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Nonhomogeneous tension–torsion of neo-Hookean and Mooney–Rivlin materials

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THE EXACT solutions for a nonlinear elastic layer being non-uniformly extended and then twisted are presented in the cases of both the neo-Hookean and Mooney–Rivlin materials.

Ścisłe rozwiązania dla nieliniowej sprężystej warstwy poddanej nierównomiernemu rozciąganiu, a następnie skręcaniu, przedstawiono w przypadku materiału typu „neohookean", jak i materiału Mooneya–Rivlina.

Точные решения для нелинейного упругого слоя, подвергнутого неравномерному растяжению, а затем скручиванию, представлены в случае материала типа „neohookean", как и материала Мунея-Ривлина.

1. Introduction

THERE ARE few nonhomogeneous exacts solutions that are available within the context of isotropic nonlinear elasticity. However, in recent years, there has been a great deal of interest in the study of such deformations, (cf. CURRIE and HAYES [1], RAJAGOPAL and WINEMAN [2], RAJAGOPAL, TROY and WINEMAN [3]). A discussion on the relevance and resurgence of interest in such investigations, and the lack thereof in previous years, can be found in the recent paper on the existence of nonhomogeneous deformations in incompressible isotropic elastic materials by MCLEOD, RAJAGOPAL and WINEMAN [4].

RAJAGOPAL and WINEMAN [2] have extended the classical problem of homogeneous uniaxial extension of an elastic layer, by investigating the possibility of the existence of nonhomogeneous solutions. They obtain a class of explicit exact solutions to the problem. The classical solution belongs to this class. To support these nonhomogeneous solutions, certain tractions have to be provided at infinity. Nonetheless, these surface tractions can be precisely determined and such a nonhomogeneous solution induced and supported by the application of such tractions.

In this short note, we study the possibility of a nonlinear layer being non-uniformly extended and then twisted. It is found that the problem admits exact solutions in the case of both neo-Hookean and Mooney–Rivlin materials. The equations of equilibrium reduce to a set of nonlinear equations which at first glance looks hopelessly complicated. However, these equations can be integrated and the solution expressed implicitly in terms of standard integrals.

2. Kinematics

Consider the deformation (cf. Fig. 1)

$$x = \frac{1}{\sqrt{\lambda'}}\{X\cos\Omega(Z) - Y\sin\Omega(Z)\}, \tag{2.1}$$

$$y = \frac{1}{\sqrt{\lambda'}}\{X\sin\Omega(Z) + Y\cos\Omega(Z)\}, \tag{2.2}$$

$$z = \lambda(Z), \tag{2.3}$$

where (x, y, z) denote the coordinates in the deformed state of a material point originally at (X, Y, Z) in the undeformed state. The above deformation corresponds to a nonhomogeneous extension followed by twisting $\hat{\Omega}(\lambda(Z)) = \Omega(Z)$. As observed in the introduction,

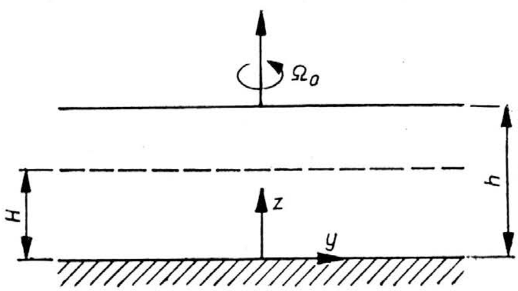

FIG. 1. Domain of deformation.

when $\Omega \equiv 0$, i.e., for the problem of nonhomogeneous extension, RAJAGOPAL and WINEMAN [2] established a one-parameter family of solutions in the case of a neo-Hookean material, and RAJAGOPAL, TROY and WINEMAN [3] extended these results in the case of a Mooney–Rivlin material.

