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Threshold crossings in a linear oscillator due to a Poissonian train of general pulses(*)

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THE DYNAMIC response of a linear oscillator to a Poisson distributed train of general pulses is considered. The complete expansion of the joint probability density of the response and its time derivative in the series of generalized Hermite polynomials is presented in explicit form. The cumulants of the response, its time derivative and the joint cumulants are evaluated for the stationary response and are discussed. The truncated series is used to evaluate approximately the expected rate of threshold upcrossings. The effect of the pulse duration and of the expected arrival rate of pulses on the mean upcrossing rate is investigated. The reliability estimation is also discussed.

Rozważane są drgania liniowego oscylatora pod wpływem poissonowskiej serii impulsów rozłożonych w czasie. Pełne rozwinięcie łącznej gęstości prawdopodobieństwa procesu odpowiedzi i jego pierwszej pochodnej w szereg uogólnionych wielomianów Hermite'a jest przedstawione w jawnej postaci. Kumulanty procesu odpowiedzi, jego pochodnej oraz łączne kumulanty wyznaczono oraz zanalizowano w przypadku stacjonarnego procesu odpowiedzi. Średnią liczbę przekroczeń w jednostce czasu wyznaczono na podstawie szeregu o ograniczonej liczbie wyrazów. Zbadano wpływ czasu trwania impulsu oraz średniego natężenia pojawiania się impulsów na średnią liczbę przekroczeń. Omówiono także oszacowanie funkcji niezawodności układu.

Рассматриваются линейные колебания осциллятора под влиянием пуассоновской серии импульсов распределенных во времени. Полное разложение совместной плотности вероятности процесса ответа и его первой производной в ряд обобщенных эрмитовых многочленов представлено в явном виде. Кумулянты процесса ответа, его производной и совместные кумулянты определены и анализируются в случае стационарного процесса ответа. Среднее количество превышений в единице времени определено на основе ряда с ограниченным количеством членов. Исследовано влияние времени продолжительности импульса и средней интенсивности появления импульсов на среднее количество превышений. Обсуждена также оценка функции надежности системы.

1. Introduction

THE PROBLEM of vibration under excitations consisting of a train of events occurring at random times (e.g. random pulses) has attracted the attention of investigators for many years. Based on the theory of stochastic point processes [1, 2], the approach proved appropriate to this problem. The papers by LIN [3], ROBERTS [4] and SRINIVASAN *et al.* [5] are some of the first dealing with the dynamic response to random trains of pulses. Later, many aspects of this problem were examined within the framework of the mean-square analysis [6–10].

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Nevertheless the evaluation of higher-order statistics of the response appears to be a much more difficult problem. For example, the equation governing the response one dimensional probability density has been given only in the case of Dirac delta impulses [11, 12]. The evaluation of the average rate of threshold crossings is not straightforward, either. An approach based on the approximate determination of the joint probability density of the response and its time derivative in the case of Dirac delta impulses was presented by ROBERTS [13].

The objective of the present paper is to evaluate approximately the average rate of threshold upcrossings in a linear oscillator subject to a Poisson distributed train of general pulses. The approach used is based on the series expansion of the joint probability density of the response and its time derivative in terms of the generalized Hermite polynomials. The series for the joint probability density is presented in systematic and explicit form. Then the truncated series is used to determine the average rate of threshold upcrossings. Various cumulants and joint cumulants up to the fourth order are evaluated for the stationary response to square pulses and are shown in figures. The effect of the pulse duration and of the average rate of pulses occurrences on the average upcrossings rate is investigated. The estimation of the system reliability is also discussed.

2. Statement of the problem

Consider the dynamic response of a linear oscillator to a random train of pulses, governed by the equation

$$\ddot{q}+2\alpha\omega\dot{q}+\omega^2 q = \sum_{i=1}^{N(t)} F_i s(t, t_i), \tag{2.1}$$

where $s(t, t_i)$ is the pulse shape function vanishing for $t < t_i$ and $t > t_i+T$ and T denotes the pulse duration. The occurrences of pulses are assumed to be the Poisson events with the expected rate $\nu(t)$; $N(t)$ denotes the random number of the occurrences in the time interval $(0, t]$. The magnitudes of pulses are given by the random variables F_i, mutually independent and independent of the counting process $N(t)$.

From the principle of linear superposition it follows that [14]

$$q(t) = \sum_{i=1}^{N(t)} F_i z(t, t_i, T), \tag{2.2}$$

where $z(t, t_i, T)$ is the response at time t to the pulse which originated at time t_i. The equivalent integral form of the expression for the response is (cf. [5])

$$q(t) = \int_0^t z(t, \tau, T)F(\tau)dN(\tau). \tag{2.3}$$

This representation is crucial for the evaluation of the response statistical moments.

