Arch. Mech., 39, 6, pp. 619–632, Warszawa 1987

Some existence results in dynamical thermoelasticity Part II. Linear case

A. CHRZĘSZCZYK (KIELCE)

THIS PAPER is a continuation of the paper [1] concerning the existence of solutions in nonlinea. thermoelastodynamics and is devoted to the investigation of corresponding linear equations The proofs are similar to those in [2] and therefore will be only sketched.

Praca dotyczy istnienia, jednoznaczności oraz ciągłej zależności rozwiązań od danych dla pewnego liniowego układu równań obejmującego w szczególności równania klasycznej dynamicznetermosprężystości. Przyjęte założenia dotyczące gładkości współczynników i danych są na tyle słabe, że wyniki pracy mogły być zastosowane w części pierwszej (patrz [1]) do dowodu istnienia rozwiązań dla nieliniowej dynamicznej termosprężystości.

Работа посвящена существованию, единственности и непрерывной зависимости от данных для некоторой линейной системы уравнений, охватывающих в частности классические уравнения динамической термоупругости. Предположения гладкости коэффициентов и данных на столько слабые, что результаты работы могли быть применены в первой части (см. [1]) в доказательстве существования решений для нелинейной динамической термоупругости.

1. Formulation of the problem

IN OPPOSITION to the nonlinear case, the literature concerning the well-posedness of initial boundary value problems of linear thermoelastodynamics is relatively extensive, see for example [3]–[6].

It seems, however, that the existing papers do not contain the results sufficiently strong to be applicable in the nonlinear case(¹). Therefore we have decided to write a separate paper devoted to the investigation of the linear equations used in [1]. These linear equations are slightly more general than the classical equations of the linear thermoelasticity and contain the last ones as a special case.

To be more precise, in the present paper we investigate the system of linear operator equations of the form

$$\ddot{\chi}+A\{\chi, \theta\}=f, \tag{1.1}$$

$$b\dot{\theta}+B\{\chi, \theta\}=g \quad \text{in} \quad \mathscr{B}\times]0, T[, \tag{1.2}$$

(¹) In particular, in the linear theory the coefficients of equations are smooth functions of the position of the point while in the nonlinear case they are functions of solutions and therefore depend additionally on the time and belong to appropriate Sobolev spaces.

where

$$A\{\chi, \theta\} = A_1\chi + A_2\theta, \tag{1.3}$$

$$B\{\chi, \theta\} = B_1\theta + B_2\chi + B_3\dot{\chi} \tag{1.4}$$

and $\mathscr{B}$ is a given regular domain in R^3, $]0, T[$ is a finite interval of R^1, χ, θ are unknown functions defined on $\mathscr{B}\times]0, T[$ with values in R^3 and R^1 respectively, $\ddot{\chi}$, $\dot{\theta}$ denote corresponding derivatives with respect to $t \in]0, T[$, f, g are given functions defined on $\mathscr{B}\times]0, T[$ with values in R^3 and R^1, respectively.

The operators A_i, $i = 1, 2$, B_i, $i = 1, 2, 3$, the coefficient b and the functions f, g are equal to the operators $A_i(\bar{\chi}, \bar{\theta})$, $i = 1, 2$, $B_i(\bar{\chi}, \bar{\theta})$, $i = 1, 2, 3$, the coefficient $b(\bar{\chi}, \bar{\theta})$ and the functions $f(\bar{\chi}, \bar{\theta})$, $g(\bar{\chi}, \bar{\theta})$ of the paper [1], respectively, and have the properties described in Lemma 1 and Lemma 2 of [1].

To make the paper accessible to readers who are not interested in the nonlinear case, we show that the system (1.1), (1.2) obeys the classical equations for a linear nonhomogeneous thermoelastic body. To this end one has to use the relation $\chi(X, t) = X + u(X, t)$ where $\chi(X, t)$ is the position of the point $X \in \mathscr{B}$ at the time $t \in]0, T[$ and $u(X, t)$ is the displacement vector, and next to put

$$A_1\chi = \frac{1}{\varrho}\operatorname{div} C[E], \qquad A_2\theta = \frac{1}{\varrho}\operatorname{div}(\theta-\theta_0)M, \qquad f = \frac{1}{\varrho}\mathsf{b}, \tag{1.5}$$

$$\begin{gathered} B_1\theta = \operatorname{div}(K\nabla\theta), \qquad B_2\chi = 0, \qquad B_3\dot{\chi} = \theta_0 M\cdot\dot{E}, \\ b = c, \qquad g = r, \end{gathered} \tag{1.6}$$

where $\theta = \theta(X, t)$ the temperature function, $\varrho = \varrho(X) > 0$ — the density, $E = \frac{1}{2}(\nabla u + \nabla u^T)$ — the infinitesimal strain tensor, $C = C(X)$ — the elasticity tensor, $M = M(X)$ — the stress-temperature tensor, θ_0 — the reference temperature, $c = c(X) > 0$ — the specific heat, $K = K(X)$ — the conductivity tensor, $\mathsf{b} = \mathsf{b}(X, t)$ — the body force, $r = r(X, t)$ — the heat supply. As a consequence we obtain the known equations, see [7], p. 310,

$$\varrho\ddot{u} = \operatorname{div}\big(C[E] + (\theta-\theta_0)M\big) + \mathsf{b}, \tag{1.7}$$

$$c\dot{\theta} = \operatorname{div}(K\nabla\theta) + \theta_0 M\cdot\dot{E} + r. \tag{1.8}$$

The reader interested in Eqs. (1.7) and (1.8) only may read the present paper independently of [1].

In a formulation of the main results of this paper we shall use the notations concerning the function spaces introduced in [1]. For easy reference we record briefly these notations

First of all let us recall that in [1] we have put

$$\mathsf{H}_\gamma \equiv W^{\gamma,2}(\mathscr{B}, R^3), \qquad H_\gamma \equiv W^{\gamma,2}(\mathscr{B}, R^1), \qquad \gamma \in R^1,$$

where the right hand sides are Sobolev spaces of functions defined on $\mathscr{B}$ and with values in R^3 and R^1, respectively. The norm of these spaces is denoted by $\|\cdot\|_\gamma$. The spaces V, V are defined as follows:

$$\mathsf{V} = \{\chi \in \mathsf{H}_1 : \chi = 0 \text{ on } \partial\mathscr{B}\}, \qquad V = \{\theta \in H_1 : \theta = 0 \text{ on } \partial\mathscr{B}\}.$$

We have also introduced the notations $X_\gamma \equiv \mathsf{H}_\gamma \cap \mathsf{V}$, $Y_\gamma \equiv H_\gamma \cap V$.

