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Shape design sensitivity analysis in nonlocal elasticity

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SHAPE DESIGN sensitivity analysis of nonlocal elastic solids has been peformed by using the material derivative idea of continuum mechanics. The variation of a general displacement stress functional whose integrand involves integral expressions is evaluated with respect to shape variation through the direct and adjoint variable methods of sensitivity analysis. It is found that for the calculation of the functional variation, "deformations" are needed in the whole physical domain, while only boundary perturbations are required in the adjoint variable method.

Analizę wrażliwości na kształtowanie ciał nielokalnie sprężystych przeprowadza się z wykorzystaniem pojęcia pochodnej materialnej. Wariację funkcjonału przemieszczeniowo-naprężeniowego przeprowadza się względem zmiany kształtu metodą zmiennych bezpośrednich i sprzężonych. Stwierdzono, że dla obliczenia wariacji funkcjonału potrzebna jest znajomość „odkształceń" w całym obszarze fizycznym, podczas gdy metoda zmiennych sprzężonych wymaga jedynie znajomości perturbacji na brzegu obszaru.

Анализ чувствительности к формированию нелокально упругих тел проводится с использованием понятия материальной производной. Вариация общего функционала в перемещениях и напряжениях проводится по отношению к изменению формы методом непосредственных и сопряженных переменных. Констатировано, что для расчета вариации функционала необходимо знание „деформаций" в целой физической области, тогда как метод сопряженных переменных требует только знания пертурбаций на границе области.

1. Introduction

IN THE CLASSICAL elasticity, the constitutive equations are differential in nature and the stress field at a point is given by the strain tensor evaluated at that point (i.e., a local theory). In contrast to this local approach which excludes the action at a distance, the nonlocal elasticity theory, developed independently by several researches in the field (see, for example, Refs. [1, 2]) studies the behaviour of structures whose constitutive equations are integro-differential equations. In other words, in such structures it is postulated that the local state at a point is influenced by the action of all particles in the body. Hence, the stress field in a nonlocal continuum is given by the strain field defined over the whole domain, and is mathematically expressed in terms of an integral equation whose kernel (i.e., influence) function characterizes the nonlocal effects.

Nonlocal effects are generally of minor importance in microscopic behaviour of materials. However, in some cases they may be dominant as in phonon dispersion in solids, in surface physics, in electromagnetic solids and in fracture mechanics. Indeed, it has been found that for the Griffith problem in fracture mechanics, nonlocal elasticity must be employed to determine stresses at a sharp crack tip [3, 4]. The nonlocal solution to this problem, which leads to a finite stress at the crack tip, displays a rather remarkable agreement with experimental evidence. Recently, mathematical difficulties which usually arise from material instabilities in distributed cracking problems have been resolved by describing the materials as nonlocal continua [5, 6]. In Refs. 5 and 6, the stress field is, however, also averaged over characteristic domains leading to the nonlocal theory, which is not considered in the present paper. The nonlocal theory, closing the gap between the classical continuum limit and the atomic theory of lattice, can also be a very useful tool in investigating the motions and deformations of bodies in wave propagation problems in a wide-range of frequencies and wave lengths [7, 8].

The shape design sensitivity analysis (SDSA), that is, finding the variation of functionals of the system's response with respect to variations of boundaries of the structure, has been developed and reasonably completed for local elastic solids in the literature [9, 12]. The SDSA, which may be extremely useful in shape (design) optimization or shape identification problems, has not been considered for nonlocal elastic structures in the literature so far (to the best knowledge of the present author). The present paper investigates the first variation of a general displacement and stress functional for a nonlocal linear elastic solid body with respect to shape variations. Only static conditions are treated, while geometric discontinuities in the domain and on the boundary of the solid are considered in the study. As such, the present SDSA may be useful for shape inverse (i.e., optimization or identification) problems in nonlocal fracture mechanics.

In the present investigation, the material derivative (MD) concept from continuum mechanics, which has been previously applied to the SDSA of local elastic structures, will be applied to the SDSA of nonlocal structures. The SDSA expressions will be derived by using both the direct and adjoint variable methods [13–16]. It has basically been found that the "deformation" velocity field characterizing shape variations is required in the whole solid domain in the case of the direct method, while only boundary perturbation information is necessary for the case of the adjoint variable method.

2. Primary problem

In the absence of nonlocal (i.e., residual) body forces, the equations of equilibrium for a homogeneous, isotropic nonlocal solid body may be written as

$$\text{in } V-\sigma: \quad \sigma_{ij,j} = -b_i, \tag{2.1}$$

where V is the physical domain of the structure to be varied; σ is a closed and regular discontinuity surface (i.e., an interface) within V; σ_{ij} is the stress tensor and b_i are the distributed body forces which may depend upon the structural shape. Mixed boundary conditions may be imposed such that

$$\text{on } S_u: \quad u_i = u_i^0, \tag{2.2}$$

$$\text{on } S_t: \quad t_i = t_i^0, \tag{2.3}$$

where S_u and S_t are parts of the varying boundary S of V; u_i and t_i are the displacements and tractions, respectively; the superscript 0 indicates prescribed quantities. The surface tractions are also given by $t_i = \sigma_{ij}n_j$, where n_i is the unit vector normal to S.

