Arch. Mech., 39, 5, pp. 461–483, Warszawa 1987

Noncharacteristic mixed problems for ideal incompressible magnetohydrodynamics

W. M. ZAJĄCZKOWSKI (WARSZAWA)

THE EQUATIONS of magnetohydrodynamics describing a motion of an ideal incompressible and infinite conductive fluid are considered. First, we replace these equations by two kinds of equations: 1) a system of symmetric hyperbolic equations and 2) a Poisson equation and then the well-posed mixed problems are formulated. Next, using the results about the existence of solutions of symmetric hyperbolic equations, the existence of local solutions to the above problems is proved by using the method of successive approximations. Moreover, these solutions belong to such spaces that equations of magnetohydrodynamics are satisfied classically.

W pracy badane są równania magnetohydrodynamiki opisujące ruch idealnej, nieściśliwej i nieskończenie przewodzącej cieczy. Najpierw, zastępując te równania przez dwa rodzaje równań 1) układ symetrycznie hiperboliczny i 2) równanie Poissona, znaleziono dobrze postawione problemy mieszane. Następnie używając rezultatów dotyczących istnienia rozwiązań dla układu symetrycznego hiperbolicznego, pokazano istnienie lokalnych rozwiązań powyższych problemów stosując metodę kolejnych przybliżeń. Ponadto otrzymane rozwiązania należą do przestrzeni, w których równania magnetohydrodynamiki są spełnione klasycznie.

В работе исследуются уравнения магнетогидродинамики описывающие движение идеальной, несжимаемой и с бесконечной проводимостью жидкости. Сначала, заменяя эти уравнения двумя родами уравнений: 1) симметрически гиперболической системой и 2) уравнением Пуассона, найдены корректно поставленные смешанные задачи. Затем используя результаты о существовании решений для симметричной гиперболической системы, показано существование локальных решений вышеупомянутых задач, применяя метод последовательных приближений. Кроме этого полученные решения принадлежат к таким пространствам, что уравнения магнетогидродинамики удовлетворены классически.

1. Introduction

IN THIS PAPER we consider initial-boundary value problems to equations of magnetohydrodynamics describing the motion of an ideal incompressible fluid (see [1]):

$$B_t + v\cdot\nabla B - B\cdot\nabla v = 0, \tag{1.1}$$

$$v_t + v\cdot\nabla v + \nabla p + \frac{1}{4\pi\varrho_0} B x \operatorname{rot} B = f, \tag{1.2}$$

$$\operatorname{div} B = 0, \tag{1.3}$$

$$\operatorname{div} v = 0 \tag{1.4}$$

in a bounded domain $\Omega \subset \mathrm{R}^3$, where B is the magnetic induction, v is the velocity, p is the pressure, f is the external force and ϱ_0 is the constant density. As initial conditions we assume

(1.5) $$v|_{t=0} = v_0(x),$$

(1.6) $$B|_{t=0} = B_0(x),$$

hence (1.3) and (1.4) imply

(1.7) $$\operatorname{div} v_0 = 0, \quad \operatorname{div} B_0 = 0.$$

Our aim is to add such boundary conditions to the problem (1.1) ÷ (1.6) that the obtained problems will be well-posed (for which theorems of uniqueness are valid). To do this we replace the problem by a system of problems for which well-posedness is well known, so we can prescribe suitable boundary data. Then, using the method of successive approximations we shall prove the existence and uniqueness of solutions.

Using the identity $Bx\operatorname{rot}B = \frac{1}{2}\nabla B^2 - B\cdot\nabla B$, introducing the total pressure

(1.8) $$q = p + \frac{1}{8\pi\varrho_0}B^2$$

and the new quantities

(1.9) $$\omega = \frac{B}{\sqrt{4\pi\varrho_0}}, \quad \omega_0 = \frac{B_0}{\sqrt{4\pi\varrho_0}},$$

we write Eqs. (1.1) and (1.2) in the following matrix form:

1.10) $$Eu_t + \sum_{i=1}^{3} A_i u_{x_i} = F,$$

where $u = (v, \omega)$ and

$$E = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad A_i = \begin{pmatrix} v_i & 0 & 0 & -\omega_i & 0 & 0 \\ 0 & v_i & 0 & 0 & -\omega_i & 0 \\ 0 & 0 & v_i & 0 & 0 & -\omega_i \\ -\omega_i & 0 & 0 & v_i & 0 & 0 \\ 0 & -\omega_i & 0 & 0 & v_i & 0 \\ 0 & 0 & -\omega_i & 0 & 0 & v_i \end{pmatrix},$$

$$i = 1, 2, 3, \quad F = \begin{pmatrix} f - \nabla q \\ 0 \end{pmatrix}.$$

Moreover, Eqs. (1.5) and (1.6) imply the initial conditions

(1.11) $$u|_{t=0} = u_0 \equiv \begin{pmatrix} v_0 \\ \omega_0 \end{pmatrix}.$$

For a given q Eqs. (1.10) are symmetric and nonlinear. To formulate boundary conditions for Eqs. (1.10) and (1.11) we use the results of [2].

However, at first we have to formulate an equation for q. Using Eqs. (1.8) and (1.9) in Eq. (1.2) one has

$$v_t+v\cdot\nabla v-\omega\cdot\nabla\omega = -\nabla q+f. \tag{1.12}$$

Applying the divergence operator to the equation and using Eqs. (1.3) and (1.4), one gets

$$\Delta q = \operatorname{div} f-\sum_{i,j=1}^{3}(v_{i,x_j}v_{j,x_i}-\omega_{i,x_j}\omega_{j,x_i}), \tag{1.13}$$

so the boundary conditions to the Poisson equation (1.13) must also be determined.

Therefore, we replace Eqs. (1.1) and (1.2) by Eqs. (1.10) and (1.13). Inversely, we see that Eqs. (1.10) imply Eqs. (1.1) and (1.2). But Eqs. (1.3) and (1.4) must be not satisfied. Hence we have to find equations which imply Eqs. (1.3) and (1.4). To do this we apply the divergence operator to the system (1.10) and using Eq. (1.13) we get

$$\chi_t+\sum_{i=1}^{3}B_i\chi_{x_i} = 0, \tag{1.14}$$

where $\chi = (\eta, \vartheta)$, $\eta = \operatorname{div} v$, $\vartheta = \operatorname{div}\omega$, $B_i = \begin{pmatrix} v_i, & -\omega_i \\ -\omega_i, & v_i\end{pmatrix}$, $i = 1, 2, 3$. Moreover Eq. (1.7) gives

$$\chi|_{t=0} = 0. \tag{1.15}$$

Therefore we have to consider our problem in the case when Eqs. (1.14) and (1.15) have only zero solution, because then Eqs. (1.3) and (1.4) are satisfied. In the case of the Cauchy problem ($\Omega = \mathsf{R}^3$) the problem (1.14), (1.15) has only zero solution if v_i, ω_i, $i = 1, 2, 3$, satisfy the assumptions of Lemma 4.2 (see Theorem 4.1). In the case of a bounded Ω our aim is to find all and such possible boundary conditions to Eqs. (1.10), (1.11) and (1.13) that the problem (1.14), (1.15) would have only zero solutions. This is equivalent to the fact that the obtained mixed problems to Eqs. (1.1) and (1.2) are well posed.

The paper is organized in the following way. In Sect. 2 four different well-posed mixed problems (A_1), ..., (A_4) are formulated. The problems are obtained by replacing the basic equations $(1.1)\div(1.4)$ by a system of two problems: (1) hyperbolic mixed problems (2.3), (2) Dirichlet–Neumann problems (2.9), (2.12), (2.20), (2.21) to the Poisson equation (1.13), where we have added boundary data $(2.3)_3$ and (2.9). Next the integral type compatibility conditions (2.22) implied by Eqs. (1.3), (1.4) give us correct well posed problems for our equations of magneto-hydrodynamics. We choose two problems: (P_1) and (P_2). In Sect. 4 using the results of [2] the existence and uniqueness of solutions of the problem (1) is proved. At last, in Sect. 5, using the method of successive approximations the existence and uniqueness of solutions to the problems (P_1), (P_2) is shown. We prove the existence of classical solutions. Section 3 has an auxiliary character.

The author thanks the referee for very essential remarks.

2. Boundary conditions

To formulate boundary conditions for the problem (1.1)÷(1.6) we have to find boundary data not only for the hyperbolic problem (1.10), (1.11) but also to the problem (1.14), (1.15), simultaneously. As it follows from [2], to find boundary conditions to the hyperbolic problems (1.10), (1.11) and (1.14), (1.15) we must analyse characteristic polynomials of the matrices

$$-A_n = -\sum_{i=1}^{3} A_i n_i, \quad -B_n = -\sum_{i=1}^{3} B_i n_i, \quad \text{where } n_i,\, i = 1,2,3,$$

are coordinates of $\bar{n}$ which is the unit outward vector normal to the boundary. These characteristic polynomials have the forms

$$\begin{aligned} \det(-A_n - \lambda E) &\equiv [(\lambda+v_n)^2 - \omega_n^2]^3 = 0, \\ \det(-B_n - \lambda I) &\equiv (\lambda+v_n)^2 - \omega_n^2 = 0, \end{aligned} \tag{2.1}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $v_n = v \cdot \bar{n}$, $\omega_n = \omega \cdot \bar{n}$. Therefore we have the following eigenvalues:

$$\lambda_{\mp} = -v_n \mp \omega_n. \tag{2.2}$$

We shall restrict our considerations to the noncharacteristic boundary only so $\det A_n \neq 0$ in the neighbourhood of the boundary what is equivalent to that $\lambda_{\mp} \neq 0$ in the neighbourhood of the boundary. The case of characteristic boundary for linearized magnetohydrodynamics was considered in [3].

