-symmetric, i.e. $\mathbf{n}_g = -\mathbf{F}\times\boldsymbol{\omega}$. Such terms describe, for example, the influence of other gyroscopic objects on the body in question. Obviously they are never isotropic (hyperrotations-invariant).

4. Analysis of potential isotropic models

From now on we concentrate on *elastic isotropic* models. We assume that there are only potential isotropic moments, thus $\mathbf{n} = \mathbf{N} = -\frac{V'(k)}{k}\mathbf{k}$. We assume also that inertial properties of the body are isotropic, $I_1 = I_2 = I_3 = I$. In other words, we have a Lagrangian model $L = T-V = \frac{I}{2}\omega^2-V(k)$, and the equations of motion (3.1) have the form

$$I\frac{d\boldsymbol{\omega}}{dt} = -\frac{V'(k)}{k}\mathbf{k}, \tag{4.1}$$

or, equivalently,

$$I\frac{d\mathbf{\Omega}}{dt} = -\frac{V'(k)}{k}\mathbf{k}. \tag{4.2}$$

Subtracting these equations and substituting the formulas (2.10), we obtain

$$\frac{d}{dt}\left(4k^{-2}\sin^2\frac{k}{2}\mathbf{k}\times\frac{d\mathbf{k}}{dt}\right) = 0, \tag{4.3}$$

i.e. just the conservation law of hyperspin, $d\mathbf{M}/dt = 0$, expressing the invariance of our model under hyperrotations. Another evident constant of motion is the energy $H = T+V$.

Obviously the system is integrable in the sense that there exist three (i.e. as many as degrees of freedom) independent constants of motion with pairwise vanishing Poisson brackets, e.g.

$$\begin{aligned} H &= T+V = \frac{I}{2}\left(\frac{dk}{dt}\right)^2 + \frac{1}{8I}M^2\sin^{-2}\frac{k}{2}+V,\\ M^2 &= \mathbf{M}\cdot\mathbf{M} = 16I^2\sin^4\frac{k}{2}\left(\left(\frac{d\vartheta}{dt}\right)^2+\sin^2\vartheta\left(\frac{d\varphi}{dt}\right)^2\right) = p_\vartheta^2+\sin^{-2}\vartheta p_\varphi^2,\\ M_3 &= 4I\sin^2\frac{k}{2}\sin^2\vartheta\frac{d\varphi}{dt}. \end{aligned} \tag{4.4}$$

With a fixed value D of the constant of motion M, the rotation angle k (radius in the $\mathbf{k}$-space) satisfies the autonomous equation

$$I\frac{d^2k}{dt^2} = -V'_{\mathrm{ef}\,D}(k), \tag{4.5}$$

where the effective potential $V_{\mathrm{ef}\,D}$ is given by

$$V_{\mathrm{ef}\,D} := V+\frac{1}{8I}D^2\sin^{-2}\frac{k}{2}, \tag{4.6}$$

$\frac{1}{8I}D^2\sin^{-2}k/2$ is the *cnetrifugal* term in $\mathbf{k}$-space. Orbits characterized by $D = 0$ are subsets of one-parameter subgroups of G, i.e. they correspond to torsional vibrations about axes fixed in space. If $D \neq 0$, then instantaneous rotation axes move in space; they perform torsional precession.

Just as in central problems of material point mechanics, for any $V(k)$ motion is at least *once degenerate* because there exists at least one additional constant of motion independent of the constants (4.4), e.g. M_2. This degeneracy reflects the flatness of motion. It enables us to reduce the problem to two degrees of freedom. Namely, it is sufficient to find, or at least describe qualitatively, all trajectories placed on the plane $k_3 = 0$, i.e. $\vartheta = \pi/2$. All other solutions can be obtained from those by applying all possible hyperrotaions about the axes k_1, k_2.