Across the discontinuity surface σ, the jump conditions are given in the form of continuous displacements and tractions as

$$\text{on } \sigma: \quad [\![u_i]\!] = 0, \tag{2.4}$$

$$\text{on } \sigma: \quad [\![t_i]\!] = [\![\sigma_{ij}]\!] m_j = 0, \tag{2.5}$$

where symbol $[\![\ ]\!]$ represents the discontinuity of the enclosed quantity, calculated as a difference of the respective values in the domains V^- and V^+, where $V^- + V^+ = V - \sigma$, with $[\![u_i]\!] = u_i^- - u_i^+$ for $x_i \varepsilon \sigma$. The unit vector m_i is normal to σ, pointing from V^- into V^+. It may also be shown that the gradient of the displacements and tractions satisfy the following jump conditions on σ due to Eqs. (2.4) and (2.5):

$$\text{on } \sigma: \quad [\![u_{i,k}]\!] = [\![u_{i,m}]\!] m_k, \tag{2.6}$$

$$\text{on } \sigma: \quad [\![t_{i,k}]\!] = [\![t_{i,m}]\!] m_k. \tag{2.7}$$

The constitutive law associated with an isotropic, homogeneous, linear nonlocal elastic solid body is taken as follows [1–4]:

$$\sigma_{ij}(x_k) = \int_{V-\sigma} \alpha(|\bar{x}_i - x_i|;\ \varepsilon)[\lambda \bar{u}_{k,k}\delta_{ij} + \mu'(\bar{u}_{i,j} + \bar{u}_{j,i})] d\bar{V}, \tag{2.8}$$

where x_i is the observation point where σ_{ij} is evaluated; α is a two-point influence (i.e., kernel) function which represents the nonlocal character of the solid body; ε is an attenuation parameter V denoting the characteristic length of nonlocal influence; $\bar{x}_i$ is an arbitrary point in $V - \sigma$ (i.e., a dummy variable of integration); λ and μ are Lamé's constants. Any barred quantity refers to that quantity evaluated at the field point $\bar{x}_i$.

It is noted that for the homogeneous material at hand, the constitutive equations are invariant under arbitrary translations of the material of reference so that they depend on $\bar{x}_i$ and x_i only through their distance $|\bar{x}_i - x_i|$, hence $\bar{\alpha} = \alpha$ [7]. In the limit $\varepsilon \to 0$, the nonlocal theory must revert to the classical elasticity theory, i.e., α must become a Dirac-delta measure as $\varepsilon \to 0$. This requires that

$$\int_{V-\sigma} \alpha d\bar{V} = 1.$$

In the present analysis, the influence function α will be assumed to be continuous and differentiable throughout V, including σ, hence $\alpha = 0$ on σ. However, the dependence of α on the structural shape will not be supressed. The influence function α and attenuation parameter ε are usually determined with dispersive wave experiments in solid state physics. In the present analysis, no considerations will be given as to the specific functional form of α for the purpose of generality [3, 4].

The nonlocal constitutive equations (2.8) may also be written as

$$\sigma_{ij} = \int_{V-\sigma} \alpha \bar{G}_{ij} d\bar{V}, \tag{2.9}$$

where G_{ij} is simply defined by the classical Hooke's law as

$$G_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \tag{2.10}$$

Integration by parts considering discontinuities will be frequently used in the present study, and hence is given for two general differentiable functions u and v as follows:

$$(2.11)\qquad \int_{V-\sigma} uv_{,i}dV = \int_{S-\Gamma} uvn_i dS - \int_{V-\sigma} vu_{,i}dV + \int_{\sigma} [\![uv]\!] m_i d\sigma,$$

where σ and Γ represent discontinuity surface and curve in V and on S, respectively.

