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Leakage problem for ideal incompressible fluid motion in bounded domain with nonsmooth boundary and vorticity prescribed in inflow

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AN IDEAL incompressible fluid motion in a bounded domain in R^3 described by the Euler equations is considered. A boundary built up of one part is divided into three submanifolds S_0, S_1, S_2, between each two of them there is a dihedral angle. The fluid enters the domain through S_1, leaves it through S_2 and moves tangentially to S_0. Initial and boundary data (vorticity on S_1, normal component of velocity on all boundaries) are prescribed such that the obtained problem is well posed. The existence and uniqueness of local solutions in Hölder spaces is proved, so the equations and data are satisfied classically. Upper bounds of magnitudes of dihedral angles are found which follow from the theory of elliptic boundary problems in domains with edges.

W pracy jest rozważany ruch idealnego nieściśliwego płynu w obszarze ograniczonym w R^3 opisanym przez równania Eulera. Granica, złożona z jednej części, jest podzielona na trzy podrozmaitości S_0, S_1, S_2, tak, że między każdymi dwoma z nich jest kąt dwuścienny. Płyn wpływa do obszaru przez S_1, opuszcza go przez S_2 i płynie stycznie do S_0. Przyjęte są dane początkowe i brzegowe (wirowość na S_1, normalna składowa prędkości na całej granicy) takie, że otrzymany problem jest dobrze postawiony. Udowodniono istnienie i jednoznaczność lokalnych rozwiązań w przestrzeniach Höldera, równania i dane są więc spełnione klasycznie. Znaleziono górne ograniczenie na wielkość kątów dwuściennych, które wynika z teorii zagadnień brzegowych dla równań eliptycznych w obszarach z krawędziami.

В работе рассматривается движение идеальной несжимаемой жидкости в ограниченной области в R^3, описанное уравнениями Эйлера. Граница, состоящая из одной части, разделена на три подмногообразия S_0, S_1, S_2, так, чтобы между двумя каждыми из них имелся двугранный угол. Жидкость втекает в область через S_1, покидает ее через S_2 и течет касательно к S_0. Заданы начальные и краевые данные (завихренность на S_1, нормальная составляющая скорости на целой границы), такие, что задача является корректно поставленной. Доказано существование и единственность локальных решений в гёльдеровых пространствах, значит уравнения и данные удовлетворены классически. Найдено верхнее ограничение на величину двугранных углов, которое вытекает из теории краевых задач для эллиптических уравнений в областях с гранями.

1. Introduction

IN THIS PAPER we consider the following leakage problem in a bounded doubly connected domain $\Omega \subset \mathsf{R}^3$:

$$v_t + v\cdot\nabla v + \nabla p = f, \tag{1.1}$$

$$\operatorname{div} v = 0, \tag{1.2}$$

$$v|_{t=0} = a(x), \quad \operatorname{div} a = 0, \tag{1.3}$$

$$v_n|_{\partial\Omega} = b(s,t), \quad \int\limits_{\partial\Omega} b(s,t)\,ds = 0, \tag{1.4}$$

(1.5) $$\omega|_{S_1} = \chi,$$

where v—velocity, p—pressure, ω—vorticity, f—exterior forces and $v_n = v\cdot\bar{n}$, where $\bar{n}$ is the unit outward vector normal to the boundary.

The boundary of Ω consists of three parts: $\partial\Omega = S_0\cup S_1\cup S_2$ such that the condition (1.4) can be written in the form

(1.6) $$v_n|_{S_0} = 0, \quad v_n|_{S_1} = b_1 < 0, \quad v_n|_{S_2} = b_2 > 0$$

and

(1.7) $$-\int\limits_{S_1} b_1\,ds = \int\limits_{S_2} b_2\,ds.$$

We assume that the domain Ω has two edges $L_i = \bar{S}_i\cap\bar{S}_0$, $i = 1, 2$, and $\bar{S}_1\cap\bar{S}_2 = \emptyset$. Therefore there exist dihedral angles between S_i and S_0, $i = 1, 2$. For $x\in L_i$, the magnitude of the dihedral angle is equal to $\vartheta_i(x)$, $i = 1, 2$. Moreover, $\vartheta_0 = \max\limits_{i=1,2}\max\limits_{x\in L_i}\vartheta_i(x)$.

Our aim is to prove the existence and uniqueness of solutions of the problem (1.1)–(1.5) in the domain Ω with edges and to obtain a restriction on the maximal dihedral angle ϑ_0.

Generally the problem (1.1)–(1.5) hasn't any solution until a certain restriction on χ is not imposed. In Sect. 2 we shall obtain this restriction.

The paper is divided in the following way. In Sect. 2 we shall formulate the problem (1.1)–(1.5) in the form of a system of three well-posed problems. In Sects. 3 and 4 some notations and auxiliary results are formulated. In Sect. 5, using Schauder's fixed point theorem, the existence of solutions of the problem (1.1)–(1.5) is proved. The method was used in [2].

2. Statement of the problem

At first we introduce some notations.

Let $S_\nu\in C^2$, $\nu = 0, 1, 2$, $L_i\in C^2$, $i = 1, 2$. Then by T_xS_ν, $x\in S_\nu$, $\nu = 0, 1, 2$, and by T_xL_i, $x\in L_i$, $i = 1, 2$ we denote the tangent spaces.

In a neighbourhood of S_1 we introduce a curvilinear system of orthonormal coordinates $\tau_1(x)$, $\tau_2(x)$, $n(x)$ such that S_1 is determined by $n(x) = 0$ and $\tau_1(x)$, $\tau_2(x)$, $x\in S_1$ are tangent coordinates on S_1. Moreover, by $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$, $\bar{n}(x)$ we denote the orthonormal basis corresponding to the coordinate system such that for $x\in S_1$, $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$ are vectors tangent to S_1 and $\bar{n}(x)$ is the outward unit vector normal to S_1.

