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On a certain method of synthesis of ordinary differential equations by means of construction of sliding sets

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THE PAPER contains the solution of the following synthesis problem: for a differential equation of the specified structure

$$(*)\ \dot{x} = Ax+B(x)\varphi, \qquad x \in R^n,$$

where φ is a two-valued control parameter, the structural parameters of the equation (*) are sought, i.e. matrices A and B and vector function φ, which would transfer the solution from point $p \in R^n$ to the origin of the coordinate set O, along a straight line and according to the given law of motion. It is assumed that the straight line is a set of discontinuity points of the function φ, and the segment $[\overline{p, O}]$ belongs to the set of slides of solutions of the differential equation (*). An example of the synthesis on a plane is given.

Praca zawiera rozwiązanie następującego zadania syntezy: dla równania różniczkowego o danej strukturze

$$(*)\ \dot{x} = Ax+B(x)\varphi, \qquad x \in R^n,$$

gdzie φ jest dwuwartościowym parametrem sterowania, określić parametry struktury równania (*), a więc macierze A i B oraz funkcję wektorową φ, aby przeprowadzić jego rozwiązanie z punktu $p \in R^n$ do początku układu współrzędnych O po prostej i według danego prawa ruchu. Przyjmuje się, że prosta jest zbiorem punktów nieciągłości funkcji φ, a odcinek $[\overline{p, O}]$ należy do zbioru poślizgów rozwiązań równania różniczkowego (*). Podano przykład syntezy na płaszczyźnie.

Работа содержит решение следующей задачи синтеза: для дифференциального уравнения с заданной структурой

$$(*)\ \dot{x} = Ax+B(x)\varphi, \qquad x \in R^n,$$

где φ — двузначный параметр управления, определить параметры структуры уравнения (*), значит матрицы A и B, а также векторную функцию φ, чтобы провести его решение из точки $p \in R^n$ в начало системы координат O вдоль прямой и согласно заданному закону движения. Принимается, что прямая является множеством точек разрыва функции φ, а отрезок $[\overline{p, O}]$ принадлежит к множеству скольжений решений дифференциального уравнения (*). Приведен пример синтеза на плоскости.

1. Introduction

WE SHALL consider the problem of synthesis of the differential equation

$$\dot{x} = Ax+\varkappa, \tag{1.1}$$

where $x = \mathrm{col}[x_1, \ldots, x_n]$, $A = [a_{ij}]$ is a given constant square matrix $n \times n$, $\varkappa: R^n \to R^n$ is a sought function; Eq. (1.1) realizes the transition of the solution $x(p, t)$, $p \in R^n$, with the initial condition $x(p, 0) = p$, from the state p to p', $p' \neq p$, along the given line δ $(p, p' \in \delta)$ and according to the given motion law

$$\dot{x} = P(x). \tag{1.2}$$

This problem has the trivial solution

$$\varkappa(x) = -Ax + P(x). \tag{1.3}$$

Obviously, among the assumptions the condition should be secured that δ be a trajectory of Eq. (1.2).

In general, the trivial solution is not acceptable in technical problems, since it is easy to notice that the solution (1.3) consists in attaching to the existing structure, represented by the matrix A, the "opposite" structure: $-A$, and a new structure, represented by the function P.

In this paper we shall give the solution to the problem of synthesis of Eq. (1.1) in the class of piecewise continuous functions, with the condition that δ is a trajectory of sliding motions of Eq. (1.1). We shall also assume some modification of the conditions, supposing that the matrix A is not defined a priori. In this case it is said about the differential Eq. (1.1), that it is of the given structure.

2. Statement of the problem of synthesis

Let us take the differential equation in the form

$$\dot{x} = Ax + B\varphi \qquad (= f(x)), \tag{2.1}$$

where $x = \mathrm{col}[x_1, \ldots, x_n]$, $A = [a_{ij}]$ is a constant square matrix $n \times n$, $B = [b_{ij}(x)]$ is a matrix $n \times (n-1)$, b_{ij}: $R^n \to R$, $b_{ij} \in C^0$, $\varphi:(R^n \setminus \bar{S}) \to \{-1, 1)\}$, $\varphi = \mathrm{col}[\varphi_1, \ldots, \varphi_{n-1}]$, $\mu\bar{S} = 0$, φ_i are constant in the regions of continuity.

