Archives of Mechanics • Archiwum Mechaniki Stosowanej • 30, 4–5, pp. 657–661, Warszawa 1978

Bifurcation of stationary solutions for the quasi-geostrophic equation with nonlinear boundary conditions

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THE BOUNDARY value problem with nonlinear boundary conditions is transformed into an integral equation with nonlinear boundary terms in it. It is shown that under certain conditions, which are satisfied in the applications, the nonlinear integral operator is a completely continuous operator in the space of continuous functions, and its Frechet derivative exists. This permits the application of a theorem by Krasnoselski which gives criteria for the existence of bifurcating solutions. The proof can be applied to other problems where an integral formulation is possible.

Problem brzegowy z nieliniowymi warunkami brzegowymi został sprowadzony do równania całkowego z członami nieliniowymi. Wykazano, że przy pewnych warunkach, które zwykle w zastosowaniach praktycznych są spełnione, nieliniowy operator całkowy jest operatorem pełnociągłym i istnieje jego pochodna Frécheta. Fakt ten pozwala na zastosowanie twierdzenia Krasnosielskiego, które dostarcza kryteriów na istnienie rozwiązań bifurkacyjnych. Niniejsze podejście może być wykorzystane również w innych zagadnieniach, w których możliwe jest sformułowanie całkowe.

Краевая задача с нелинейными граничными условиями сведена к интегральному оператору с нелинейными граничными членами. Показано, что при определенных усовиях, которые выполняются в приложениях, нелинейный интегральный оператор и его производная Фреша, являются вполне непрерывными операторами в пространстве непрерывных функций. Это позваляет применять теорему Красносельского, которая устанавливает критерий существования разветвленных решений. Доказательство может быть применено и к другим задачам, допускающим интегральную постановку.

WE SHALL examine CHARNEY'S formulation [1] of the quasi-geostrophic problem describing the large scale atmospheric flow in middle latitudes. It will be shown that if a linear problem to be defined later does have eigenvalues of odd multiplicity, then bifurcation will occur at these eigenvalues in the nonlinear problem.

In order to simplify the calculations, the nonlinear boundary conditions will be extended to the entire boundary, but the proof can be modified in such a way that it carries over to Charney's boundary conditions. The problem will be reformulated as an integral equation, whereby the domain of the unknown streamfunction will be extended from the usual set of infinitely often differentiable functions. The result derived concern the integral equation.

Given two strictly positive and infinitely often differentiable functions $k_1(z)$ and $k_2(z)$ (z is the vertical coordinate) on $0 \leqslant z \leqslant 1$.

Let $\psi(x, y, z)$ be an infinitely often differentiable function defined on the domain

$$D: \quad 0 < x < l, \quad 0 < y < 1, \quad 0 < z < 1.$$

We denote the boundary of D by ∂D.

On ∂D the stream function ψ must satisfy prescribed conditions of the form

$$\partial\psi/\partial\mathbf{n} = B\psi + 2(\psi), \tag{1}$$

where $B \neq 0$ is a uniformly bounded function defined on ∂D, $\mathbf{n}$ is the direction of the exterior unit normal vector to D on ∂D. h is a function *of* ψ and the coordinates of the boundary. It has Frechet derivatives of up to and including second order with $h(0) = h'(0) \equiv 0$ and $h''(0) \neq 0$. From Charney's problem the following nonlinear elliptic differential equation on D for the unknown stationary streamfunction ψ can easily be derived.

$$k_1(\partial^2/\partial x^2 + \partial^2/\partial y^2)\psi + \partial/\partial z(k_2\,\partial\psi/\partial z) = k_1 f(\psi, z), \tag{2}$$

where the Frechet derivatives of f of up to and including second order exist with $f(0, z) = = \theta \equiv 0$. It is easily seen that $\psi = \theta \equiv 0$ is a solution of the problem under these conditions.