Substituting in Eq. (2.3) $z(t, \tau, T) = \int_\tau^t h(t-\theta)s(\theta-\tau)d\theta$ and considering the domain of integration reveals, as was shown by KAWCZYŃSKI [15], the splitting of the function $z(t, \tau, T)$ into two parts:

$$z(t,\tau,T)=\begin{cases} z_1(t,\tau,T)=\int\limits_{\tau}^{t} h(t-\theta)s(\theta-\tau)\,d\theta, & t-T<\tau<t,\\ z_2(t,\tau,T)=\int\limits_{\tau}^{\tau+T} h(t-\theta)s(\theta-\tau)\,d\theta, & 0<\tau\leqslant t-T,\end{cases}\tag{2.4}$$

and consequently

$$q(t)=\int\limits_{t-T}^{t} z_1(t,\tau,T)F(\tau)\,dN(\tau)+\int\limits_{0}^{t-T} z_2(t,\tau,T)F(\tau)\,dN(\tau).\tag{2.5}$$

The cumulants of the response process (displacement response) can be evaluated by making use of the expressions given by LIN [14]. However, in order to determine the cumulants of the response process time derivative (velocity response) and joint cumulants of the response and its time derivative, it is expedient to follow the procedure due to ROBERTS [13]. Then the obvious relationship

$$\dot q(t)=\int\limits_{0}^{t} \dot z(t,\tau,T)F(\tau)\,dN(\tau)\tag{2.6}$$

must be used (cf. [16]).

Taking into account the splitting (2.4), the expressions for the cumulants become (cf. [13])

$$\varkappa_{n0}(t)=\int\limits_{0}^{t-T} z_2^n(t,\tau,T)\nu(\tau)E[F^n(\tau)]\,d\tau+\int\limits_{t-T}^{t} z_1^n(t,\tau,T)\nu(\tau)E[F^n(\tau)]\,d\tau.\tag{2.7}$$

$$\varkappa_{0n}(t)=\int\limits_{0}^{t-T} \dot z_2^n(t,\tau,T)\nu(\tau)E[F^n(\tau)]\,d\tau+\int\limits_{t-T}^{t} \dot z_1^n(t,\tau,T)\nu(\tau)E[F^n(\tau)]\,d\tau,\tag{2.8}$$

$$\begin{aligned}\varkappa_{mn}(t)=&\int\limits_{0}^{t-T} z_2^m(t,\tau,T)\dot z_2^n(t,\tau,T)\nu(\tau)E[F^{m+n}(\tau)]\,d\tau\\ &+\int\limits_{t-T}^{t} z_1^m(t,\tau,T)\dot z_1^n(t,\tau,T)\nu(\tau)E[F^{m+n}(\tau)]\,d\tau,\end{aligned}\tag{2.9}$$

where $\varkappa_{n0}(t)$, $\varkappa_{0n}(t)$ and $\varkappa_{mn}(t)$ denote the n-th cumulant of the response process, the n-th cumulant of its time derivative and the m, n-th joint cumulant, respectively.

LONGUET–HIGGINS [17] gave the series expansion for the joint probability density of the random process and its first time derivative. This is an expansion in terms of the joint cumulants and the generalized Hermite polynomials. Following the procedure due to LONGUET–HIGGINS, let us derive this expansion in a complete and explicit form allowing the systematic generating and truncating of the series.

Expressing the joint probability density function $p(x_1, x_2)$ as the inverse two-fold Fourier transform of the characteristic function $\Phi(it_1, it_2)$

$$p(x_1, x_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(it_1, it_2) \exp[-i(x_1 t_1 + x_2 t_2)] dt_1 dt_2 \tag{2.10}$$

substituting

$$\Phi(it_1, it_2) = \exp\left\{ \sum_{\substack{j,k \\ (j+k=1,2,3,\ldots)}}^{\infty} \frac{(it_1)^j (it_2)^k}{j!\,k!} \varkappa_{jk} \right\} \tag{2.11}$$

and expanding the exponential in the series yields

$$p(x_1, x_2) = \frac{1}{(2\pi)^2 \sqrt{\varkappa_{20}\varkappa_{02}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{ -i(\xi_1 s_1 + \xi_2 s_2) - \frac{1}{2}(s_1^2 + 2\lambda_{11} s_1 s_2 + s_2^2) \right\} \times \left\{ 1 + \sum_{\substack{j,k \\ (j+k=3,4,5,\ldots)}}^{\infty} \frac{\lambda_{jk}}{j!\,k!} (is_1)^j (is_2)^k + \frac{1}{2!} \sum_{\substack{j,k,l,m \\ (j+k=3,4,5,\ldots)}}^{\infty} \frac{\lambda_{jk}\lambda_{lm}}{j!\,k!\,l!\,m!} (is_1)^{j+l} (is_2)^{k+m} + \ldots \right\} ds_1\, ds_2, \tag{2.12}$$

where

$$t_1 = s_1/\sqrt{\varkappa_{20}}, \quad \frac{x_1 - \varkappa_{10}}{\sqrt{\varkappa_{20}}} = \xi_1,$$

$$t_2 = s_2/\sqrt{\varkappa_{02}}, \quad \frac{x_2 - \varkappa_{01}}{\sqrt{\varkappa_{02}}} = \xi_2;$$

$$\lambda_{jk} = \varkappa_{jk}/\sqrt{\varkappa_{20}^j \varkappa_{02}^k}. \tag{2.13}$$