According to the formalism introduced in Sect. 1, matrices A and B and vector function φ are the elements of the (given) structure of the differential Eq. (2.1). Synthesis of this equation will be understood as such an evaluation of the elements of its structure, that the solutions (in particular, the specified solution) would fulfill the conditions imposed beforehand.

REMARK 1. Because the synthesis does not include the uniqueness condition, the evaluation of structural elements of the differential equation is understood as finding the sufficient condition for these elements, which would secure the assumed properties of the solutions (solution).

REMARK 2. The above defined synthesis is different from the one known in the theory of control (e.g. [1]), which consists in finding the control parameter φ: $R^n \to R^{n-1}$ for the given matrices A and B.

THE AIM OF SYNTHESIS. Synthesize the Eq. (2.1) in such a way that the given point $p \in R^n$ would be translated to the origin O of the coordinate set along a straight line, according to the assumed law of motion

$$\dot{x} = P(x), \qquad P:[\overline{p, O}] \to R^n. \tag{2.2}$$

REMARK 3. It is assumed that the law (2.2) displaces point p to the point O, i.e. that the segment $[\overline{p, O}]$ is a subset of the trajectory of Equation (2.2).

REMARK 4. The origin of coordinates does not have to be the singular point of Eq. (2.2).

In the solution of the defined problem the following conditions will be assumed:

(i) The closed segment $[\overline{p, O}]$ is a subset of a straight line S being the intersection of $(n-1)$-dimensional, mutually non-coinciding hyperplanes S_i $(i = 1, \ldots, n-1)$. Its equation is

$$Kx = 0; \tag{2.3}$$

$K = [s'_{ij}]$ is a constant matrix $(n-1)\times n$, such that $Kp = 0$.

(ii) Every hyperplane S_i $(i = 1, \ldots, n-1)$ is a set of discontinuities of the function f. We shall denote $\bar{S} = \bigcup_{i=1}^{n-1} S_i$.

(iii) The motion defined by the rule (2.2) on the segment $[\overline{p, O}[$ is always "towards O", i.e. the following condition holds:

$$\forall i \in \{1, \ldots, n\}\, \forall x_i \in]0, p_i], \qquad x_i \dot{x}_i < 0, \tag{2.4}$$

where $p = [p_1, \ldots, p_n]$.

In the consequence of the assumption (i) it is

$$\operatorname{rank} K = n-1,$$

what means that the solution x of Eq. (2.3) contains one parameter ξ among its coordinates $x_1, \ldots, x_n$; to set our attention assume

$$x = \operatorname{col}[x_1(\xi), \ldots, x_{n-1}(\xi), \xi]. \tag{2.5}$$

It is easy to show that the necessary and sufficient condition for the inequality (2.4) to hold has the form

$$\exists i \in \{1, \ldots, n\}\, \forall x_i \in]0, p_i], \qquad x_i \dot{x}_i < 0. \tag{2.6}$$

3. The conditions of choice of the motion law on S

On the basis of Condition (i) we assume

$$S = \{x : s(x) = 0\}, \qquad s: R^n \to R^{n-1} \tag{3.1}$$

where

$$s(x) = Kx.$$

The condition of motion on the manifold S is that the derivative of s should equal zero according to Eq. (2.2)

$$\dot{s}(x) = KP(x),$$

i.e. the condition

$$\forall x \in [\overline{p, O}], \qquad KP(x) = 0. \tag{3.2}$$

This condition and the inference (2.5) about the rank of the matrix K yields the linear interdependence of the functions P_i $(i = 1, \ldots, n)$

$$P(x) = \operatorname{col}[P_1(\psi(x)), \ldots, P_{n-1}(\psi(x)), \psi(x)], \tag{3.3}$$

which constitute the solution of (3.2) at every point x

$$\forall x \in [\overline{p, O}[\ \forall i \in \{1, \ldots, n-1\}, \quad P_i(x) = C_i(s'_{jk}) \cdot \psi(x), \tag{3.4}$$

where $C_i: R^{n-1} \times R^n \to R^n$ are the functions of the elements s'_{jk} $(j = 1, \ldots, n-1, k = 1, \ldots, n)$ of the matrix K.

REMARK. Arbitrariness of the motion law, to which reference has been made in the problem of synthesis (cf. Sect. 2), is understood in the sense of relation (3.3): ψ is a parameter of this law.