At last in a neighbourhood of L_1 we introduce an orthonormal basis $\bar{\sigma}(x)$, $\bar{\varkappa}(x)$ such that $\bar{\varkappa}(x)$ is a vector tangent to L_1 and $\bar{\sigma}(x)$ is the unit outward vector normal to L_1. The vectors $\bar{\sigma}(x)$, $\bar{\varkappa}(x)$, $x\in S_1$ belong to T_xS_1.

Moreover, we introduce the Lamé's coefficients determined by $\partial/\partial\tau_i = H_i\bar{\tau}_i\cdot\nabla$, $i = 1, 2$, $\partial/\partial n = H_n\bar{n}\cdot\nabla$.

Now we shall obtain the restriction on χ mentioned in Section 1. To show this we repeat shortly the considerations from [3, 7]. Hence to prove the existence and uniqueness of solutions of the problem (1.1)–(1.5), we replace it by a system of problems

$$\text{(A)}\qquad \begin{aligned} \text{rot} v &= \omega, \\ \text{div} v &= 0, \\ v_n|_{\partial\Omega} &= b, \end{aligned}$$

and

$$\text{(B)}\qquad \begin{aligned} \omega_t + v\cdot\nabla\omega - \omega\cdot\nabla v &= F \equiv \text{rot} f, \\ \omega|_{t=0} &= \omega_0 \equiv \text{rot} a, \\ \omega|_{S_1} &= \chi \equiv \eta + \pi\bar{n}, \end{aligned}$$

where $\eta \in TS_1$. Now we would like to use the method of iteration. To do this we must know that a solution of the problem (B) satisfies

$$\text{(2.1)}\qquad \text{div}\omega = 0,$$

because (2.1) is the solvability condition for the problem (A). Introducing the curves $y = y(x, t; s)$ determined by

$$\text{(2.2)}\qquad \begin{aligned} \frac{dy(x, t; s)}{ds} &= v\big(y(x, t; s), s\big), \\ y(x, t; t) &= x, \end{aligned}$$

from the condition (1.6) we see that there are two kinds of curves such that

(a) $y(x, t; s) \in \Omega$ for $\forall s \in [0, t]$:

(b) there exists a moment $t_*(x, t)$ such that $y(x, t; t_*(x, t)) \in S_1$.

Using the curves (2.2), the equations (B_1) have the form of ordinary differential equations:

$$\text{(2.3)}\qquad \frac{d\omega\big(y(x, t; s), s\big)}{ds} - \omega^k\big(y(x, t; s), s\big)v_{,y^k}\big(y(x, t; s), s\big) = F\big(y(x, t; s), s\big),$$

where the conditions $(\text{B})_2$ and $(\text{B})_3$ play the role of initial conditions on curves of the set (a) and (b), respectively. Therefore, on curves from (a) the condition $(\text{B})_2$ implies (2.1). But to satisfy the condition (2.1) on curves from (b), we have to impose the following restriction on χ:

$$\text{(2.4)}\qquad \text{div}\omega|_{S_1} = 0.$$

Using the relations (1.2) and $(\text{B})_1$, the unknown quantity $\omega_{n,n}|_{S_1}$ is eliminated, so instead of the restriction (2.4) we obtain

$$\text{(2.5)}\qquad \pi_t + \sum_{\mu=1}^{2}\left[\frac{1}{H_\mu} v_\mu \pi_{,\tau_\mu} + \pi\left(\frac{1}{H_\mu} v_{\mu,\tau_\mu} + v_\mu \text{rot}_n(\bar{n}\times\bar{\tau}_\mu)\right)\right] = \varphi(x', t), \quad x' \in S_1,$$

where

$$\eta_\mu = \eta\cdot\bar{\tau}_\mu, \quad F_n = F\cdot\bar{n}, \quad v_\mu = v\cdot\bar{\tau}_\mu$$

and

$$\text{(2.6)}\qquad \varphi(x', t) = \sum_{\mu=1}^{2}\left[\frac{1}{H_\mu}(\eta_\mu b_1)_{,\tau_\mu} + b_1\eta_\mu \text{rot}_n(\bar{n}\times\bar{\tau}_\mu)\right] + F_n.$$

Therefore, for $\pi = 0$ the condition (2.4) implies that

$$\varphi(x, t) = 0, \tag{2.7}$$

which is the restriction on the tangent to S_1 components η of χ. In this case the problem (A, B) was considered in [6] for a doubly connected domain with a smooth boundary. For arbitrary tangent to S_1 components of χ, the problem (A, B) was formulated without any proof of existence in [3] for a non-doubly connected domain with a smooth boundary. The well-posedness of the last problem in the case of doubly connected domains with both a smooth and non-smooth boundary was analyzed in [7, chapt. 3].

From the above considerations we see that to prove the existence of solutions of the problem (1.1)–(1.5), it is necessary to consider additionally the problem for π besides the problems (A) and (B) for v and ω, respectively. Therefore, to formulate the problem on π we have to get initial and boundary conditions for Eq. (2.5). From the relations (1.3) and $(B)_2$ as the initial condition we have

$$\pi|_{t=0} = \omega_0 \cdot \bar{n}|_{S_1}. \tag{2.8}$$

To determine the boundary condition for Eq. (2.5) we consider the characteristic curves for this equation determined by the vector field $v^{\tau} = \sum_{\mu=1}^{2} v_\mu \tau_\mu|_{S_1}$ qiven by the following equations:

$$\frac{d\delta_\mu}{ds} = \frac{v_\mu(\delta_1(\tau, t; s), \delta_2(\tau, t; s), s)}{H_\mu}, \qquad \delta_\mu(\tau, t; t) = \tau_\mu, \tag{2.9}$$

where $\tau = (\tau_1, \tau_2)$, $\mu = 1, 2$, $0 \leqslant s \leqslant t$. Using the curves (2.9), Eq. (2.5) can be written in the form

$$\frac{d\pi}{ds} + \sum_{\mu=1}^{2} \left[\frac{1}{H_\mu} v_{\mu,\tau_\mu} + v_\mu \operatorname{rot}_n(\bar{n} \times \bar{\tau}_\mu)\right]\pi = \varphi(x', t), \qquad x' \in S_1. \tag{2.10}$$

Comparing Eq. (2.9) with Eq. (2.2), we see that in this case there are also two kinds of curves determined by Eq. (2.9):

(a′) $\delta(\tau, t; s) \in S_1$ for $\forall s \in [0, t]$,

(b′) there exists a moment $\tau_*(\tau, t)$ such that $\delta(\tau, t; \tau_*(\tau, t)) \in L_1$.