Substituting into Eq. (2.12) the relationship

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{ -i(\xi_1 s_1 + \xi_2 s_2) - \frac{1}{2}(s_1^2 + 2\lambda_{11} s_1 s_2 + s_2^2) \right\} (is_1)^m (is_2)^n ds_1\, ds_2 = \frac{1}{\sqrt{1-\lambda_{11}^2}} \exp\left\{ -\frac{1}{2}(\xi_1^2 - 2\lambda_{11}\xi_1\xi_2 + \xi_2^2)/(1-\lambda_{11}^2) \right\} H_{mn}(\xi_1, \xi_2), \tag{2.14}$$

where $H_{mn}(\xi_1, \xi_2)$ is the generalized, or bivariate, Hermite polynomial, gives the result

$$p(x_1, x_2) = \frac{1}{2\pi\sqrt{\varkappa_{20}\varkappa_{02} - \varkappa_{11}^2}} \exp\left\{ -\frac{1}{2}(\xi_1^2 - 2\lambda_{11}\xi_1\xi_2 + \xi_2^2)/(1-\lambda_{11}^2) \right\} \cdot \left\{ 1 + \sum_{\substack{j,k \\ (j+k=3,4,5,\ldots)}}^{\infty} \frac{\lambda_{jk}}{j!\,k!} H_{jk}(\xi_1, \xi_2) + \frac{1}{2!} \sum_{j,k,l,m}^{\infty} \frac{\lambda_{jk}\lambda_{lm}}{j!\,k!\,l!\,m!} H_{j+l,k+m}(\xi_1, \xi_2) + \frac{1}{3!} \sum_{j,k,l,m,n,r}^{\infty} \frac{\lambda_{jk}\lambda_{lm}\lambda_{nr}}{j!\,k!\,l!\,m!\,n!\,r!} H_{j+l+n,k+m+r}(\xi_1, \xi_2) + \ldots \right\}. \tag{2.15}$$

The generalized Hermite polynomial $H_{mn}(\xi_1, \xi_2)$ satisfies the relationship

$$H_{mn}(\xi_1, \xi_2)\exp\left[-\frac{1}{2}(\xi_1^2-2\lambda_{11}\xi_1\xi_2+\xi_2^2)/(1-\lambda_{11}^2)\right] = (-1)^{m+n}\frac{\partial^{m+n}}{\partial\xi_1^m\partial\xi_2^n}\exp\left[-\frac{1}{2}(\xi_1^2-2\lambda_{11}\xi_1\xi_2+\xi_2^2)/(1-\lambda_{11}^2)\right]. \tag{2.16}$$

The form (2.16) of the expansion is complete and general and allows the systematic generation of the series up to any order required and its truncation.

3. Steady-state response to a Poissonian train of square pulses. Joint cumulants of the response and its time derivative

Confining our attention to the steady-state response to the stationary train of square pulses, we have: $s(t) = 1$ for $t_i < t < t_i+T$, $\nu(t) = \nu =$ const, $E[F^n(t)] = E[F^n] =$ const.

After the change of variable $u = t-\tau$, the expressions (2.7), (2.8) and (2.9) for the cumulants take, respect vely, the forms

$$\varkappa_{n0} = \nu E[F^n]\int\limits_T^\infty z_2^n(u)\,du+\nu E[F^n]\int\limits_0^T z_1^n(u)\,du, \tag{3.1}$$

$$\varkappa_{0n} = \nu E[F^n]\int\limits_T^\infty \dot z_2^n(u)\,du+\nu E[F^n]\int\limits_0^T \dot z_1^n(u)\,du, \tag{3.2}$$

$$\varkappa_{mn} = \nu E[F^{m+n}]\int\limits_T^\infty z_2^m(u)\dot z_2^n(u)\,du+\nu E[F^{m+n}]\int\limits_0^T z_1^m(u)\dot z_1^n(u)\,du, \tag{3.3}$$

where

$$z_1(u) = \omega^{-2}\left[1-e^{-\alpha\omega u}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\sin\zeta u+\cos\zeta u\right)\right], \tag{3.4}$$

$$z_2(u) = \omega^{-2}s(\alpha,\omega,T)e^{-\alpha\omega u}\sin\zeta u+\omega^{-2}c(\alpha,\omega,T)e^{-\alpha\omega u}\cos\zeta u. \tag{3.5}$$

$$s(\alpha,\omega,T) = -\frac{\alpha}{\sqrt{1-\alpha^2}}+\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\cos\zeta T+\sin\zeta T\right)e^{\alpha\omega T}, \tag{3.6}$$

$$c(\alpha,\omega,T) = -1+\left(-\frac{\alpha}{\sqrt{1-\alpha^2}}\sin\zeta T+\cos\zeta T\right)e^{\alpha\omega T}, \tag{3.7}$$

$$\dot z_1(u) = \frac{1}{\zeta}e^{-\alpha\omega u}\sin\zeta u, \tag{3.8}$$

$$\dot z_2(u) = \frac{v_s}{\zeta}e^{-\alpha\omega u}\sin\zeta u+\frac{v_c}{\zeta}e^{-\alpha\omega u}\cos\zeta u, \tag{3.9}$$

$$v_s = 1-e^{\alpha\omega T}\cos\zeta T, \tag{3.10}$$

$$v_c = \sin\zeta T e^{\alpha\omega T} \tag{3.11}$$

and $\zeta = \omega\sqrt{1-\alpha^2}$ is damped natural frequency.

The same results for $z_1(u)$ and $z_2(u)$ are given in the reference [8].

It is interesting to note that in the case of the stationary response (steady-state response to the stationary train of pulses) the following cumulants vanish: $\varkappa_{01} = 0$, $\varkappa_{11} = 0$, $\varkappa_{21} = 0$, $\varkappa_{31} = 0$.