The primary problem may now be given in terms of G_{ij} by substituting Eq. (2.9) into (2.1)–(2.5) and utilizing integration by parts, Eq. (2.11), in the following form:

$$(2.12)\qquad \text{in } V-\sigma: \quad \int_{V-\sigma} \alpha\bar{G}_{ij,j}d\bar{V} - \int_{S-\Gamma} \alpha\bar{G}_{ij}\bar{n}_j d\bar{S} - \int_{\sigma} \alpha[\![\bar{G}_{ij}]\!]\bar{m}_j d\bar{\sigma} = -b_i;$$

$$(2.13)\qquad \text{on } S_u: \quad u_i = u_i^0;$$

$$(2.14)\qquad \text{on } S_t: \quad \Big[\int_{V-\sigma} \alpha\bar{G}_{ij}d\bar{V}\Big] n_j = t_i^0;$$

$$(2.15)\qquad \text{on } \sigma: \quad [\![u_i]\!] = 0;$$

$$(2.16)\qquad \text{on } \sigma: \quad \Big[\int_{V-\sigma} \alpha[\![\bar{G}_{ij}]\!]d\bar{V}\Big] m_j = 0;$$

where the fact that

$$(2.17)\qquad \frac{\partial\alpha}{\partial\bar{x}_i} = -\frac{\partial\alpha}{\partial x_i}$$

has been employed, along with $\alpha = 0$ on σ. It is noted that surface and interface integrals are involved in the field equation (2.12) due to the nonlocal character of the primary problem.

3. A general performance criterion

In inverse problems, for example, shape design, optimization or identification problems, integral functionals of the system's response play an important role. A general integral functional (i.e., the performance criterion) may now be defined, which could serve as a functional to be minimized or simply as an integral behavioural constraint to be satisfied. The general performance criterion I is thus given as follows:

$$(3.1)\qquad I = \int_{V-\sigma} f(u_i, \sigma_{ij})dV + \int_{S-\Gamma} g(u_i, t_i)dS + \int_{\sigma} [\![h(u_i, t_i)]\!]d\sigma,$$

where f, g and h are continuous and differentiable functions with respect to their arguments in their domains of integration; Γ denotes boundary surface curves on S across which discontinuities of boundary data or geometry occur. In particular, for only boundary data discontinuity on a smooth surface S, $S-\Gamma$ would be equal to S_u+S_t, where Γ is the boundary surface curve between S_u and S_t.

It is now desired to find the effects of shape variation of S and σ on the functional I, while the primary problem defined in the last section is satisfied, i.e., the so-called SDSA.

Once the SDSA is completed, iterative mathematical programming methods can be utilized for any structural optimization or shape identification problem at hand in nonlocal linear elasticity.

4. Material derivative concept

The problem of any sensitivity analysis is to compute explicitly derivatives of performance criteria with respect to decision variables. In the present study, the structural shape itself represents the decision variable. This type of problems are inherently more complex than the structural optimization problems where the shapes are defined by cross-section and/or thickness variables, which appear explicitly in the system's (primary) equations and performance criteria.

It is noted that the integral functional I, Eq. (3.1), has a nonlocal character in that it has integrals as well as differentials as its arguments, as dictated by the nonlocal constitutive equations (2.8).

The SDSA for local structures has been investigated throughly in a recent book [1]. An excellent interpretation for the SDSA has also been given in a recent paper [10]. In the present SDSA, the material derivative (MD) concept (or interpretation) of Refs. [11 and 12] will be utilized. Discontinuities across σ and Γ, and their variations will also be considered [9, 14].

Since the shape of domain V of the nonlocal elastic solid body is treated as the decision (or design) variable, it is convenient to think of V as a continuous medium and utilize the MD idea from continuum mechanics. Thus, the general formula pertaining to the MD of a domain (or volume) integral ψ_1 containing a discontinuity surface σ and defined by

$$\psi_1 = \int_{V-\sigma} u\,dV \tag{4.1}$$

is given as follows:

$$\dot{\psi}_1 = \int_{V-\sigma} u'\,dV + \int_{S-\Gamma} uV_n\,dS + \int_{\sigma} [\![u]\!]V_m\,d\sigma, \tag{4.2}$$

where u is a general (differentiable) function; $(\dot{\ })$ and $(\ )'$ denote the material and partial derivatives of $(\ )$, respectively [11, 12]; V_n and V_m are the normal components of the design perturbation velocity V_n on S and σ, respectively. It is noted that the pointwise MD of a general function u is defined by

$$\dot{u} = u' + u_{,k}V_k, \tag{4.3}$$

where the partial derivative (PD) operator commutes with the space derivatives, i.e.

$$(u_{,i})' = (u')_{,i}. \tag{4.4}$$

The MD of a general (piecewise smooth) surface integral ψ_2 defined by

$$\psi_2 = \int_{S-\Gamma} u\,dS \tag{4.5}$$

is given in the following form:

$$(4.6)\qquad \dot{\psi}_2 = \int_{S-\Gamma} [u' + (u_{,n} + H_s u) V_n] dS + \oint_{\Gamma} [\![u]\!] V_\mu d\Gamma + \sum_k \int_{\Gamma_k} (u^+ V_{\mu^+} + u^- V_{\mu^-}) d\Gamma_k,$$

where $(\,)_{,n}$ represents the normal derivative of $(\,)$ on S; H_S is the curvature of the boundary S in R^2 and twice the mean curvature of S in R^3; V is the unit vector normal to Γ and tangent to S; the symbol $[\![\;]\!]$ in this case indicates a discontinuity across $\Gamma \in S$; the plus and minus signs attached to quantities denote that they are evaluated at the plus and minus sides of Γ, respectively; the summation is taken over all the surface curves Γ_k bounding piecewise regular surfaces of S [14].