The mixed problems to the hyperbolic equations (1.10) and (1.14) are formulated in the following forms:

$$\begin{aligned} \mathrm{Lu} &\equiv \mathrm{Eu}_t + \sum_{i=1}^{3} A_i(u) u_{x_i} = F, \\ u|_{t=0} &= u_0, \\ \mathrm{Mu}|_{\partial\Omega} &= g \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} K\chi &\equiv I\chi_t + \sum_{i=1}^{3} B_i(u) \chi_{x_i} = 0, \\ \chi|_{t=0} &= 0, \\ N\chi|_{\partial\Omega} &= h. \end{aligned} \tag{2.4}$$

Now, analysing the signs of eigenvalues (2.2) and using the results of [2] we find the form of boundary matrices M and N. We can distinguish the following possibilities:

$$\lambda_- > 0, \quad \lambda_+ > 0 \quad \text{on } \partial\Omega, \tag{C_1}$$

what can be satisfied if $v_n|_{\partial\Omega} < -|\omega_n||_{\partial\Omega}$. A particular case is $v_n|_{\partial\Omega} < 0$, $\omega_n|_{\partial\Omega} = 0$. Therefore $M = E$, $N = I$:

$$\lambda_- < 0, \quad \lambda_+ > 0 \quad \text{on } \partial\Omega, \tag{C_2}$$

what can be satisfied if $-\omega_n|_{\partial\Omega} < v_n|_{\partial\Omega} < \omega_n|_{\partial\Omega}$ and $\omega_n|_{\partial\Omega} > 0$. As a particular case we have $v_n|_{\partial\Omega} = 0$, $\omega_n|_{\partial\Omega} > 0$. Knowing that the boundary data which belong to eigenspaces corresponding to positive eigenvalues of matrices either $-A_n$ or $-B_n$, respectively, must be prescribed only, we consider $\ker(A_n+\lambda^+ E) = \{e_1, e_2, e_3\}$, where $e_1 = \{1, 0, 0, 1, 0, 0)$, $e_2 = (0, 1, 0, 0, 1, 0)$, $e_3 = (0, 0, 1, 0, 0, 1)$, so $M = \sum_{i=1}^{3} e_i \otimes e_i$. Moreover, $\ker(B_n + \lambda_+ I) = \{(1, 1)\}$, so $N = (1, 1)$.

$$\text{(C}_3\text{)} \qquad \lambda_- > 0, \quad \lambda_+ < 0 \quad \text{on } \partial\Omega,$$

what is satisfied for $\omega_n|_{\partial\Omega} < v_n|_{\partial\Omega} < -\omega_n|_{\partial\Omega}$, so $\omega_n|_{\partial\Omega} < 0$. As a particular case we have $v_n|_{\partial\Omega} = 0, \omega_n|_{\partial\Omega} < 0$. In this case we describe $\ker(A_n+\lambda_- E) = \{e_4, e_5, e_6\}$, where $e_4 = (1, 0, 0, -1, 0, 0)$, $e_5 = (0, 1, 0, 0, -1, 0)$, $e_6 = (0, 0, 1, 0, 0, -1)$, so $M = \sum_{i=4}^{6} e_i \otimes e_i$. Moreover, $\ker(B_n + \lambda_- I) = \{(1, -1)\}$, so $N = (1, -1)$. At last we consider the case of negative eigenvalues

$$\text{(C}_4\text{)} \qquad \lambda_- < 0, \quad \lambda_+ < 0,$$

which takes place for $v_n|_{\partial\Omega} > |\omega_n|_{\partial\Omega}|$. As a particular case we have $v_n|_{\partial\Omega} > 0$, $\omega_n|_{\partial\Omega} = 0$. In this case $M = 0$, $N = 0$, so no boundary data must be prescribed. Summarizing the above considerations, we get

$$\text{(2.5)} \qquad \begin{aligned} &M = E && \text{for (C}_1\text{)},\\ &M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} && \text{for (C}_2\text{)},\\ &M = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix} && \text{for (C}_3\text{)},\\ &M = 0 && \text{for (C}_4\text{)}. \end{aligned}$$

Hence using the problem (2.5) instead of the problem $(2.3)_3$, we obtain

$$\text{(2.6)} \qquad \begin{aligned} &u|_{\partial\Omega} = (b, d) && \text{for} \quad (\text{C}_1)\\ &(v+\omega)|_{\partial\Omega} = \alpha && \text{for} \quad (\text{C}_2),\\ &(v-\omega)|_{\partial\Omega} = \beta && \text{for} \quad (\text{C}_3), \end{aligned}$$

where g is equal to b, d, α, and β, respectively. Moreover, we have

$$\text{(2.7)} \qquad \begin{aligned} &N = I && \text{for} \quad (\text{C}_1),\\ &N = (1, 1) && \text{for} \quad (\text{C}_2),\\ &N = (1, -1) && \text{for} \quad (\text{C}_3),\\ &N = 0 && \text{for} \quad (\text{C}_4), \end{aligned}$$

therefore, assuming $h = 0$ in Eq. $(2.4)_3$, we have

$$\begin{aligned} &\operatorname{div} v|_{\partial\Omega} = \operatorname{div}\omega|_{\partial\Omega} = 0 \quad \text{for} \quad (C_1),\\ (2.8)\qquad &\operatorname{div}(v+\omega)|_{\partial\Omega} = 0 \quad \text{for} \quad (C_2).\\ &\operatorname{div}(v-\omega)|_{\partial\Omega} = 0 \quad \text{for} \quad (C_3). \end{aligned}$$

Consequently, the problem (2.4), (2.7), (2.8) ($h = 0$) implies $\chi = 0$ (see Theorem 4.1), so Eqs. (1.3) and (1.4) are satisfied.

However, the boundary conditions (2.8) are not justified. We shall obtain them by formulating boundary data for the Poisson equation (1.13).

In the case (C_4) we haven't any conditions in Eqs. (2.8) so we have some arbitrariness in prescribing boundary data for Eq. (1.13). Therefore we assume that

$$(2.9) \qquad q|_{\partial\Omega} = \pi,$$

where π is a given function.

To formulate boundary conditions to Eq. (1.13) for the cases (C_1), (C_2), (C_3), curvilinear coordinates in the neighbourhood of $\partial\Omega$ must be introduced. Let $\bar{n}(x)$, $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$ be orthonormal vectors determined in the neighbourhood of $\partial\Omega$ such that for $x \in \partial\Omega$ $\bar{n}(x)$ is a unit outward vector normal to $\partial\Omega$ and $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$ are tangent to $\partial\Omega$. Let $n(x)$, $\tau_1(x)$, $\tau_2(x)$ be an orthonormal system of curvilinear coordinates corresponding to the above vectors such that $n(x) = 0$ describes $\partial\Omega$ locally and then $\tau_1(x)$, $\tau_2(x)$ are locally parameters on $\partial\Omega$. Moreover, the following relations are valid:

$$\bar{n}\cdot\nabla = \frac{1}{\mathscr{H}_n}\frac{\partial}{\partial n}, \quad \bar{\tau}_i\cdot\nabla = \frac{1}{\mathscr{H}_i}\frac{\partial}{\partial \tau_i}, \quad i = 1, 2,$$

where $\mathscr{H}_i$, $i = 1, 2$, $\mathscr{H}_n$ are Lame's coefficients. Using the curvilinear coordinates, we can write the equation $\operatorname{div} a = 0$ in the form

$$(2.10) \qquad \bar{n}\cdot\nabla a_n + a_n \operatorname{div}\bar{n} + \sum_{i=1}^{2}(\bar{\tau}_i\cdot\nabla a_{\tau_i} + a_{\tau_i}\operatorname{div}\bar{\tau}_i) = 0,$$

where $a_n = a\cdot\bar{n}$, $a_{\tau_i} = a\cdot\bar{\tau}_i$, $i = 1, 2$.

Let us consider the case (C_1). Projecting the normal component of Eq. (1.12) on $\partial\Omega$, using Eq. $(2.6)_1$ and the projection of Eq. (2.10) on $\partial\Omega$ for $a = v$ and $a = \omega$, we obtain

$$\begin{aligned} (2.11)\qquad \frac{1}{\mathscr{H}_n}\frac{\partial q}{\partial n}\bigg|_{\partial\Omega} &= f_n|_{\partial\Omega} - b_{n,t} + \sum_{k=1}^{3}(b\cdot\nabla n_k b_k - d\cdot\nabla n_k d_k)\\ &- \sum_{i=1}^{2}(b_{\tau_i}\bar{\tau}_i\cdot\nabla b_n - d_{\tau_i}\bar{\tau}_i\cdot\nabla d_n) + b_n\Big[b_n\operatorname{div}\bar{n} + \sum_{i=1}^{2}(\bar{\tau}_i\cdot\nabla b_{\tau_i} + b_{\tau_i}\operatorname{div}\bar{\tau}_i)\\ &- \operatorname{div} v|_{\partial\Omega}\Big] - d_n\Big[d_n\operatorname{div}\bar{n} + \sum_{i=1}^{2}(\bar{\tau}_i\cdot\nabla d_{\tau_i} + d_{\tau_i}\operatorname{div}\bar{\tau}_i) - \operatorname{div}\omega|_{\partial\Omega}\Big]. \end{aligned}$$

To determine Eq. $(2.8)_1$, we assume the following condition:

$$(2.12)\qquad \frac{1}{\mathcal{H}_n}\frac{\partial q}{\partial n}\bigg|_{\partial\Omega} = f_n|_{\partial\Omega} - b_{n,t} + \sum_{k=1}^{3}(b\cdot\nabla n_k b_k - d\cdot\nabla n_k d_k)$$
$$- \sum_{i=1}^{2}(b_{\tau_i}\overline{\tau}_i\cdot\nabla b_n - d_{\tau_i}\overline{\tau}_i\cdot\nabla d_n) + b_n\Big[b_n\operatorname{div}\overline{n} + \sum_{i=1}^{2}(\overline{\tau}_i\cdot\nabla b_{\tau_i} + b_{\tau_i}\operatorname{div}\overline{\tau}_i)\Big]$$
$$- d_n\Big[d_n\operatorname{div}\overline{n} + \sum_{i=1}^{2}(\overline{\tau}_i\cdot\nabla d_{\tau_i} + d_{\tau_i}\operatorname{div}\overline{\tau}_i)\Big].$$

Comparing Eq. (2.11) with Eq. (2.12), we see that Eq. $(2.8)_1$ is not implied, so we project the normal component of Eq. (1.1) on $\partial\Omega$ and use Eqs. $(2.6)_1$ and (2.10). Therefore we get

$$(2.13)\qquad d_{n,t} + \sum_{i=1}^{2}(b_{\tau_i}\overline{\tau}_i\cdot\nabla d_n - d_{\tau_i}\overline{\tau}_i\cdot\nabla b_n) - b\cdot\nabla\overline{n}d + d\cdot\nabla\overline{n}b$$
$$+ b_n\Big[\operatorname{div}\omega|_{\partial\Omega} - \Big(d_n\operatorname{div}\overline{n} + \sum_{i=1}^{2}(\overline{\tau}_i\cdot\nabla d_{\tau_i} + d_{\tau_i}\operatorname{div}\overline{\tau}_i)\Big)\Big]$$
$$- d_n\Big[\operatorname{div}v|_{\partial\Omega} - \Big(b_n\operatorname{div}\overline{n} + \sum_{i=1}^{2}(\overline{\tau}_i\cdot\nabla b_{\tau_i} + b_{\tau_i}\operatorname{div}\tau_i)\Big)\Big] = 0.$$

Assuming the compatibility condition

$$(2.14)\qquad d_{n,t} + \sum_{i=1}^{2}(b_{\tau_i}\overline{\tau}_i\cdot\nabla d_n - d_{\tau_i}\overline{\tau}_i\cdot\nabla b_n) - b\cdot\nabla\overline{n}d + d\cdot\nabla\overline{n}b$$
$$+ \sum_{i=1}^{2}[\overline{\tau}_i\cdot(d_n\nabla b_{\tau_i} - b_n\nabla d_{\tau_i}) + (d_n b_{\tau_i} - b_n d_{\tau_i})\operatorname{div}\overline{\tau}_i] = 0,$$

by comparison of Eqs. (2.12) and (2.14) with Eqs. (2.11) and (2.13), respectively, we get

$$(2.15)\qquad \begin{pmatrix} b_n, & -d_n \\ -d_n, & b_n \end{pmatrix}\begin{pmatrix}\operatorname{div}v|_{\partial\Omega} \\ \operatorname{div}\omega|_{\partial\Omega}\end{pmatrix} = 0$$

so Eq. $(2.8)_1$ is satisfied because $b_n^2 - d_n^2 \neq 0$ (see the relation (C_1)).