The non-zero coefficients λ_{jk} corresponding to all the cumulants of the order up to $j+k = 4$ are expressed as

$$\lambda_{30} = \frac{\varkappa_{30}}{(\varkappa_{20})^{3/2}} = \frac{16\alpha\sqrt{\alpha}}{3(1+8\alpha^2)}\sqrt{\frac{\omega}{\nu}}\,\frac{E[F^3]}{\{E[F^2]\}^{3/2}}\,\tilde{\lambda}_{30}, \tag{3.12}$$

$$\lambda_{12} = \frac{\varkappa_{12}}{(\varkappa_{20})^{1/2}\varkappa_{02}} = \frac{8\alpha\sqrt{\alpha}}{3(1+8\alpha^2)}\sqrt{\frac{\omega}{\nu}}\cdot\frac{E[F^3]}{\{E[F^2]\}^{3/2}}\,\tilde{\lambda}_{12}, \tag{3.13}$$

$$\lambda_{03} = \frac{\varkappa_{03}}{(\varkappa_{02})^{3/2}} = \frac{32\alpha^2\sqrt{\alpha}}{3(1+8\alpha^2)}\sqrt{\frac{\omega}{\nu}}\,\frac{E[F^3]}{\{E[F^2]\}^{3/2}}\,\tilde{\lambda}_{03}, \tag{3.14}$$

$$\lambda_{40} = \frac{\varkappa_{40}}{(\varkappa_{20})^2} = \frac{3\alpha}{2(1+3\alpha^2)}\,\frac{\omega}{\nu}\,\frac{E[F^4]}{\{E[F^2]\}^2}\,\tilde{\lambda}_{40}, \tag{3.15}$$

$$\lambda_{22} = \frac{\varkappa_{22}}{\varkappa_{20}\varkappa_{02}} = \frac{\alpha}{2(1+3\alpha^2)}\,\frac{\omega}{\nu}\,\frac{E[F^4]}{\{E[F^2]\}^2}\,\tilde{\lambda}_{22}, \tag{3.16}$$

$$\lambda_{13} = \frac{\varkappa_{13}}{(\varkappa_{20})^{1/2}\varkappa_{02}^{3/2}} = \frac{\alpha^2}{1+3\alpha^2}\,\frac{\omega}{\nu}\,\frac{E[F^4]}{\{E[F^2]\}^2}\,\tilde{\lambda}_{13}, \tag{3.17}$$

$$\lambda_{04} = \frac{\varkappa_{04}}{\varkappa_{02}^2} = \frac{3\alpha(1+4\alpha^2)}{2(1+3\alpha^2)}\,\frac{\omega}{\nu}\,\frac{E[F^4]}{\{E[F^2]\}^2}\,\tilde{\lambda}_{04}. \tag{3.18}$$

It may be shown that as $T \to 0$ in such a way that $T^2E[F^2] = \text{const}$, $T^3E[F^3] = \text{const}$, $T^4E[F^4] = \text{const}$, the expressions for the cumulants $\varkappa_{jk}$ and coefficients λ_{jk} approach the respective expressions for Dirac delta impulses. The expressions (3.12)–(3.18) are normalized in such a way that $\lim\limits_{T\to 0}\tilde{\lambda}_{jk} = 1$.

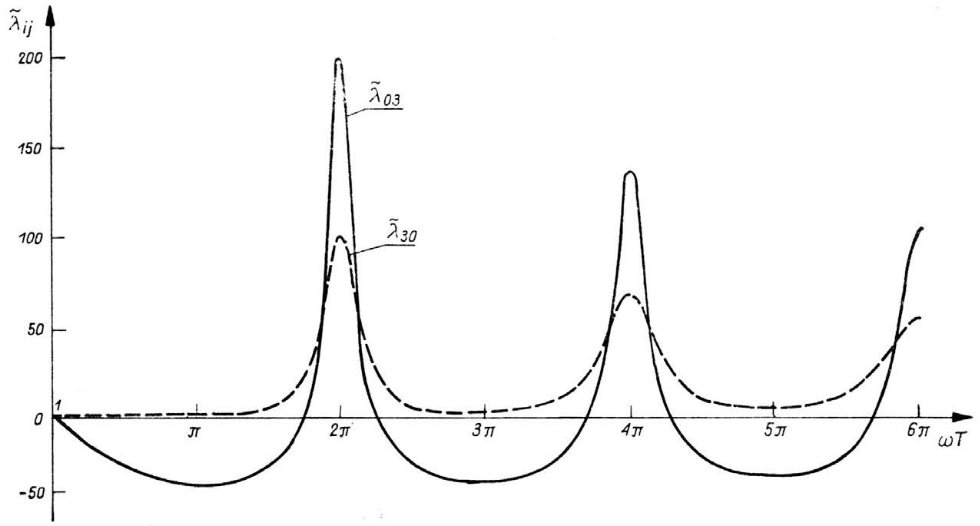

FIG. 1.