If we have only the curves (a′), the problem (2.10), (2.8) is well posed, but if the curves (b′) appear, it is not. For the second case we must prescribe additionally

$$\pi|_{L_1} = \varrho(x'', t), \qquad x'' \in L_1. \tag{2.11}$$

For domains with edges we have the following relation:

$$-v(x) \cdot \bar{\sigma}(x) = d(x) \operatorname{ctg} \vartheta_1(x), \qquad x \in L_1, \tag{2.12}$$

where $d(x) = -b_1(x) > 0$. For $\vartheta_1(x) \geqslant \frac{\pi}{2} : v \cdot \bar{\sigma}|_{L_1} \geqslant 0$ and the curves (a′) can appear only, but for $\vartheta_1(x) < \frac{\pi}{2}$ we must consider the curves (a′) and (b′).

Therefore for $\vartheta_1(x) < \frac{\pi}{2}$ we shall consider the problem (2.5), (2.8), (2.11) and for $\vartheta_1(x) \geqslant \frac{\pi}{2}$ the problem (2.5), (2.8). These problems will be denoted by (C).

At last we formulate

LEMMA 2.1

Let v, ω, π be a solution of the problem (A, B, C). Then v, p is a solution of the problem (1.1)–(1.4) with rot $v|_{S_1}$ prescribed by Eqs. $(B)_3$ and (2.11), where p is determined by the following elliptic problem:

(2.13)
$$\Delta p = \operatorname{div} f - v^i_{x^k} v^k_{x^i},$$
$$\frac{1}{H_n}\frac{\partial p}{\partial n}\bigg|_{S_\nu} = (f_n - v\cdot\nabla v\cdot\bar{n} - b_t)|_{S_\nu}, \quad \nu = 0, 1, 2.$$

3. Notations

We introduce some notations concerning the Hölder spaces. By $C(\Omega)$, $C(\Omega^t)$ we denote spaces of continuous functions with the norms

$$||u||_{C(\Omega)} = \sup_{\Omega}|u| = |u|_\Omega, \quad ||u||_{C(\Omega^t)} = \sup_{\Omega}|u| = |u|_{\Omega^t}, \quad \text{where} \quad \Omega^t = \Omega\times[0, t].$$

By $C^k(\Omega)$ and $C^{k+\alpha}(\Omega)$ we denote spaces with the norms

$$||u||_{C^k(\Omega)} = \sum_{|\alpha|\leqslant k}\sup_{\Omega}|D^\alpha u| \equiv |u|_{k,\Omega}, \quad ||u||_{C^{k+\alpha}(\Omega)} = |u|_{k,\Omega} + \langle D^k u\rangle_{\alpha,\Omega} \equiv |u|_{k+\alpha,\Omega},$$

where

$$\langle u\rangle_{\alpha,\Omega} = \sup_{x',x''\in\Omega}\frac{|u(x')-u(x'')|}{|x'-x''|^\alpha}, \quad \alpha\in(0,1).$$

To consider the time-dependent problems, we have to introduce the following spaces:

$$||u||_{C^{k+\alpha,l+\alpha}(\Omega^t)} \equiv |u|_{k,l,\alpha,\Omega^t} = \sum_{\substack{i\leqslant k\\ j\leqslant l}}|D^i_x D^j_t u|_{\Omega^t} + \langle D^k_x u\rangle_{\alpha,x,\Omega^t}$$
$$+\langle D^k_x u\rangle_{\alpha,t,\Omega^t} + \langle D^l_t u\rangle_{\alpha,x,\Omega^t} + \langle D^l_t u\rangle_{\alpha,t,\Omega^t},$$

where k, l—positive integers and

$$\langle u\rangle_{\alpha,t,\Omega^t} = \sup_{\Omega}\sup_{t',t''\in[0,T]}\frac{|u(x,t')-u(x,t'')|}{|t'-t''|^\alpha},$$

$$\langle u\rangle_{\alpha,x,\Omega^t} = \sup_{t\in[0,T]}\sup_{x',x''\in\Omega}\frac{|u(x'\,t)-u(x''\,t)|}{|x'-x''|^\alpha}.$$

We shall denote $C^{\alpha,\alpha}(\Omega^t) = C^\alpha(\Omega^t)$ and $||u||_{C^\alpha(\Omega^t)} = |u|_{\alpha,\Omega^t}$. Moreover, we introduce the space $C(0, t; C^{k+\alpha}(\Omega))$ with the norm

$$||u||_{C(0,t;C^{k+\alpha}(\Omega))} \equiv |u|_{k,\alpha,\Omega^t} = \sum_{i\leqslant k}|D^i_x u|_{\Omega^t} + \langle D^k_x u\rangle_{\alpha,x,\Omega^t}.$$

4. Auxiliary results

At first we recall some properties of the solutions of (2.2). Integrating the Eq. (2.2), we obtain

(4.1) $$y(x, t; s) = x + \int_t^s v(y(x, t; \tau), \tau) d\tau.$$

Moreover, we have

(4.2) $$\frac{d}{ds} y^i_{x^k} = v^i_{y^j} y^j_{x^k},$$

where the summation convention is used. Introducing $J(x, t; s) = \det\left(\frac{\partial y^i(x, t; s)}{\partial x^j}\right)$, we have $J(x, t; s) = \exp \int_t^s \frac{\partial v^k(y(x, t; \tau), \tau)}{\partial y^k} d\tau = 1$, because $\operatorname{div} v = 0$. Therefore, there exists an inverse transformation to Eq. (4.1)