Moreover, the following spaces of operators are used:

$$(1.9)\qquad \begin{gathered} \mathscr{L}_1 = \bigcap_{j=-1}^{m-1} \mathscr{L}(\mathsf{Z}_{j+1}\mathsf{Z}_j), \qquad \mathscr{L}_2 = \bigcap_{j=-1}^{m-2} \mathscr{L}(\mathsf{Z}_{j+2}, \mathsf{Z}_j), \\ \mathscr{L}_k = \bigcap_{j=0}^{m-k} \mathscr{L}(\mathsf{Z}_{j+k}, \mathsf{Z}_j), \qquad k = 3, \ldots, m, \qquad m \geqslant 4, \end{gathered}$$

where $Z_j = \mathsf{H}_j$ or $Z_j = H_j$ and $\mathscr{L}(Z^1, Z^2)$ denotes the space of bounded linear operators from the Banach space Z^1 to the Banach space Z^2.

Finally the usual notations concerning spaces of functions defined on $[0, T]$ with values in a Banach space Z are used as for example $L^p([0, T], Z)$, $W^{k,p}([0, T], Z)$, $C^k([0, T], Z)$, $1 \leqslant p \leqslant +\infty$, $k = 0, 1, 2, \ldots$ (see [8], Chapt, I Sect. 3). If ψ is an element of $W^{k,p}([0, T], Z)$ or $C^k([0, T], Z)$, then $\overset{(i)}{\psi}$ denotes the i-th derivative of ψ with respect to $t \in [0, T]$.

Now we are ready to formulate precise assumptions concerning the operators A_i, $i = 1, 2$, B_i, $i = 1, 2, 3$ and other data.

First of all let us assume that an integer m, $m \geqslant 4$, is given and $\mathscr{B}$ is a bounded domain of class C^m.

Let the following inclusions be satisfied:

$$(1.10)\qquad \begin{gathered} A_1 \in \bigcap_{k=1}^{m-1} W^{k,\infty}([0, T], \mathscr{L}_{k+1}), \\ B_1, B_2 \in \bigcap_{k=1}^{m-2} W^{k,\infty}([0, T], \mathscr{L}_{k+1}), \\ \overset{(m-1)}{B_1} \in L^2\big([0, T], \mathscr{L}(H_{m-1}, H_{-1}) \cap \mathscr{L}(H_m, H_0)\big), \\ \overset{(m-1)}{B_2} \in L^2\big([0, T], \mathscr{L}(\mathsf{H}_m, H_0)\big), \\ A_2, B_3 \in \bigcap_{k=1}^{m-1} W^{k,\infty}([0, T], \mathscr{L}_k), \\ b \in \bigcap_{k=0}^{m-1} W^{k,\infty}([0, T], H_{m-1-k}). \end{gathered}$$

Also let the operators A_1 and B_1 satisfy the following conditions:

(1.11) For any $k = 0, 1, \ldots, m-2$ and any χ, θ the inclusions $\chi \in X_k$, $A_1\chi \in \mathsf{H}_k$ (resp. $\theta \in Y_k$, $B_1\theta \in H_k$) imply $\chi \in X_{k+2}$ (resp. $\theta \in Y_{k+2}$).

Furthermore assume that there exist positive constants μ_i, $\varkappa_i$, λ_i, $i = 1, 2$, c_0, such that

$$(1.12)\qquad \begin{aligned} \|\chi\|_{k+2} &\leqslant \mu_1(\|\chi\|_k + \|A_1\chi\|_k) \quad \text{for all} \quad \chi \in X_{k+2}, \\ \|\theta\|_{k+2} &\leqslant \mu_2(\|\theta\|_k + \|B_1\theta\|_k) \quad \text{for all} \quad \theta \in Y_{k+2}, \end{aligned}$$

$$(1.13)\qquad \begin{aligned} \langle A_1\chi, \chi\rangle + \varkappa_1\|\chi\|_0^2 &\geqslant \lambda_1\|\chi\|_1^2 \quad \text{for all} \quad \chi \in X_1, \\ \langle B_1\theta, \theta\rangle + \varkappa_2\|\theta\|_0^2 &\geqslant \lambda_2\|\theta\|_1^2 \quad \text{for all} \quad \theta \in Y_1, \end{aligned}$$

$$(1.14)\qquad b \geqslant c_0 \quad \text{on} \quad \overline{\mathscr{B}} \times [0, T].$$

For the general equations (1.1), (1.2) the conditions (1.10)–(1.14) are motivated by Lemma 1 and Lemma 2 of the paper [1].

In the special case of Eqs. (1.7), (1.8) where the operators A_1, A_2, B_1, B_3 and the coefficient b are defined in the relations (1.5), (1.6) and are time-independent, the relations (1.10) reduce to

$$\begin{aligned} &A_1, B_1 \in \mathscr{L}_2, \qquad A_2, B_3 \in \mathscr{L}_1, \\ &b \in H_{m-1}. \end{aligned} \tag{1.15}$$

To satisfy the relations (1.15) it is sufficient to assume that the functions c, ϱ and the components of the tensors C, M, K belong to the space H_{m-1}. The relations (1.11)–(1.14) are in this case the consequences of the strong ellipticity of the tensors C, K and the positivity of the functions c, ϱ.

As in [1] the right hand sides of Eqs. (1.1), (1.2) are assumed to satisfy the conditions

$$\begin{aligned} &f \in \bigcap_{k=0}^{m-2} C^k([0, T], \mathsf{H}_{m-k-2}), \qquad \overset{(m-1)}{f} \in L^2([0, T], \mathsf{H}_0), \\ &g \in \bigcap_{k=0}^{m-2} C^k([0, T], H_{m-k-2}), \qquad \overset{(m-1)}{g} \in L^2([0, T], H_{-1}). \end{aligned} \tag{1.16}$$

Equations (1.1), (1.2) will be considered together with the initial conditions

$$\chi(0) = \overset{0}{\chi}, \quad \dot{\chi}(0) = \overset{1}{\chi}, \quad \theta(0) = \overset{0}{\theta} \quad \text{on} \quad \mathscr{B}, \tag{1.17}$$

where the right hand sides are given functions satisfying the relations

$$\overset{0}{\chi} \in \mathsf{H}_m, \quad \overset{1}{\chi} \in \mathsf{H}_{m-1}, \quad \overset{0}{\theta} \in H_m \tag{1.18}$$

and with the boundary conditions

$$\chi = 0, \quad \theta = 0 \quad \text{on} \quad \partial\mathscr{B} \times [0, T]. \tag{1.19}$$

To satisfy the compatibility conditions at $t = 0$, we define for $k = 0, 1, \dots, m-2$

$$\overset{k+2}{\chi} = \overset{(k)}{f}(0) - \sum_{i=0}^{k} \binom{k}{i} [\overset{(i)}{A_1}(0) \overset{k-i}{\chi} + \overset{(i)}{A_2}(0) \overset{k-i}{\theta}], \tag{1.20}$$

$$b(0) \overset{k+1}{\theta} = \overset{(k)}{g}(0) - \sum_{i=1}^{k} \binom{k}{i} \overset{(i)}{b}(0) \overset{k+1-i}{\theta} - \sum_{i=0}^{k} \binom{k}{i} [\overset{(i)}{B_1}(0) \overset{k-i}{\theta} + \overset{(i)}{B_2}(0) \overset{k-i}{\chi} + \overset{(i)}{B_3}(0) \overset{k+1-i}{\chi}]^{(2)} \tag{1.21}$$

and assume that

$$\begin{aligned} &\overset{k}{\chi} \in X_{m-k}, \qquad k = 2, \dots, m, \\ &\overset{k}{\theta} \in Y_{m-k}, \qquad k = 1, \dots, m-1. \end{aligned} \tag{1.22}$$

([2]) In the case of the time-independent operators defined in the relations (1.5), (1.6) the relations (1.20), (1.21) simplify in an obvious manner.