To obtain boundary conditions for the cases (C_2) and (C_3), we must reformulate Eqs. (1.1) and (1.12). Taking the sum and difference of Eqs. (1.1) and (1.12), we obtain, respectively,

$$(2.16)\qquad (v+\omega)_t + (v-\omega)\cdot\nabla(v+\omega) = f - \nabla q,$$

$$(2.17)\qquad (v-\omega)_t + (v+\omega)\cdot\nabla(v-\omega) = f - \nabla q.$$

In the case (C_2), in order to obtain the condition $(2.8)_2$ we have to use Eq. (2.16). Projecting the normal components of Eq. (2.16) on $\partial\Omega$ and using Eq. $(2.6)_2$, we get

$$(2.18)\qquad \frac{1}{\mathcal{H}_n}\frac{\partial q}{\partial n}\bigg|_{\partial\Omega} = f_n|_{\partial\Omega} - \alpha_{n,t} + \sum_{k=1}^{3}(v-\omega)\cdot\nabla n_k\alpha_k$$

$$- \sum_{i=1}^{2} (v-\omega)_{\tau_i} \bar{\tau}_i \cdot \nabla \alpha_n - (v-\omega)_n \bar{n} \cdot \nabla (v+\omega)_n. \tag{2.18 [cont.]}$$

Using Eq. (2.10) for $a = v+\omega$ and Eq. $(2.6)_2$, the last term in Eq. (2.18) can be replaced by expressions with α. Hence one gets

$$\frac{1}{\mathcal{H}_n} \frac{\partial q}{\partial n}\Big|_{\partial\Omega} = f_n|_{\partial\Omega} - \alpha_{n,t} + \sum_{k=1}^{3} (v-\omega) \cdot \nabla n_k \alpha_k - \sum_{i=1}^{2} (v-\omega)_{\tau_i} \bar{\tau}_i \cdot \nabla \alpha_n$$
$$- (v-\omega)_n \Big[\operatorname{div}(v+\omega)|_{\partial\Omega} - \alpha_n \operatorname{div} \bar{n} - \sum_{i=1}^{2} (\bar{\tau}_i \cdot \nabla \alpha_{\tau_i} + \alpha_{\tau_i} \operatorname{div} \bar{\tau}_i) \Big]. \tag{2.19}$$

Demanding the boundary condition in the form

$$\frac{1}{\mathcal{H}_n} \frac{\partial q}{\partial n}\Big|_{\partial\Omega} = f_n|_{\partial\Omega} - \alpha_{n,t} + \sum_{k=1}^{3} (v-\omega) \cdot \nabla n_k \alpha_k - \sum_{i=1}^{2} (v-\omega)_{\tau_i} \bar{\tau}_i \cdot \nabla \alpha_n$$
$$+ (v-\omega)_n \Big[\alpha_n \operatorname{div} \bar{n} + \sum_{i=1}^{2} (\bar{\tau}_i \cdot \nabla \alpha_{\tau_i} + \alpha_{\tau_i} \operatorname{div} \bar{\tau}_i) \Big], \tag{2.20}$$

we obtain that Eq. $(2.8)_2$ is satisfied because $(v_n - \omega_n)|_{\partial\Omega} \neq 0$, what follows from the relation (C_2). In the case (C_3) similarly as above by using Eqs. (2.17) and $(2.6)_3$, we get

$$\frac{1}{\mathcal{H}_n} \frac{\partial q}{\partial n}\Big|_{\partial\Omega} = f_n|_{\partial\Omega} - \beta_{n,t} + \sum_{k=1}^{3} (v+\omega) \cdot \nabla n_k \beta_k - \sum_{i=1}^{2} (v+\omega)_{\tau_i} \bar{\tau}_i \cdot \nabla \beta_n$$
$$+ (v+\omega)_n \Big[\beta_n \operatorname{div} \bar{n} + \sum_{i=1}^{2} (\bar{\tau}_i \cdot \nabla \beta_{\tau_i} + \beta_{\tau_i} \operatorname{div} \bar{\tau}_i) \Big] \tag{2.21}$$

and $(v+\omega)_n \operatorname{div}(v-\omega)|_{\partial\Omega} = 0$ so by (C_3) it follows that Eq. $(2.8)_3$ is satisfied.

Summarizing, our problem is replaced by a system of two problems: mixed problems for symmetric hyperbolic equations (2.3), (2.5), (2.6) and Dirichlet–Neumann problems (2.9), (2.12), (2.20), (2.21) for the Poisson equation (1.13). Moreover, the boundary conditions (2.12), (2.20), (2.21) imply Eq. (2.8) so from the hyperbolic problems (2.4), (2.7), (2.8) it follows that Eqs. (1.3) and (1.4) are satisfied. Therefore we have obtained the following types of mixed problems:

(A_1) (2.3), $(2.5)_1$, $(2.6)_1$, (1.13), (2.12), (2.14);

(A_2) (2.3), $(2.5)_2$, $(2.6)_2$, (1.13), (2.20);

(A_3) (2.3), $(2.5)_3$, $(2.6)_3$, (1.13), (2.21);

(A_4) (2.3), $(2.5)_4$, $(2.6)_4$, (1.13), (2.9).

In order to prove the existence of solutions of our problem, the above formulation suggests the method of successive approximations. However, Eqs. (1.3) and (1.4), which do not explicitly occur in the problems $(A_1) \div (A_4)$, must be satisfied. They imply the following compatibility conditions:

$$\int_{\partial\Omega} v_n(s,t)\,ds = 0, \qquad \int_{\partial\Omega} \omega_n(s,t)\,ds = 0, \tag{2.22}$$

which have a global character. Now, considering the problems $(C_1) \div (C_4)$ we see that

$$\begin{aligned} v_n|_{\partial\Omega} &< 0 \quad \text{for} \quad (C_1), & v_n|_{\partial\Omega} &> 0 \quad \text{for} \quad (C_4), \\ \omega_n|_{\partial\Omega} &> 0 \quad \text{for} \quad (C_2), & \omega_n|_{\partial\Omega} &< 0 \quad \text{for} \quad (C_3), \end{aligned} \tag{2.23}$$

therefore Eq. (2.22) implies that we have to consider domains with a boundary which consists of at least two disjoint parts with different kinds of boundary data coorresponding to different initial-boundary value problems $(A_1) \div (A_4)$. Considering the boundary composed of disjoined parts helps us to omit real difficulties connected with a jump of either v_n or ω_n which must appear in the case of a connected boundary (see Eq. (2.23)). The last possibility implies that we have to look for solutions v, ω in a class of noncontinuous functions because they have jumps on the boundary, but it is not possible because proving the existence the nonlinearity of our problems needs more regularity. These jumps can be avoided if we consider domains with edges (dihedral angles). However, in this case we are faced with difficult problems solving boundary problems to the elliptic equation and mixed problems to hyperbolic equations in domains with dihedral angles (see for example [4, 5]).

In the end we have to examine a condition which selects the problems from the set $(A_1) \div (A_4)$ which may be considered simultaneously. We recall that the problems $(A_1) \div$ $\div (A_3)$ enclose the Neumann problem to the Poisson equation (1.13). Therefore, to solve such a problem a compatibility condition is needed. By some examples we show why this condition cannot be satisfied. Let us assume that on a part S_ν of the boundary the condition (A_ν) is given, $\nu = 1, 2, 3, 4$. Let us consider the problem (A_2), (A_3), so $\partial\Omega = S_2 \cup S_3$ and $S_2 \cap S_3 = \emptyset$. In this case the condition (2.22) can be satisfied (see Eq. (2.23)). From Eq. (1.13) we have

$$\int_{\Omega} \operatorname{div} f - \sum_{i,j=1}^{3} (v_{i,x_j} v_{j,x_i} - \omega_{i,x_j} \omega_{j,x_i}) = \int_{S_2 \cap S_3} \bar{n} \cdot \nabla q. \tag{2.24}$$

Using Eqs. (2.20) and (2.21), we see that Eq. (2.24), besides the given boundary data contains traces of unknown quantities v and ω. Hence we do not known how to satisfy Eq. (2.24). Similar considerations can be done in other cases in which the problems $(A_1) \div (A_3)$ appear only. Therefore we assume:

(2.25) Equation (1.13) with the Neumann condition only will not be considered in this paper.

Summarizing the above considerations, we shall restrict ourselves to the following two kinds of problems:

(P_1) (A_1, A_4), $\partial\Omega = S_1 \cup S_4$, $S_1 \cap S_4 = \emptyset$,

(P_2) (A_2, A_3, A_4), $\partial\Omega = S_2 \cup S_3 \cup S_4$, $S_2 \cap S_3 = \emptyset$, $S_2 \cap S_4 = \emptyset$, $S_3 \cap S_4 = \emptyset$,

where the boundary data are prescribed in such a way that Eq. (2.22) is satisfied.

3. Notations

To simplify the next considerations we introduce the following spaces and notations. At first we introduce the ordinary Sobolev spaces $H^S(\Omega^T)$ with the norm

$$||u||_{H^s(\Omega^T)} = \Big(\sum_{|\nu|\leqslant s} \int_0^T \int_\Omega |D^\nu_{t,x} u|^2 dx\, dt\Big)^{1/2},$$

where

$$D^\nu_{t,x} = \frac{\partial^{\nu_0}}{\partial t^{\nu_0}} \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \dots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}}, \qquad |\nu| = \nu_0+\nu_1+\dots+\nu_n, \qquad \Omega^T = \Omega\times[0, T],$$

and ν is the multi-index. Similarly spaces $H^s(\Omega)$, $H^s(\partial\Omega^T)$ can be introduced. The norms of spaces $H^s(\Omega)$, $H^s(\Omega^T)$, $H^s(\partial\Omega^T)$ will be denoted by $||\ ||_{s,\Omega}$, $||\ ||_{s,\Omega^T}$, $||\ ||_{s,\partial\Omega^T}$, respectively. Moreover, we introduce $L_2(\Omega^T) = H^0(\Omega^T)$, $L_2(\partial\Omega^T) = H^0(\partial\Omega^T)$ and so on with the norms $||\ ||_{\Omega^T}$, $||\ ||_{\partial\Omega^T}$, respectively.

Using the above notations we define $L_p(0, T; H^s(\Omega))$, $L^j_p(0, T; H^s(\Omega))$, where $p \geqslant 1$ is real, by the following norms:

$$\Big(\int_0^T ||u||^p_{s,\Omega} dt\Big)^{1/p}, \qquad \Big(\int_0^T ||D^j_t u||^p_{s,\Omega} dt\Big)^{1/p}.$$

Therefore we can introduce

$$\Pi^l_{k,p}(\Omega^T) = \bigcap_{i=k}^{l} L^{l-i}_p(0, T; H^i(\Omega)) \quad \text{with the norm } |\ |_{lk,p,\Omega^T}.$$

For $p = 2$ we have $\Pi^l_k(\Omega^T) = \Pi^l_{k,2}(\Omega^T)$ and $|\ |_{l,k,\Omega^T}$.

Introducing the space of traces $H^{s-1/2}(\partial\Omega)$, $H^{s-1/2}(\partial\Omega^T)$, we can define

$$\Pi^{l-1/2}_{k,p}(\partial\Omega^T) = \bigcap_{i=k}^{l} L^{l-i}_p(0, T; H^{i-1/2}(\partial\Omega)),\ k \geqslant 1, \quad \text{with the norm } |\ |_{l-1/2,k,p,\partial\Omega^T}.$$

Moreover, we introduce the weighted Sobolev spaces $H^s_\alpha(\Omega^T)$, $\alpha \geqslant 0$, with the norm

$$||u||_{H^s_\alpha(\Omega^T)} = ||u||_{s,\alpha,\Omega^T} = \Big(\sum_{|\nu|\leqslant s} \int_{\Omega^T} |D^\nu_{t,x} u|^2 e^{-2\alpha t} dx\, dt\Big)^{1/2}.$$

We also define $H^0_\alpha(\Omega^T) = L_{2,\alpha}(\Omega^T)$, and so on.