(4.3) $$x(y, t; \tau) = y - \int_t^\tau v(y(x, t; s), s) ds$$

such that

(4.4) $$\frac{d}{d\tau} x^i_{y^k} = -v^i_{x^j} x^j_{y^k}.$$

From [1, Chapt. 4, § 4 p.3] we have

LEMMA 4.1

Let $v \in C^{1+\alpha, \alpha}(\Omega^T)$, $\alpha \in (0, 1)$, then the following estimates are valid:

(4.5) $$|y_x(x, t; s)| \leqslant c e^{t|v_x|_{\Omega^t}}, \quad |y_t(x, t; s)| \leqslant c|v|_{\Omega^t} e^{t|v_x|_{\Omega^t}},$$
$$\forall x, t \in \Omega^T, s \in [\bar{t}(x, t), t].$$

$\bar{t}(x, t)$ is defined in Lemma 4.3.

(4.6) $$\langle y_x(x, t; s)\rangle_{\alpha, x} \leqslant c e^{(2+\alpha)t|v_x|_{\Omega^t}} t \langle v_x\rangle_{\alpha, x, \Omega^t}, \quad \forall x, t \in \Omega^T,$$
$$\langle y_x(x, t; s)\rangle_{\alpha, t} \leqslant c g_1(|v|_{1, 0, \alpha, \Omega^t}, t) e^{ct|v_x|_{\Omega^t}} t^{1-\alpha}, \quad \forall x, t \in \Omega^T,$$

where

$$\langle f(x, t)\rangle_{\alpha, x} = \frac{|f(x', t) - f(x'', t)|}{|x' - x''|^\alpha}, \quad \langle f(x, t)\rangle_{\alpha, t} = \frac{|f(x, t') - f(x, t'')|}{|t' - t''|^\alpha}$$

and $0 < g_1(x, t) \sim x$ as $x \to 0$.

From [1, Chapt. 4 § 4, p. 3] we also have

LEMMA 4.2

Let $v \in C^{1+\alpha, \alpha}(\Omega^T)$, $\alpha \in (0, 1)$ such that $|v_n|_{S_1}| \geqslant C_0 > 0$, and S_1 be of class C^1 described by $\Phi(x) = 0$. Then the following estimates are valid:

(4.7) $$\langle y_x(x, t; t_*(x, t))\rangle_{\alpha, x} \leqslant c e^{ct|v_x|_{\Omega^t}} (1+t)|v|_{1, 0, \alpha, \Omega^t} t^{1-\alpha},$$

(4.8) $$\langle y_x(x, t: t_*(x, t))\rangle_{\alpha, t} \leqslant c g_2(|v|_{1, 0, \alpha, \Omega^t}, t) t^{1-\alpha} e^{ct|v_x|_{\Omega^t}},$$

where $0 < g_2(x, t) \sim x$ as $x \to 0$.

Proof. From the definition of $t_*(x, t)$ we have that $\Phi(y(x, t; t_*(x, t))) = 0$. Therefore $\Phi_{y^k}(y^k_{x^l}+v^k t_{*,x^l}) = 0$ and $\Phi_{y^k}(y^k_t+v^k t_{*,t}) = 0$.

Hence we get

$$(4.9) \qquad t_{*,x^l} = -\frac{\Phi_{y^k} y^k_{x^l}}{v^k \Phi_{y^k}}, \qquad t_{*,t} = -\frac{\Phi_{y^k} y^k_t}{v^k \Phi_{y^k}}.$$

Assuming that $t_*(x', t) \leqslant t_*(x'', t)$, we have

$$(4.10) \qquad \langle y_x(x, t; t_*(x, t))\rangle_{\alpha, x} \leqslant \frac{|y_x(x', t; t_*(x', t))-y_x(x', t; t_*(x'', t))|}{|x'-x''|^\alpha} + \frac{|y_x(x', t; t_*(x'', t))-y_x(x'', t; t_*(x'', t))|}{|x'-x''|^\alpha}$$
$$\leqslant \sup_{x', x'', t} [\sup_\sigma |v_y(y(x', t; \sigma), \sigma) y_x(x', t; \sigma)|_{\sigma\in[t_*(x', t), t_*(x'', t)]}| \cdot t^{1-\alpha}|t_{*,x}|^\alpha] + cte^{ct|v_x|_{\Omega^t}}\langle v\rangle_{\alpha, x, \Omega^t},$$

where we used the inequality (4.6) and $|t_1-t_2| = |t_1-t_2|^\alpha |t_1-t_2|^{1-\alpha} \leqslant ct^{1-\alpha}|t_{*,x}|^\alpha|x'-x''|^\alpha$ where $t_1 = t_*(x', t)$, $t_2 = t_*(x'', t)$. Using the relations (4.9) and (4.5) in the inequality (4.10) and the assumptions of the Lemma, we get the estimate (4.7). Assuming that $t_*(x, t') \leqslant t_*(x, t'')$, we get

$$(4.11) \qquad \langle y_x(x, t; t_*(x, t))\rangle_{\alpha, t} \leqslant \frac{|y_x(x, t'; t_*(x, t'))-y_x(x, t''; t_*(x, t'))|}{|t'-t''|^\alpha} + \frac{|y_x(x, t''; t_*(x, t'))-y_x(x, t''; t_*(x, t''))|}{|t'-t''|^\alpha} \leqslant cg_1(|v|_{1,0,\alpha,\Omega^t}, t)$$
$$\cdot e^{ct|v_x|_{\Omega^t}}t^{1-\alpha}+\sup_x \sup_{t', t''}|t_{*,t}|^\alpha t^{1-\alpha}\cdot \sup_\sigma |v_y(y(x, t''; \sigma), \sigma) y_x(x, t''; \sigma)|_{\sigma\in[t_*(x, t'), t_*(x, t'')]}|.$$

Using the relations (4.5) and (4.9) in the inequality (4.11) and the assumptions of the lemma, we get the inequality (4.8). This concludes the proof.