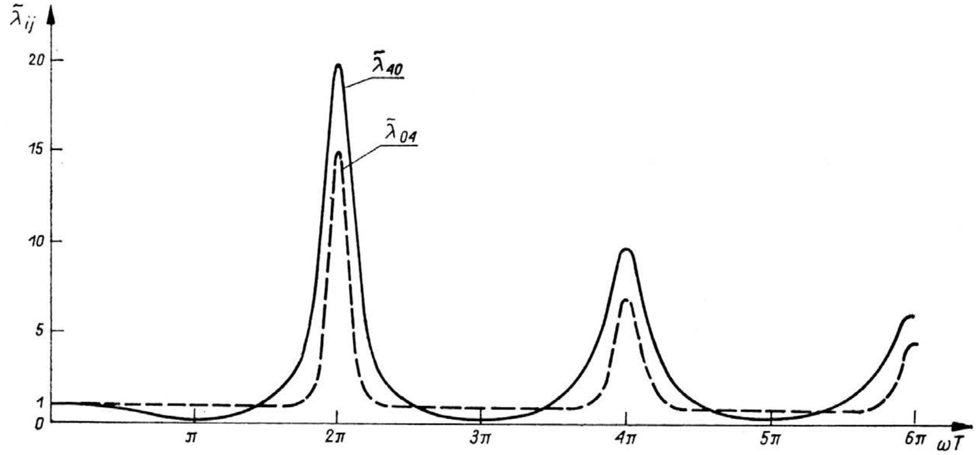

FIG. 2.

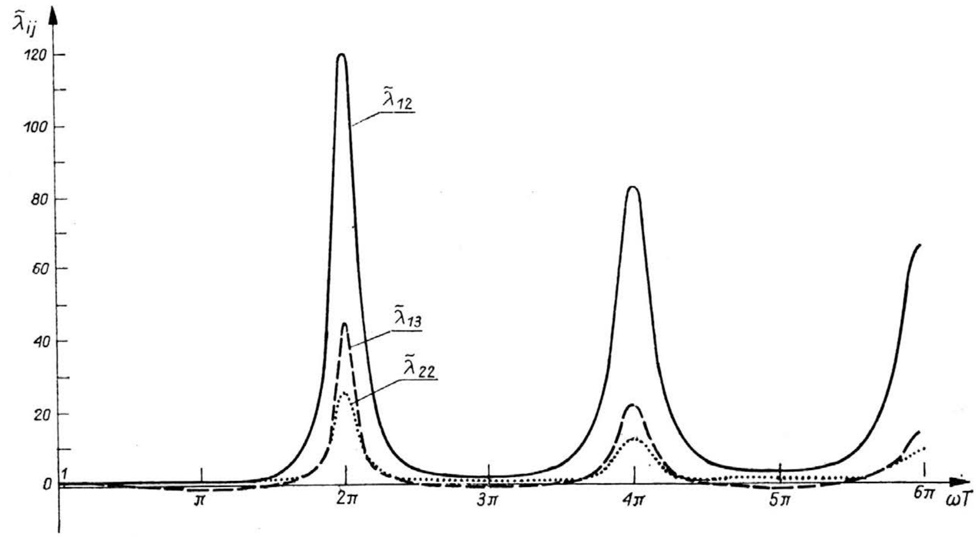

FIG. 3.

The coefficients $\tilde{\lambda}_{jk}$ evaluated for the damping ratio $\alpha = 0.01$, plotted against the pulse duration (ωT) are shown in Figs. 1, 2 and 3.

It is seen that the skewness coefficient $\tilde{\lambda}_{30}$ of the response process (dashed line in Fig. 1) is always positive and greater than in the case of Dirac delta impulses ($\omega T \to 0$). The behaviour of the skewness coefficient of the response time derivative is different (solid line in Fig. 1); it assumes positive values only at the values of ωT in the neighbourhood of $n \cdot 2\pi$ ($n = 1, 2, 3, \ldots$), otherwise it is negative. This means that while the marginal probability density curve of the response process has always positive skewness, the skewness of the distribution of the velocity process may be either positive or negative. At the value of $\omega T = n \cdot 2\pi$ both coefficients reveal very much pronounced maxima.

The coefficients of excess $\tilde{\lambda}_{40}$ (for the response) and $\tilde{\lambda}_{04}$ (for the response time derivative) shown in Fig. 2 are both positive. They also reveal high maxima at $\omega T = n \cdot 2\pi$, moreover the coefficient $\tilde{\lambda}_{40}$ assumes the minima at $(\omega T = (2n+1)\pi,\ n = 0, 1, 2, \ldots)$ the values of which are lower than in the asymptotic case $\omega T \to 0$.

The coefficients $\tilde{\lambda}_{12}$ and $\tilde{\lambda}_{22}$ corresponding to the joint cumulants are again always positive. The coefficient $\tilde{\lambda}_{13}$ assumes, however, also negative values. Also these coefficients attain high maxima at $\omega T = 2\pi n$, $n = 1, 2, 3, \ldots$.

It may be concluded that the departure of the joint probability density $p(x_1, x_2)$ from the Gaussian distribution is the largest when the pulse duration is equal to the natural period of the structure; $\omega T = 2\pi$.

4. Analysis of the average upcrossings rate for the stationary response

The series (2.12) for the Poisson distributed train of pulses can be shown to be the expansion in powers of $(\omega/\nu)^{1/2}$ (cf. [13]). The approximate solution for $p(x_1, x_2)$ is obtained herein by retaining the terms of the order $(\omega/\nu)^{1/2}$ and $(\omega/\nu)^1$only. It is worth noting that in the stationary case the series (2.12) simplifies because some cumulants vanish. In particular, $\lambda_{11} = 0$ which implies that the generalized two-dimensional Hermite polynomials split into the product forms

$$H_{mn}(\xi_1, \xi_2) = H_m(\xi_1) H_n(\xi_2).$$

The expected rate of upcrossings (i.e. crossings with the positive slope) of a threshold $x_1 = a$ is given by the formula due to RICE [18]:

$$\mu_a^+ = \int_0^\infty x_2 p(a, x_2) dx_2. \tag{4.1}$$

Substituting into Eq. (4.1) the truncated series obtained from Eq. (2.15) and integrating yields

$$\begin{aligned}\mu_y^+ = \frac{1}{2\pi}\sqrt{\frac{\varkappa_{02}}{\varkappa_{20}}}\exp(-y^2/2)\Big\{1 &+ \frac{1}{6}[\lambda_{30} H_3(y) + 3\lambda_{12} H_1(y)] \\ &+ \frac{1}{24}[\lambda_{40} H_4(y) + 6\lambda_{22} H_2(y) - \lambda_{04}] \\ &+ \frac{1}{72}\lambda_{30}^2 H_6(y) - \frac{1}{8}\lambda_{12}^2 H_2(y) + \frac{1}{12}\lambda_{30}\lambda_{12} H_4(y) + \frac{1}{24}\lambda_{03}^2\Big\},\end{aligned} \tag{4.2}$$

where

$$y = \frac{a - \varkappa_{10}}{\sqrt{\varkappa_{20}}} \quad \text{and} \quad \sqrt{\frac{\varkappa_{02}}{\varkappa_{20}}} = \omega\sqrt{\frac{\tilde{\varkappa}_{02}}{\tilde{\varkappa}_{20}}}.$$

In the subsequent analysis the pulses magnitudes are assumed to be Gaussian distributed with $\sigma_F/E[F] = 1$, hence $\frac{E[F^3]}{\{E[F^2]\}^{3/2}} = \sqrt{2}$ and $\frac{E[F^4]}{\{E[F^2]\}^2}$ 2.5. The light damping is assumed, i.e. $\alpha = 0.01$.

The non-dimensionalized expected rate of threshold upcrossings $2\pi\mu_y^+/\omega$ is plotted against the pulse duration (ωT) in Figs. 4 through 10 where the solid line represents the Gaussian asymptotic case ($\omega/\nu = 0$, i.e. $\nu \to \infty$), the dotted line is for $\omega/\nu = 2$ and the dashed line—for $\omega/\nu = 5$.

It is of importance to notice (cf. [19]) that when the pulse duration T approaches the multiple natural period, i.e. $T \to n \cdot 2\pi/\omega$ ($n = 1, 2, 3, \ldots$), then the response becomes quasi-static; the induced free vibrations are not essential. In the reference [19] it has also been pointed out that the first-order probability density curve reveals positive skewness. Both observations are helpful in explaining the behaviour of the expected upcrossings rate.

When the response becomes quasi-static, there are small oscillations about the relatively high level. However, these oscillations do not frequently correspond with the crossings of the zero (Fig. 4), of the low positive level ($y = 1$, Fig. 5) or negative level close to zero,

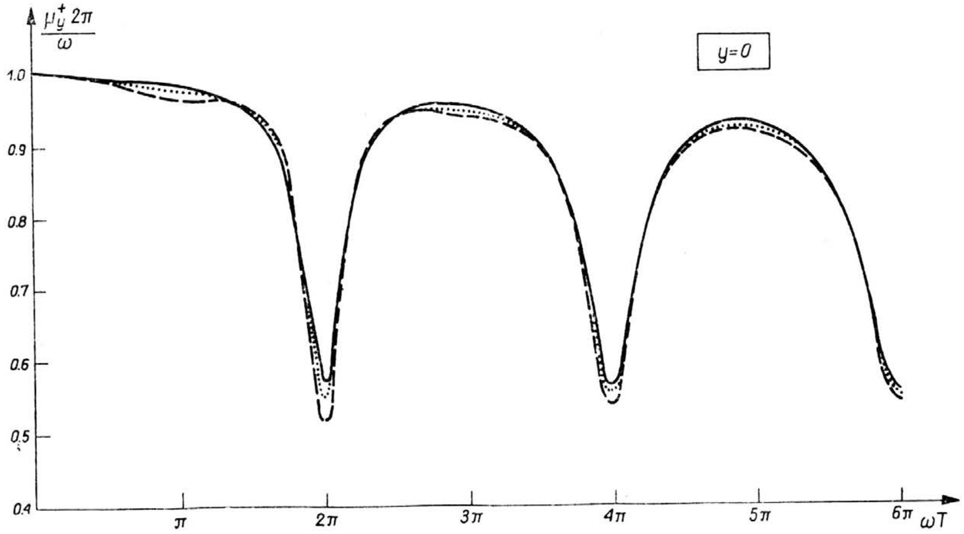

FIG. 4.

($y = -1$, Fig. 7). Therefore, as ωT approaches $n \cdot 2\pi$, the expected upcrossings rate μ_y^+ decreases (Figs. 4, 5, 7). The behaviour of μ_y^+ is different in the case of the high positive threshold $y = 3$ (Fig. 6). Since in the quasi-static case the oscillations are about a certain relatively high positive level, the crossings of the high threshold become more frequent, hence the expected upcrossings rate increases as $\omega T \to n \cdot 2\pi$.

The comparison made between Figs. 4, 5 and 6 shows that as the threshold height increases, the average rate of upcrossings decreases, what might have been intuitively expected, except for the value in the vicinity of $\omega T = n \cdot 2\pi$.