Obviously, after reduction to the plane $\vartheta = \pi/2$, we have

$$\begin{aligned} M|(\vartheta = \pi/2) &= 4I\sin^2\frac{k}{2}\left|\frac{d\varphi}{dt}\right| = |M_3|,\\ M_3 &= 4I\sin^2\frac{k}{2}\frac{d\varphi}{dt}. \end{aligned} \tag{4.7}$$

The constants of motion H, M_3 reduce the problem to the following two-dimensional dynamical system:

$$(4.8)\qquad \begin{aligned} \frac{dk}{dt} &= \left[\frac{2}{I}(E-V) - \frac{1}{4} I^{-2} D^2 \sin^{-2}\frac{k}{2}\right]^{1/2} = \left[\frac{2}{I}(E-V_{\mathrm{ef}})\right]^{1/2},\\ \frac{d\varphi}{dt} &= \frac{1}{4} D I^{-1} \sin^{-2}\frac{k}{2}, \end{aligned}$$

E, D being respectively fixed values of H and M_3.

Orbits of motion without the time-dependence are described by the ratio of equations (4.8):

$$(4.9)\qquad \frac{d\varphi}{dk} = \frac{D}{4I}\sin^{-2}\frac{k}{2}\left[\frac{2}{I}(E-V_{\mathrm{ef}})\right]^{-\frac{1}{2}}.$$

Thus the problem has been reduced to calculating integrals. It is natural to expect that for realistic phenomenological potentials, the convenient integration variables are $z = \operatorname{tg} k/2$, $w = \operatorname{ctg} k/2$, $u = \operatorname{ctg}^2 k/2$, etc. This is suggested by the philosophy of Cayley transforms and $\boldsymbol{\theta}$-vectors presented above. If we use the variable $w = \operatorname{ctg} k/2 = 2/\theta$, then the formula for trajectories reads

$$(4.10)\qquad \varphi = \frac{D}{I}\int_{w_0}^{w}\left[-D^2 I^{-2} w^2 + \frac{8}{I}\left(\left(\mathrm{E} - \frac{D^2}{8I}\right) - V\right)\right]^{-\frac{1}{2}} dw.$$

It is almost identical with the corresponding formula for material points moving in central fields:

$$(4.11)\qquad \varphi = \frac{L}{m}\int_{w_0}^{w}\left[-L^2 m^{-2} w^2 + \frac{8}{m}(E-V)\right]^{-\frac{1}{2}} dw,$$

where now $w = 2/r$, L denotes the magnitude of angular momentum and m — the mass. It is seen that Eq. (4.10) results from Eq. (4.11) under replacing $m \mapsto I$, $L \mapsto D$, $E \mapsto \left(E - \frac{D^2}{8I}\right)$. This enables us to apply directly the numerous explicit results and formulas from the theory of central spatial motion. For example, it is well-known that for material points there are only two completely degenerate central potentials, namely, the isotropic harmonic oscillator $V = \frac{\varkappa}{2} r^2$ and attractive Kepler–Coulomb problem, $V = -\alpha/r$, $\alpha > 0$. With these and only these central potentials all bounded orbits are closed (all orbits in the oscillatory case and $E < 0$ orbits in the Kepler case). If we consider the problem in projective space, then also free motion is completely degenerate because straight-lines are closed at infinity. By analogy we have exactly three completely degenerate rigid-body potentials invariant under hyperrotations. They are:

$(4.12)_1$
1. Trivial potential $V = 0$; free rigid body.
2. Torsional oscillator

$$V_{\mathrm{osc}} = \frac{\varkappa}{2}\theta^2 = 2\varkappa \operatorname{tg}^2\frac{k}{2}, \quad \varkappa > 0. \tag{4.12$_2$}$$

3. Torsional "Kepler" problem

$$V_{\mathrm{Kp}} = -\frac{\alpha}{\theta} = -\frac{\alpha}{2}\operatorname{ctg}\frac{k}{2}, \quad \alpha > 0. \tag{4.12$_3$}$$

Model (1), i.e. the free rigid body fails to be the regular special case of models (2, 3) when $\varkappa = 0$, $\alpha = 0$ because there is no continuous limit transition with phase portraits when $\varkappa \to 0$, $\alpha \to 0$.