5. Direct method of SDSA

It is possible to use different methods of SDSA, one of which is the direct method. In this method, the local variations (i.e., the PD's with respect to τ) of functions are directly evaluated in terms of an auxiliary problem, and then substituted into the MD form of the performance criterion I. The procedure for the direct method of SDSA may be given in the following form:

Step 1: Take the MD of I.
Step 2: Take the PD of the field equations.
Step 3: Take the MD of the boundary and jump conditions.
Step 4: Take the PD of the constitutive equations.
Step 5: Define an auxiliary problem.
Step 6: Obtain the MD of I in terms of the primary and auxiliary variables.

It is noted that, although the PD forms of the functions may be expressed in terms of variations of some decision parameters, such an approach will not be adopted at the outset. The above-outlined procedure will now be applied to the performance criterion I, Eq. (3.1), subject to the primary problem equations given in Sect. 2.

5.1. The MD of I

Step 1: The MD of I, Eq. (3.1), can be formally taken by employing the general MD formulas (4.1), (4.2), (4.5) and (4.6) as

$$(5.1)\qquad \dot{I} = \int_{V-\sigma} \left(\frac{\partial f}{\partial u_i} u_i' + \frac{\partial f}{\partial \sigma_{ij}} \sigma_{ij}' \right) dV + \int_{S-\Gamma} \left[(f + g_{,n} + H_s g) V_n + \frac{\partial g}{\partial u_i} u_i' + \frac{\partial g}{\partial t_i} t_i' \right] dS$$
$$+ \int_{\sigma} \left\| (f + h_{,m} + H_\sigma h) V_m + \frac{\partial h}{\partial u_i} u_i' + \frac{\partial h}{\partial t_i} t_i' \right\| d\sigma$$
$$+ \oint_{\Gamma} [\![g]\!] V_\mu d\Gamma + \sum_k \int_{\Gamma_k} (g^+ V_{\mu^+} + g^- V_{\mu^-}) d\Gamma_k,$$

where $(\,)_{,m}$ is the normal derivative of $(\,)$ on σ; H_σ is the curvature of σ. The functions

u'_i, t'_i and σ'_{ij} may be taken as denoting the auxiliary displacements, tractions and stresses, respectively. For the evaluation of $\dot{I}$, Eq. (5.1), the auxiliary variables must be evaluated in the direct method.

Step 2: The PD of the equilibrium equation (2.1) may simply be given by

$$\text{in } V-\sigma: \quad \sigma'_{ij,j} = -b'_i, \tag{5.2}$$

where the commutative property of the PD operator is understood. It is also assumed that the distributed body forces b_i depend on the structural shape.

5.2. The MD of the boundary and jump conditions

Step 3: By using the general MD formula for continuous functions, Eq. (4.3), the MD of the boundary conditions (2.2) and (2.3) yield the following expressions:

$$\text{on } S_u: \quad u'_i = \dot{u}^0_i - u_{i,k}V_k, \tag{5.3}$$

$$\text{on } S_t: \quad t'_i = \dot{t}^0_i - t_{i,k}V_k. \tag{5.4}$$

Since $t_i = \sigma_{ij}\, n_j = t^0_i$ on S_t, the MD of this equation also gives

$$\text{on } S_t: \quad \dot{t}_i = \dot{\sigma}_{ij}n_j + \sigma_{ij}\dot{n}_j = \dot{t}^0_i. \tag{5.5}$$

Introducing the MD of n_j in the form [10, 14]

$$\dot{n}_j = (n_j n_l - \delta_{jl})n_k V_{k,l} \tag{5.6}$$

it may be shown that

$$\text{on } S_t: \sigma'_{ij}n_j = \dot{t}^0_i + \sigma_{ij}(\delta_{jl} - n_j n_l)n_k V_{k,l} - \sigma_{ij,k}n_j V_k. \tag{5.7}$$

Similar expressions may be obtained for the MD forms of the jump conditions (2.4) and (2.5) resulting in

$$\text{on } \sigma: \quad [\![u'_i]\!] = -[\![u_{i,m}]\!]V_m; \tag{5.8}$$

$$\text{on } \sigma: \quad [\![t'_i]\!] = -[\![t_{i,m}]\!]V_m; \tag{5.9}$$

$$\text{on } \sigma: \quad [\![\sigma'_{ij}]\!]m_j = [\![\sigma_{ij}]\!](\delta_{jl} - m_j m_l)m_k V_{k,l} - [\sigma_{ij,k}]m_j V_k, \tag{5.10}$$

where Eqs. (2.6) and (2.7) have been utilized.