Finally we introduce the auxiliary expressions

$$\mu = \{\mu_1, \mu_2\}, \quad \varkappa = \{\varkappa_1, \varkappa_2\}, \quad \lambda = \{\lambda_1, \lambda_2\}, \tag{1.23}$$

$$L^0 = \|A_1(0)\|_{\mathscr{L}_2} + \|b(0)\|_{m-\varepsilon}, \qquad \varepsilon \in \left]0, \frac{1}{2}\right[\text{— sufficiently small,} \tag{1.24}$$

$$\begin{aligned} L = \operatorname*{ess\,sup}_{t\in[0,T]} \Big\{ & \|A_1(t)\|_{\mathscr{L}_2} + \sum_{k=1}^{m-1} \|\overset{(k)}{A_1}(t)\|_{\mathscr{L}_{k+1}} + \|A_2(t)\|_{\mathscr{L}_1} + \sum_{k=1}^{m-1} \|\overset{(k)}{A_2}(t)\|_{\mathscr{L}_k} \\ & + \|B_1(t)\|_{\mathscr{L}_2} + \sum_{k=1}^{m-2} \|\overset{(k)}{B_1}(t)\|_{\mathscr{L}_{k+1}} + \|B_2(t)\|_{\mathscr{L}_2} + \sum_{k=1}^{m-2} \|\overset{(k)}{B_2}(t)\|_{\mathscr{L}_{k+1}} + \|B_3(t)\|_{\mathscr{L}_1} \\ & + \sum_{k=1}^{m-1} \|\overset{(k)}{B_3}(t)\|_{\mathscr{L}_k} + \sum_{k=0}^{m-} \|\overset{(k)}{b}(t)\|_{m-k-1-\varepsilon} + \|\overset{(m-1)}{b}(t)\|_0 \Big\} \\ & + \Big(\int_0^T \|\overset{(m-1)}{B_1}(t)\|^2_{\mathscr{L}(H_{m-1}, H_{-1})} dt \Big)^{1/2} + \Big(\int_0^T \|\overset{(m-1)}{B_1}(t)\|^2_{\mathscr{L}(H_m, H_0)} dt \Big)^{1/2} \\ & + \Big(\int_0^T \|\overset{(m-1)}{B_2}(t)\|^2_{\mathscr{L}(H_m, H_0)} dt \Big)^{1/2}, \end{aligned} \tag{1.25}$$

$$\begin{aligned} N = & \sum_{k=0}^{m} \|\overset{k}{\chi}\|^2_{m-k} + \max_{t\in[0,T]} \sum_{k=0}^{m-2} \|\overset{(k)}{f}(t)\|^2_{m-2-k} + \int_0^T \|\overset{(m-1)}{f}(t)\|^2_0 dt \\ & + \sum_{k=0}^{m-1} \|\overset{k}{\theta}\|^2_{m-k} + \max_{t\in[0,T]} \sum_{k=0}^{m-2} \|\overset{(k)}{g}(t)\|^2_{m-2-k} + \int_0^T \|\overset{(m-1)}{g}(t)\|^2_{-1} dt. \end{aligned} \tag{1.26}$$

Now we formulate our main result:

THEOREM. *Let the assumptions* (1.10)–(1.14), (1.16), (1.18), (1.22) *be satisfied. Then for any positive number* T *the problem* (1.1), (1.2), (1.17), (1.19) *has a unique solution* $\{\chi, \theta\}$ *with the properties*

$$\chi \in \bigcap_{k=0}^{m} C^k([0, T], \mathsf{H}_{m-k}), \tag{1.27}$$

$$\theta \in \bigcap_{k=0}^{m-2} C^k([0, T], H_{m-k}), \qquad \overset{(m-1)}{\theta} \in C^0([0, T], H_0) \cap L^2([0, T], H_1). \tag{1.28}$$

Furthermore, the following estimate holds:

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in[0,T]} \Big(\sum_{k=0}^{m} \|\overset{(k)}{\chi}(t)\|^2_{m-k} + \sum_{k=0}^{m-2} \|\overset{(k)}{\theta}(t)\|^2_{m-k} + \|\overset{(m-1)}{\theta}(t)\|^2_0 \Big) + \int_0^T \|\overset{(m-1)}{\theta}(t)\|^2_1 dt \\ \leqslant (C_1 N + C_2 LN)\big(1 + (1 + TC_3)\exp(TC_3)\big), \end{aligned} \tag{1.29}$$

where

$$C_i = \hat{C}_i(c_0, \mu, \varkappa, \lambda, L^0) \quad \text{for} \quad i = 1, 2, \qquad \hat{C}_3 = C_3(c_0, \mu, \varkappa, \lambda, L^0, L)$$

and the functions $\hat{C}_1, \hat{C}_2, \hat{C}_3$ *depend continuously on their arguments.*

The proof of this theorem will be divided into several steps.

2. Regularization

It is well known that the operators A_i, $i = 1, 2$, B_i, $i = 1, 2, 3$ can be extended to the interval $[-T, 2T]$ in a smooth manner, see [9] Chapt. 1, Sect. 2.2 and therefore the sequences

$$(2.1)\qquad \begin{aligned} A_{in}(t) &= (\varrho_n \ast A_i)(t), \quad i = 1, 2 \\ B_{in}(t) &= (\varrho_n \ast B_i)(t), \quad i = 1, 2, 3 \quad n = 1, 2, \ldots \end{aligned}$$

can be constructed, where ϱ_n is a usual sequence of regularizing kernels and the symbol $\ast$ denotes convolution, i.e.

$$(\varrho_n \ast \psi)(t) = \int\limits_{-\infty}^{\infty} \varrho_n(t-\sigma)\psi(\sigma)d\sigma, \qquad t \in [0, T], \qquad \psi = A_i \text{ or } B_i.$$