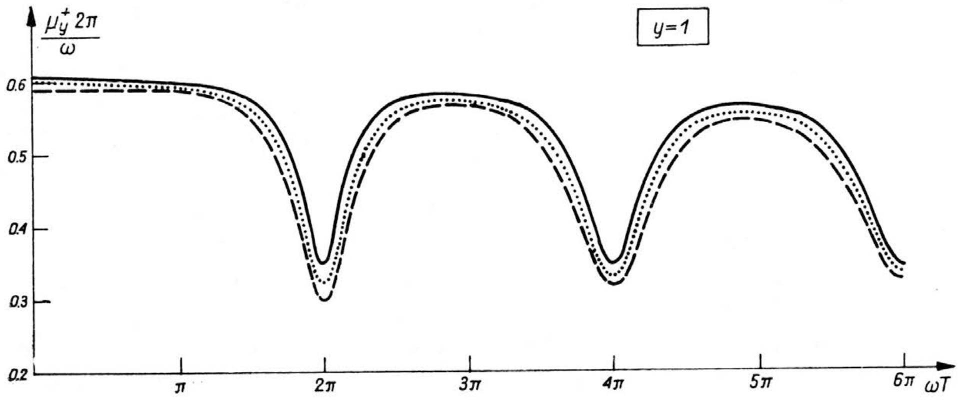

FIG. 5.

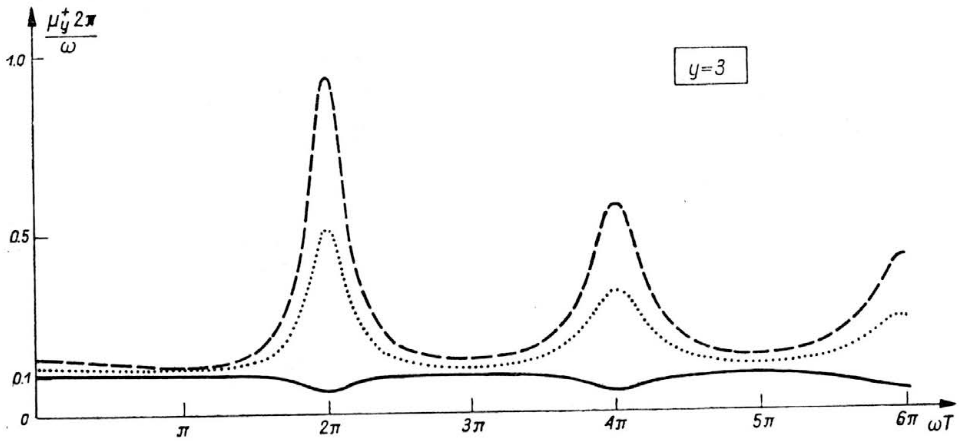

FIG. 6.

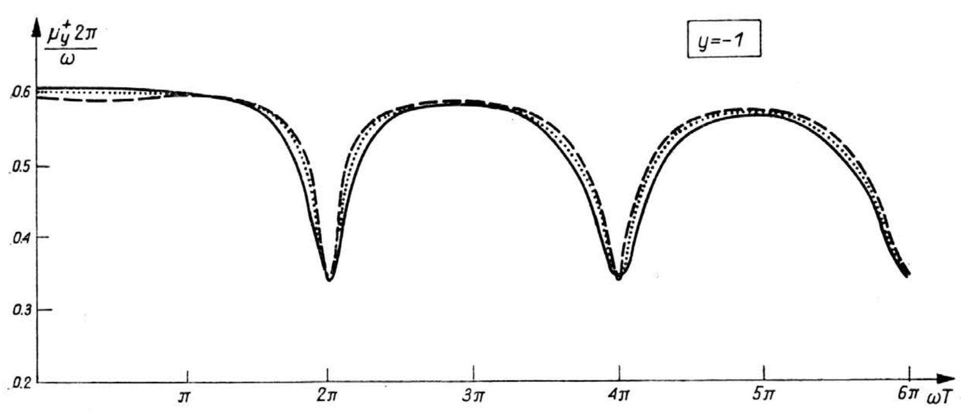

FIG. 7.

The mean rate of upcrossings of a threshold $y = -3$ (Fig. 8) is very low; unfortunately in this case the results are poorly interpretable because the rate μ_y^+ assumes negative values in some regions. This should be regarded as the result of an insufficient number of terms of expansion.

As the ratio ω/ν increases (the mean occurrence rate ν decreases), so does the skewness of the probability distribution. This means that the probability of small positive values

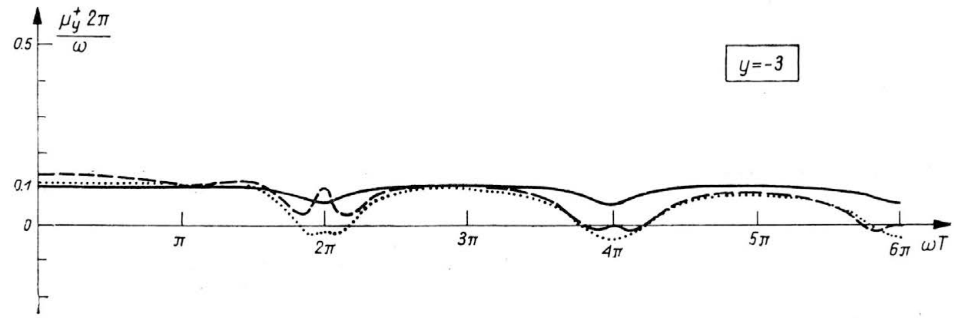

FIG. 8.

of the response (e.g. $y = 1$) decreases and the probability of large positive values increases (cf. [19]). These large positive values of the response may or may not correspond to the crossings of the zero or low positive threshold. Einally it appears that as ω/ν increases, the crossings become less frequent; the expected rate μ_y^+ for thresholds $y = 0$ (Fig. 4) and $y=1$ (Fig. 5) decreases. On the other hand the large values of the response (displacements) correspond to the crossings of the high threshold; the rate $\mu_y^+(y = 3$ Fig. 6) increases. At the same time the probability of close to zero negative values of the response increases, consequently the average upcrossings rate of the negative level close to zero increases (Fig. 7).