Define $C \equiv \cos\Omega$, $S \equiv \sin\Omega$, and

$$f \equiv f(X, Y, Z) = -\left[\frac{1}{2}\frac{\lambda''}{(\lambda')^{3/2}}(XC - YS) + \frac{\Omega'}{\sqrt{\lambda'}}(XS + YC)\right], \tag{2.4}$$

$$g \equiv g(X, Y, Z) = -\left[\frac{1}{2}\frac{\lambda''}{(\lambda')^{3/2}}(XS + YC) - \frac{\Omega'}{\sqrt{\lambda'}}(XC - YS)\right]. \tag{2.5}$$

It follows from (2.1)–(2.3) that the deformation gradient $\mathbf{F}$ has the structure

$$\mathbf{F} = \begin{pmatrix} \frac{C}{\sqrt{\lambda'}} & -\frac{S}{\sqrt{\lambda'}} & f \\ \frac{S}{\sqrt{\lambda'}} & \frac{C}{\sqrt{\lambda'}} & g \\ 0 & 0 & \lambda' \end{pmatrix}, \tag{2.6}$$

thus we find that the Cauchy–Green strain tensor **B**, and $\mathbf{B}^{-1}$ have the following matrix representations:

$$\mathbf{B} = \begin{pmatrix} \frac{1}{\lambda'}+f^2 & fg & \lambda' f \\ fg & \frac{1}{\lambda'}+g^2 & \lambda' g \\ \lambda' f & \lambda' g & (\lambda')^2 \end{pmatrix}, \tag{2.7}$$

$$\mathbf{B}^{-1} = \begin{pmatrix} \lambda' & 0 & -f \\ 0 & \lambda' & -g \\ -f & -g & \left(\frac{1}{\lambda'}\right)^2 + \frac{1}{\lambda'}(f^2+g^2) \end{pmatrix}. \tag{2.8}$$

Thus, the principal invariants of **B** are

$$\begin{aligned} \mathrm{I}_B &= \operatorname{trace}\mathbf{B} = \frac{2}{\lambda'} + (\lambda')^2 + f^2 + g^2, \\ \mathrm{II}_B &= \frac{1}{2}\{(\operatorname{trace}\mathbf{B})^2 - \operatorname{trace}(\mathbf{B}^2)\} = 2\lambda' + \left(\frac{1}{\lambda'}\right)^2 + \frac{1}{\lambda'}(f^2+g^2), \\ \mathrm{III}_B &= \det\mathbf{B} = 1. \end{aligned} \tag{2.9}$$

3. Equations of equilibrium

The Cauchy stress **T** in the Mooney–Rivlin theory of elasticity is given by (cf. TRUESDELL and NOLL [5])

$$\mathbf{T} = -p\mathbf{1} + \mu\left(\frac{1}{2}+\bar{\beta}\right)\mathbf{B} - \mu\left(\frac{1}{2}-\bar{\beta}\right)\mathbf{B}^{-1}, \tag{3.1}$$

where $-p\mathbf{1}$ is the spherical part of the stress due to the constraint of incompressibility. The material constants μ and $\bar{\beta}$ are such that $\mu > 0$, $-\frac{1}{2} \leqslant \bar{\beta} \leqslant \frac{1}{2}$. When $\bar{\beta} = \frac{1}{2}$, the constitutive relation reduces to the neo-Hookean case

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{B}. \tag{3.2}$$

First, let us consider the equations of equilibrium in the case of the neo-Hookean material. On neglecting the body forces, the equations of equilibrium reduce to

$$\operatorname{div}\mathbf{T} = 0. \tag{3.3}$$

We shall find it convenient to express the equations of equilibrium in terms of the reference configuration and thus

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{\partial T_{ij}}{\partial X_p} F_{pj}^{-1} = 0. \tag{3.4}$$

Replacing $-(p/\mu)$ by P, a lengthy but straightforward calculation yields

$$C\sqrt{\lambda'}\left(\frac{\partial P}{\partial X}+f\frac{\partial f}{\partial X}+f\frac{\partial g}{\partial Y}\right)-S\sqrt{\lambda'}\left(\frac{\partial P}{\partial Y}+f\frac{\partial f}{\partial Y}-f\frac{\partial g}{\partial X}\right)+\frac{\lambda''}{\lambda'}f+\frac{\partial f}{\partial Z}=0, \tag{3.5}$$

$$C\sqrt{\lambda'}\left(\frac{\partial P}{\partial Y}+g\frac{\partial f}{\partial X}+g\frac{\partial g}{\partial Y}\right)+S\sqrt{\lambda'}\left(\frac{\partial P}{\partial X}-g\frac{\partial f}{\partial Y}+g\frac{\partial g}{\partial X}\right)+\frac{\lambda''}{\lambda'}g+\frac{\partial g}{\partial Z}=0, \tag{3.6}$$

$$C\lambda'^{3/2}\left(\frac{\partial f}{\partial X}+\frac{\partial g}{\partial Y}\right)+S\lambda'^{3/2}\left(\frac{\partial g}{\partial X}-\frac{\partial f}{\partial Y}\right)-\lambda'^{-1/2}(Sg+Cf)\frac{\partial P}{\partial X}$$
$$-\lambda'^{-1/2}(Cg-Sf)\frac{\partial P}{\partial Y}+\frac{1}{\lambda'}\frac{\partial P}{\partial Z}+2\lambda''=0. \tag{3.7}$$