In this paper we shall consider the related problem

(3) for
$$u(P) = \int\limits_D k_1 Gf(u(Q), Q)\,dQ - \int\limits_{\partial D_v} k_1 Gh(u(Q), Q)\,dQ - \int\limits_{\partial D_H} k_2 Gh(u(Q), Q)\,dQ,$$
$$u = \mathbf{G}u = \mathbf{F}u + \mathbf{H}_1 u + \mathbf{H}_2 u,$$

where Q is a point in $\bar{D}$, and G is the Green's function that is a solution of the boundary value problem

$$k_1(\partial^2/\partial x^2 + \partial^2/\partial y^2)G + \partial/\partial z(k_2 \partial G/\partial z) = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) \tag{4}$$

and

$$\partial G/\partial\mathbf{n} - BG = 0 \quad \text{on} \quad \partial D. \tag{5}$$

∂D_v denotes the vertical boundaries and ∂D_H the horizontal boundaries. The function u is continuous $(u \in C(\bar{D}))$, and the operators $\mathbf{G}$, $\mathbf{F}$, $\mathbf{H}_1$ and $\mathbf{H}_2$ map $C \times \bar{D}$ and $C \times \partial D$ respectively into C.

We make the following assumptions:

H.1. $\mathbf{F}$, $\mathbf{H}_1$ and $\mathbf{H}_2$ have Frechet derivatives $\mathbf{F}'$, $\mathbf{H}_1'$ and $\mathbf{H}_2'$, respectively, at $u = \theta \equiv 0$, and the operators $\mathbf{G}$, $\mathbf{F}$, $\mathbf{H}_1$ and $\mathbf{H}_2$ satisfy the conditions $\mathbf{G}\theta = \theta$, $\mathbf{F}\theta = \theta$, $\mathbf{H}_1\theta = \theta$ and $\mathbf{H}_2\theta = \theta$, respectively.

H.2. For every fixed $u \in C(\bar{D})$ f and h are continuous functions of Q, and they are uniformly bounded.

It is not difficult to show that these assumptions are satisfied for a large number of quasi-geostrophic flow topographies.

We want to prove that the operator $\mathbf{G}$ is completely continuous. In order to do so we shall first prove the complete continuity of the operator $\mathbf{H}_2$

$$\mathbf{H}_2 u = -\int\limits_{\partial D_H} k_2 G(P, Q)h(u(Q), Q)\,dQ. \tag{6}$$

Let us also consider the linear operator

$$\mathbf{L}u = -\int\limits_{\partial D_H} k_2 G(P, Q)\phi(Q)\,dQ, \quad \phi \in C(\partial D_H) \tag{7}$$

for an arbitrary fixed $P \in \bar{D}$.

We now define the distance between the points P and Q

$$r = |P-Q|$$

and split up the kernel G in the following way:

$$G_1(P,Q) = \begin{cases} G(P,Q) & \text{for } r \geqslant \eta_n, \\ 0 & \text{for } r < \eta_n, \end{cases}$$

$$G_2(P,Q) = \begin{cases} 0 & \text{for } r \geqslant \eta_n, \\ G(P,Q) & \text{for } r < \eta_n. \end{cases}$$

We denote the domain $\Omega_n = \partial D \cap \{P \in \bar{D}, Q \in \partial D_H | r > \eta_n\}$, where η_n is a positive real number. We can write

$$(8) \qquad -\int_{\partial D_H} k_2 G(P,Q)\phi(Q)dQ = -\int_{\partial D_H} k_2 G_1(P,Q)\phi(Q)dQ - \int_{\partial D_H} k_2 G_2(P,Q)\phi(Q)dQ$$
$$= \mathbf{G}_1 u + \mathbf{G}_2 u.$$

We start out to prove that the operator $\mathbf{L}$ is completely continuous as an operator from $C(\Omega_n) \to C(\Omega_n)$.

Proof. Let $\{\phi_n(Q)\}$ be a set of uniformly bounded functions defined on a set of domains $\{\Omega_n\}$, $||\phi_n|| \leqslant k$.

Since G has a simple pole at $r = 0$, and is a uniformly continuous function everywhere else, the functión

$$\sigma_n(P) = \int_{\Omega_n} k_2 G(P,Q)\phi_n(Q)dQ$$

is uniformly bounded. If, for example, the area of the domain of integration is S_n then

$$||\sigma_n(P)|| \leqslant M \cdot k \cdot S_n,$$

where $M = \max G(P,Q) \cdot k_2$, $P, Q \in \Omega_n$.

Since $G(P,Q)$ is uniformly continuous in $\bar{D} \times \Omega_n$, then a $\delta > 0$ exists such that for any prescribed $\varepsilon > 0$

$$||k_2||\,||G(P_1,Q) - G(P_2,Q)|| < \frac{\varepsilon}{Sk}$$

for $|P_1 - P_2| < \delta$ and every $Q \in \Omega_n$. S is the area of ∂D_H.