LEMMA 4.3

Let $v \in C^{1+\alpha,\alpha}(\Omega^t)$, $\eta \in C^\alpha(S_1^t)$, $\pi \in C^\alpha(S_1^t)$, $\omega_0 \in C^\alpha(\Omega)$, $S_1 \in C^1$, $\alpha \in (0, 1)$, $t \leqslant T$ and $(B)_{2,3}$ be satisfied. Then the following estimate is valid:

$$(4.12) \qquad \langle \omega(y(x, t; \bar{t}(x, t)), \bar{t}(x, t))\rangle_{\alpha, x}+\langle \omega(y(x, t; \bar{t}(x, t)), \bar{t}(x, t))\rangle_{\alpha, t}$$
$$\leqslant c[g_3(|v|_{1,0,\alpha,\Omega^t}, t)t^{1-\alpha}+1]e^{ct|v_x|_{\Omega^t}}(|\omega_0|_{\alpha,\Omega}+|\eta|_{\alpha, S_1^t}+|\pi|_{\alpha, S_1^t}+t^{1-\alpha}|F|_{\alpha,\Omega^t})$$

for $\forall t \leqslant T$ and $0 < g_3(x, t) \sim x$ as $x \to 0$, where $\bar{t}(x, t) = 0$ for curves (a) and $\bar{t}(x, t) = t_*(x, t)$ for curves (b).

Proof. Let $\bar{t}(x', t) \leqslant \bar{t}(x'', t)$, then

$$(4.13) \qquad \langle \omega(y(x, t; \bar{t}(x, t)), \bar{t}(x, t))\rangle_{\alpha, x}$$
$$\leqslant \frac{|\omega(y(x', t; \bar{t}(x', t)), \bar{t}(x', t))-\omega(y(x'', t; \bar{t}(x'', t)), \bar{t}(x', t))|}{|x'-x''|^\alpha} + \frac{|\omega(y(x', t; \bar{t}(x'', t)), \bar{t}(x', t))-\omega(y(x'', t; \bar{t}(x'', t)), \bar{t}(x'', t))|}{|x'-x''|^\alpha}$$
$$\leqslant (\langle \omega(x, t)\rangle_{\alpha, x, S_1^t}+\langle \omega_0(x)\rangle_{\alpha, x, \Omega})\langle y(x, t; \bar{t}(x, t))\rangle_{\alpha, x}+\langle \omega(x, t)\rangle_{\alpha, t, S_1^t}|t_{*,x}|^\alpha.$$

Let $\bar{t}(x,t') \leqslant \bar{t}(x,t'')$, then

$$(4.14)\quad \langle\omega\big(y(x,t;\bar{t}(x,t)),\bar{t}(x,t)\big)\rangle_{\alpha,t} \leqslant \frac{|\omega\big(y(x,t';\bar{t}(x,t')),\bar{t}(x,t')\big)-\omega\big(y(x,t'',\bar{t}(x,t'')),\bar{t}(x,t')\big)|}{|t'-t''|^\alpha} + \frac{|\omega\big(y(x,t'';\bar{t}(x,t'')),\bar{t}(x,t')\big)-\omega\big(y(x,t'',\bar{t}(x,t'')),\bar{t}(x,t'')\big)|}{|t'-t''|^\alpha}$$

$$\leqslant \big(\langle\omega(x,t)\rangle_{\alpha,x,S_1^t}+\langle\omega_0(x)\rangle_{\alpha,x,\Omega}\big)\langle y(x,t;\bar{t}(x,t))\rangle_{\alpha,t}+\langle\omega(x,t)\rangle_{\alpha,t,S_1^t}|t_{*,t}|^\alpha.$$

From the relations (4.13) and (4.14) we have the relation (4.12). This concludes the proof.

From Eq. (2.9) we have

$$(4.15)\qquad \delta_\mu(\tau,t,\sigma) = \tau_\mu+\int_t^\sigma v_\mu\big(\delta(\tau,t;s),s\big)ds$$

and

$$(4.16)\qquad \frac{d\delta_{\mu,\tau_\nu}}{ds} = v_{\mu,\delta_\varrho}\delta_{\varrho,\tau_\nu},\qquad \mu,\nu,\varrho = 1,2$$

and the summation convention is used. We formulate

LEMMA 4.4

Let $v\in C^{1+\alpha,\alpha}(\Omega)^T$, $\alpha\in(0,1)$, then the following estimates are valid for $t\leqslant T$:

$$(4.17)\qquad |\delta_{,\tau}(\tau,t;s)| \leqslant ce^{t|v_x|_{\Omega^t}},\qquad |\delta_{,t}(\tau,t;s)| \leqslant c|v|_{\Omega^t}e^{t|v_x|_{\Omega^t}},$$

$$(4.18)\qquad \begin{aligned}&\langle\delta_{,\tau}(\tau,t;s)\rangle_{\alpha,\tau} \leqslant ce^{ct|v_x|_{\Omega^t}}t\langle v_x\rangle_{\alpha,x,\Omega^t},\\ &\langle\delta_{,\tau}(\tau,t;s)\rangle_{\alpha,t} \leqslant cg_1(|v|_{1,0,\alpha,\Omega^t},t)e^{ct|v_x|_{\Omega^t}}.\end{aligned}$$

P r o o f. The proof is the same as the proof of Lemma 4.1.