The operators A_{in}, $i = 1, 2$, B_{in}, $i = 1, 2, 3$ have the following regularity properties:

$$(2.2)\qquad \begin{aligned} &A_{1n}, B_{1n}, B_{2n} \in C^m([0, T], \mathscr{L}_2), \\ &A_{2n}, B_{3n} \in C^m([0, T], \mathscr{L}_1). \quad n = 1, 2, \ldots \end{aligned}$$

and the following convergence properties if $n \to +\infty$:

$$(2.3)\qquad \begin{aligned} &A_{1n} \to A_1 \quad \text{in} \quad \bigcap_{k=1}^{m-1} W^{k,\infty}([0, T], \mathscr{L}_{k+1}), \\ &B_{1n} \to B_1, B_{2n} \to B_2 \quad \text{in} \quad \bigcap_{k=1}^{m-2} W^{k,\infty}([0, T], \mathscr{L}_{k+1}), \\ &\overset{(m-1)}{B_{1n}} \to \overset{(m-1)}{B_1} \quad \text{in} \quad L^2([0, T], \mathscr{L}(H_{m-1}, H_{-1}) \cap \mathscr{L}(H_m, H_0)), \\ &\overset{(m-1)}{B_{2n}} \to \overset{(m-1)}{B_2} \quad \text{in} \quad L^2([0, T], \mathscr{L}(H_m, H_0)), \\ &A_{2n} \to A_2, \quad B_{3n} \to B_3 \quad \text{in} \quad \bigcap_{k=1}^{m-1} W^{k,\infty}([0, T], \mathscr{L}_k). \end{aligned}$$

Since the construction does not generally preserve compatibility conditions (1.22), we approximate $\overset{2}{\chi}, \ldots, \overset{m}{\chi}, \overset{1}{\theta}, \ldots, \overset{m-1}{\theta}, f, g$ in such a way that the compatibility holds true. To this end we put

$$(2.4)\qquad \overset{m}{\chi}_n = \overset{m}{\chi}, \qquad \overset{m-1}{\chi}_n = \overset{m-1}{\chi}, \qquad \overset{m-1}{\theta}_n = \overset{m-1}{\theta},$$

and note that for sufficiently large $r > 0$ and for $\varkappa_1, \varkappa_2, n > r$ the operators

$$(2.5)\qquad A_{1n}(0) + \varkappa_1 - A_{2n}(0)\big(B_{1n}(0) + \varkappa_2\big)^{-1} B_{2n}(0)$$

and

$$(2.6)\qquad B_{1n}(0) + \varkappa_2$$

are invertible from X_{k+2} to X_k and from Y_{k+2} to Y_k, respectively, if $k = 0, 1, \ldots, m-2$. Thus, a recursive determination of $\overset{k}{\chi}_n$, $\overset{k}{\theta}_n$, $k = 0, 1, \ldots, m-2$ from the equations

$$(2.7)\quad \overset{k+2}{\chi}_n + [A_{1n}(0)+\varkappa_1 - A_{2n}(0)\big(B_{1n}(0)+\varkappa_2\big)^{-1}B_{2n}(0)]\overset{k}{\chi}_n$$
$$+A_{2n}(0)\big(B_{1n}(0)+\varkappa_2\big)^{-1}\big(-b(0)\overset{k+1}{\theta}_n - B_{3n}(0)\overset{k+1}{\chi} + b(0)\overset{k+1}{\theta} + B_1(0)\overset{k}{\theta}$$
$$+\varkappa_2\overset{k}{\theta}+B_2(0)\overset{k}{\chi}+B_3(0)\overset{k+1}{\chi}\big) = \overset{k+2}{\chi} + A(0)\{\overset{k}{\chi},\overset{k}{\theta}\}+\varkappa_1\overset{k}{\chi},$$

$$(2.8)\quad b(0)\big(\overset{k+1}{\theta}_n + B_{1n}(0)+\varkappa_2\big)\overset{k}{\theta}_n + B_{2n}(0)\overset{k}{\chi}_n + B_{3n}(0)\overset{k+1}{\chi}_n = b(0)\overset{k+1}{\theta} + B(0)\{\overset{k}{\chi},\overset{k}{\theta}\}+\varkappa_2\overset{k}{\theta}$$

is possible. Here

$$A(0)\{\overset{k}{\chi},\overset{k}{\theta}\} = A_1(0)\overset{k}{\chi}+A_2(0)\overset{k}{\theta}$$

and

$$B(0)\{\overset{k}{\chi},\overset{k}{\theta}\} = B_1(0)\overset{k}{\theta}+B_2(0)\overset{k}{\chi}+B_3(0)\overset{k+1}{\chi}.$$

It is not difficult to check that the convergence properties (2.3) imply

$$(2.9)\quad \begin{aligned} &\overset{k}{\chi}_n \to \overset{k}{\chi} \quad \text{in} \quad X_{m-k}, \quad k = 0, 1, \ldots, m-2,\\ &\overset{k}{\theta}_n \to \overset{k}{\theta} \quad \text{in} \quad Y_{m-k}, \quad n \to \infty. \end{aligned}$$

Now let us rewrite the formulas (2.7), (2.8) in the abbreviate form:

$$(2.10)\quad \begin{aligned} \overset{k+2}{\chi}_n + A_n(0)\{\overset{k}{\chi}_n,\overset{k}{\theta}_n\}+\varkappa_1\overset{k}{\chi}_n &= \overset{k+2}{\chi} + A(0)\{\overset{k}{\chi},\overset{k}{\theta}\}+\varkappa_1\overset{k}{\chi},\\ b(0)\overset{k+1}{\theta}_n + B_n(0)\{\overset{k}{\chi}_n,\overset{k}{\theta}_n\}+\varkappa_2\overset{k}{\theta}_n &= b(0)\overset{k+1}{\theta} + B(0)\{\overset{k}{\chi},\overset{k}{\theta}\}+\varkappa_2\overset{k}{\theta}, \end{aligned}$$

where

$$A_n(0)\{\overset{k}{\chi}_n,\overset{k}{\theta}_n\} = A_{1n}(0)\overset{k}{\chi}_n + A_{2n}(0)\overset{k}{\theta}_n,$$
$$B_n(0)\{\overset{k}{\chi}_n,\overset{k}{\theta}_n\} = B_{1n}(0)\overset{k}{\theta}_n + B_{2n}(0)\overset{k}{\chi}_n + B_{3n}(0)\overset{k+1}{\chi}_n$$

and let us define the sequences

$$(2.11)\quad \begin{aligned} p_n^0 &= \varkappa_1(\overset{0}{\chi}-\overset{0}{\chi}_n),\\ q_n^0 &= \varkappa_2(\overset{0}{\theta}-\overset{0}{\theta}_n), \end{aligned}$$

$$(2.12)\quad p_n^k = \varkappa_1(\overset{k}{\chi}-\overset{k}{\chi}_n)+\sum_{i=1}^{k}\binom{k}{i}[\overset{(i)}{A}_n(0)\{\overset{k-i}{\chi}_n,\overset{k-i}{\theta}_n\}-\overset{(i)}{A}(0)\{\overset{k-i}{\chi},\overset{k-i}{\theta}\}],$$