It is interesting to note that the potentials $(4.12)_2$, $(4.12)_3$ are singular; this proves that the unboundedness of the harmonic oscillator and Kepler potential is not a consequence of the non-compact character of R^3.

The explicit solution of Eqs. (4.8) for the oscillatory potential $(4.12)_2$ is given by elementary functions, namely

$$\begin{aligned}\sin 2(\varphi-\varphi_0) &= -\frac{2D^2}{I^2\sqrt{\Delta}}\operatorname{ctg}^2\frac{k}{2}+\frac{1}{\sqrt{\Delta}}\left(\frac{8}{I}E-\frac{D^2}{I^2}\right),\\ \sin(2\Omega t-\delta) &= \frac{8\Omega^2}{\sqrt{\Delta}}\sin^2\frac{k}{2}-\frac{1}{\sqrt{\Delta}}\left(\frac{2E}{I}+\frac{D^2}{4I^2}\right),\end{aligned} \tag{4.13}$$

where

$$\Delta = I^{-2}\left(8\mathrm{E}-\frac{D^2}{I}\right)^2-\frac{64\varkappa^2D^2}{I^3}, \quad \Omega = \sqrt{\omega^2+\frac{E}{2I}}, \quad \omega = \sqrt{\frac{\varkappa}{\mu}}.$$

Obviously our oscillator is non-harmonic (and, as mentioned, there is no harmonic torsional oscillator at all). For example, the solutions (4.13) are not isochronic; their frequency Ω depends on the trajectory. As it is always the case with completely degenerate systems, Ω depends on initial conditions only through the energy E. When $E \to 0$, Ω tends to infinitesimal frequency ω. The dynamical model $(4.12)_2$ is, roughly speaking, "as harmonic as possible" with gyroscopic degrees of freedom.

The "Kepler–Coulmb potential" $(4.12)_3$ is not a good technical model of rotational vibrations about the standard configuration $k = 0$. Indeed, this configuration is not a proper equilibrium state of the potential (4.12), instead, it is a negative (attractive) singularity of V. Orbits of the model $(4.12)_3$ are given by

$$\sin(\varphi-\varphi_0) = -2\frac{D^2}{I^2\sqrt{\Delta}}\operatorname{ctg}\frac{k}{2}+\frac{4\alpha}{I\sqrt{\Delta}}, \tag{4.14}$$

where

$$\Delta = \frac{16\alpha^2}{I^2}+32\frac{D^2}{I^3}\left(E-\frac{D^2}{8I}\right).$$

Time-dependence is non-elementary.

Let us notice that if we use $SO[3, R]$ as the configuration space, then only trajectories with $E < D^2/8I$ are correctly described by the above formula (4.14). The reason is that trajectories above the energy threshold $E_{\mathrm{th}} = D^2/8I$ approach the sphere of straight-angle rotations $k = \pi$; thus the analytical formula (4.14) must be combined with antipodal jumping $(\pi\mathbf{n}) \mapsto (-\pi\mathbf{n})$, usually violating periodicity of motion. But even if we do this, the problem is not completely well-defined because, strictly speaking, the function $\operatorname{ctg} k/2$, considered as a function on $SO[3, R]$ fails to be differentiable at half-rotations $R(\pi\mathbf{n})$, $\mathbf{n}\cdot\mathbf{n} = 1$; thus there are some doubts as to the well-definiteness of our differential equations of motion. Indeed, for any isotropic function $f(\mathbf{k}) = V(k)$, antipodal identification gives two, in general different, values for the gradient ∇f at $R(\pi\mathbf{n})$, namely, $V'(\pi)\mathbf{n}$ and $-V'(\pi)\mathbf{n}$. They coincide, i.e. f is differentiable at $R(\pi\mathbf{n})$, if $V'(\pi) = 0$. But for $\operatorname{ctg}\frac{k}{2}$ this is not the case. There is no such problem if we use the $SU[2]$-description; however, with the potential $V_{\mathrm{Kp}} = -\frac{\alpha}{2}\operatorname{ctg}\frac{k}{2}$ this description is non-physical for macroscopic physical objects. The reason is that the function V_{Kp}, in contrast to V_{osc}, is not projectable from $SU[2]$ to $SO[3, R]$; after projection it becomes two-valued. If we project $SU[2]$-trajectories of V_{Kp} onto physical configuration space $SO[3, R]$, we obtain smooth curves which do not obey equations of motion corresponding to the single potential $V_{\mathrm{Kp}} = -\frac{\alpha}{2}\operatorname{ctg}\frac{k}{2}$. Instead, they consist of segments separated from each other by points $R(\pi\mathbf{n})$, and corresponding to different signs of α, i.e. behaving as influenced subsequently by the attractive and repulsive "Kepler" potential $(4.12)_3$.