It follows from the definitions of f and g, i.e., (2.4) and (2.5), that (3.5)–(3.7) can be re-written as

$$\frac{\partial P}{\partial X}=-\frac{1}{\sqrt{\lambda'}}\left(C\frac{\partial f}{\partial Z}+S\frac{\partial g}{\partial Z}\right), \tag{3.8}$$

$$\frac{\partial P}{\partial Y}=\frac{1}{\sqrt{\lambda'}}\left(S\frac{\partial f}{\partial Z}-C\frac{\partial g}{\partial Z}\right), \tag{3.9}$$

$$\frac{\partial P}{\partial Z}=-\lambda'\lambda''+C\left(\sqrt{\lambda'}\right)\left[\frac{1}{2}\frac{\lambda''}{\lambda'^2}X\frac{\partial f}{\partial Z}+\frac{\Omega'}{\lambda'}Y\frac{\partial f}{\partial Z}+\frac{1}{2}\frac{\lambda''}{\lambda'^2}Y\frac{\partial g}{\partial Z}\right.$$
$$\left.-\frac{\Omega'}{\lambda'}X\frac{\partial g}{\partial Z}\right]+S\left(\sqrt{\lambda'}\right)\left[-\frac{1}{2}\frac{\lambda''}{\lambda'^2}Y\frac{\partial f}{\partial Z}\right.$$
$$\left.+\frac{\Omega'}{\lambda'}X\frac{\partial f}{\partial Z}+\frac{1}{2}\frac{\lambda''}{\lambda'^2}X\frac{\partial g}{\partial Z}+\frac{\Omega'}{\lambda'}Y\frac{\partial g}{\partial Z}\right]. \tag{3.10}$$

It follows from (3.8) (2.4) and (2.5) that

$$\frac{\partial^2 P}{\partial X\partial Y}=-\frac{1}{\sqrt{\lambda'}}\left[\left(\frac{1}{2}\frac{\lambda''}{\lambda'^{3/2}}\Omega'\right)-\frac{d}{dZ}\left(\frac{\Omega'}{\sqrt{\lambda'}}\right)\right], \tag{3.11}$$

while from (3.9), (2.4) and (2.5) that

$$\frac{\partial^2 P}{\partial X\partial Y}=\frac{1}{\sqrt{\lambda'}}\left[\left(\frac{1}{2}\frac{\lambda''}{\lambda'^{3/2}}\Omega'\right)-\frac{d}{dz}\left(\frac{\Omega'}{\sqrt{\lambda'}}\right)\right]. \tag{3.12}$$

Equations (3.11) and (3.12) imply that

$$\left[\left(\frac{1}{2}\frac{\lambda''}{\lambda'^{3/2}}\Omega'\right)-\frac{d}{dZ}\left(\frac{\Omega'}{\sqrt{\lambda'}}\right)\right]=0, \tag{3.13}$$

which in turn implies that

$$\frac{d}{dZ}\left(\frac{\Omega'}{\lambda'}\right)=0. \tag{3.14}$$

We assume that the bottom layer at $Z = 0$ is held fixed while that which was at $Z = H$ is twisted by angle Ω_0. Then, by integrating (3.14) we obtain

$$\Omega(Z) = \frac{\Omega_0}{\lambda(H)}\lambda(Z) = \frac{\Omega_0}{h}\lambda(Z), \tag{3.15}$$

On substituting (3.15) into (3.10), and using (2.4) and (2.5), we obtain

$$\frac{\partial P}{\partial Z} = -\lambda'\lambda'' + \frac{1}{2}\frac{\lambda''}{\lambda'^{3/2}}(X^2+Y^2)Q(Z), \tag{3.16}$$

where

$$Q(Z) = \frac{d^2}{dZ^2}[(\lambda')^{-1/2}] - \frac{\Omega'^2}{\sqrt{\lambda'}}. \tag{3.17}$$

Next, note that (3.8) can be rewritten in virtue of (2.4), (2.5) and (3.17) as

$$\frac{\partial P}{\partial X} = -\frac{1}{\sqrt{\lambda'}}XQ. \tag{3.18}$$

It follows from (3.16) and (3.18) that

$$\frac{1}{\sqrt{\lambda'}}\frac{dQ}{dZ} = Q\frac{d}{dZ}\left(\frac{1}{\sqrt{\lambda'}}\right). \tag{3.19}$$

It immediately follows that

$$Q(Z) = A_1\frac{1}{\sqrt{\lambda'}}, \tag{3.20}$$

where A_1 is a constant.