5.3. The PD of the constitutive equations

Step 4: The stress σ_{ij} at a point x_i is given by Eq. (2.9) in terms of a volume integral. Hence, due to this nonlocal property of σ_{ij}, its PD form will be evaluated by the general MD formula for a volume integral, i.e., Eqs. (4.1) and (4.2), in contrast to the local theory. Thus,

$$\sigma'_{ij} = \int_{V-\sigma} (\alpha\overline{G}'_{ij} + \alpha'\overline{G}_{ij})d\overline{V} + \int_{S-\Gamma} \alpha\overline{G}_{ij}\overline{V}_n d\overline{S} + \int_{\sigma} \alpha[\![\overline{G}_{ij}]\!]\overline{V}_m d\overline{\sigma}. \tag{5.11}$$

The PD form of G_{ij} is, in turn, given by

$$G'_{ij} = \lambda u'_{k,k}\delta_{ij} + \mu(u'_{i,j} + u'_{j,i}), \tag{5.12}$$

where $\lambda' = \mu' = 0$ have been taken.

5.4. Auxiliary problem

Step 5: The auxiliary stress σ'_{ij}, given by Eq. (5.11), is now inserted into the auxiliary equations (5.2), (5.7) and (5.10). Using integration by parts, Eq. (2.11), and Eq. (2.17), the auxiliary problem is stated in terms of u'_i (and G'_{ij}) as follows:

$$(5.13)\quad \text{in } V-\sigma:\quad \int_{V-\sigma} \alpha\overline{G}'_{ij,j}d\overline{V} - \int_{S-\Gamma} \alpha\overline{G}'_{ij}n_j d\overline{S} - \int_{\sigma} \alpha[\![\overline{G}'_{ij}]\!]\overline{m}_j\, d\overline{\sigma}$$
$$= -b'_i - \int_{V-\sigma} \alpha'\overline{G}_{ij,j}d\overline{V} + \int_{S-\Gamma} (\alpha'\overline{G}_{ij} - \alpha_{,k}\overline{G}_{ik}\overline{V}_j)\overline{n}_j d\overline{S}$$
$$+ \int_{\sigma} [\![\alpha'\overline{G}_{ij} - \alpha_{,k}\overline{G}_{ik}\overline{V}_j]\!]\overline{m}_j d\overline{\sigma};$$

$$(5.14)\quad \text{on } S_u:\quad u'_i = \dot{u}^0_i - u_{i,k}V_k;$$

$$(5.15)\quad \text{on } S_t:\quad \Big[\int_{-\sigma} \alpha\overline{G}'_{ij}d\overline{V}\Big] n_j = \dot{t}^0_i + \sigma_{ij}(\delta_{jl} - n_j n_l)n_k V_{k,l} - \sigma_{ij,k}n_j V_k$$
$$-\Big[\int_{V-\sigma} \alpha'\overline{G}_{ij}d\overline{V} + \int_{S-\Gamma} \alpha\overline{G}_{ij}\overline{V}_n d\overline{S} + \int_{\sigma} \alpha[\![\overline{G}_{ij}]\!]\overline{V}_m d\overline{\sigma}\Big]n_j;$$

$$(5.16)\quad \text{on } \sigma:\quad [\![u'_i]\!] = -[\![u_{i,m}]\!]V_m;$$

$$(5.17)\quad \text{on } \sigma:\quad \Big[\int_{V-\sigma} \alpha[\![G'_{ij}]\!]d\overline{V}\Big]m_j = [\![\sigma_{ij}]\!](\delta_{jl} - m_j m_l)m_k V_{k,l} - [\![\sigma_{ij,k}]\!]m_j V_k$$
$$-\Big[\int_{V-\sigma} \alpha'[\![\overline{G}_{ij}]\!]d\overline{V} + \int_{S-\Gamma} \alpha[\![\overline{G}_{ij}]\!]\overline{V}_n d\overline{S}\Big]m_j.$$

It is noted that the expressions on the righthand sides of Eqs. (5.13)–(5.17) are treated as known, since the primary problem is solved first for a structural shape configuration. For the solution of the auxiliary problem, the V_i distribution in the whole domain is necessary, along with b'_i, α', $\dot{u}^0_i$ and $\dot{t}^0_i$.

If the auxiliary variables are assumed to be given in terms of some decision parameters ϕ_k, they can be expressed as, for example, [13]

$$(5.18)\qquad u'_i = \frac{\partial u_i}{\partial \phi_k}\delta\phi_k = u^k_i\,\delta\phi_k.$$

Introducing similar expressions for the other auxiliary variables, the auxiliary problem can be expressed and solved in terms of u^k_i. It is important to realize that in that case for each decision parameter ϕ_k, one auxiliary problem will have to be constructed and solved.