At last we introduce $L_p(\Omega)$ with the norm $||\ ||_{0,p,\Omega}$ and a Banach space $\Gamma^l_k(\Omega)$ with the norm

$$||u||_{\Gamma^l_k(\Omega)} \equiv |u|_{l,k,\Omega} = \sum_{i=k}^{l} ||D^{l-i}_t u||_{i,\Omega}.$$

For convenience the following spaces will be introduced:

$$\tilde\Pi^l_{k,p}(\Omega^T) = \bigcap_{i=k}^{l} L^i_p(0, T, H^{l-i}(\Omega)), \qquad \tilde\Pi^l_{k,2}(\Omega^T) = \tilde\Pi^l_k(\Omega^T) \quad \text{and} \quad \tilde\Gamma^l_k(\Omega)$$

with the norm

$$||u||_{\tilde{\Gamma}^l_k(\Omega)} = \sum_{i=k}^{l} ||D^i_t u||_{l-i,\Omega}.$$

Further, by $|u|$ we denote the Euclidean norm, where u may be a vector or matrix.

4. Existence of solutions of the problem (2.3)

In this section we shall consider the following problem:

$$\begin{aligned} \mathrm{L}u \equiv \mathrm{E}u_t + \sum_{i=k}^{l} A_i(x,t)u_{x_i} + D(x,t)u &= F(x,t), \\ u|_{t=0} &= u_0(x), \qquad (4.1)\\ \mathrm{M}u|_{\partial\Omega} &= g(x',t), \qquad x' \in \partial\Omega, \end{aligned}$$

where M is described by Eq. (2.5). By [2] the proof of the existence of solutions to the problem (4.1) can be restricted to getting an a priori estimate only. Moreover, using a partition of unity it is sufficient to obtain this estimate in a half space only. Therefore, in this section we assume that $\Omega = \{x \in \mathrm{R}^3 : x_1 > 0\}$, $\partial\Omega = \{x \in \mathrm{R}^3 : x_1 = 0\}$; moreover, we denote $x' = (x_2, x_3)$. To obtain an a priori estimate we assume that a solution of the problem (4.1) is sufficiently smooth.

LEMMA 4.1.

(a) Let Λ^- and Λ^+ be sets of negative and positive eigenvalues of the matrix $-A_n$, respectively. Let $u^- \in \ker(A + \lambda^- E)$ for each $\lambda^- \in \Lambda^-$ and $u^+ \in \ker(A_n + \lambda^+ E)$ for each $\lambda^+ \in \Lambda^+$.

Let

$$0 < c \leqslant \min_{\partial\Omega^t} \min_{\Lambda^-} (-\lambda^-), \tag{4.2}$$

and

$$\max_{\partial\Omega^T} \max_{\Lambda^+} \lambda^+ \leqslant c_1, \tag{4.3}$$

where c_0, c_1 are constants.

(b) Let $D \in L_\infty(0,t; H^2(\Omega))$, $A_i \in L_\infty(0,t; H^3(\Omega))$, $i = 1, 2, 3$, and there exists a constant α such that

$$\max_{\Omega^t}\Big(2|D| + \sum_{i=1}^{3} |A_{i,x_i}|\Big) \leqslant c\Big(\sum_{i=1}^{3} |A_i|_{3,3,\infty,\Omega^t} + |D|_{2,2,\infty,\Omega^t}\Big) \leqslant \frac{\alpha}{2}. \tag{4.4}$$

(c) Let $F \in L_{2,\alpha}(\Omega^t)$. Then the following estimate

$$(4.5) \qquad \int_{\Omega} u^2 e^{-2\alpha t} dx \Big|_{t=0}^{t} + \alpha \int_{\Omega^t} u^2 e^{-2\alpha t} dx\, dt + c_0 \int_{\partial\Omega^t} u^2 e^{-2\alpha t} dx' dt$$

(4.5) [cont.]
$$\leqslant (c_0+c_1)\int\limits_{\partial\Omega^t}(u^+)^2e^{-2\alpha t}dx'dt+\frac{2}{\alpha}\int\limits_{\Omega^t}|F|^2e^{-\alpha t}dx\,dt$$

holds, when $u^+ = Mu$.

P r o o f. Multiplying Eq. $(4.1)_1$ by $ue^{-2\alpha t}$ and integrating by parts, we obtain

$$\frac{d}{dt}\int\limits_{\Omega}u^2e^{-2\alpha t}dx+2\alpha\int\limits_{\Omega}u^2e^{-2\alpha t}dx+\int\limits_{\partial\Omega}A_nu\cdot ue^{-2\alpha t}dx'$$
$$=\int\limits_{\Omega}\Big(2Du\cdot u-\sum_{i=1}^{3}A_{i,x_i}u\cdot u\Big)e^{-2\alpha t}dx+2\int\limits_{\Omega}F\cdot ue^{-2\alpha t}dx.$$

Using the Young inequality and the assumptions (b), (c) we get

$$\int\limits_{\Omega}u^2e^{-2\alpha t}dx|_{t=0}^{t}+\alpha\int\limits_{\Omega^t}u^2e^{-2\alpha t}dx\,dt+\int\limits_{\partial\Omega^t}A_nu\cdot ue^{-2\alpha t}dx'dt\leqslant\frac{2}{\alpha}\int\limits_{\Omega^t}|F|^2e^{-2\alpha t}dx\,dt.$$

At last the assumption (a) implies the estimate (4.5). This concludes the proof.

As it was mentioned in Sect. 2, we have to use a method of successive approximations to prove the existence of solutions of the problem $(A_1)\div(A_4)$. The nonlinearity of these problems needs sufficiently high regularity which must be such that $x \to u(t,\cdot)\in H^3(\Omega)$. Therefore the aim of this section is to get an a priori estimate for solutions of the problem (4.1) in H^3. To do this we consider the problems

$$LD^{\sigma}_{t,x'}u = D^{\sigma}_{t,x'}F+(LD^{\sigma}_{t,x}u-D^{\sigma}_{t,x'}Lu),$$
$$(4.6)\qquad D^{\sigma}_{t,x'}u|_{t=0} = D^{\sigma}_{t,x'}u|_{t=0},$$
$$MD^{\sigma}_{t,x'}u|_{\partial\Omega} = D^{\sigma}_{t,x'}g,$$

where

$$D^{\sigma}_{t,x'} = \frac{\partial^{\sigma_0}}{\partial t^{\sigma_0}}\frac{\partial^{\sigma_1}}{\partial x_2^{\sigma_1}}\frac{\partial^{\sigma_2}}{\partial x_3^{\sigma_2}},\quad |\sigma| = \sigma_0+\sigma_1+\sigma_2,\quad \sigma' = (\sigma_2,\sigma_3),\quad |\sigma'| = \sigma_2+\sigma_3,$$
$$|\sigma| = 1,2,3$$

and the right-hand side of Eq. $(4.6)_2$ must be calculated from Eq. $(4.1)_1$.

LEMMA 4.2.

Suppose that $\tilde{L} = (A_1,A_2,A_3,D)$, $A_i\in\Pi^3_{0,\infty}(\Omega^t)$, $i = 1,2,3$, $D\in\Pi^3_{0,\infty}(\Omega^t)$ and that there exist numbers δ, δ_1 such that

$$(4.7)\qquad \sum_{i=1}^{3}|A_i|_{3,0,\infty,\Omega^t}+|D|_{3,0,\infty,\Omega^t}\leqslant\delta,$$

$$(4.8)\qquad \max_{\Omega^t}|A_1^{-1}|\leqslant\delta_1.$$

Suppose

(4.9) $F\in H^{\sigma}_{\alpha}(\Omega^t)$, $g\in H^{\sigma}_{\alpha}(\partial\Omega^t)$, where $\sigma = 1,2,3$, $u_0\in H^3(\Omega)$.

Then there exist polynomials $p = p_\sigma(\delta_1, \delta)$ and $q = q_\sigma(\delta_1, \delta)$ such that the following estimate

$$(4.10)\qquad |u|^2_{\sigma,0,\Omega} e^{-2\alpha t} + \frac{\alpha}{2} ||u||^2_{\sigma,\Omega^t,\alpha} + c_0 ||u||^2_{\sigma,\partial\Omega^t,\alpha} \leqslant p_\sigma(\delta_1, \delta)(||F||^2_{\sigma,\Omega^t,\alpha} + |F|^2_{\sigma-1,0,\Omega}\big|_{t=0}) + q_\sigma(\delta_1, \delta)||g||^2_{\sigma,\partial\Omega^t,\alpha} + |u|^2_{\sigma,,0,\Omega}\big|_{t=0}$$

is valid, where $\sigma = 1, 2, 3$ and there exists a polynomials $r_\sigma(\delta)$, $\varrho_\sigma(\delta_1, \delta)$ such that $\varrho_\sigma(\delta_1, \delta) < \alpha$ and

$$(4.11)\qquad |u|_{\sigma,0,\Omega}\big|_{t=0} \leqslant r_\sigma(\delta)\,[||u_0||_{\sigma,\Omega} + |F|_{\sigma-1,0,\Omega}\big|_{t=0}].$$

Proof. Let us consider the case $\sigma = 1$. Considering the problem (4.6) for $\sigma = 1$ from the estimate (4.5), we get

$$(4.12)\qquad \int\limits_{\Omega} |D^1_{t,x'}u|^2 e^{-2\alpha t} dx|^t_{t=0} + \alpha \int\limits_{\Omega^t} |D^1_{t,x'}u|^2 e^{-2\alpha t} dx\,dt + c_0 \int\limits_{\partial\Omega^t} |D^1_{t,x'}u|^2 e^{-2\alpha t} dx'\,dt \leqslant (c_0 + c_1) \int\limits_{\partial\Omega^t} |D^1_{t,x'}g|^2 e^{-2\alpha t} dx'\,dt + \frac{2}{\alpha} \int\limits_{\Omega^t} |D^1_{t,x'}F|^2 e^{-2\alpha t} dx\,dt + \frac{2}{\alpha} \int\limits_{\Omega^t} |LD^1_{t,x'}u - D^1_{t,x'}Lu|^2 e^{-2\alpha t} dx\,dt,$$

where the last term is estimated by

$$(4.13)\qquad \frac{2}{\alpha} \int\limits_{\Omega^t} |D^1_{t,x'}\tilde{L}|^2 (|D^1_{t,x}|^2 + |u|^2) e^{-2\alpha t} dx\,dt.$$

Using

$$(4.14)\qquad u_{x_1} = A_1^{-1}(F - A'u_{x'} - Du)$$

and the assumptions (4.7) and (4.8), the expression (4.13) is bounded by

$$(4.15)\qquad \frac{c}{\alpha}\delta^2 \Big[(\delta_1^2\delta^2 + 1) \int\limits_{\Omega^t} (|D^1_{t,x'}u|^2 + |u|^2) e^{-2\alpha t} dx\,dt + \int\limits_{\Omega^t} |F|^2 e^{-2\alpha t} dx\,dt\Big].$$