Let us introduce a function $\eta(Z)$ through

$$\eta(Z) = \frac{1}{\sqrt{\lambda'}}. \tag{3.21}$$

Equations (3.19), (3.20) and (3.21) imply that

$$\eta'' = A_1\eta + \frac{1}{\eta^3}\left(\frac{\Omega'}{\lambda'}\right)^2, \tag{3.22}$$

$$= A_1\eta + \frac{A^2}{\eta^3},$$

where the second equality is true in virtue of (3.15) and $A = \Omega_0/(\lambda(H))$ is a constant. If $A = 0$, the governing equation (3.22) reduces to that which governs the non-uniform extension of a neo-Hookean material studied by RAJAGOPAL and WINEMAN [2] .We shall now proceed to show that the nonlinear equation (3.22) can be integrated twice, and the solution exhibited explicitly.

Multiplying (3.22) by η' and integrating it once, we obtain

$$(\eta')^2 - A_1\eta^2 + A^2\eta^{-2} = A_2, \tag{3.23}$$

where A_2 is a constant. Solving for η', we obtain

$$\eta' = \pm[A_1\eta^2 + A_2 - A^2\eta^{-2}]^{1/2}. \tag{3.24}$$

Introducing a function

$$\xi(Z) = \eta^2(Z), \tag{3.25}$$

(3.24) can be rewritten as

$$\pm \frac{1}{2} \frac{d\xi}{\sqrt{(A_1\xi^2+A_2\xi-A^2)}} = dZ. \tag{3.26}$$

Let us define

$$\Delta = -4A_1A^2-A_2^2. \tag{3.27}$$

Depending on the respective values of the constants A_1, A_2 and A, different explicit expression for the integral of (3.26) are possible. The only physically acceptable solution corresponds to $A_1 < 0$, $\Delta < 0$, and it follows that (cf. [6])

$$\pm \frac{1}{2} \frac{-1}{\sqrt{-A_1}} \sin^{-1}\left(\frac{2A_1\xi+A_2}{\sqrt{-\Delta}}\right) = Z+A_3, \tag{3.28}$$

where A_3 is a constant. Thus

$$\lambda'(Z) = 2A_1\{-A_2 \mp (\sqrt{-\Delta})\sin[2(\sqrt{-A_1})(Z+A_3)]\}^{-1} \tag{3.29}$$

and thus

$$\lambda(Z) = \int 2A_1\{-A_2 \mp (\sqrt{-\Delta})\sin[2(\sqrt{-A_1})(Z+A_3)]\}^{-1} dZ, \tag{3.30}$$

where A_3 is a constant.

4. Mooney–Rivlin material

Let us now turn our attention to the equations of equilibrium of a Mooney–Rivlin material. In this case a lengthy but straightforward calculation yields

$$C\sqrt{\lambda'}\frac{\partial P}{\partial X} - S\sqrt{\lambda'}\frac{\partial P}{\partial Y} + \left(\frac{1}{2}+\bar{\beta}\right)\frac{\partial f}{\partial Z} + \left(\frac{1}{2}-\bar{\beta}\right)\left\{\frac{S}{\sqrt{\lambda'}}\left(f\frac{\partial f}{\partial Y} - g\frac{\partial f}{\partial X}\right) - \frac{C}{\sqrt{\lambda'}}\left(f\frac{\partial f}{\partial X} + g\frac{\partial f}{\partial Y}\right) + \frac{1}{\lambda'}\frac{\partial f}{\partial Z}\right\} = 0, \tag{4.1}$$