Then we have

$$(9) \qquad ||\sigma_n(P_2) - \sigma_n(P_1)|| \leqslant \int_{\Omega_n} ||k_2||\,||G(P_1,Q) - G(P_2,Q)||\,||\phi_n(Q)||dQ < \varepsilon$$

for all functions $\sigma_n(P)$ when only $|P_2 - P_1| < \delta$.

Arzela's theorem then shows that the set $\{\sigma_n(P)\}$ is compact. Since the operator $\mathbf{G}_1$ maps every bounded set $\{\phi_n(Q)\}$ into a compact set in $C(\Omega_n)$, then $\mathbf{G}_1$ is *completely continuous.*

As the next step we choose a sequence of operators $\{\mathbf{G}_{1n}\}$ which converges towards $\mathbf{G}$ — i.e. a sequence of radii $\{\eta_n\}$ in which $\eta_n \to 0$ for $n \to \infty$.

In the neighbourhood of $r = 0$ we can express the kernel G in the following way:

$$G(P, Q) = \frac{\Phi(P, Q)}{r} + R(P, Q),$$

where as well $\Phi(P, Q)$ as $R(P, Q)$ are uniformly bounded for $r \to 0$ (i.e. $P \to Q$);

$$(k_2 \Phi(P, Q) \leqslant C_1,\ k_2 R(P, Q) \leqslant C_2).$$

We now have

$$(10) \quad \|\mathbf{L}\phi - \mathbf{G}_{1n}\phi\| = \max\Big(\int\limits_{\partial D_H} (k_2 G(P, Q)\phi(Q)dQ - \int\limits_{\Omega_n} k_2 G_{1n}(P, Q)\phi(Q)dQ \Big)$$
$$\leqslant \max\Big(\int\limits_{r<\eta_n} k_2 G_2(P, Q)\phi(Q)dQ \Big) = \max\Big(\int\limits_{r<\eta_n} k_2 \frac{\Phi(P, Q)}{r} \phi(Q)dQ$$
$$+ \int\limits_{r<\eta_n} k_2 R(P, Q)\phi(Q)dQ \Big) \leqslant C_1 \cdot 2\pi \int\limits_0^{\eta_n} \phi(Q)dr + C_2 \cdot 2\pi \int\limits_0^{\eta_n} \phi(Q)rdr$$
$$\leqslant \pi k (2C_1 \eta_n + C_2 \eta_n^2) < \varepsilon$$

if only

$$\eta_n < \frac{\varepsilon}{2\pi(C_1 + C_2)k} < 1.$$

We have now shown that

$$\|\mathbf{L}\phi - \mathbf{G}_{1n}\phi\| \to 0 \quad \text{for} \quad n \to \infty,$$

and since all $\mathbf{G}_{1n}$ are completely continuous in the norm of the space C, so is $\mathbf{L}$ according to Theorem 2 on page 146 in LYUSTERNIK and SOBOLEV [2].

Since the linear operator $\mathbf{L}$ $(C(\partial D_H) \to C(\partial D_H))$ is completely continuous and the operator $\mathbf{h}$ $(\mathbf{h}\phi = h(\phi, Q))$ acts from $C(\partial \bar{D}_H)$ to $C(\partial D_H)$ and is both continuous and bounded according to the assumption *H.2.*, then the operator $\mathbf{H} = \mathbf{Lh}$ acts in $C(\partial D_H)$ and is completely continuous according to the argument on page 46 in KRASNOSIELSKI [3].

The proof of complete continuity of the operators $\mathbf{H}_1$ and $\mathbf{F}$ is analogous. Since the sum of $\mathbf{F}$, $\mathbf{H}_1$ and $\mathbf{H}_2$ is the operator $\mathbf{G}$ and the sum of three completely continuous operators is itself a completely continuous operator, then $\mathbf{G}$ is completely continuous. Now, Theorem (2.1) on page 196 in KRASNOSIELSKI [3] applies to the operator $\mathbf{G}$. Thus the existence of bifurcating solutions is proven provided the linearized integral operator defined in the theorem has eigenvalues of odd multiplicity.

Acknowledgement

This work was done during the author's stay at the California Institute of Technology. It was supported by a grant awarded under the NATO Science Fellowship Programme and grant nr 511-5024 from the Danish Natural Science Research Council. I should like to thank Professor G. Iooss for a comment that led to an improvement in the original manuscript.

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Received September 27, 1977.