Assuming

$$(4.16)\qquad c\delta^2(\delta_1^2\delta^2 + 1) \leqslant \frac{\alpha^2}{2}$$

by Eqs. (4.5), (4.12), (4.13) ad (4.15), we obtain

$$(4.17)\qquad \int\limits_{\Omega} (|D^1_{t,x'}u|^2 + |u|^2) dx\, e^{-2\alpha t}|^t_{t=0} + \frac{\alpha}{2} \int\limits_{\Omega^t} (|D^1_{t,x'}u|^2 + |u|^2) e^{-2\alpha t} dx\,dt + c_0 \int\limits_{\partial\Omega^t} (|D^1_{t,x'}u|^2 + |u|^2)^{-2\alpha t} dx'\,dt \leqslant (c_0 + c_1) \int\limits_{\partial\Omega^t} (|D^1_{t,x'}g|^2 + |g|^2) e^{-2\alpha t} dx'\,dt$$

(4.17) [cont.]
$$+c\int_{\Omega^t}(|D^1_{t,x'}F|^2+|F|^2)e^{-2\alpha t}dx\,dt.$$

From Eq. (4.14)

(4.18)
$$\|u_{x_1}\|^2_{0,\Omega}\leqslant c[\|F\|^2_{0,\Omega}+\delta^2(\|D^1_{t,x'}u\|^2_{0,\Omega}+\|u\|^2_{0,\Omega})],$$
$$\|u_{x_1}\|^2_{0,\Omega^t,\alpha}\leqslant c[\|F\|^2_{0,\Omega^t,\alpha}+\delta^2(\|D^1_{t,x'}u\|^2_{0,\Omega^t,\alpha}+\|u\|^2_{0,\Omega^t,\alpha})]$$

so Eqs. (4.17) and (4.18) imply the estimate (4.10) for $\sigma=1$ where $c(\alpha+1)\leqslant p_1(\delta^2)$, $c(c_0+c_1)\leqslant q_1(\delta)$ hence p_1, q_1 are polynomials of the first and sixth order, respectively, because $c_1\leqslant c\max|\det A_1|\leqslant c\delta^6$.

Let us consider the case $\sigma=2$. From Eqs. (4.5) and (4.6) we have

(4.19)
$$\int_{\Omega}|D^2_{t,x'}u|^2dx\,e^{-2\alpha t}|^t_{t=0}+\alpha\int_{\Omega^t}|D^2_{t,x'}u|^2e^{-2\alpha t}dx'dt$$
$$+c_0\int_{\partial\Omega^t}|D^2_{t,x'}u|^2e^{-2\alpha t}dx'dt\leqslant(c_0+c_1)\int_{\partial\Omega^t}|D^2_{t,x'}g|^2e^{-2\alpha t}dx'dt$$
$$+\frac{2}{\alpha}\int_{\Omega^t}|D^2_{t,x'}F|^2e^{-2\alpha t}dx\,dt+\frac{2}{\alpha}\int_{\Omega^t}|D^2_{t,x'}Lu-LD^2_{t,x'}u|^2e^{-2\alpha t}dx\,dt,$$

where the last term is estimated by

(4.20)
$$\frac{2}{\alpha}\int_{\Omega^t}[|D^2_{t,x'}A|^2\,|D^1_{t,x}u|^2+|D^1_{t,x'}A|^2\,|D^1_{t,x'}D^1_{t,x}u|^2+|D^2_{t,x'}D|^2\,|u|^2$$
$$+|D^1_{t,x'}D|^2\,|D^1_{t,x'}u|^2]e^{-2\alpha t}dx\,dt\leqslant\frac{c}{\alpha}\delta^2[\|D^2_{t,x}u\|^2_{0,\Omega^t,\alpha}+\|u\|^2_{1,\Omega^t,\alpha}].$$

To estimate the first term at the right-hand side of the inequality (4.20), we shall consider the following inequalities (where Eq. (4.14) is used):

(4.21)
$$\|D^1_{t,x'}D^1_{x_1}u\|^2_{0,\Omega^t,\alpha}\leqslant c\delta_1^2\delta^2\|D^2_{t,x'}u\|^2_{0,\Omega^t,\alpha}+cr_2(\delta_1^2,\delta^2)(\|u\|^2_{1,\Omega^t,\alpha}+\|F\|^2_{1,\Omega^t,\alpha}),$$

where r_k is a polynominal of the k degree.

(4.22)
$$\|D^2_{x_1}u\|^2_{0,\Omega^t,\alpha}\leqslant c\delta_1^4\delta^2(\delta^2+1)\|D^2_{t,x'}u\|^2_{0,\Omega^t,\alpha}+cr_4(\delta_1^2,\,\delta^2)\|u\|^2_{1,\Omega^t,\alpha}$$
$$+cr_3(\delta_1^2,\,\delta^2)\|F\|^2_{1,\Omega^t,\alpha}.$$

Using

(4.23)
$$c\delta_1^2(\delta_1^2+1)\,\delta^4(\delta^2+1)\leqslant\frac{\alpha^2}{2}$$

from Eqs. (4.19) ÷ (4.22) we have

(4.24)
$$\int_{\Omega}|D^2_{t,x'}u|^2dx\,e^{-2\alpha t}\Big|^t_{t=0}+\frac{\alpha}{2}\int_{\Omega^t}|D^2_{t,x'}u|^2e^{-2\alpha t}dx\,dt$$
$$+c_0\int_{\partial\Omega^t}|D^2_{t,x'}u|^2e^{-2\alpha t}dx'dt\leqslant(c_0+c_1)\|D^2_{t,x'}g\|^2_{0,\partial\Omega^t,\alpha}$$
$$+cr_4(\delta_1^2,\,\delta^2)[\|F\|^2_{2,\Omega^t,\alpha}+\|u\|^2_{1,\Omega^t,\alpha}].$$

Moreover, we have to calculate

$$(4.25)\qquad ||D^1_{t,x'}D^1_{x_1}u||^2_{0,\Omega}+||D^2_{x_1}u||^2_{0,\Omega}\leqslant \mathrm{cr}_3(\delta^1_2,\delta^2)[|F|^2_{1,0,\Omega}+\sum_{s\leqslant 2}||D^s_{t,x'}u||^2_{0,\Omega}].$$

Therefore from Eqs. (4.24) and (4.25) we get

$$(4.26)\qquad ||D^2_{t,x}u||^2_{0,\Omega}e^{-2\alpha t}\Big|_{t=0}^{t}+\frac{\alpha}{2}||D^2_{t,x}u||^2_{0,\Omega^t,\alpha}+c_0||D^2_{t,x'}u||^2_{0,\partial\Omega^t,\alpha}$$
$$\leqslant (c_0+c_1)||D^2_{t,x'}g||^2_{0,\partial\Omega^t,\alpha}+cr_3(\delta^2_1,\delta^2)(||F||^2_{2,\Omega^t,\alpha}+|F|^2_{1,0,\Omega}e^{-2\alpha t}$$
$$+cr_5(\delta^2_1,\delta^2)||u||^2_{1,\Omega^t,\alpha}.$$

From the energy inequality for the operator $\partial/\partial t$ we have

$$(4.27)\qquad |F|^2_{\nu,0,\Omega}e^{-2\alpha t}\leqslant \frac{c}{\alpha}|F|^2_{\nu+1,\Omega^t,\alpha}+|F|^2_{\nu,0,\Omega}|_{t=0},\qquad \nu=1,2,$$

so by Eqs. (4.26) and (4.10) for $\sigma=1$, we obtain the estimate (4.10) for $\sigma=2$, where $c\alpha^{-1}r_5(\delta^2_1,\delta)q_1(\delta_1,\delta)\leqslant q_2(\delta_1,\delta)$, $c[r_3(\delta^2_1,\delta^2)+\alpha^{-1}r_5(\delta^2_1,\delta^2)p_1(\delta_1,\delta)]\leqslant p_2(\delta_1,\delta)$.

Let us consider the case $\sigma=3$. From Eqs. (4.5) and (4.6) we obtain

$$(4.28)\qquad ||D^3_{t,x'}u||^3_{0,\Omega}e^{-2\alpha t}|^t_{t=0}+\alpha||D^3_{t,x'}u||^2_{0,\Omega^t,\alpha}+c_0||D^3_{t,x'}u||^2_{0,\partial\Omega^t,\alpha}$$
$$\leqslant (c_0+c_1)||D^3_{t,x'}g||^2_{0,\partial\Omega^t,\alpha}+\frac{2}{\alpha}||D^3_{t,x'}F||^2_{0,\Omega^t,\alpha}+\frac{2}{\alpha}||D^3_{t,x'}Lu-LD^3_{t,x'}u||^2_{0,\Omega^t,\alpha},$$

where the last term is estimated by

$$(4.29)\qquad \frac{c\delta^2}{\alpha}\left(\sum_{\nu=0}^{3}||D^{3-\nu}_{t,x'}D^\nu_{x_1}u||^3_{0,\Omega^t,\alpha}+||u||^2_{2,\Omega^t,\alpha}\right)\leqslant c\frac{\delta^8\delta^6_1}{\alpha}||D^3_{t,x'}u||^2_{0,\Omega^t,\alpha}$$
$$+\frac{c}{\alpha}\mu_1(\delta_1,\delta)||F||^2_{2,\Omega^t,\alpha}+\frac{c}{\alpha}\mu_2(\delta_1,\delta)||u||^2_{2,\Omega^t,\alpha},$$

where μ_1,μ_2 are polynomials. Assuming that

$$(4.30)\qquad c\delta^8\delta^6_1\leqslant\frac{\alpha^2}{2}$$

Eq. (4.28) and (4.29) imply

$$(4.31)\qquad ||D^3_{t,x'}u||^2_{0,\Omega}e^{-2\alpha t}|^t_{t=0}+\frac{\alpha}{2}||D^3_{t,x'}u||^3_{0,\Omega^t,\alpha}+c_0||D^3_{t,x'}u||^2_{0,\partial\Omega^t,\alpha}$$
$$\leqslant (c_0+c_1)||D^3_{t,x'}g||^2_{0,\partial\Omega^t,\alpha}+\mu_3(\delta_1,\delta)||F||^2_{3,\Omega^t,\alpha}+\mu_4(\delta_1,\delta)||u||^2_{0,\Omega^t,\alpha},$$

where μ_3, μ_4 are polynomials. At last we have

$$(4.32)\qquad \sum_{\nu=1}^{3}||D^{3-\nu}_{t,x'}D^\nu_{x_1}u||^2_{0,\Omega}\leqslant\mu_5(\delta_1,\delta)(||D^3_{t,x'}u||^2_{0,\Omega}+|F|^2_{2,0,\Omega})+\mu_6(\delta_1,\delta)|u|^2_{2,0,\Omega},$$

where μ_5,μ_6 are polynomials. Using Eq. (4.27) for $\nu=2$, from Eqs. (4.29), (4.31) and (4.32) we get Eq. (4.10) for $\sigma=3$. This concludes the proof.

Using [2] we get

THEOREM 4.1.

Let the assumptions of Lemma 4.2 be satisfied and $u_0|_{\partial\Omega} = 0$. Then there exists a unique solution of the problem (4.1) such that $u \in \Pi^{\sigma}_{0,\infty}(\Omega^t)\cap H^{\sigma}_{\alpha}(\Omega^t)\cap H^{\sigma}_{\alpha}(\partial\Omega^t)$, $\sigma = 1, 2, 3$ and the estimate (4.10) is valid.

In the end we add considerations about the relations between the problems (4.1) in a bounded domain and in the half-space.

REMARK 4.1.