Orbits of both degenerate models $(4.12)_{2,3}$ are situated with respect to the centre $k = 0$ just as the corresponding orbits of material points with respect to the centre of forces.

Thus, for V_{osc} we have circular orbits $k = \text{const}$ or elliptic (in $\boldsymbol{\theta}$-space) orbits with the point $k = 0$ placed in the geometric centre. This means that there exist two turning points of each kind (two pericentral and two apocentral points). For V_{Kp} the centre $k = 0$ is placed asymmetrically with respect to non-circular orbits, at their focal points; thus there exists only one turning point of each kind. This difference is reflected by the formulas (4.13), (4.14). In Eq. (4.14) there is no factor 2 at the angle φ.

It is instructive to describe these results in terms of action-angle variables. In the coordinates k, ϑ, φ, the stationary Hamilton–Jacobi equation has the form

$$(4.15)\qquad \left(\frac{\partial S}{\partial k}\right)^2 + \frac{1}{4}\sin^{-2}\frac{k}{2}\left(\frac{\partial S}{\partial \vartheta}\right)^2 + \frac{1}{4}\sin^{-2}\frac{k}{2}\sin^{-2}\vartheta\left(\frac{\partial S}{\partial \varphi}\right)^2 = 2I(E-V),$$

E being a fixed energy value.

According to the Stäckel theorem, this equation is separable for all potentials of the form

$$(4.16)\qquad V = V(k) + A(\vartheta)\sin^{-2}\frac{k}{2} + B(\varphi)\sin^{-2}\frac{k}{2}\sin^{-2}\vartheta.$$

We consider only isotropic models, thus $A = B = 0$.

Solutions with separated variables,

$$S = \alpha_\varphi \varphi + S_\vartheta(\vartheta) + S_k(k)$$

are characterized by three separation constants α_φ, α_ϑ, E where

$$p_\vartheta = \frac{dS_\vartheta}{d\vartheta} = (\alpha_\vartheta^2 - \alpha_\varphi^2 \sin^{-2}\varphi)^{1/2}, \quad p_k = \frac{dS_k}{dk} = \left(2I(E-V) - \frac{1}{4}\alpha_\vartheta^2 \sin^{-2}\frac{k}{2}\right)^{1/2}.$$

The action variables have the form

$$J_\varphi = \oint p_\varphi d\varphi = 2\pi\alpha_\varphi, \quad J_\vartheta = \oint p_\vartheta d\vartheta = 2\pi(\alpha_\vartheta - \alpha_\varphi),$$

thus

$$\alpha_\vartheta = D = \frac{1}{2\pi}(J_\vartheta + J_\varphi),$$

D denoting a fixed value of the magnitude of $\mathbf{M}$,

$$J_k = \oint p_k dk = \oint \left(2I(E-V) - \frac{1}{4} D^2 \sin^{-2}\frac{k}{2}\right)^{-\frac{1}{2}} dk. \tag{4.17}$$

The dynamical structure of isotropic models is reflected by the last formula (4.17). Solving it with respect to E, we, in principle, can express the Hamilton function through action variables. Obviously the explicit formula is possible only if we know the functional form of V. However, it is seen that for any isotropic potential, H depends on J_ϑ, J_φ through their sum $2\pi D$,

$$H = E(J_k, J_\vartheta, J_\varphi) = f(J_k, J_\vartheta + J_\varphi).$$

This corresponds to the one-fold degeneracy, characteristic of all isotropic problems. Denoting the fundamental frequencies by

$$\nu_p = \frac{\partial E}{\partial J_p}, \quad p = k, \vartheta, \varphi,$$

we obviously have

$$\nu_\vartheta = \nu_\varphi.$$

For completely degenerate models quoted above (the potentials (4.12)) the expression (4.17) can be explicitly found.