Step 6: If the solutions of the primary and auxiliary problems are introduced into Eq. (5.1), the MD (i.e., the total variation) of the performance criterion I may be obtained, thus concluding the direct method of SDSA.

6. Adjoint variable method of SDSA

The local variations of the primary variables in a SDSA may be eliminated by using adjoint variables in the so-called AVM of SDSA, instead of directly calculating them as in the previously studied direct method. The procedure for the AVM may be outlined in the following form for nonlocal elastic structures:

Step 1: Augment the performance criterion I by incorporating the equilibrium equations.

Step 2: Integrate by parts.

Step 3: Take the MD of the augmented functional.

Step 4: Substitute the PD form of the constitutive equations.

Step 5: Interchange the order of integration and rename the dummy variables of integration.

Step 6: Integrate by parts again.

Step 7: Substitute the MD forms of the boundary and jump conditions.

Step 8: Define the relevant adjoint problem.

Step 9: Obtain the MD of I.

In the following subsections, the above given procedure will be employed for the present nonlocal elasticity problem.

6.1. Augmentation of I and integration by parts

Step 1 and 2: In the AVM of optimization, the equilibrium Eqs. (2.1) are incorporated into I in terms of the adjoint displacements u_i^* as follows:

$$\hat{I} = I + \int\limits_{V-\sigma} u_i^*(\sigma_{ij,j}+b_i)\,dV. \tag{6.1}$$

Integration by parts, Eq. (2.11), is then used for the stress term in the above equation yielding $\hat{I}$ as

$$\hat{I} = \int\limits_{V-\sigma} (f-\sigma_{ij}u_{i,j}^*+b_iu_i^*)\,dV + \int\limits_{S-\Gamma} (g+t_iu_i^*)\,dS + \int\limits_{\sigma} [\![h+t_iu_i^*]\!]\,d\sigma, \tag{6.2}$$

where Eq. (3.1) has been substituted.

6.2. The MD of $\hat{I}$

Step 3: Using the general MD formulas, Eqs. (4.1), (4.2), (4.5) and (4.6), the MD of $\hat{I}$ may be written as follows:

$$\begin{aligned}\dot{\hat{I}} = &\int\limits_{V-\sigma} \left[\frac{\partial f}{\partial u_i}u_i' - \left(u_{i,j}^{*\prime} - \frac{\partial f}{\partial \sigma_{ij}}\right)\sigma_{ij}' - \sigma_{ij}u_{i,j}^{*\prime} + b_iu_i^{*\prime} + u_i^*b_i'\right]dV \\ &+ \int\limits_{S-\Gamma}\Big\{[f-\sigma_{ij}u_{i,j}^* + b_iu_i^* + (g+t_iu_i^*)_{,n} + H_s(g+t_iu_i^*)]V_n \\ &+ \frac{\partial g}{\partial u_i}u_i' + \left(\frac{\partial g}{\partial t_i} + u_i^*\right)t_i' + t_iu_i^{*\prime}\Big\}\,dS \\ &+ \int\limits_{\sigma} \Big[\!\!\Big[[f-\sigma_{ij}u_{i,j}^* + b_iu_i^* + (h+t_iu_i^*)_{,m} + H_\sigma(h+t_iu_i^*)]V_m\end{aligned} \tag{6.3}$$

(6.3) [cont.]

$$+\frac{\partial h}{\partial u_i}u_i'+\left(\frac{\partial h}{\partial t_i}+u_i^*\right)t_i'+t_i u_i^{*\prime}\Big]\!\Big]\, d\sigma$$

$$+\oint_{\Gamma}[\![g+t_i u_i^*]\!]V_\mu d\Gamma+\sum_k\int_{\Gamma_k}[(g+t_i u_i^*)^+V_{\mu^+}+(g+t_i u_i^*)^-V_{\mu^-}]d\Gamma_k,$$

where the discontinuity surface (i.e. interface) σ in V has been assumed as smooth and closed, as before.