In the light of this remark and considering the form of the condition (2.5), condition (2.6) can be expressed as follows:

$$\forall \xi \in]0, p], \quad \xi \cdot \psi(x(\xi)) < 0. \tag{3.5}$$

Conclusion. The law (2.2), in which the function P fulfils the conditions (3.3), (3.4) and (3.5), defines the motion $x(p, t)$ that fulfills the condition (iii).

4. Conditions for choice of the matrix B

To find the elements of matrix B we shall apply the necessary condition of existence of the sliding solutions of Eq. (2.1) on segment $[\overline{p, O}]$, making use of the condition, that the differential Equation (2.2) is the law of motion on this segment.

Assuming that the segment $[\overline{p, O}]$ is the set of slides of Eq. (2.1), according to the form of Eq. (2.1) and the conditions (i) and (iii), we obtain a vector field on $[\overline{p, O}]$ ([2])

$$\bar{f}: [\overline{p, O}] \to R^n, \quad \bar{f}(x) = Ax - B(x) \cdot [KB(x)]^{-1} KAx, \tag{4.1}$$

where K is a matrix which fulfills the condition $Kp = 0$ (see Sect. 2).

Considering (2.5) and denoting

$$\bar{\varphi}(x) = [K \cdot B(x)]^{-1} KAx,$$

we obtain a field equivalent to (4.1)

$$\begin{cases} \bar{f}(x(\xi)) = Ax(\xi) - B(x(\xi) \cdot \bar{\varphi}(x(\xi)), \\ \xi \in [0, p_n]. \end{cases}$$

We have assumed in Sect. 2 (assumption (i)), that S is a one-dimensional intersection of $(n-1)$ hyperplanes S_i in R^n—Fig. 1. Every one of them divides the space into two disjoint parts, and $(n-1)$ hyperplanes divide R^n into 2^{n-1} disjoint parts.

Let $x \in S$. Then, there exist in x exactly 2^{n-1} boundaries $\hat{f}(x)$ of the function f

$$\hat{f}_j(x) = \lim_{\substack{y \to x \\ y \in \Omega_j}} f(y), \quad j = 1, \ldots, 2^{n-1},$$

where Ω_j $(j = 1, \ldots, 2^{n-1})$ are the largest regions of R^n, which do not contain the points of $\bar{S}$. Hence, according to Eq. (2.1), the continuous functions are defined,

$$\hat{f}_j: S \to R^n, \quad \hat{f}_j(x) = Ax + B(x) \cdot \operatorname{col}[\hat{\varphi}_j(x)], \quad j = 1, \ldots, 2^{n-1}, \tag{4.2}$$

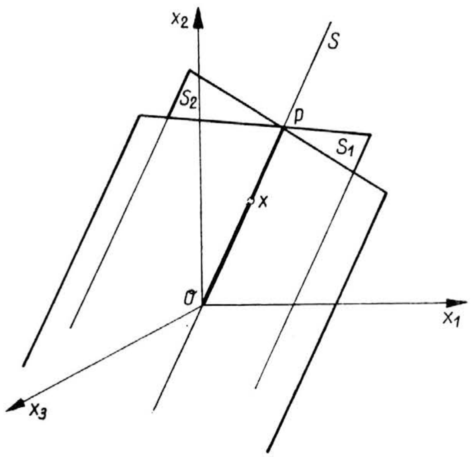

FIG. 1.

where $[\hat{\varphi}_j]: S \to R^{n-1}$, $[\hat{\varphi}_j] \in C^0(S)$. Thus, every index $j \in \{1, \ldots, 2^{n-1}\}$ corresponds to a certain vector function $[\hat{\varphi}_j]$:

$$[\hat{\varphi}_j(x)] = \lim_{\substack{y \to x \\ y \in \Omega_j}} \varphi(y), \quad j = 1, \ldots, 2^{n-1}.$$

We shall denote the sequence of function with elements $[\hat{\varphi}_j]$ by $\{[\hat{\varphi}_j]\}_{2^{n-1}}$. Let us denote the sections of the functions φ to the regions Ω_j by

$$\varphi|_{\Omega_j} = \varphi^j. \tag{4.3}$$

Then, evidently

$$[\hat{\varphi}_j(x)] = \lim_{y \to x} \varphi^j(y).$$

In view of the assumptions from Sect. 2, concerning Eq. (2.1) $[\hat{\varphi}_j]$ are constant vectors. It is easy to notice that

$$\forall x \in R^n \setminus \bar{S}\ \dot{\exists} j \in \{1, \ldots, 2^{n-1}\}, \quad \varphi(x) = \varphi^j(x). \tag{4.4}$$