LEMMA 4.5

Let $v\in C^{1+\alpha,\alpha}(\Omega^T)$, $\alpha\in(0,1)$, such that $|v\cdot\sigma|_{L_1}|\geqslant c_0>0$ and L_1 be a curve of class C^1 described by $\psi(x)=0$. Then the following estimates are valid:

$$(4.19)\qquad \langle\delta_{,\tau}\big(\tau,t;\tau_*(\tau,t)\big)\rangle_{\alpha,x} \leqslant cg_4(|v|_{1,0,\alpha,\Omega^t},t)t^{1-\alpha}e^{ct|v_x|_{\Omega^t}},$$

$$(4.20)\qquad \langle\delta_{,\tau}\big(\tau,t;\tau_*(\tau,t)\big)\rangle_{\alpha,t} \leqslant cg_5(|v|_{1,0,\alpha,\Omega^t},t)t^{1-\alpha}e^{ct|v_x|_{\Omega^t}},$$

where $g_i(x,t)\sim x$ as $x\to 0$, $i=4,5$.

From the definition of $\tau_*(\tau,t)$ we have $\psi(\delta(\tau,t;\tau_*(\tau,t)))=0$, so

$$(4.21)\qquad \tau_{*,\tau} = -\frac{\psi_{,\delta_\mu}\delta_{\mu,\tau}}{v_\mu\psi_{,\delta_\mu}},\qquad \tau_{*,t} = -\frac{\psi_{,\delta_\mu}\delta_{\mu,t}}{v_\mu\psi_{,\delta_\mu}}.$$

The remaining part of the proof is the same as the corresponding one in the proof of Lemma 4.2. This concludes the proof.

LEMMA 4.6

Let $v\in C^{1+\alpha,\alpha}(\Omega^t)$, $\omega_0\in C^\alpha(\Omega)$, $\varrho\in C^\alpha(L_1^t)$, $L_1\in C^1$, $\alpha\in(0,1)$, $t\leqslant T$ and the relations (2.8) and (2.11) be satisfied. Then the following estimate holds:

$$(4.22)\quad \langle\pi\big(\delta(\tau,t;\bar{\tau}(\tau,t)),\bar{\tau}(\tau,t)\big)\rangle_{\alpha,\tau}+\langle\pi\big(\delta(\tau,t;\bar{\tau}(\tau,t)),\bar{\tau}(\tau,t)\big)\rangle_{\alpha,t} \leqslant c[g_6(|v|_{1,0,\alpha,\Omega^t},t)t^{1-\alpha}+1]e^{ct|v_x|_{\Omega^t}}(|\omega_0|_{\alpha,\Omega}+|\varrho|_{\alpha,S_1^t}),$$

where $g_6(x, t) \sim x$ as $x \to 0$, $\bar{\tau}(\tau, t) = 0$ for the curves (a′) and $\bar{\tau}(\tau, t) = \tau_*(\tau, t)$ for the curves (b′).

Proof. The proof is the same as the proof of Lemma 4.3.

We recall from [4].

THEOREM 4.1

Let $\omega \in C^\alpha(\Omega^T)$, $b \in C^\alpha(S_i^T)$, $i = 1, 2$, $\alpha \in (0, 1)$, $S_\nu \in C^2$, $\nu = 0, 1, 2$ *and*

$$2+\alpha < \Lambda(\vartheta_0), \tag{4.23}$$

where $\Lambda(\vartheta_0) = \dfrac{\pi}{\vartheta_0} - 1$ *because* $\vartheta_0 < \pi$. *Then there exists a unique solution of the problem* (A) *such that* $v \in C^{1+\alpha,\alpha}(\Omega^T)$ *and*

$$|v|_{1,0,\alpha,\Omega^T} \leqslant c(|\omega|_{\alpha,\Omega^T} + \sum_{i=1}^{2} |b|_{\alpha,S_i^T}). \tag{4.24}$$

For the dihedral angle equal to π/n, $n > 1$—natural, the condition (4.23) can be omitted but b satisfies the compatibility condition which is described in Theorem 3.2 of [5].

5. Existence of solutions of the problem (A, B, C)

At first we obtain an *a priori* estimate. Integrating Eq. (2.3) we have

$$\omega^k(x, t) = A_j^k\Big(y_x\big(x, t; \bar{t}(x, t)\big)\Big)\omega^j\Big(y\big(x, t; \bar{t}(x, t); \bar{t}(x, t)\big)\Big) + \int_{\bar{t}(x,t)}^{t} A_j^k(y_x(x, t; s))F^j\big(y(x, t; s), s\big)\,ds, \tag{5.1}$$

where $A_j^k = (\partial J)/(\partial y_{x^j}^k)$, $\bar{t}(x, t) = 0$ and $\bar{t}(x, t) = t_*(x, t)$ for curves of the family (a) and (b), respectively. Similarly integrating Eqs. (2.10), we get

$$\pi(\tau, t) = \Big\{\exp \int_{\bar{\tau}(\tau,t)}^{t} \sum_{\mu=1}^{2} \Big[H_\mu^{-1} v_{\mu,\delta_\mu}\big(\delta(\tau, t; \sigma), \sigma\big) + v_\mu\big(\delta(\tau, t; \sigma), \sigma\big) \cdot \operatorname{rot}_n(\bar{n}\times\bar{\tau})\Big]\,d\sigma\Big\}\Big[\int_{\bar{\tau}(\tau,t)}^{t} \varphi\big(\delta(\tau, t; \sigma), \sigma\big)d\sigma + \pi\Big(\delta\big(\tau, t; \bar{\tau}(\tau, t)\big), \bar{\tau}(\tau, t)\Big)\Big], \tag{5.2}$$

where $\bar{\tau}(\tau, t) = 0$ and $\bar{\tau}(\tau, t) = \tau_*(\tau, t)$ for curves of the family (a′) and (b′), respectively.

Now we estimate $\omega(x, t)$.