$$q_n^k = \varkappa_2(\overset{k}{\theta}-\overset{k}{\theta}_n)+\sum_{i=1}^{k}\binom{k}{i}[\overset{(i)}{B}_n(0)\{\overset{k-i}{\chi}_n,\overset{k-i}{\theta}_n\}-\overset{(i)}{B}(0)\{\overset{k-i}{\chi},\overset{k-i}{\theta}\}]$$
$$+\sum_{i=1}^{k}\binom{k}{i}\overset{(i)}{b}(0)\,(\overset{k+1-i}{\theta}_n - \overset{k+1-i}{\theta}),$$

where

$$k = 1, 2, \dots, m-2, \qquad \overset{(i)}{A}_n(0)\{\overset{k-i}{\chi}_n, \overset{k-i}{\theta}_n\} = \overset{(i)}{A}_{1n}(0)\overset{k-i}{\chi}_n + \overset{(i)}{A}_{2n}(0)\overset{k-i}{\theta}_n$$

and the symbols

$$\overset{(i)}{A}(0)\{\overset{k-i}{\chi}, \overset{k-i}{\theta}\}, \qquad \overset{(i)}{B}_n\{\overset{k-i}{\chi}_n, \overset{k-i}{\theta}_n\}, \qquad \overset{(i)}{B}(0)\{\overset{k-i}{\chi}, \overset{k-i}{\theta}\}$$

are defined analogously.

From the properties (2.3) and (2.9) it follows that

$$\text{(2.13)} \qquad \begin{aligned} &p_n^k \to 0 \quad \text{in} \quad X_{m-k}, \quad k = 0, 1, \dots, m-2, \\ &q_n^k \to 0 \quad \text{in} \quad Y_{m-k}, \quad \text{as} \quad n \to \infty. \end{aligned}$$

By the trace theorem, see [9], Chapt. I, Sect. 3.2, there exist sequences $\{p_n\}, \{q_n\}$, $n > r$ such that

$$\text{(2.14)} \qquad \begin{aligned} &p_n \in \bigcap_{k=0}^{m-1} W^{k,2}([0, T], \mathsf{H}_{m-1-k}), \qquad q_n \in \bigcap_{k=0}^{m-1} W^{k,2}([0, T], H_{m-1-k}), \\ &\overset{(k)}{p}_n(0) = p_n^k, \qquad \overset{(k)}{q}_n(0) = q_n^k, \\ &p_n \to 0 \quad \text{in} \quad \bigcap_{k=0}^{m-1} W^{k,2}([0, T], \mathsf{H}_{m-1-k}), \\ &q_n \to 0 \quad \text{in} \quad \bigcap_{k=0}^{m-1} W^{k,2}([0, T], H_{m-1-k}), \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore if we define f_n, g_n by

$$\text{(2.15)} \qquad \begin{aligned} f_n &= f + p_n, \\ g_n &= g + q_n, \end{aligned}$$

then it is not difficult to prove that

$$\text{(2.16)} \qquad \begin{aligned} &f_n \to f \quad \text{in} \quad \bigcap_{k=0}^{m-2} C^k([0, T], \mathsf{H}_{m-2-k}), \\ &\overset{(m-1)}{f_n} \to \overset{(m-1)}{f} \quad \text{in} \quad L^2([0, T], \mathsf{H}_0), \quad \text{as } n \to \infty, \end{aligned}$$

$$\text{(2.17)} \qquad \begin{aligned} &g_n \to g \quad \text{in} \quad \bigcap_{k=0}^{m-2} C^k([0, T], H_{m-2-k}), \\ &\overset{(m-1)}{g_n} \to \overset{(m-1)}{g} \quad \text{in} \quad L^2([0, T], H_{-1}), \quad \text{as } n \to \infty \end{aligned}$$

and the compatibility conditions

$$\text{(2.18)} \qquad \begin{aligned} &\overset{k+2}{\chi}_n = \overset{(k)}{f}_n(0) - \sum_{i=0}^{k} \binom{k}{i} \overset{(i)}{A}_n(0)\{\overset{k-i}{\chi}_n, \overset{k-i}{\theta}_n\} \in X_{m-k-2}, \\ &b(0)\overset{k+1}{\theta}_n = \overset{(k)}{g}(0) - \sum_{i=0}^{k} \binom{k}{i} \overset{(i)}{B}_n(0)\{\overset{k-i}{\chi}_n, \overset{k-i}{\theta}_n\} - \sum_{i=1}^{k} \binom{k}{i} \overset{(i)}{b}(0)\overset{k+1-i}{\theta}_n \in Y_{m-k-2}, \\ &n > r, \quad k = 0, 1, \dots, m-2, \quad \text{hold true.} \end{aligned}$$

3. Existence of solutions for the regularized problem

In the present section we shall show that if the conditions (1.10) are replaced by the stronger conditions

$$\begin{gathered} A_1, B_1, B_2 \in C^m([0, T], \mathscr{L}_2), \\ A_2, B_3 \in C^m([0, T], \mathscr{L}_1), \end{gathered} \tag{3.1}$$

then the existence of solutions of the problem (1.1), (1.2), (1.17), (1.19) can be proved by use of the Faedo-Galerkin method. In a consequence we shall prove that the regularized problem from Sect. 2 has at least one solution.

Let us differentiate Eqs. (1.1), (1.2) $m-1$ times with respect to t to obtain

$$\begin{aligned} \overset{(m+1)}{\chi} + A_1 \overset{(m-1)}{\chi} &= -\sum_{i=1}^{m-1} \binom{m-1}{i} \overset{(i)}{A_1} \overset{(m-1-i)}{\chi} - \sum_{i=0}^{m-1} \binom{m-1}{i} \overset{(i)}{A_2} \overset{(m-1-i)}{\theta} + \overset{(m-1)}{f} \equiv F, \\ b\overset{(m)}{\theta} + B_1 \overset{(m-1)}{\theta} &= -\sum_{i=1}^{m-1} \binom{m-1}{i} (\overset{(i)}{b} \overset{(m-i)}{\theta} + \overset{(i)}{B_1} \overset{(m-1-i)}{\theta}) \\ &\quad - \sum_{i=0}^{m-1} \binom{m-1}{i} (\overset{(i)}{B_2} \overset{(m-1-i)}{\chi} + \overset{(i)}{B_3} \overset{(m-i)}{\chi}) + \overset{(m-1)}{g} \equiv G. \end{aligned} \tag{3.2}$$

We shall show the existence of a weak solution of the problem (3.2) with the initial conditions

$$\begin{gathered} \chi(0) = \overset{0}{\chi}, \quad \dot{\chi}(0) = \overset{1}{\chi}, \ldots, \quad \overset{(m)}{\chi}(0) = \overset{m}{\chi}, \\ \theta(0) = \overset{0}{\theta}, \quad \dot{\theta}(0) = \overset{1}{\theta}, \ldots, \quad \overset{(m-1)}{\theta}(0) = \overset{m-1}{\theta}. \end{gathered} \tag{3.3}$$

Let us note that the compatibility conditions (1.22) imply that a solution of the problem (3.2), (3.3) is also a solution of Eqs. (1.1), (1.2), (1.17). The boundary conditions (1.19) will be accounted for by appropriate choice of the function spaces used in the proof.