Let us fix a sequence $\{S_i\}_{n-1}$, and form a sequence $\{\eta_i\}_{n-1}$

$$\eta_i: R^n \setminus S_i \to R, \quad \eta_i(x) = \begin{cases} \eta_i^-(x), & s_i(x) < 0, \\ \eta_i^+(x), & s_i(x) > 0, \end{cases} \quad i = 1, \ldots, n-1,$$

where s_i $(i = 1, \ldots, n-1)$ are the linear functions defined by (3.1). It is easily seen that

$$\forall x \in R^n \setminus \bar{S}\ \dot{\exists} \{l_i\}_{n-1}, \quad \varphi(x) = \operatorname{col}[\eta_1^{l_1}(x), \ldots, \eta_{n-1}^{l_{n-1}}(x)],$$

where $l_i \in \{-, +\}$ $(i = 1, \ldots, n-1)$. From this and from (4.4) it follows that

$$\forall j \in \{1, \ldots, 2^{n-1}\}\ \dot{\exists} \{l_i\}_{n-1}, \quad \varphi^j = \operatorname{col}[\eta_1^{l_1}, \ldots, \eta_{n-1}^{l_{n-1}}]. \tag{4.5}$$

By $\{\{l_i\}_{n-1}(j)\}_{n-1}$ we shall denote the sequence the elements of which are the sequences consisting of $(n-1)$ elements. Since $\varphi^j: \Omega_j \to R^{n-1}$, then, in view of (4.3) and (4.5)

$$\forall j \in \{1, \ldots, 2^{n-1}\}\ \dot{\exists} \{l_i\}_{n-1} \forall x \in \Omega_i, \quad \varphi(x) = \operatorname{col}[\eta_1^{l_1}, \ldots, \eta_{n-1}^{l_{n-1}}],$$

so that

$$\forall\, x \in R^n \setminus \bar{S}, \quad \varphi(x) = \mathrm{col}[\eta_1(x), \ldots, \eta_{n-1}(x)].$$

The necessary condition for the Eq. (2.2) to represent the law of motion on S is (cf. [2]):

$$\forall\, x \in S\, \exists\, \{\varkappa_j\}_{2^{n-1}}, \quad \sum_{j=1}^{2^{n-1}} \varkappa_j \cdot \hat{f}_j(x) = P(x) \tag{4.6}$$

with the conditions

$$\forall\, j \in \{1, \ldots, 2^{n-1}\}, \quad \varkappa_j \in R^+, \tag{4.7}$$

$$\sum_{j=1}^{2^{n-1}} \varkappa_j = 1. \tag{4.8}$$

Due to Eqs. (4.2) and (4.5) we have from (4.6)

$$\forall\, x \in S\, \exists\, \{\varkappa_j\}_{2^{n-1}}, \quad \sum_{j=1}^{2^{n-1}} \varkappa_j [Ax + B(x) \cdot \varphi^j(x)] = P(x), \quad i = 1, \ldots, n-1$$

together with the conditions (4.7) and (4.8). Since, from (4.8),

$$\sum_{j=1}^{2^{n-1}} \varkappa_j \cdot Ax = Ax \tag{4.8}$$

then, in view of (4.5)

$$\forall\, x \in S\, \exists\, \{\varkappa_j\}_{2^{n-1}}, \quad \sum_{j=1}^{2^{n-1}} \varkappa_j \cdot B(x) \cdot \mathrm{col}[\hat{\eta}_i^{\{l_i\}(j)}(x)] = -Ax + P(x), \quad i = 1, \ldots, n-1. \tag{4.9}$$