First, to prove Theorem 4.1 in a bounded domain it must be assumed that $\partial\Omega \in C^3$. Next, to find the relation between the problems (4.1) in a bounded Ω and in R^3_+ we restrict the problem (4.1) to the neighbourhood Q of the boundary in which $\partial\Omega\cap Q$ is described by the equation $x_1 = F(x')$ (where $x = (x_1, x_2, x_3)$ is the local coordinate system centered in the middle of $\partial\Omega\cap Q$, such that points with $x_1 > 0$ belong to Ω). By transformation

$$y' = x', \quad y_1 = x_1 - F(x'), \tag{4.33}$$

Q is transformed into the half-space $y_1 > 0$. Then $-A\cdot\bar{n} = \sum_{t=1}^{3} A_s \cdot \frac{\partial y_1}{\partial x_s}|y_{1,x}|^{-1} = A_1'$, because $\bar{n} = (-1, F_{x'})/\sqrt{1+F^2_{x'}} = -\left(\frac{\partial y_1}{\partial x_1}, \frac{\partial y_1}{\partial x_2}, \frac{\partial y_1}{\partial x_3}\right)\left(\sum_{i=1}^{3} y^2_{1,x_i}\right)^{-1/2}$. The characteristic polynomial for the matrix $-A_n$ has the form

$$\left[\left(\lambda' - \sum_{i=1}^{3} v_i \frac{\partial y_1}{\partial x_i}\right)^2 - \left(\sum_{i=1}^{3} \omega_i \frac{\partial y_1}{\partial x_i}\right)^2\right]^3 = 0 \quad \text{so} \quad \lambda'_{\mp} = -\sum_{i=1}^{3}[-v_i \pm \omega_i]\frac{\partial y_1}{\partial x_i},$$

has the same signs as $\lambda_{\mp}$ determined by the eigenvalues (2.2) because $(y_{1,x_1}, y_{1,x_2}, y_{1,x_3})$ has the opposite direction to $\bar{n}$ (however, we have to assume additionally that the size of Q is sufficiently small and $\partial\Omega\cap Q$ is sufficiently smooth). Therefore the boundary condition to the problem (4.1) and eigenvectors of the matrix $-A_n$ (M remains unchanged) do not change after the transformation (4.33).

5. Existence of solutions

To prove the existence of solutions of the mixed problems (P_1), (P_2) formulated in Section 2 we shall use the following method of successive approximations:

$$\begin{gathered} E\overset{m+1}{u}_t + \sum_{i=1}^{3} A_i(\overset{m}{v}, \overset{m}{\omega})\overset{m+1}{u}_{x_i} = \begin{pmatrix} f - \nabla\overset{m}{q} \\ 0 \end{pmatrix}, \\ \overset{m+1}{u}\Big|_{t=0} = u_0, \\ M_\nu \overset{m+1}{u}\Big|_{S_\nu} = g_\nu, \quad \nu = 1, \ldots, 4, \end{gathered} \tag{5.1}$$

where $\overset{m+1}{u} = (\overset{m+1}{v}, \overset{m+1}{\omega})$ and

$$\Delta \overset{m}{q} = \operatorname{div} f - \sum_{i,j=1}^{3} (\overset{m}{v}_{i,x_j} \overset{m}{v}_{j,x_i} - \overset{m}{\omega}_{i,x_j} \overset{m}{\omega}_{j,x_i}),$$

(5.2)
$$\left.\frac{\partial \overset{m}{q}}{\partial n}\right|_{S_\nu} = h_\nu, \quad \nu = 1, 2, 3,$$

$$\overset{m}{q}\Big|_{S_4} = \pi,$$

where ν corresponds to the problem (A_ν), $\nu = 1, \ldots, 4$. The forms of M_ν, g_ν, h_ν are described in the definition of (A_ν), $\nu = 1, \ldots, 4$. Moreover, we assume

(5.3)
$$\overset{0}{u} = u_0 = (v_0, \omega_0).$$

The existence of solutions of the problems (5.1), (5.2) is well known. Therefore we can restrict our considerations to get an a priori estimate (independent of m) and convergence of the constructed sequence of solutions of the problems (5.1), (5.2).

Let us consider the problem (P_1). At first we shall obtain an a priori estimate. From Eqs. (5.1) ($\nu = 1,4$) and (4.10) we have

(5.4)
$$|\overset{m+1}{u}|^2_{3,0,\infty,\Omega^t} e^{-2\alpha t} + \frac{\alpha}{2} ||\overset{m+1}{u}||^2_{3,\Omega^t,\alpha} + c_0 ||\overset{m+1}{u}||^2_{3,\partial\Omega^t,\alpha}$$
$$\leqslant s_1(\delta_1, |\overset{m}{u}|_{3,0,\infty,\Omega^t}) [|\overset{m+1}{u}|^2_{3,0,\Omega}|_{t=0} + ||g_1||^2_{3,S_1^t,\alpha} + ||\nabla \overset{m}{q}||^2_{3,\Omega^t,\alpha}$$
$$+ |\nabla \overset{m}{q}|^2_{2,0,\infty,\Omega^t} + ||f||^2_{3,\Omega^t,\alpha} + |f|^2_{2,0,\infty,\Omega^t}],$$

where s_1 is a polynomial determined by Eq. (4.10). From Eqs. (5.2) ($\nu = 1,4$) we obtain

(5.5)
$$|\overset{m}{q}|_{4,2,\infty,\Omega^t} \leqslant c(|\operatorname{div} f|_{2,0,\infty,\Omega^t} + |\overset{m}{u}|^2_{3,1,\infty,\Omega^t} + |h_1|_{3-1/2,1,\infty,S_1^t} + |\pi|_{4-1/2,2,\infty,S_4^t}).$$

To estimate $D_t^3 \overset{m}{q}$ we introduce a quantity $\tilde{\pi}$ which is an extension of π such that $\tilde{\pi}$ vanishes in the neighbourhood of S_1 and

(5.6)
$$|\tilde{\pi}|_{4,1,\infty,\Omega^t} \leqslant c|\pi|_{4-1/2,1,\infty,S_4^t}.$$

Then $\overset{m}{\tilde{q}} = \overset{m}{q} - \pi$ vanishes on S_4 and instead of Eqs. (5.2) we shall consider the problem for $\overset{m}{\tilde{q}}$ which will be denoted by Eq. (5.2)′. Taking the third derivative with respect to t of Eq. (5.2)′, multiplying the result by $D_t^3 \overset{m}{\tilde{q}}$ and integrating over Ω^t, one gets

(5.7)
$$||\nabla D_t^3 \overset{m}{\tilde{q}}||^2_{\Omega^t} = \int_{S_1^t} \left(\frac{\partial}{\partial n} D_t^3 \overset{m}{\tilde{q}} D_t^3 \overset{m}{\tilde{q}} - \bar{n} \cdot D_t^3 f D_t^3 \overset{m}{\tilde{q}} \right) + \int_{\Omega^t} D_t^3 f \cdot \nabla D_t^3 \overset{m}{\tilde{q}}$$
$$- \int_{\Omega^t} \nabla D_t^3 \tilde{\pi} \nabla D_t^3 \overset{m}{\tilde{q}} + \int_{\Omega^t} D_t^3 (\nabla v \nabla v - \nabla \omega \nabla \omega) D_t^3 \overset{m}{\tilde{q}}.$$

Using

$$\sum_{i,j=1}^{3}\int_{\Omega^t} D_t^3 \overset{m}{v}_{i,x_j} \overset{m}{v}_{j,x_i} D_t^3 \overset{m}{\tilde{q}} = \sum_{i,j=1}^{3}\left[\int_{S_1^t} n_j D_t^3 \overset{m}{v}_i \overset{m}{v}_{j,x_i} D_t^3 \overset{m}{\tilde{q}} - \int_{\Omega^t} D_t^3 \overset{m}{v}_i \overset{m}{v}_{j,x_i} D_{x_j} D_t^3 \overset{m}{\tilde{q}}\right],$$

theorems of imbeddings and the Poincaré inequality (because $\overset{m}{\tilde{q}}|_{S_4} = 0$), one has

$$(5.8)\qquad |D_t^3 \overset{m}{\tilde{q}}|^2_{1,1,\infty,\Omega^t} \leqslant c\left(\left\|\frac{\partial}{\partial n} D_t^3 \overset{m}{\tilde{q}}\right\|^2_{S_1^t} + \|D_t^3 f\|^2_{S_1^t} + \|D_t^3 f\|^2_{\Omega^t} + \|\nabla D_t^3 \pi\|^2_{\Omega^t}\right) + c|\overset{m}{u}|^2_{3,2,\infty,\Omega^t}(|\overset{m}{u}|^2_{3,0,\infty,\Omega^t} + \|D_t^3 \overset{m}{u}\|^2_{\partial\Omega^t}).$$

Therefore from Eqs. (5.5), (5.6) and (5.8) we obtain

$$(5.9)\qquad |\overset{m}{q}|^2_{4,1,\infty,\Omega^t} \leqslant c(\|D_t^3 h_1\|^2_{S_1^t} + |h_1|^2_{3-1/2,1,\infty,S_1^t} + |\pi|^2_{4-1/2,1,\infty,S_4^t} + \|D_t^3 f\|^2_{S_1^t} + |f|^2_{3,0,\infty,\Omega^t}) + c|\overset{m}{u}|^2_{3,1,\infty,\Omega^t}(|\overset{m}{u}|^2_{3,0,\infty,\Omega^t} + \|D_t^3 \overset{m}{u}\|^2_{\partial\Omega^t}).$$

Introducing the notations

$$(5.10)\qquad \begin{aligned} \overset{m}{x}{}^2 &= |\overset{m}{u}|^2_{3,0,\infty,\Omega^t} + \|\overset{m}{u}\|^2_{3,\Omega^t} + \|\overset{m}{u}\|^2_{3,\partial\Omega^t},\\ K_1 &= |\overset{m}{u}|^2_{3,0,\Omega}\Big|_{t=0} + \|f\|^2_{3,\Omega^t} + |f|^2_{3,0,\infty,\Omega^t} + |\pi|_{4-1/2,1,\infty,S_4^t},\\ K_2 &= \|D_t^3 f\|^2_{S_1^t} + \|g_1\|^2_{3,S_1^t} + \|D_t^3 h_1\|^2_{S_1^t} + |h_1|^2_{3-1/2,1,\infty,S_1^t}, \end{aligned}$$

from Eqs. (5.4) and (5.9) one gets

$$(5.11)\qquad \overset{m+1}{x}{}^2 \leqslant s_1(\delta_1, \overset{m}{x})[K_1 + K_2 + t\overset{m}{x}{}^4]e^{s_2(\delta_1, \overset{m}{x})t},$$

where $\alpha = s_2(\delta_1, \overset{m}{x})$. Hence we have proved

LEMMA 5.1.

Suppose

a) $|\overset{m}{u}|_{3,0,\Omega}|_{t=0}$ are bounded for each m;

b) $f \in H^3(\Omega^t) \cap \Pi^3_{2,\infty}(\Omega^t)$, $\pi \in \Pi^{4-1/2}_{1,\infty}(S_4^t)$, $\partial\Omega \subset C^4$;

c) $D_t^3 f \in L_2(S_1^t)$, $g_1 \in H^3(S_1^t)$, $h_1 \in \Pi^{3-1/2}_{1,\infty}(S_1^t)$, $D_t^3 h_1 \in L_2(S_1^t)$.