LEMMA 5.1

Let $\eta \in C^\alpha(S_1^t)$, $\pi \in C^\alpha(S_1^t)$, $F \in C(0, t; C^\alpha(\Omega))$, $v \in C^{1+\alpha,\alpha}(\Omega^t)$, $\omega_0 \in C^\alpha(\Omega)$, $S_1 \in C^1$, then the following estimate

$$|\omega|_{\alpha,\Omega^t} \leqslant c(|\eta|_{\alpha,S_1^t} + |\pi|_{\alpha,S_1^t} + |\omega_0|_{\alpha,\Omega} + t|F|_{0,\alpha,\Omega^t}) \cdot e^{ct|v_x|_{\Omega^t}}[1 + t^{1-\alpha}G_1(|v|_{1,0,\alpha,\Omega^t}, t)], \tag{5.3}$$

is valid, where $G_1(x, t) \sim x$ as $x \to 0$.

Proof. From Eq. (5.1) we have

$$|\omega|_{\Omega_t} \leqslant c(|\eta|_{S_1^t} + |\pi|_{S_1^t} + |\omega_0|_\Omega + t|F|_{\Omega^t})e^{2t|v_x|_{\Omega^t}}. \tag{5.4}$$

Let $\bar{t}(x', t) \leqslant \bar{t}(x'', t)$. From Eq. (5.1) we also have

$$(5.5)\quad \langle \omega \rangle_{\alpha, x, \Omega^t} \leqslant c|y_x(x', t; \bar{t}(x', t))|_{\Omega^t}|\omega(y(x', t; \bar{t}(x', t)), \bar{t}(x', t))|_{\Omega^t} \cdot \langle y_x(x, t, t(x, t))\rangle_{\alpha, x, \Omega^t} + c|y_x|^2_{\Omega^t}\langle \omega(y(x, t; \bar{t}(x, t)), \bar{t}(x, t))\rangle_{\alpha, x, \Omega^t} + |x'-x''|^{-\alpha}\left|\int_{\bar{t}(x', t)}^{\bar{t}(x'', t)} A(y_x(x', t; s))F(y(x', t; s), s)ds\right| + c\int_{\bar{t}(x'', t)}^{t} |y_x(x', t; s)| \cdot |F(y(x', t; s), s)|\langle y_x(x, t; s)\rangle_{\alpha, x} ds + c\int_{\bar{t}(x'', t)}^{t} |y_x(x'', t; s)|^2 \langle F(y(x, t; s), s)\rangle_{\alpha, x, \Omega^t} ds.$$

Using Lemmas 4.1–4.3 we get

$$(5.6)\quad \langle \omega \rangle_{\alpha, x, \Omega^t} \leqslant c[|\omega_0|_{\alpha, \Omega} + |\eta|_{\alpha, S_1^t} + |\pi|_{\alpha, S_1^t} + t|F|_{0, \alpha, \Omega^t}] \cdot e^{ct|v_x|_{\Omega^t}}(1 + t^{1-\alpha}g_7(|v|_{1, 0, \alpha, \Omega^t}, t)),$$

where $g_7(x, t) \sim x$ as $x \to 0$. Now we shall estimate $\langle \omega \rangle_{\alpha, t, \Omega^t}$. From Eq. (5.1) we have

$$(5.7)\quad \langle \omega \rangle_{\alpha, t, \Omega^t} \leqslant ce^{ct|v_x|_{\Omega^t}}[(|\omega_0|_\Omega + |\eta|_{S_1^t} + |\pi|_{S_1^t}) \cdot \langle y_x(x, t; \bar{t}(x, t))\rangle_{\alpha, t, \Omega^t} + \langle \omega(y(x, t; \bar{t}(x, t)), \bar{t}(x, t))\rangle_{\alpha, t, \Omega^t} + |F|_{\Omega^t}t^{1-\alpha} + t^2\langle v_x\rangle_{\alpha, t, \Omega^t}|F|_{\Omega^t} + t\langle F\rangle_{\alpha, t, \Omega^t}].$$

Using Lemmas 4.1–4.3 in the relation (5.7) we obtain that $\langle \omega \rangle_{\alpha, t, \Omega^t}$ is estimated by the right-hand side of the relation (5.6). Therefore, we obtained the relation (5.3). This concludes the proof.

LEMMA 5.2

Let $v \in C^{1+\alpha, \alpha}(\Omega^t)$, $\varphi \in C^\alpha(S_1^t)$, $\omega_0 \in C^\alpha(\Omega)$, $\varrho \in C^\alpha(L_1^t)$, $L_1 \in C^1$, $S_1 \in C^2$, then the following estimate holds:

$$(5.8)\quad |\pi|_{\alpha, S_1^t} \leqslant ce^{ct|v|_{1, 0, \alpha, \Omega^t}}[t|\varphi|_{\alpha, S_1^t} + |\omega_0|_{\alpha, \Omega} + |\varrho|_{\alpha, S_1^t}] \cdot [1 + G_2(|v|_{1, 0, \alpha, \Omega^t}, t)t^{1-\alpha}],$$

where $G_2(x, t) \sim x$ as $x \to 0$.

P r o o f. From Eq. 5.2 we have

$$(5.9)\quad |\pi|_{S_1^t} \leqslant ce^{ct|v|_{1, 0, \alpha, \Omega^t}}[t|\varphi|_{S_1^t} + |\omega_0|_\Omega + |\varrho|_{L_1^t}].$$

Denoting the expression under exp in Eq. (5.2) by $K(\tau, t)$, from Eq. (5.2) we have

$$(5.10)\quad \langle \pi \rangle_{\alpha, \tau, S_1^t} \leqslant ce^{ct|v|_{1, 0, \alpha, \Omega^t}}\{\langle K(\tau, t)\rangle_{\alpha, \tau, S_1^t}[t|\varphi|_{S_1^t} + |\omega_0|_\Omega + |\varrho|_{L_1^t}] + |\varphi|_{S_1^t}\langle \bar{\tau}(\tau, t)\rangle_{\alpha, \tau, S_1^t} + t\sup_\sigma\langle \varphi(\delta(\tau, t; \sigma), \sigma)\rangle_{\alpha, \tau, S_1^t} + \langle \pi(\delta(\tau, t; \bar{\tau}(\tau, t)), \bar{\tau}(\tau, t))\rangle_{\alpha, \tau, S_1^t}.$$