Let us observe that $\{\{l_i\}_{n-1}(j)\}_{2^{n-1}}$ contains all the possible sequences $\{l_i\}_{n-1}$, resulting from the division of the space. Let us also notice that, for every index i, index l in the sequence $\{\{l_i\}_{n-1}(j)\}_{2^{n-1}}$ assumes the value $(+)$ exactly 2^{n-2} times, and the value $(-)$ the same number of times. Because of this, and considering the assumptions for the values of control functions $\varphi_i(x) \in \{-1, 1\}$ (cf. Sect. 2), the statement (4.9) can be expressed as follows:

$$\forall\, x \in S, \quad B(x) \cdot \phi = P(x) - Ax, \tag{4.10}$$

where $\phi = \mathrm{col}\,[\phi_i]$, $i = 1, \ldots, n-1$,

$$\phi_i = \sum_{j=1}^{2^{n-1}} (\pm \varkappa_j), \quad -1 < \phi_i < 1.$$

Therefore, Eq. (4.10) constitutes the condition of choice of the matrix B. Note that, in general, condition (4.10) does not define B uniquely, representing the set of (n) identities, while $n(n-1)$ elements of B are sought. More precisely, (4.10) is the set of n identities, combining $n(n-1)$ elements of matrix B, n^2 elements of matrix A, $2^{n-1}-1$ independent coefficients $\varkappa_j$ fulfilling the conditions (4.7) and (4.8), and the parameters of the vector function P. From this set, $m(m \leqslant n(n-1))$ elements b_{ij} of B can be found as the functions of ξ and of the remaining parameters.

Equations (2.5) and (3.2) yield the condition (4.10) the form

$$\forall \xi \in [0, p_n], \quad B(x(\xi)) \cdot \phi = P(\psi(x(\xi))) - Ax(\xi). \tag{4.11}$$

5. Additional conditions for the parameters of Eq. (2.1)

The law of motion on the segment $[\overline{p, O}]$, under the assumption that this segment is a sliding set of the solutions of Eq. (2.1), has according to [2] the form

$$\dot{x} = \bar{f}(x), \tag{5.1}$$

where function $\bar{f}$ is defined by Eq. (4.1). This means that for every solution $x(x^0, t)$ of Eq. (4.1) with the condition $x^0 \in [p, O[$:

$$\exists \bar{t} > 0 \ \ \forall t \in [0, \bar{t}], \quad x(x^0, t) \in [\overline{p, O}]$$

and $x(x^0, t)$ is the sliding solution of Eq. (2.1).

On the other hand, the given motion law on the segment $[\overline{p, O}]$ is the differential equation (2.2). Comparing the laws (5.1) and (2.2) and taking into account (2.5) and (3.3), we obtain

$$\forall \xi \in [0, p_n], \quad Ax(\xi) - B[KB]^{-1} \cdot KAx(\xi) = P(\psi(x(\xi))), \tag{5.2}$$

being the set of n scalar identities interrelating $n(n-1)-m$ elements b_{ij} of B, n^2 elements a_{ij} of A, $n(n-1)$ elements s'_{ij} of the matrix K and $2(n-1)-1$ coefficients $\varkappa_j$ fulfilling the conditions (4.7) and (4.8).

6. Conclusions

It can be shown that the set of two vector identities (4.11) and (5.2) is the necessary and sufficient condition for existence of the sliding solutions on $[\overline{p, O}]$, which would move the point p to the origin of coordinates set O according to the law of motion (2.2).

A question arises whether, within the considered structure of the differential Eq. (2.1), the synthesis is possible with the motion law (2.2) arbitrary in the sense of the remark made in Sect. 3, i.e. whether the problem of synthesis from Sect. 2 is correctly formulated in the general case. In other words, is it possible that function P could lead to the set of equations being contradictory with respect to the parameters of Eq. (2.1). This question will be left without answer here.

7. An example of synthesis of the differential equation on the plane

Assume that we have a set of two differential equations of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + b_1 \varphi, \\ \dot{x}_2 &= a_{21} x_1 + a_{22} x_2 + b_2 \varphi, \end{aligned} \tag{7.1}$$

where $a_{21}, a_{22} \in R$, $b_1: R^2 \to R$, $b_2: R^2 \to R$, $\varphi:(R^2 \setminus \bar{S}) \to \{-1, 1\}$, $\mu\bar{S} = 0$, which is a particular form of Eq. (2.1)

$$x = \mathrm{col}[x_1, x_2], \quad A = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \mathrm{col}[b_1, b_2], \quad \varphi:(R^2 \setminus \bar{S}) \to R.$$