$$S\sqrt{\lambda'}\frac{\partial P}{\partial X} + C\sqrt{\lambda'}\frac{\partial P}{\partial Y} + \left(\frac{1}{2}+\bar{\beta}\right)\frac{\partial g}{\partial Z} + \left(\frac{1}{2}-\bar{\beta}\right)\left\{\frac{S}{\sqrt{\lambda'}}\left(f\frac{\partial g}{\partial Y} - g\frac{\partial g}{\partial X}\right) - \frac{C}{\sqrt{\lambda'}}\left(g\frac{\partial g}{\partial Y} + f\frac{\partial g}{\partial X}\right) + \frac{1}{\lambda'}\frac{\partial g}{\partial Z}\right\} = 0, \tag{4.2}$$

$$\frac{1}{\lambda'}\frac{\partial P}{\partial Z} - \frac{1}{\sqrt{\lambda'}}(Cg-Sf)\frac{\partial P}{\partial Y} - \frac{1}{\sqrt{\lambda'}}(Sg+Cf)\frac{\partial P}{\partial X} + \left(\frac{1}{2}+\bar{\beta}\right)\lambda'' + \left(\frac{1}{2}-\bar{\beta}\right)\left(-\frac{\lambda''}{\lambda'} + \frac{2\lambda''}{\lambda'^4} - \frac{1}{\lambda'^2}\left(2f\frac{\partial f}{\partial Z} + 2g\frac{\partial g}{\partial Z}\right)\right) = 0. \tag{4.3}$$

It follows from (4.1), (4.2), (4.3), (2.4) and (2.5) that

$$\sqrt{\lambda'}\,\frac{\partial P}{\partial X} = -\left(\frac{1}{2}+\bar{\beta}\right)\left(C\frac{\partial f}{\partial Z}+S\frac{\partial g}{\partial Z}\right)-\left(\frac{1}{2}-\bar{\beta}\right)\left\{\frac{f}{\sqrt{\lambda'}}(\alpha C-\gamma S)+\frac{g}{\sqrt{\lambda'}}(\alpha S+\gamma C)+\frac{C}{\lambda'}\frac{\partial f}{\partial Z}+\frac{S}{\lambda'}\frac{\partial g}{\partial Z}\right\}, \tag{4.4}$$

$$\sqrt{\lambda'}\,\frac{\partial P}{\partial Y} = -\left(\frac{1}{2}+\bar{\beta}\right)\left(C\frac{\partial g}{\partial Z}-S\frac{\partial f}{\partial Z}\right)-\left(\frac{1}{2}-\bar{\beta}\right)\left\{\frac{f}{\sqrt{\lambda'}}(-\alpha S-\gamma C)+\frac{g}{\sqrt{\lambda'}}(\alpha C-\gamma S)+\frac{C}{\lambda'}\frac{\partial g}{\partial Z}-\frac{S}{\lambda'}\frac{\partial f}{\partial Z}\right\}, \tag{4.5}$$

$$\frac{\partial P}{\partial Z} = \left(\frac{1}{2}+\bar{\beta}\right)[(\alpha Q+R\gamma)(X^2+Y^2)-\lambda'\lambda'']+\left(\frac{1}{2}-\bar{\beta}\right)\left(\frac{1}{\sqrt{\lambda'}}(-\alpha^3-\alpha\gamma^2)(X^2+Y^2)-\frac{2\lambda''}{\lambda'^3}+\lambda''-\frac{1}{\lambda'}(\alpha Q+R\gamma)(X^2+Y^2)\right), \tag{4.6}$$

where

$$\alpha = \frac{1}{2}\frac{\lambda''}{\lambda'^{3/2}}, \qquad \gamma = \frac{\Omega'}{\lambda'},$$

$$R = \left(\frac{1}{2}\frac{\lambda''}{\lambda'^{3/2}}\Omega'\right)-\frac{d}{dZ}\left(\frac{\Omega'}{\sqrt{\lambda'}}\right). \tag{4.7}$$

Notice that (4.4) and (4.5) reduce to Eqs. (3.8) and (3.9), when $\bar{\beta} = \frac{1}{2}$. A simple and straightforward computation using (4.4) and (4.5) yields

$$\left(\frac{1}{2}+\bar{\beta}\right)\left(\frac{\lambda''\Omega'-\lambda'\Omega''}{\lambda'^{3/2}}\right) = \left(\frac{1}{2}-\bar{\beta}\right)\left(\frac{\Omega''}{\lambda'^{3/2}}\right). \tag{4.8}$$