From Lemmas 4.4–4.6 we have

$$(5.11)\quad \langle K(\tau, t)\rangle_{\alpha, \tau, S_1^t} \leqslant ce^{ct|v|_{1, 0, \alpha, \Omega^t}}g_8(|v|_{1, 0, \alpha, \Omega^t}, t)t^{1-\alpha},$$

where $g_8\ (x, t) \sim x$ as $x \to 0$.

$$(5.12)\quad \langle \bar{\tau}(\tau, t)\rangle_{\alpha, \tau, S_1^t} \leqslant ce^{ct|v|_{1, 0, \alpha, \Omega^t}}t^{1-\alpha},$$

$$(5.13)\quad \langle \varphi(\delta(\tau, t; \sigma), \sigma)\rangle_{\alpha, \tau, S_1^t} \leqslant c\langle \varphi\rangle_{\alpha, \tau, S_1^t}e^{ct|v|_{1, 0, \alpha, \Omega^t}}t^\alpha.$$

Moreover, the last term in the right-hand side of the relation (5.10) is estimated by $(\langle\omega_0\rangle_{\alpha,\Omega}+\langle\varrho\rangle_{\alpha,\varkappa,L_1^t})g_8(|v|_{1,0,\alpha,\Omega^t}, t)t^{1-\alpha}$, where $\varkappa$ is the parameter along L_1 connected with $\bar{\varkappa}$, which is the tangent vector to L_1.

Therefore we obtain

$$(5.14)\qquad \langle\pi\rangle_{\alpha,\tau,S_1^t} \leqslant ce^{ct|v|_{1,0,\alpha,\Omega^t}}[t|\varphi|_{\alpha,S_1^t}+|\omega_0|_{\alpha,\Omega}+|\varrho|_{\alpha,L_1^t}]\cdot g_9(|v|_{1,0,\alpha,\Omega^t}, t)t^{1-\alpha},$$

where $g_9(x, t) \sim x$ as $x \to 0$. Similarly we obtain

$$(5.15)\qquad \langle\pi\rangle_{\alpha,t,S_1^t} \leqslant ce^{ct|v|_{1,0,\alpha,\Omega^t}}[t|\varphi|_{\alpha,S_1^t}+|\omega_0|_{\alpha,\Omega}+|\varrho|_{\alpha,L_1^t}]\cdot G_4(|v|_{1,0,\alpha,\Omega^t}, t)t^{1-\alpha}.$$

From the relations (5.9), (5.14) and (5.15) we get the relation (5.8). This concludes the proof.

From Lemma 5.1 and 5.2 we have that

$$(5.16)\qquad |\omega|_{\alpha,\Omega^t} \leqslant ce^{ct|v|_{1,0,\alpha,\Omega^t}}[t|\varphi|_{\alpha,S_1^t}+|\omega_0|_{\alpha,\Omega}+|\varrho|_{\alpha,L_1^t} + |\eta|_{\alpha,S_1^t}+t|F|_{0,\alpha,\Omega^t}]\big(1+G(|v|_{1,0,\alpha,\Omega^t}, t)t^{1-\alpha}\big),$$

where $G(x, t) \sim x$ as $x \to 0$. Assuming that $\beta = c[t|\varphi|_{\alpha,S_1^t}+|\omega_0|_{\alpha,\Omega}+|\varrho|_{\alpha,L_1^t}+|\eta|_{\alpha,S_1^t}+ +t|F|_{0,\alpha,\Omega^t}]$ and using Theorem (4.1), we get

$$(5.17)\qquad |\omega|_{\alpha,\Omega^t} \leqslant e^{ct(|\omega|_{\alpha,\Omega^t}+|b|_{1,0,\alpha,\partial\Omega^t})}\beta[1+G(|\omega|_{\alpha,\Omega^t}+|b|_{1,0,\partial\Omega^t}, t)t^{1-\alpha}]$$

and

$$(5.18)\qquad \vartheta_0 < \frac{\pi}{3+\alpha}, \quad \alpha \in (0, 1).$$

From the relation (5.17) for $t \leqslant t_0$, where t_0 is sufficiently small, we obtain the estimate

$$(5.19)\qquad |\omega|_{\alpha,\Omega^t} \leqslant G_0(\beta, |b|_{1,0,\alpha,\partial\Omega^t}, t_0),$$

where $G_0(\beta, y, t_0) \sim \beta$ as $t_0 \to 0$.

Let $N \subset C^\alpha(\Omega_1^t)$ be a set determined by the inequality (5.19). The elliptic problem (B), (5.1) and (5.2) determine an operator $S: N \to N$, which is continuous in $C(\Omega^t)$. But $N \subset C(\Omega^t)$ is compact, therefore, by the Schauder theorem we have (similarly as in [2]:

THEOREM 5.1

Let $\omega_0 \in C^\alpha(\Omega)$, $\eta \in C^{1+\alpha,\alpha}(S_1^t)$, $\varrho \in C^\alpha(L_1^t)$, $b \in C^{1+\alpha,\alpha}(S_i^t)$, $F \in C(0, t; C^\alpha(\Omega))$, $S_i \in C^2$, $i = 0, 1, 2$, $L_1 \in C^1$, $\alpha \in (0, 1)$, $t \leqslant t_0$, *where* t_0 *is so small that the inequality* (5.19) *is satisfied. Let the maximal angle* ϑ_0 *be less than* $\pi/3$. *Then there exists a solution of the problem* (A, B, C) *such that* $\omega \in C^\alpha(\Omega^t)$, $v \in C^{1+\alpha,\alpha}(\Omega^t)$, $\pi \in C^\alpha(S_1^t)$.

Uniqueness can be proved in the standard way.

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