6.3. The PD form of the constitutive equations

Step 4 and 5: The second integral term in Eq. (6.1) can be treated separately for convenience, thus define π' as

$$\pi'=\int_{V-\sigma}\left(u_{i,j}^*-\frac{\partial f}{\partial\sigma_{ij}}\right)\sigma_{ij}'dV. \tag{6.4}$$

The PD form of the constitutive equations, Eq. (5.11), is then substituted into π' yielding

$$\pi'=\int_{V-\sigma}\left(u_{i,j}^*-\frac{\partial f}{\partial\sigma_{ij}}\right)\left[\int_{V-\sigma}(\alpha\bar{G}_{ij}'+\alpha'\bar{G}_{ij})d\bar{V}+\int_{S-\Gamma}\alpha\bar{G}_{ij}\bar{V}_n d\bar{S}+\int_{\sigma}\alpha[\![\bar{G}_{ij}]\!]\bar{V}_m d\bar{\sigma}\right]dV. \tag{6.5}$$

Now, changing the orders of integration and renaming the dummy variables of integration, π' can be transformed by employing Eq. (5.12) into the following form:

$$\pi'=\int_{V-\sigma}\left\{\sigma_{ij}^*u_{i,j}'+u_{i,j}\left[\int_{V-\sigma}\alpha'\bar{G}_{ij}^*d\bar{V}\right]\right\}dV+\int_{S-\Gamma}u_{i,j}\sigma_{ij}^*V_n dS+\int_{\sigma}[\![u_{i,j}\sigma_{ij}^*]\!]V_m d\sigma, \tag{6.6}$$

where the adjoint stress tensor σ_{ij}^* is defined by the equations given below:

$$\sigma_{ij}^*=\int_{V-\sigma}\alpha\bar{G}_{ij}^*d\bar{V}, \tag{6.7}$$

$$G_{ij}^*=\lambda\left(u_{k,k}^*-\frac{\partial f}{\partial\sigma_{kk}}\right)\delta_{ij}+\mu\left[\left(u_{i,j}^*-\frac{\partial f}{\partial\sigma_{ij}}\right)+\left(u_{j,i}^*-\frac{\partial f}{\partial\sigma_{ji}}\right)\right]. \tag{6.8}$$

6.4. Integration by parts

Step 6: Integration by parts, Eq. (2.11), is utilized for the third integral term in Eq. (6.3) and for the first term of π' in Eq. (6.6) yielding $\dot{\hat{I}}$ as follows:

$$\dot{\hat{I}}=\int_{V-\sigma}\left\{(\sigma_{ij,j}+b_i)u_i^{*\prime}+\left(\sigma_{ij,j}^*+\frac{\partial f}{\partial u_i}\right)u_i'+u_i^*b_i'-u_{i,j}\left[\int_{V-\sigma}\alpha'\bar{G}_{ij}^*d\bar{V}\right]\right\}dV \tag{6.9}$$

$$+\int_{S-\Gamma}\left\{[f-\sigma_{ij}u_{i,j}^*-u_{i,j}\sigma_{ij}^*+b_iu_i^*+(g+t_iu_i^*)_{,n}+H_S(g+t_iu_i^*)]V_n\right.$$

$$\left.+\left(\frac{\partial g}{\partial u_i}-t_i^*\right)u_i'+\left(\frac{\partial g}{\partial t_i}+u_i^*\right)t_i'\right\}dS$$

(6.9) [cont.]
$$+\int_{\sigma}\left[\!\left[[f-\sigma_{ij}u^*_{i,j}-u_{i,j}\sigma^*_{ij}+b_iu^*_i+(h+t_iu^*_i)_{,m}+H_\sigma(h+t_iu^*_i)]V_m + \left(\frac{\partial h}{\partial u_i}-t^*_i\right)u'_i+\left(\frac{\partial h}{\partial t_i}+u^*_i\right)t'_i\right]\!\right]d\sigma+\oint_{\Gamma}[\![g+t_iu^*_i]\!]V_\mu d\Gamma + \sum_k\int_{\Gamma_k}[(g+t_iu^*_i)^+V_{\mu^+}+(g+t_iu^*_i)^-V_{\mu^-}]d\Gamma_k,$$

where $t^*_i = \sigma^*_{ij}n_j$, denoting the adjoint tractions.

6.5. Adjoint problem

Step 7 and 8: The MD form of the boundary conditions, Eqs. (5.3) and (5.4), and of the jump conditions, Eqs. (5.8) and (5.9), may be inserted into Eq. (6.9). On the discontinuity surface σ, proper jump conditions for the adjoint displacements and tractions are also imposed.

The coefficients of the local variations of the primary variables (i.e., u'_i and t'_i) are equated to zero, hence defining the adjoint problem corresponding to I as given below:

(6.10) $$\text{in } V-\sigma:\quad \sigma^*_{ij,j} = -\frac{\partial f}{\partial u_i},$$

(6.11) $$\text{on } S_u:\quad u^*_i = -\frac{\partial g}{\partial t_i},$$

(6.12) $$\text{on } S_t:\quad t^*_i = \frac{\partial g}{\partial u_i},$$

(6.13) $$\text{on } \sigma:\quad [\![u^*_i]\!] = -\left[\!\left[\frac{\partial h}{\partial t_i}\right]\!\right],$$

(6.14) $$\text{on } \sigma:\quad [\![t^*_i]\!] = \left[\!\left[\frac{\partial h}{\partial u_i}\right]\!\right].$$