The comparison between the expected upcrossings rate of threshold $y = 1$ and $y = -1$ made for $\omega/\nu = 5$ (Fig. 9) reveals that the upcrossings of a negative threshold are more frequent than the upcrossings of a symmetric positive threshold. This is in accordance with the type of probability distribution since the small negative values are more probable (more frequent) than small positive ones. On the other hand, in the case of barriers fairly distant from zero ($y = 3$ only $y = -3$), the situation is different (Fig. 10). The upcrossings of a high positive barrier ($y = 3$) are much more frequent than those of an equally distant from zero negative barrier. The large positive values are more probable than large negative ones, hence there are also more upcrossings.

Knowledge of the expected rate of upcrossings would be sufficient to evaluate the reliability function of the system (defined as the probability of no upcrossing in the time interval $(0, t)$) if the upcrossings were Poisson (independent) events. However, the upcrossings are only asymptotically Poissonian (cf. e.g. [20]).

For a highly reliable system the following rough estimate of the lower bound of the reliability function [20] can be made

$$R(t) > 1 - E[M(t)], \tag{4.3}$$

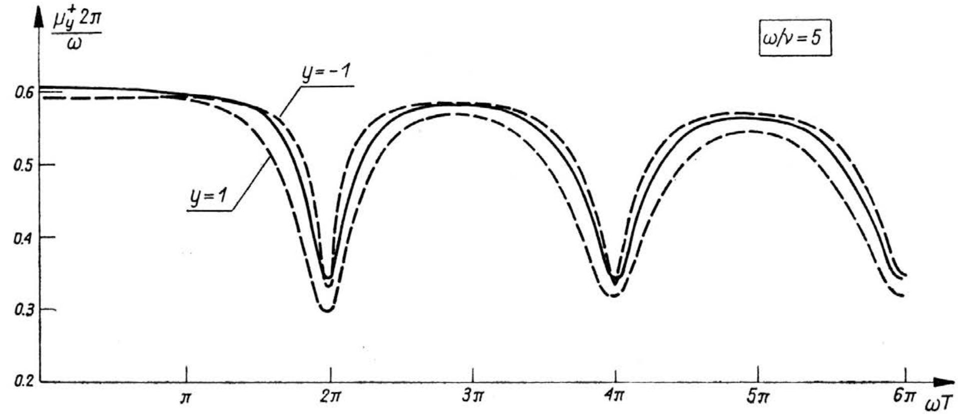

FIG. 9.

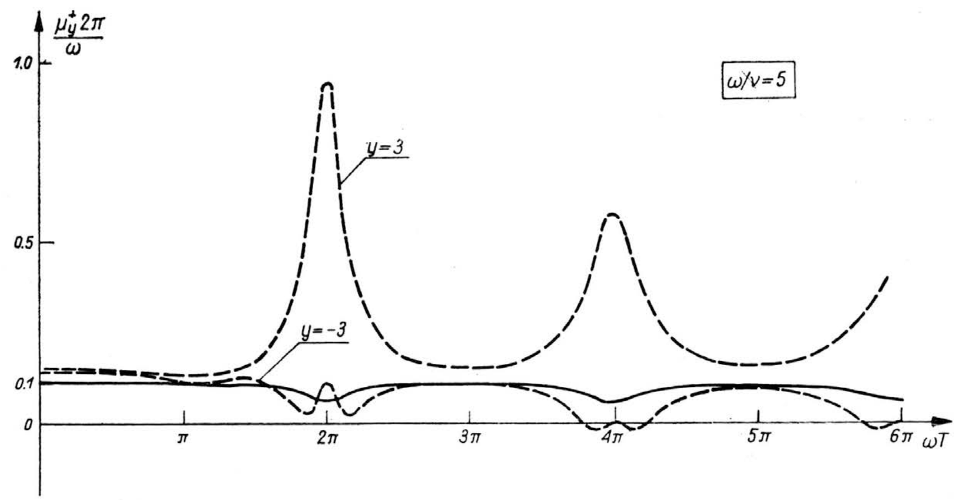

FIG. 10.

where

$$E[M(t)] = \int_0^t \mu_y^+(\tau)\, d\tau.$$

Otherwise a better estimation may be used [20]:

$$(4.4) \quad 1-\frac{11}{6}E[M(t)]+E[M^2(t)]-\frac{1}{6}E[M^3(t)] \leqslant R(t)$$

$$\leqslant 1-\frac{25}{12}E[M(t)]+\frac{35}{24}E[(M^2(t)]-\frac{5}{12}E[M^3(t)]+\frac{1}{24}E[M^4(t)].$$

The use of this estimation requires, however, knowledge of high-order statistics of the upcrossing process (two-point, three-point, e.t.c. time statistics).

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