For the isotropic degenerate oscillator we have

$$E = \frac{J^2}{32\pi^2 I} + \frac{\omega}{2\pi} J, \tag{4.18}$$

where

$$\omega = \sqrt{\varkappa/I}, \quad J = 2J_k + J_\vartheta + J_\varphi. \tag{4.19}$$

The factor 2 in Eq. (4.19) corresponds to the existence of two turning points of each kind. Degeneracy of frequencies has the form

$$\nu_k = 2\nu_\vartheta = 2\nu_\varphi = \nu,$$

where

$$\nu = \frac{dE}{dJ} = \frac{1}{2\pi}\sqrt{\omega^2 + \frac{E}{2I}} = \boldsymbol{\Omega}/2\pi,$$

in agreement with expression (4.13).

For the torsional Kepler problem $(4.12)_3$ we obtain

$$E = \frac{J^2}{32\pi^2 I} - \frac{2\pi^2 I\alpha^2}{J^2}, \tag{4.20}$$

where now,

$$J = J_k + J_\vartheta + J_\varphi, \quad \nu_k = \nu_\vartheta = \nu_\varphi = \frac{dE}{dJ}, \tag{4.21}$$

thus all frequencies are identical — there is only one turning point of each kind.

There are some delicate points concerning the free rigid body, $V = 0$. If we use the proper, classical model of the configuration space, $SO[3, R]$, then

$$E = \frac{J^2}{8\pi^2 I}, \quad J = J_k + J_\vartheta + J_\varphi. \tag{4.22}$$

However, calculations performed formally for $SU[2]$, give

$$E = \frac{J^2}{32\pi^2 I}, \quad J = J_k + J_\vartheta + J_\varphi. \tag{4.23}$$

This shows that the free spherical top with the physical configuration space $SO[3, R]$ fails to be the special case of V_{osc}, V_{Kp} with $\varkappa \to 0$, $\alpha \to 0$. If we put formally $\alpha \to 0$, then Eq. (4.20) asymptotically becomes Eq. (4.23), i.e. the formula for the spherical top with $SU[2]$-range of angular parameters. Seemingly, the oscillatory formula (4.18) also gives $SU[2]$-top when $\varkappa \to 0$; nevertheless there is a qualitative discontinuity of the limit transition because in Eq. (4.23) there is no factor 2 characteristic of Eq. (4.18). This qualitative gap follows from the fact that V_{osc} $(4.12)_2$, when considered as as function on $SU[2]$, divides it into two disjoint parts separated at $k = \pi$ by the potential barrier which is inpenetrable for any nonvanishing value of $\varkappa$, no matter how small.

Let us observe that the second terms in Eqs. (4.18) and (4.20) are identical with the action-angle expressions of energy of a material point, respectively for the isotropic harmonic oscillator and Kepler attraction. Thus the energy of our isotropic degenerate torsional models consists additively of two parts: free rotations in $SU[2]$ i.e. in the doubled angular range (4π), and degenerate central vibrations in three-dimensional Euclidean space. This is a rather unexpected and interesting phenomenon. In particular, Eq. (4.18) means that the potential $V_{osc} = \frac{\varkappa}{2}\theta^2 = 2\varkappa \operatorname{tg}^2 \frac{k}{2}$ is really "as harmonic as possible" with rigid body degrees of freedom and that the whole nonlinearity of the model is absorbed by free doubled rotations; these rotations are in some sense imposed upon true harmonic oscillation.

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