Then there exist sufficiently small t_0, K_1^0, K_2^0 and $X_1(t_0, K_1^0, K_2^0)$ such that for $t \leqslant t_0$, $K_i \leqslant K_0^i$, $i = 1, 2$, and for solutions of the problem (P_1), we have

$$(5.12)\qquad \overset{m+1}{x} \leqslant X_1(t_0, K_1^0, K_2^0), \quad m = 0, 1, \ldots .$$

Let us consider the problem (P_2). From Eqs. (5.1) ($\nu = 2, 3, 4$) and (4.10) we have

$$(5.13)\qquad |\overset{m+1}{u}|^2_{3,0,\infty,\Omega^t}e^{-2\alpha t} + \frac{\alpha}{2}\|\overset{m+1}{u}\|^2_{3,\Omega^t,\alpha} + c_0\|\overset{m+1}{u}\|^2_{3,\partial\Omega^t,\alpha} \leqslant s_1(\delta_1, |\overset{u}{u}|_{3,0,\infty,\Omega^t})\Big[|\overset{m+1}{u}|_{3,0,\Omega}|_{t=0} + \sum_{i=2}^{3}\|g_i\|^2_{3,S_i^t,\alpha} + \|\nabla\overset{m}{q}\|^2_{3,\Omega^t,\alpha} + |\nabla\overset{m}{q}|^2_{2,0,\infty,\Omega^t} + \|f\|^2_{3,\Omega^t,\alpha} + |f|^2_{2,0,\infty,\Omega^t}\Big].$$

Similarly as above we get

$$(5.14)\qquad |\overset{m}{q}|^2_{4,1,\infty,\Omega^t} \leqslant c\Bigg[\sum_{i=2,3}(||D_t^3 h_i||^2_{S_i^t}+|h_i|^2_{3-1/2,1,\infty,S_i^t} + ||D_t^3 f||^2_{S_i^t})+|f|^2_{3,0,\infty,\Omega^t}+|\pi|^2_{4-1/2,1,\infty,S_4^t}\Bigg]+c|\overset{m}{u}|^2_{3,1,\infty,\Omega^t}(|\overset{m}{u}|^2_{3,0,\infty,\Omega^t}+||D_t^3\overset{m}{u}||^2_{\partial\Omega^t}).$$

Equations (2.20) and (2.21) imply

$$(5.15)\qquad ||D_t^3 h_\nu||^2_{S_\nu^t}+|h_\nu|^2_{3-1/2,1,\infty,S^t} \leqslant c[||D_t^3 f||^2_{S_\nu^t}+|f|^2_{3,0,\infty,\Omega^t} + (|\overset{m}{u}|^2_{3,1,\infty,\Omega^t}+||D_t^3\overset{m}{u}||^2_{\partial\Omega^t})|e_\nu|^2_{4-1/2,1,\infty,S_\nu^t}+||D_t^4 e_\nu||^2_{S_\nu^t}],$$

where $\nu = 2, 3$, $e_2 = \alpha$, $e_3 = \beta$. Using the notations $(5.10)_{1,2}$ and

$$(5.16)\qquad K_3 = \sum_{\nu=2,3}(||D_t^3 f||^2_{S_\nu^t}+||D_t^4 e_\nu||^2_{S_\nu^t}+|e_\nu|^2_{4-1/2,1,\infty,S_\nu^t}+||g_\nu||^2_{3,S_\nu^t})$$

from Eqs. (5.13), (5.14) and (5.15) we obtain

$$(5.17)\qquad \overset{m+1}{x^2} \leqslant s_1(\delta_1,\overset{m}{x})[K_1+K_3+t\overset{m}{x}{}^2(1+K_3\overset{m}{x}{}^2)]e^{s_2(\delta_1,\overset{m}{x})t}.$$

Therefore we have carried out the proof.

LEMMA 5.2.

Let the assumptions a, b of Lemma 5.1 be satisfied and

d. $D_t^3 f\in L_2(S_\nu^t)$, $e_\nu\in\Pi^{4-1/2}_{1,\infty},(S_\nu^t)$, $D_t^4 e_\nu\in L_2(S_\nu^t)$, $g_\nu\in H^3(S_\nu^t)$, $\nu = 2,3$, $e_2=\alpha$, $e_3=\beta$.

Then there exist sufficiently small t_0', K_1', K_3' and $X_2(t_0', K_1', K_2')$ such that for $t\leqslant t'$, $K_1\leqslant K_1'$, $K_3\leqslant K_3'$, and for solutions of the problem (P_2) the following estimate

$$(5.18)\qquad \overset{m+1}{x} \leqslant X_2(t'\,K_1', K_3'),\qquad m = 0,1,\dots,$$

holds.

Now to estimate $|\overset{m+1}{u}|_{3,0,\Omega}|_{t=0}$ we must consider the following problems obtained from Eqs. (5.1) and (5.2):

$$(5.19)\qquad D_t^{s+1}\overset{m+1}{u}|_{t=0} = D_t^s\Big(-\sum_{i=1}^3 A_i(\overset{m}{u})\overset{m+1}{u}_{x_i}+f-\nabla\overset{m}{q}\Big)\Big|_{t=0},$$

and

$$\Delta D_t^s\overset{m}{q}|_{t=0} = D_t^s(\operatorname{div} f-(\nabla\overset{m}{v}\nabla\overset{m}{v}-\nabla\overset{m}{\omega}\nabla\overset{m}{\omega}))|_{t=0},$$

$$(5.20)\qquad \frac{\partial}{\partial n}D_t^s\overset{m}{q}|_{t=0} = D_t^s h_\nu|_{t=0}\quad\text{on}\quad S_\nu,\ \nu=1,2,3,$$

$$D_t^s\overset{m}{q}|_{t=0} = D_t^s\pi|_{t=0}\quad\text{on}\quad S_4,$$

where $s = 0, 1, 2$. We consider simultaneously the problems (P_1) and (P_2) where the expressions in brackets { } are for (P_2) and replace the expression before { }. It is sufficient to consider only the time derivatives in $|u|_{3,0,\Omega}$. From Eq. (5.19) we have

$$(5.21)\qquad ||D_t^3 \overset{m+1}{u}(0)||_\Omega \leqslant \sigma(||D_t^2 f(0)||_\Omega, ||D_t^2 \nabla \overset{m}{q}(0)||_\Omega, ||D_t^2 \overset{m+1}{u_x}(0)||_\Omega),$$

where $\chi(0) = \chi|_{t=0}$ and σ describes polynomial type dependence. Now from the problem (5.20) we have

$$(5.22)\qquad ||\nabla D_t^2 \overset{m}{q}(0)||_\Omega \leqslant c\big(||D_t^2 h_1(0)||_{S_1}\{||D_t^2 h_2(0)||_{S_2} + ||D_t^3 h_3(0)||_{S_3}\} + ||D_t^2 f(0)||_{S_1} + ||D_t^2 f(0)||_\Omega + ||\nabla D_t^2 \tilde{\pi}(0)||_\Omega + ||D_t^2(\nabla \overset{m}{u} \nabla \overset{m}{u})(0)||_\Omega\big)$$

and from Eq. (5.19) we get

$$(5.23)\qquad ||D_t^2 \nabla \overset{m+1}{u}(0)||_\Omega \leqslant \sigma(||D_t^1 \nabla f(0)||_\Omega, ||D_t^1 \nabla^2 \overset{m}{q}(0)||_\Omega, ||D\nabla^2 \overset{m+1}{u}(0)||_\Omega).$$

Therefore we have to estimate the second and third norms in the right-hand side of the inequality (5.23). By Eq. (5.20) we have

$$(5.24)\qquad ||D_t^1 \overset{m}{q}(0)||_{2,\Omega} \leqslant c\big(||D_t^1 f(0)||_{1,\Omega} + ||D_t^1(\nabla \overset{m}{u} \nabla \overset{m}{u})(0)||_\Omega + ||D_t^1 h_1(0)||_{1/2,S_1} \cdot \{||D_t^1 h_2(0)||_{1/2,S_2} + ||D_t^1 h_3(0)||_{1/2,S_3}\} + ||D_t^1 \pi(0)||_{3/2,S_4}\big)$$

and by Eq. (5.19)

$$(5.25)\qquad ||D_t^1 \overset{m+1}{u}(0)||_{2,\Omega} \leqslant \sigma(||f(0)||_{2,\Omega}, ||\overset{m}{q}(0)||_{3,\Omega}, ||\nabla^3(\overset{m}{u}\overset{m}{u})(0)||_\Omega).$$

At last we have to estimate

$$(5.26)\qquad ||\overset{m}{q}(0)||_{3,\Omega} \leqslant c\big(||f(0)||_{2,\Omega} + ||u_0||_{3,\Omega} + ||\pi(0)||_{5/2,S_4} + ||h_1(0)||_{3/2,S_1} \cdot \{||h_2(0)||_{3/2,S_2} + ||h_3(0)||_{3/2,S_3}\}\big).$$

Hence we have obtained

LEMMA 5.3

Let a) $\partial\Omega \in C^4$, $D_t^2 f(0) \in L_2(S_1)$, $f(0) \in \Gamma_0^2(\Omega)$, $u_0 \in H^3(\Omega)$, $\pi \in \Gamma_1^{5/2}(S_4)$, b) $h_i(0) \in \Gamma_2^{3/2}(S_i)$, $D_t^2 h_i(0) \in L_2(S_i)$, where $i = 1$ for (P_1) and $i = 2, 3$ for (P_2).

Then $|u|_{3,0,\Omega}|_{t=0}$ is estimated by norms of quantities described in the assumptions a and b.

REMARK 5.1

To satisfy the assumption b of Lemma 5.3 we have to assume that $b(0), d(0) \in \Gamma_1^{3-1/2}(S_1)$, $D_t^3 b(0)$, $D_t^3 d(0) \in L_2(S_1)$ for $i = 1$ and $e_i(0) \in \Gamma_1^3(S_i)$, $D_t^3 e_i(0) \in L_2(S_i)$ for $i = 2, 3$, where $e_2 = \alpha$, $e_3 = \beta$.