Let us introduce a constant β through

$$\beta = \frac{\frac{1}{2}-\bar{\beta}}{\frac{1}{2}+\bar{\beta}}. \tag{4.9}$$

Then

$$\lambda''\Omega'-(\lambda'+\beta)\Omega'' = 0. \tag{4.10}$$

It immediately follows that

$$\Omega(Z) = \left[\frac{\Omega_0}{\lambda(H)+\beta H}\right](\lambda(Z)+\beta Z). \tag{4.11}$$

Using the expression for $\Omega(Z)$ in (4.11) and substituting this into (4.4) and (4.6), it follows by cross-differentiating and eliminating the pressure that

$$(4.12)\qquad \eta' Q-\eta Q'+4B^2\eta\eta'\beta\left(\frac{1}{\eta^2}+\beta\right) = \beta(4B^2\eta\eta'\beta+4B^2\eta^3\eta'\beta^2+3\eta^2\eta'\eta''+\eta^3\eta'''),$$

where

$$(4.13)\qquad B \equiv \frac{\Omega_0}{\lambda(H)+\beta H}.$$

On dividing both the sides of (4.12) by η^2, rearranging the terms and subsequently integrating it, we find

$$(4.14)\qquad \frac{Q}{\eta}+F = -2B^2\frac{\beta}{\eta^2}-2B^2\eta^2\beta^3-\beta\eta'^2-\beta\eta\eta'',$$

where F is a constant.

It follows from (3.17) and (4.11) that

$$(4.15)\qquad Q = \eta''-B^2\eta\left(\frac{1}{\eta^2}+\beta\right)^2.$$

Substituting (4.15) into (4.14), we get

$$(4.16)\qquad \eta''-\frac{B^2}{\eta^3}-F\eta = B^2\beta^2(-2\beta\eta^3+\eta)-\beta\eta\eta'^2-\beta\eta^2\eta''.$$

On multiplying the above equation by η' and integrating one obtains

$$(4.17)\qquad (\eta')^2+\frac{B^2}{\eta^2}+F\eta^2 = B^2\beta^2(-\eta^4\beta+\eta^2)-\beta\eta^2\eta'^2+D,$$

where D is a constant.

It follows from equation (3.25) that (4.17) can be rewritten as

$$(4.18)\qquad \frac{1}{4}\xi'^2(1+\beta\xi) = B^2\beta^2\xi^2(1-\beta\xi)+F\xi^2+D\xi-B^2.$$

Thus

$$(4.19)\qquad \xi' = \pm 2\left(\frac{B^2\beta^2\xi^2(1-\beta\xi)+F\xi^2+D\xi-B^2}{1+\beta\xi}\right)^{1/2},$$

which can be integrated to yield

$$(4.20)\qquad \pm\frac{1}{2}\int\left(\frac{B^2\beta^2\xi^2(1-\beta\xi)+F\xi^2+D\xi-B^2}{1+\beta\xi}\right)^{-1/2}d\xi = Z+E,$$

where E is a constant. We can thus in principle determine λ'.

In our derivation we have made the tacit assumption that $\beta\neq 0$, i.e., $\bar{\beta}\neq-\frac{1}{2}$. If $\bar{\beta}=-\frac{1}{2}$, the constitutive expression reduces to

$$(4.21)\qquad \mathbf{T} = -p\mathbf{1}-\mu\mathbf{B}^{-1}$$

In this case it is easy to show that explicit exact solutions can be exhibited, in a manner identical to that of the neo-Hookean case.

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References

1. P. K. CURRIE and M. HAYES, Proc. the IUTAM Symposium on Finite Elasticity, D. E. CARLSON and R. T. SHIELD (Eds.), Martinus Nijhoff Publishers, The Hague, Boston, London 1981.
2. K. R. RAJAGOPAL, W. C. TROY and A. S. WINEMAN, *A note on non-universal deformations of nonlinear elastic layers* [to appear in Proc. Irish Acad].
3. K. R. RAJAGOPAL and A. S. WINEMAN, Intl. J. Engng. Sci., **23**, 217, 1985.
4. J. B. MCLEOD, K. R. RAJAGOPAL and A. S. WINEMAN, *On the existence of a class of deformation in elastic material* [to appear].
5. C. TRUESDELL and W. NOLL, *The nonlinear field theories of mechanics*, Handbuch der Physik (Ed. Flügge), III/3, Berlin, Heidelberg, New York, Springer 1965.
6. I. S. GRADSHTEYN and I. M. RYZHIK, *Table of integrals, series and products*, Academic Press Inc., New York 1980.

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