As usual in the Faedo-Galerkin method let $\{\xi_\mu\}_{\mu=1}^\infty$ be a base of V and $\{\zeta_\mu\}_{\mu=1}^\infty$ a base of V. Let V_ν and V_ν be the subspaces of V and V spanned by $\{\xi_1, \ldots, \xi_\nu\}$ and $\{\zeta_1, \ldots, \zeta_\nu\}$, respectively. We seek a Faedo-Galerkin approximate solution in the form

$$\chi_\nu = \sum_{\mu=1}^{\nu} \varphi_{\nu\mu} \xi_\mu, \quad \theta_\nu = \sum_{\mu=1}^{\nu} \psi_{\nu\mu} \zeta_\mu, \quad \nu = 1, 2, \ldots,$$

where $\varphi_{\nu\mu}$, $\psi_{\nu\mu}$ are real-valued functions defined on $[0, T]$ such that the ordinary differential equations

$$\begin{gathered} \langle \overset{(m+1)}{\chi_\nu}, \xi_\mu \rangle + \langle A_1 \overset{(m-1)}{\chi_\nu}, \xi_\mu \rangle = \langle F, \xi_\mu \rangle, \\ \langle b \overset{(m)}{\theta_\nu}, \zeta_\mu \rangle + \langle B_1 \overset{(m-1)}{\theta_\nu}, \zeta_\mu \rangle = \langle G, \zeta_\mu \rangle, \quad \mu = 1, 2, \ldots, \nu, \end{gathered} \tag{3.4}$$

with the initial conditions

$$\begin{gathered} \chi_\nu(0) = \overset{0}{\chi}_\nu, \ldots, \overset{(m)}{\chi_\nu}(0) = \overset{m}{\chi}_\nu, \\ \theta_\nu(0) = \overset{0}{\theta}_\nu, \ldots, \overset{(m-1)}{\theta_\nu}(0) = \overset{m-1}{\theta}_\nu \end{gathered} \tag{3.5}$$

are satisfied, where $\overset{0}{\chi}_\nu, \dots, \overset{m}{\chi}_\nu, \overset{0}{\theta}_\nu, \dots, \overset{m-1}{\theta}_\nu$ are the projections of $\overset{0}{\chi}, \dots, \overset{m}{\chi}$ and $\overset{0}{\theta}, \dots, \overset{m-1}{\theta}$ on V_ν and V_ν, respectively.

It is easy to check that the following a priori equality for the approximate solution $\{\chi_\nu, \theta_\nu\}$ holds:

$$
\begin{aligned}
(3.6)\qquad & \|\overset{(m)}{\chi}_\nu\|_0^2 + \|b^{1/2}\overset{(m-1)}{\theta}_\nu\|_0^2 + \langle A_1 \overset{(m-1)}{\chi}_\nu, \overset{(m-1)}{\chi}_\nu\rangle + \int_0^t \langle B_1 \overset{(m-1)}{\theta}_\nu, \overset{(m-1)}{\theta}_\nu\rangle d\sigma = \|\overset{m}{\chi}_\nu\|_0^2 \\
& + \|b^{1/2}(0)\overset{m-1}{\theta}_\nu\|_0^2 + \langle A_1(0)\overset{m-1}{\chi}_\nu, \overset{m-1}{\chi}_\nu\rangle + \int_0^t \langle \dot{A}_1 \overset{(m-1)}{\chi}_\nu, \overset{(m-1)}{\chi}_\nu\rangle d\sigma + \int_0^t \langle \dot{b}\overset{(m-1)}{\theta}_\nu, \overset{(m-1)}{\theta}_\nu\rangle d\sigma \\
& + 2\langle F, \overset{(m-1)}{\chi}_\nu\rangle - 2\langle F(0), \overset{m-1}{\chi}_\nu\rangle + 2\int_0^t \langle F, \overset{(m-1)}{\chi}_\nu\rangle d\sigma + 2\int_0^t \langle G, \overset{(m-1)}{\theta}_\nu\rangle d\sigma
\end{aligned}
$$

which, according to the relations (3.1) and (1.12)–(1.14) leads to

$$
\begin{aligned}
(3.7)\qquad & \|\overset{(m)}{\chi}_\nu\|_0^2 + \|\overset{(m-1)}{\theta}_\nu\|_0^2 + \|\overset{(m-1)}{\chi}_\nu\|_1^2 + \int_0^t \|\overset{(m-1)}{\theta}_\nu\|_1^2 d\sigma \\
& \leqslant \tilde{C}_1 + \tilde{C}_2 \int_0^t (\|\overset{(m)}{\chi}_\nu\|_0^2 + \|\overset{(m-1)}{\chi}_\nu\|_1^2 + \|\overset{(m-1)}{\theta}_\nu\|_0^2) d\sigma + \frac{\tilde{C}_3 \delta}{2} \int_0^t \|\overset{(m-1)}{\theta}_\nu\|_1^2 d\sigma
\end{aligned}
$$

with the arbitrary positive number δ and some constants $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$. Putting $\tilde{C}_3\delta \leqslant 1$ and using the Gronwall's inequality, we get

$$
(3.8)\qquad \|\overset{(m)}{\chi}_\nu\|_0^2 + \|\overset{(m-1)}{\theta}_\nu\|_0^2 + \|\overset{(m-1)}{\chi}_\nu\|_1^2 + \int_0^t \|\overset{(m-1)}{\theta}_\nu\|_1^2 d\sigma \leqslant \tilde{C}_4,
$$

with constant $\tilde{C}_4 > 0$.

The estimate (3.8) enables us to extract two subsequences of the sequences $\{\chi_\nu\}_{\nu=1}^\infty$ and $\{\theta_\nu\}_{\nu=1}^\infty$ weakly convergent in the space

$$
W^{m,2}([0, T], \mathrm{H}_0) \cap W^{m-1,2}([0, T], \mathrm{H}_1) \quad \text{and} \quad W^{m-1,2}([0, T], H_1),
$$

respectively. By a standard argument, the limits χ and θ of the two subsequences form a weak solution of the problem (3.2), (3.3) and therefore the solution of the relations (1.1), (1.2), (1.17).