Substituting the adjoint constitutive equations (6.7) into the above equations, the adjoint problem may also be given in expanded form for comparison purposes as

(6.15) $$\text{in } V-\sigma:\quad \int_{V-\sigma}\alpha\bar{G}^*_{ij,j}d\bar{V}-\int_{S-\Gamma}\alpha\bar{G}^*_{ij}\bar{n}_jd\bar{S}-\int_{\sigma}\alpha[\![G^*_{ij}]\!]\bar{m}_jd\bar{\sigma} = -\frac{\partial f}{\partial u_i},$$

(6.16) $$\text{on } S_u:\quad u^*_i = -\frac{\partial g}{\partial t_i},$$

(6.17) $$\text{on } S_t:\quad t^*_i = \frac{\partial g}{\partial u_i},$$

(6.18) $$\text{on } \sigma:\quad [\![u^*_i]\!] = -\left[\!\left[\frac{\partial h}{\partial t_i}\right]\!\right],$$

(6.19) $$\text{on } \sigma:\quad \left[\int_{V-\sigma}\alpha[\![\bar{G}^*_{ij}]\!]d\bar{V}\right]m_j = \left[\!\left[\frac{\partial h}{\partial u_i}\right]\!\right].$$

6.6. The MD of I

Step 9: Assuming that the primary and adjoint problems are satisfied for a current shape configuration of the structure, the MD (i.e., the total variation) of the general performance criterion I is finally given as follows:

$$(6.20)\quad \dot{I} = \int_{V-\sigma} \left\{ u_i^* b_i' - u_{i,j} \left[\int_{V-\sigma} \alpha' \overline{G}_{ij}^* d\overline{V} \right] \right\} dV$$

$$+ \int_{S-\Gamma} [f - \sigma_{ij} u_{i,j}^* - u_{i,j} \sigma_{ij}^* + b_i u_i^* + (g + t_i u_i^*)_{,n} + H_S(g + t_i u_i^*)] V_n dS$$

$$+ \int_{S_u} \left(\frac{\partial g}{\partial u_i} - t_i^* \right) (\dot{u}_i^0 - u_{i,k} V_k) dS + \int_{S_t} \left(\frac{\partial g}{\partial t_i} + u_i^* \right) (\dot{t}_i^0 - t_{i,k} V_k) dS$$

$$+ \int_{\sigma} \left[\!\left[f - \sigma_{ij} u_{i,j}^* - u_{i,j} \sigma_{ij}^* + b_i u_i^* + (h + t_i u_i^*)_{,m} \right.\right.$$

$$\left.\left. + H_\sigma (h + t_i u_i^*) - \left(\frac{\partial h}{\partial u_i} - t_i^* \right) u_{i,m} - \left(\frac{\partial h}{\partial t_i} + u_i^* \right) t_{i,m} \right]\!\right] V_m d\sigma + \oint_{\Gamma} [\![g + t_i u_i^*]\!] V_\mu d\Gamma$$

$$+ \sum_k \int_{\Gamma_k} [(g + t_i u_i^*)^+ V_{\mu^+} + (g + t_i u_i^*)^- V_{\mu^-}] d\Gamma_k .$$

It is noted that $\dot{I}$, as given by Eq. (6.20), is expressed solely in terms of boundary perturbations V_n, V_m and V_μ, requiring no assumptions on the distribution of V_i in the whole space. This fact serves as an example of the most important feature of the present AVM of SDSA regarding the efficiency of any numerical calculations. In the case of parameter-constrained variation of the boundaries, the boundary perturbations can also be expressed in terms of finite number of decision parameters, as has been done in the direct method of SDSA.

7. Concluding remarks

A few conclusive remarks regarding the SDSA procedures for nonlocal elastic solids are due at this point and are outlined in the following:

1) In the direct method of SDSA, the (local) variations of the primary field variables are evaluated explicitly by means of the solution of auxiliary problems. In the case of the AVM, however, they are not evaluated but eliminated through the introduction of adjoint variables satisfying the adjoint problem [13].

2) An auxiliary problem is defined for each decision parameter, while adjoint problems are constructed corresponding to each of the integral functionals present in a specific situation.

3) The local variations b_i' and α', and the total variations $\dot{u}_i^0$ and $\dot{t}_i^0$ are known due to assumed forms of the body forces, influence function and boundary conditions, respec-

tively. In particular, when the boundary conditions are conservative and do not depend on the surface configuration, their total variations are given only by "convective" derivatives [10].

4) Any geometric singularities of the interface surface σ can also be attended for by following a similar procedure as for the boundary surface S.

5) The SDSA procedures presented in the paper should prove to be useful for physical problems described by integral equations, as well as integro-differential equations.

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