To prove convergence of $\{\overset{m}{u}, \overset{m}{q}\}$ we introduce $\overset{m}{U} = \overset{m}{u} - \overset{m-1}{u}$, $\overset{m}{Q} = \overset{m}{q} - \overset{m-1}{q}$, and we shall show it in $\Pi^1_{0,\infty}(\Omega^t) \cap H^1(\Omega^t) \cap H^1(\partial\Omega^t)$ and $\Pi_1^2(\Omega^t)$, respectively. From Eqs. (5.1) and (5.2) the following problems for differences $\overset{m}{U}, \overset{m}{Q}$ are obtained:

$$E \overset{m+1}{U}_t + \sum_{i=1}^3 A_i(\overset{m}{u}) \overset{m+1}{U}_{x_i} + \sum_{i=1}^3 A_i(\overset{m}{U}) \overset{m}{u}_{x_i} = -\begin{pmatrix} \nabla \overset{m}{Q} \\ 0 \end{pmatrix},$$

$$(5.27)\qquad \overset{m+1}{U}|_{t=0} = 0,$$

$$M_\nu \overset{m+1}{U}|_{S_\nu} = 0, \quad \nu = 1, \ldots, 4,$$

where $A_i(\overset{m}{U})\overset{m}{u}_{x_i} = E\overset{m}{u}_{x_i}\overset{m}{\vartheta}_i + \begin{pmatrix} 0 & -E_1 \\ -E_1 & 0 \end{pmatrix} \overset{m}{u}_{x_i}\overset{m}{\Omega}_i, \overset{m}{\vartheta} = \overset{m}{v} - \overset{m-1}{v}, \overset{m}{\Omega} = \overset{m}{\omega} - \overset{m-1}{\omega}, E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$\overset{0}{U} = \overset{0}{u} = u_0$, and

$$\Delta \overset{m}{Q} = -\sum_{i,j=1}^{3} (\overset{m}{\vartheta}_{i,x_j}\overset{m-1}{v}_{j,x_i} + \overset{m}{v}_{i,x_j}\overset{m}{\vartheta}_{j,x_i} - \overset{m}{\Omega}_{i,x_j}\overset{m}{\omega}_{j,x_i} - \overset{m-1}{\omega}_{i,x_j}\overset{m}{\Omega}_{j,x_i}),$$

(5.28)
$$\left.\frac{\partial \overset{m}{Q}}{\partial n}\right|_{S_\nu} = H_\nu, \quad \nu = 1, 2, 3,$$

$$\overset{m}{Q}|_{S_4} = 0,$$

where

$$H_1 = 0,$$

(5.29)
$$H_i = \overset{m}{a}_i \sum_{k=1}^{3} \nabla n^k e_{ik} - \sum_{j=1}^{2} \overset{m}{a}_{i\tau_j}\overline{\tau}_j \cdot \nabla e_{in} + \overset{m}{a}_n \Big[e_{in} \operatorname{div} \overline{n} + \sum_{j=1}^{2} (\overline{\tau}_j \cdot \nabla e_{i\tau_j} + e_{i\tau_j} \operatorname{div} \overline{\tau}_j) \Big], \quad i = 2, 3,$$

where $\overset{m}{a}_2 = \overset{m}{\vartheta} - \overset{m}{\Omega}$, $\overset{m}{a}_3 = \overset{m}{\vartheta} + \overset{m}{\Omega}$, $e_2 = \alpha$, $e_3 = \beta$.

Using Lemma 4.2 for $\sigma = 1$ from Eq. (5.27) one has

(5.30)
$$|\overset{m+1}{U}|^2_{1,0,\infty,\Omega^t} + \frac{\alpha}{2} ||\overset{m+1}{U}||^2_{1,\Omega^t} + c_0 ||\overset{m+1}{U}||_{1,\partial\Omega^t} \leqslant c p_1(\delta_1, \delta) |\overset{m}{Q}|^2_{2,1,\Omega^t} e^{2\alpha t}$$

and from Eq. (5.28)

(5.31)
$$|\overset{m}{Q}|_{2,2,\infty,\Omega^t} \leqslant c(|\overset{m}{u}|_{3,3,\infty,\Omega^t} |\overset{m}{U}|_{1,1,\infty,\Omega^t} + K),$$

where $K = 0$ for the problem (P_1) and $K = \sum_{i=2}^{3} |H_i|_{1/2,0,\infty,S_i^t}$ for (P_2).

Comparing Eq. (5.30) with Eq. (5.31) we see that $D_t D_x \overset{m}{Q}$ must be estimated too. To do this we consider the time derivative of the problem (5.28). Then, after repeating the considerations which were necessary to get Eq. (5.8), we have

(5.32)
$$||\nabla D_t \overset{m}{Q}||_{\Omega^t} \leqslant c(|\overset{m}{u}|_{3,2,\infty,\Omega^t} + |\overset{m-1}{u}|_{3,2,\infty,\Omega^t})(|\overset{m}{U}|_{1,0,\infty,\Omega^t} + ||\overset{m}{U}||_{1,\partial\Omega^t}).$$

At last, from Eqs. (5.30) ÷ (5.32) we obtain

(5.33)
$$|\overset{m+1}{U}|^2_{1,0,\infty,\Omega^t} + ||\overset{m+1}{U}||^2_{1,\Omega^t} + ||\overset{m+1}{U}||^2_{1,\partial\Omega^t} \leqslant s_3(\delta_1, \overline{X}_i) e^{s_4(\delta_1, \overline{X}_i)t} \overline{X}_i^2 (|\overset{m}{U}|^2_{1,0,\infty,\Omega^t} + ||\overset{m}{U}||^2_{1,\partial\Omega^t}),$$

where s_3, s_4 are polynomials, $\overline{X}_i = X_i$ for the problem (P_i), $i = 1, 2$. Therefore for sufficiently small $\bar{X}_i$, $i = 1, 2$, Eq. (5.33) implies that the sequence $\{\overset{m}{u}, \overset{m}{q}\}$ converge in the above mentioned spaces. Let us assume

$$(5.34) \quad \partial\Omega \in C^4, \quad v_0, \omega_0 \in H^3(\Omega), \quad f \in \Pi_0^3(\Omega^t) \cap H^3(\Omega^t), \quad \pi \in \Pi_1^{4-1/2}(S_4^t),$$

(v_0, ω_0) satisfy the assumptions (C_i), $i = 1, \dots, 4$ (see also Remarks 5.3 and 5.6);

$$(5.35) \quad D_t^3 f \in L_2(S_1^t), \quad b, d \in \Pi_1^{4-1/2}(S_t^1) \cap \tilde{\Pi}_3^4(S_t^1),$$

$$(5.36) \quad D_t^3 f \in L_2(S_2^t) \cap L_2(S_3^t), \quad e_i \in \Pi_1^{4-1/2}(S) \cap \tilde{\Pi}_3^4(S_i^t), \quad \text{where} \quad i = 2, 3, \quad e_2 = \alpha, \quad e_3 = \beta.$$

Hence we can formulate the main result of this paper:

THEOREM. *Let Eqs.* (5.34), (5.35) (*or* (5.36)) *be satisfied. Assume that time* t_0 (*or* t_0') *and data functions are so small that Eq.* (5.12) (or *Eq.* (5.18)) *holds,* Equation (5.33) *implies the convergence of the considered sequences and Remark* 5.3 *must be taken into consideration. Moreover, let the compatibility condition* (2.22) *hold. In the case of the problem* (P_1), *we assume additionally that Eq.* (2.13) *is valid on* S_1.

Then there exists a unique solution of the problem (P_1) (*or* (P_2)) *such that*

$$(5.37) \quad u \in \Pi_{0,\infty}^3(\Omega^t) \cap H^3(\Omega^t) \cap H_3(\partial\Omega^t), \quad q \in \Pi_1^4(\Omega^t), \quad t \leqslant t_0 (\text{or } t \leqslant t_1).$$

REMARK 5.2

From Eq. (5.37) and theorems of imbedding it follows that Eqs. (1.1) and (1.2), initial and boundary conditions are satisfied classically.

REMARK 5.3

Proving the existence of solutions of the problems (P_1), (P_2) the conditions $v_n|_{S_1} < 0$, $\omega_n|_{S_1} < 0$, etc., must be assumed. Moreover, the eigenvalues (2.2) must be separated from zero also. Assuming that they are separated from zero in the initial moment (we assume stronger restrictions: $v_0 \cdot \bar{n}|_{S_1} \leqslant -a_0 < 0$, etc.) by the continuity of solutions with respect to time we can satisfy them for sufficiently small time also. This is the other restriction on the existence time.

REMARK 5.4

At each step a solution of the problems (5.1), (5.2) is such that $\operatorname{div} \overset{m}{v} \neq 0$, $\operatorname{div} \overset{m}{\omega} \neq 0$, $m = 1, 2, \dots,$. To show that

$$(5.38) \quad \lim_{m \to 0} \operatorname{div} \overset{m}{v} = \lim_{m \to 0} \operatorname{div} \overset{m}{\omega} = 0$$

we apply the divergence operator to Eq. (5.1) and use Eq. (5.2) so we obtain the problem

$$(5.39) \quad \begin{aligned} &\overset{m+1}{\chi}_t + \sum_{i=1}^{3} B_i(\overset{m}{u}) \overset{m+1}{\chi}_{x_i} = H(\overset{m}{u}, \overset{m+1}{u}), \\ &\overset{m+1}{\chi}|_{t=0} = 0, \quad N \overset{m+1}{\chi}|_{S_\nu} = 0, \quad \nu = 1, \dots, 4, \end{aligned}$$

where the boundary conditions (2.8) for $\overset{m+1}{v}$, $\overset{m+1}{\omega}$ were used. Moreover,

$$H = (H_1, H_2), \quad \text{where} \quad H_1 = \sum_{i,j=1}^{3} [-\overset{m}{v}_{i,x_j} \overset{m+1}{\vartheta}_{j,x_i} + \overset{m}{\omega}_{i,x_j} \overset{m+1}{\Omega}_{j,x_i}],$$

$$H_2 = \sum_{i,j=1}^{3} [\overset{m}{\omega}_{i,x_j} \overset{m+1}{\vartheta}_{j,x_i} - \overset{m}{v}_{i,x_j} \overset{m+1}{\Omega}_{j,x_i}].$$

Then from $\lim_{m\to\infty} H(\overset{m}{u}, \overset{m+1}{u}) = 0$ Eqs. (5.39) imply Eq. (5.38).

At last we shall discuss the uniqueness problem for solutions of $(\mathbf{P}_1)$ and $(\mathbf{P}_2)$.

REMARK 5.5

Let us assume that we have two different solutions (u_i, q_i), $i = 1, 2$, of the problems $(\mathbf{P}_1)$ and $(\mathbf{P}_2)$. Then, repeating considerations which imply Eq. (5.33), we obtain it in the form where $\overset{m+1}{U}$ and $\overset{m}{U}$ are replaced by $U = u_1 - u_2$. Therefore we get uniqueness for sufficiently small $\bar{X}_i$, $i = 1, 2$. Thus we obtain uniqueness for solutions of class $C^{1+\alpha}(\Omega^t)$ in $\Pi^1_{0,\infty}(\Omega^t) \times H^1(\Omega^t_i) \times H^1(\partial\Omega^t)$ and the bound on t is determined in the Theorem.

REMARK 5.6

Consider the problem $Lu = f$, $u|_{t=0} = u_0 = (v_0, \omega_0)$, $Mu|_{\partial\Omega} = g$, where $u|_{\partial\Omega}$ does not vanish in general. Introduce a function ω such that $u - \omega|_{\partial\Omega}$ vanishes in a neighbourhood of $\partial\Omega$. Therefore we consider the problem: $Lu' = f - L\omega \equiv f'$, $u'|_{t=0} = u_0 - \omega|_{t=0} \equiv u_0'$, $Mu'|_{\partial\Omega} = g - M\omega|_{\partial\Omega} \equiv g'$ and $u = u' + \omega$. Hence knowing that $u'|_{\partial\Omega|t=0} = 0$ some compatibility conditions on g' for $t = 0$ must be added.

References

1. L. LANDAU and E. LIFSZIC, *Magneto-electro-dynamics*, Moscow 1954 [in Russian].
2. W. M. ZAJĄCZKOWSKI, *Noncharacteristic mixed problems for nonlinear symmetric hyperbolic systems* [to be published].
3. A. MAJDA and S. OSHER, *Initial boundary value problems for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math., **28**, 607–675, 1975.
4. V. A. SOLONNIKOV and W. M. ZAJĄCZKOWSKI, *On the Neumann problem for elliptic equations of second order in domains with edges on the boundary*, Zap. Nauch. Sem. LOMI 127, 7–48, 1983 [in Russian].
5. W. M. ZAJĄCZKOWSKI, *Existence and regularity properties of solutions of some elliptic system in domains with edges* [to be published].

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received May 20, 1985.