Integrating Eqs. (3.2) with respect to t and using the compatibility conditions (1.22), we obtain the equalities

$$
\begin{aligned}
& A_1 \overset{(k)}{\chi} = -\overset{(k+2)}{\chi} - \sum_{i=1}^{k} \binom{k}{i} \overset{(i)}{A_1} \overset{(k-i)}{\chi} - \sum_{i=0}^{k} \binom{k}{i} \overset{(i)}{A_2} \overset{(k-i)}{\theta} + \overset{(k)}{f}, \\
(3.9)\qquad & B_1 \overset{(k)}{\theta} = -\sum_{i=0}^{k} \binom{k}{i} \overset{(i)}{b} \overset{(k-i+1)}{\theta} - \sum_{i=1}^{k} \binom{k}{i} \overset{(i)}{B_1} \overset{(k-i)}{\theta} - \sum_{i=0}^{k} \binom{k}{i} [\overset{(i)}{B_2} \overset{(k-i)}{\chi} + \overset{(i)}{B_3} \overset{(k+1-i)}{\chi}] + \overset{(k)}{g}, \\
& k = 0, 1, \dots, m-2.
\end{aligned}
$$

Similarly as in [2], from Eq. (3.9) and the elliptic regularity properties of the operators A_1, B_1 it follows that the solution satisfies the additional smoothness relations

$$(3.10)\qquad \chi \in \bigcap_{k=0}^{m} W^{k,2}([0,T], X_{m-k}), \qquad \theta \in \bigcap_{k=0}^{m-1} W^{k,2}([0,T], Y_{m-k}).$$

4. Regularity and uniqueness of solutions of the non-regularized problem

Now let us assume that $\{\chi.\theta\}$ is a solution of the problem (1.1), (1.2), (1.17), (1.19) with the properties (3.10) and that the hypotheses (1.10)–(1.14), (1.16), (1.18), (1.22) are satisfied ([3]).

From the Sobolev embedding theorem and from the properties (3.10) it follows that

$$(4.1)\qquad \begin{aligned} &\chi \in \bigcap_{k=0}^{m-1} C^{m-1-k}([0,T], X_k),\\ &\theta \in \bigcap_{k=1}^{m-1} C^{m-1-k}([0,T], Y_k). \end{aligned}$$

To prove that the solution $\{\chi, \theta\}$ possesses the regularity properties (1.27), (1.28) we use Eqs. (3.2), which we regard as evolution equations for $\overset{(m-1)}{\chi}$ and $\overset{(m-1)}{\theta}$ of order two or one, respectively, with the initial conditions

$$(4.2)\qquad \overset{(m-1)}{\chi}(0) = \overset{m-1}{\chi}, \quad \overset{(m)}{\chi}(0) = \overset{m}{\chi}, \quad \overset{(m-1)}{\theta}(0) = \overset{m-1}{\theta}.$$

The right hand sides F and G of Eqs. (3.2) satisfy the inclusions

$$(4.3)\qquad F \in L^2([0,T], \mathsf{H}_0), \qquad G \in L^2([0,T], H_{-1})$$

and therefore the known results concerning evolution equations (see [9], Chapt. 3, Sect. 4.4, 8.4) can be used to obtain

$$(4.4)\qquad \begin{aligned} &\overset{(m)}{\chi} \in C^0([0,T], X_0), \qquad \overset{(m-1)}{\chi} \in C^0([0,\mathrm{T}], X_1),\\ &\overset{(m-1)}{\theta} \in C^0([0,T], Y_0). \end{aligned}$$

Moreover the following energy identity is valid:

$$(4.5)\qquad \|\overset{m}{\chi}\|_0^2 + \|b^{1/2}\overset{(m-1)}{\theta}\|_0^2 + \langle A_1 \overset{(m-1)}{\chi}, \overset{(m-1)}{\chi}\rangle + \int_0^t \langle B_1 \overset{(m-1)}{\theta}, \overset{(m-1)}{\theta}\rangle d\sigma$$

$$= \|\overset{m}{\chi}\|_0^2 + \|b^{1/2}(0)\overset{m-1}{\theta}\|_0^2 + \langle A_1(0)\overset{m-1}{\chi}, \overset{m-1}{\chi}\rangle + \int_0^t \langle \dot{A}_1 \overset{(m-1)}{\chi}, \overset{(m-1)}{\chi}\rangle d\sigma$$

$$+ \int_0^t \|(\dot{b})^{1/2}\overset{(m-1)}{\theta}\|_0^2 d\sigma + 2\int_0^t \langle F, \overset{(m)}{\chi}\rangle d\sigma + 2\int_0^t \langle G, \overset{(m-1)}{\theta}\rangle d\sigma.$$

([3]) In the present and next section we do not assume that the relation (3.1) holds.

Starting from the relations (4.4), using the elliptic regularity properties (1.11) of the operators A_1 and B_1, Eqs. (3.9) and proceeding inductively as in [2] we can show that the properties (1.27), (1.28) hold true.

It is clear that a solution with the properties (1.27) and (1.28) is unique.

5. A priori estimate for the non-regularized problem

From the identity (4.5) we obtain the estimate

$$(5.1)\quad \operatorname*{ess\,sup}_{\tau\in[0,t]} \{\|\overset{(m)}{\chi}(\tau)\|_0^2+\|\overset{(m-1)}{\theta}(\tau)\|_0^2+\|\overset{(m-1)}{\chi}(\tau)\|_1^2+\int_0^\tau \|\overset{(m-1)}{\theta}(\sigma)\|_1^2 d\sigma\}$$

$$\leqslant \overline{C}_1 N+\overline{C}_2 LN+\overline{C}_3\left(L+\frac{L^2}{2\delta_1}+\frac{1}{2\delta_2}+\frac{1}{2\delta_3}\right)\int_0^t\left\{\sum_{k=0}^{m}\|\overset{(k)}{\chi}(\sigma)\|_{m-k}^2\right.$$

$$\left.+\sum_{k=0}^{m-2}\|\overset{(k)}{\theta}(\sigma)\|_{m-k}^2+\|\overset{(m-1)}{\theta}(\sigma)\|_0^2\right\}d\sigma+\frac{\delta_1}{2}\overline{C}_4\int_0^t\|\overset{(m-1)}{\theta}(\sigma)\|_1^2 d\sigma$$

$$+\frac{\delta_2}{2}\overline{C}_5\int_0^t\|\theta(\sigma)\|_m^2\|\overset{(m-1)}{B_1}(\sigma)\|_{\mathscr{L}(H_m,H_0)}^2 d\sigma+\frac{\delta_3}{2}\overline{C}_6\int_0^t\|\chi(\sigma)\|_m^2\|\overset{(m-1)}{B_2}(\sigma)\|_{\mathscr{L}(H_m,H_0)}^2 d\sigma,$$

where δ_i, $i = 1, 2, 3$ are arbitrary positive numbers and the constants $\overline{C}_i$, $i = 1, 2, \ldots, 6$ may depend on c_0, $\varkappa$, μ, λ, L^0. Using the relations (3.9) and (1.12) we also get

$$(5.2)\quad \operatorname*{ess\,sup}_{\tau\in[0,t]}\left(\sum_{k=0}^{m}\|\overset{(k)}{\chi}(\tau)\|_{m-k}^2+\sum_{k=0}^{m-2}\|\overset{(k)}{\theta}(\tau)\|_{m-k}^2+\|\overset{(m-1)}{\theta}(\tau)\|_0^2\right)+\int_0^t\|\overset{(m-1)}{\theta}(\sigma)\|_1^2 d\sigma\leqslant\overline{C}_7\overline{X},$$

where $\overline{C}_7$ is a constant depending on c_0, $\varkappa$, μ, L^0, λ and $\overline{X}$ is equal to the right hand side of the estimate (5.1).