We shall solve the problem of synthesis formulated in Sect. 2, assuming $p = (1, 1)$. Then

$$\bar{S} = S = \{x : s(x) = 0\}, \quad s(x) = s_1' x_1 + s_2' x_2, \quad s_1' + s_2' = 0 \tag{7.2}$$

and

$$K = [s_1', s_2']. \tag{7.3}$$

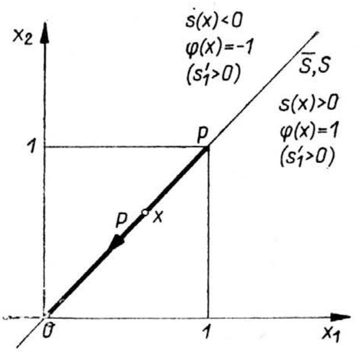

FIG. 2.

Formulae (2.3), (2.5), (7.2) and (7.3) imply that $x = \mathrm{col}\ [\xi, \xi]$ (Fig. 2), and formulae (3.1), (3.2), (7.2) and (7.3) imply that

$$\forall x \in [\overline{p, O}], \quad P(x) = \mathrm{col}[\psi(x), \psi(x)].$$

From the assumption on the point p we have

$$\forall \xi \in [0, 1], \quad P\big(x(\xi)\big) = \mathrm{col}[\psi\big(x(\xi)\big), \psi\big(x(\xi)\big)]$$

yielding the class of functions admissible by the conditions of the problem of synthesis. Assume the functions admissible by the conditions of the problem of synthesis. Assume the function ψ to be linear,

$$\psi:[0, 1] \to R, \quad \psi(\xi) = \alpha\xi + \beta.$$

Then

$$P(\xi) = \mathrm{col}[\alpha\xi + \beta, \alpha\xi + \beta]. \tag{7.4}$$

From the condition of movement "towards the point O" (see (3.4)):

$$\forall \xi \in]0, 1], \quad \xi(\alpha\xi + \beta) < 0$$

the parameters α and β are found

$$\beta < 0, \quad \alpha < -\beta$$

for the function P from (7.4).

Let us determine the matrix B according to the Sect. 4. Assuming (see Fig. 2)

$$\varphi(x) = \begin{cases} -1, & s(x) < 0, \\ 1, & s(x) > 0 \end{cases}$$

and using the condition (4.8)

$$\varkappa_1 = 1-\varkappa_2, \quad 0 < \varkappa_2 < 1$$

we find ϕ

$$\phi = \varkappa_1 \varphi^+ + \varkappa_2 \varphi^- = 1-2\varkappa_2.$$

Equation (4.11) assumes the form

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} (1-2\varkappa_2) = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \xi \\ \xi \end{bmatrix},$$

from which

$$\begin{aligned} b_1 &= \frac{1}{1-2\varkappa_2} [(\alpha-1)\xi+\beta], \\ b_2 &= \frac{1}{1-2\varkappa_2} [(\alpha-a_{21}-a_{22})\xi+\beta]. \end{aligned} \tag{7.5}$$

We shall find the additional conditions for the parameters of Eq. (7.1) according to Sect. 5. Identities (5.2) for the Eq. (7.1) are as follows

$$\forall\, \xi \in [0, 1], \quad \begin{bmatrix} 1 \\ a_{21}+a_{22} \end{bmatrix} \xi - \begin{bmatrix} b_1(\xi) \\ b_2(\xi) \end{bmatrix} \frac{1-a_{21}-a_{22}}{b_2-b_1} \xi = \begin{bmatrix} \alpha\xi+\beta \\ \alpha\xi+\beta \end{bmatrix}$$

for $b_2-b_1 \neq 0$. Taking into consideration the formula (4.9) we have

$$\left[1- \frac{1-a_{21}-a_{22}}{\alpha-a_{21}-a_{22}}(\alpha-1)\right]\xi - \frac{1-a_{21}-a_{22}}{\alpha-a_{21}-a_{22}}\beta = \alpha\xi+\beta,$$

$$(2a_{21}+2a_{22}-1)\xi - \frac{1-a_{21}-a_{22}}{\alpha-a_{21}-a_{22}}\beta = \alpha\xi+\beta$$

together with the condition $a_{21}+a_{22} \neq 1$. Comparing the coefficients for the same powers of ξ, we get the condition for the elements of the matrix A

$$a_{21}+a_{22} = \frac{\alpha+1}{2}.$$

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