If the numbers δ_i, $i = 1, 2, 3$ are sufficiently small, then the following inequalities

$$(5.3)\quad \begin{gathered}\delta_1\overline{C}_4\overline{C}_7\leqslant 1,\\ \delta_2\overline{C}_5\overline{C}_7\int_0^T\|\overset{(m-1)}{B_1}(\sigma)\|_{\mathscr{L}(H_m,H_0)}^2 d\sigma\leqslant 1,\\ \delta_3\overline{C}_6\overline{C}_7\int_0^T\|\overset{(m-1)}{B_2}(\sigma)\|_{\mathscr{L}(H_m,H_0)}^2 d\sigma\leqslant 1\end{gathered}$$

hold true, and we obtain

$$(5.4)\quad \sum_{k=0}^{m}\|\overset{(k)}{\chi}(t)\|_{m-k}^2+\sum_{k=0}^{m-2}\|\overset{(k)}{\theta}(t)\|_{m-k}^2+\|\overset{(m-1)}{\theta}(t)\|_0^2+\int_0^t\|\overset{(m-1)}{\theta}(\sigma)\|_1^2 d\sigma$$

(5.4) [cont.]
$$\leqslant \operatorname*{ess\,sup}_{\tau\in[0,t]}\Big(\sum_{k=0}^{m}||\overset{(k)}{\chi}(\tau)||^2_{m-k}+\sum_{k=0}^{m-2}||\overset{(k)}{\theta}(\tau)||^2_{m-k}+||\overset{(m-1)}{\theta}(\tau)||^2_0\Big)+\int_0^t||\overset{(m-1)}{\theta}(\sigma)||^2_1 d\sigma$$
$$\leqslant \bar{\bar{C}}_1 N+\bar{\bar{C}}_2 LN+\bar{\bar{C}}_3\int_0^t\Big(\sum_{k=0}^{m}||\overset{(k)}{\chi}(\sigma)||^2_{m-k}+\sum_{k=0}^{m-2}||\overset{(k)}{\theta}(\sigma)||^2_{m-k}+||\overset{(m-1)}{\theta}(\sigma)||^2_0\Big)d\sigma,$$

where the constants $\bar{\bar{C}}_i$, $i = 1, 2, 3$ depend on $c_0, \varkappa, \mu, \lambda, L^0$ and $\bar{\bar{C}}_3$ depends additionally on L.

Finally, using the Gronwall's inequality we arrive at the estimates

$$(5.5)\quad \begin{aligned}&\sum_{k=0}^{m}||\overset{(k)}{\chi}(t)||^2_{m-k}+\sum_{k=0}^{m-2}||\overset{(k)}{\theta}(t)||^2_{m-k}+||\overset{(m-1)}{\theta}(t)||^2_0\leqslant(\bar{\bar{C}}_1 N+\bar{\bar{C}}_2 LN)\exp(T\bar{\bar{C}}_3),\\ &\int_0^t||\overset{(m-1)}{\theta}(\sigma)||^2_1 d\sigma\leqslant(\bar{\bar{C}}_1 N+\bar{\bar{C}}_2 LN)\big(1+\bar{\bar{C}}_3 T\exp(T\bar{\bar{C}}_3)\big),\end{aligned}$$

which imply the estimate (1.29).

6. Proof of the main theorem

Let us consider a sequence of regularized problems from Sect. 2, i.e. the problems of finding solutions of the system

$$(6.1)\quad \begin{aligned}&\ddot{\chi}_n+A_n\{\chi_n,\theta_n\}=f_n,\\ &b\dot{\theta}_n+B_n\{\chi_n,\theta_n\}=g_n,\qquad n>r,\\ &\chi_n(0)=\overset{0}{\chi}_n,\quad \dot{\chi}_n(0)=\overset{1}{\chi}_n,\quad \theta(0)=\overset{0}{\theta}_n,\end{aligned}$$

where r is sufficiently large. According to Sect. 3, the problem (6.1) possesses at least one solution. From Sect. 4 it follows that the problem has only one solution.

Let $\{\chi_n, \theta_n\}$ $n > r$ be the solution of the system (6.1). By virtue of the construction of A_n, B_n, $\overset{0}{\chi}_n$, $\overset{1}{\chi}_n$, $\overset{0}{\theta}_n$, f_n, g_n and the a priori estimate (1.29) proved in Section 5 the sequence $\{\chi_n, \theta_n\}_{n>r}$ is bounded in the space

$$\bigcap_{k=0}^{m}W^{k,2}([0,T],X_{m-k})\times\bigcap_{k=0}^{m-1}W^{k,2}([0,T],Y_{m-k})$$

and therefore we can extract a subsequence which converges weakly in this space to some $\{\chi, \theta\}$. Standard arguments show that $\{\chi, \theta\}$ is a solution of Eqs. (1.1), (1.2), (1.17), (1.19). The additional regularity (1.27), (1.28) follows from Sect. 4. The proof is complete.

References

1. A. CHRZĘSZCZYK, *Some existence results in dynamical thermoelasticity. Part I. Nonlinear case*, Arch. Mech., **39**, 6, 1987.
2. C. M. DAFERMOS and W. J. HRUSA, *Energy methods for quasi-linear hyperbolic initial-boundary value problems, Applications to elastodynamics*, Arch. Rat. Mech. Anal., **87**, 267–292, 1985.

3. C. M. DAFERMOS, *On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity*, Arch. Rat. Mech. Anal., **29**, 241–271, 1968.
4. V. D. KUPRADZE, T. V. GEGELIA and M. O. BASHELEISHVILI and T. V. BURCHULADZE, *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity*, Amsterdam 1979.
5. T. KOWALSKI and A. PISKOREK, *Existenz der Lösung einer linearen Anfangs-Randwertaufgabe in der Thermoelastizitätstheorie*, ZAMM **61**, 250, 1981.
6. R. LEIS, *Anfangs-Randwertaufgaben in der Theorie thermoelastischer Schwingungen*, ZAMM, **64**, 1984 12.
7. D. E. CARLSON, *Linear thermoelasticity*, Handbuch der Physik, Bd. VI a/2, Berlin, Heidelberg, New York 1972.
8. V. BARBU and Th. PRECUPANU, *Convexity and optimization in Banach spaces*, Bucaresti 1978.
9. J. L. LIONS and E. MAGENES, *Problèmes aux limites non homogènes et applications*, vol. 1, Paris 1968.

INSTITUTE OF MATHEMATICS, WSP, KIELCE.

Received January 8, 1987.