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On the isolated liquid volume motion

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THIS work investigates some qualitative properties of an unsteady, isolated liquid volume motion bounded by a free surface. The motion arises from a pre-determined initial state. External volume forces are absent. The liquid can be viscous or ideal, with a surface tension or without it.

W pracy zbadano kilka jakościowych własności nieustalonego przepływu izolowanej objętości cieczy ograniczonej powierzchnią swobodną. Ruch powstaje z wcześniej określonego stanu początkowego. Zewnętrzne siły masowe nie występują. Ciecz może być lepka lub idealna, z napięciem powierzchniowym lub bez napięcia powierzchniowego.

В работе исследовано несколько качественных свойств неустановившегося течения изолированного объема, ограниченного свободной поверхностью. Движение возникает из заданного начального состояния. Внешние массовые силы отсутствуют. Жидкость может быть вязкой или идеальной, с поверхностным натяжением или без поверхностного натяжения.

1. Formulation of the problem

A MATHEMATICAL formulation of the problem of the isolated volume motion is as follows: it is necessary to find the region $\Omega_t \in \mathsf{R}^3$ and the solution $v(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t))$, $p(x, t)$ of the system of the Navier-Stokes equations

$$v_t + v\cdot\nabla v = -\nabla p + \nu\Delta v, \quad \nabla\cdot v = 0 \tag{1.1}$$

for this region, so that at the boundary Γ_t of the region Ω_t the boundary conditions

$$v\cdot n = \mathscr{D}, \quad x\in\Gamma_t, \tag{1.2}$$

$$pn - 2\nu S\cdot n = 2\sigma K n, \quad x\in\Gamma_t, \tag{1.3}$$

and the initial condition

$$v(x, 0) = v_0(x), \quad x\in\Omega_0 \equiv \Omega \tag{1.4}$$

for $t = 0$ are satisfied, the region Ω being prescribed and bounded. Here v denotes the velocity vector, p the pressure of the liquid, $\nu \geqslant 0$ its viscosity, $\sigma \geqslant 0$ the surface tension coefficient; ν and σ are assumed to be constant. The liquid density is assumed to be equal to unity. The velocity of the motion of the surface Γ_t in the direction of the external normal is denoted by $\mathscr{D}$ in the relation (1.2), and the unity vector of the external normal to Γ_t is denoted by n. If the surface Γ_t is given by the equation $F(x, t) = 0$, then $\mathscr{D} = -F_t/|\nabla F|$. In the condition (1.3) S denotes the deformation velocity tensor with the elements $S_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$, and K denotes a double surface curvature Γ_t. It is assumed that $K > 0$ if Γ_t is a convex outward liquid.

The equality (1.2) indicates that the surface Γ_t bounds the liquid volume Ω_t. According to Eqs. (1.1) and (1.3), this volume moves under inertia. i.e. at the absence of external volume and surface forces, and its boundary Γ_t is a free surface.

The vector field v_0 in Eqs (1.4) is assumed to be prescribed and solenoidal:

$$\nabla \cdot v_0 = 0, \quad x \in \Omega. \tag{1.5}$$

If $\nu > 0$, v_0 satisfies the condition of agreement with the condition (1.3),

$$S \cdot n - (n \cdot S \cdot n)n = 0 \quad \text{if} \quad x \in \Gamma_0, \quad t = 0. \tag{1.6}$$

The main difficulty in investigating the problem (1.1)–(1.4) is that it is necessary to find the region Ω_t. However, the specificity of the condition (1.2) allows to transform this problem to the other one in which the region of the determination of the solution is pre-fixed. It is achieved by the transition to the Lagrangian coordinates.

The trajectory of the particle which is at the point ξ at the instant $t = 0$ is given by the formula

$$x = x(\xi, t), \tag{1.7}$$

where the functions $x_i(\xi, t)$, $i = 1, 2, 3$ are determined from the Cauchy problem:

$$\frac{dx}{dt} = v(x, t), \quad x = \xi \quad \text{if} \quad t = 0. \tag{1.8}$$

The variables $\xi = (\xi_1, \xi_2, \xi_3)$ are called the Lagrangian coordinates. If the relation $F(x(\xi, t), t) \equiv f(\xi, t) = 0$ defines a free boundary, it follows from Eqs. (1.2) and (1.8) that $f_t = 0$, and the free boundary equation in the Lagrangian coordinates is simply $f(\xi) = 0$. Therefore, if Γ_t is constructed as the image of $\Gamma_0 \equiv \Gamma$ at the mapping (1.7), the condition (1.2) will be fulfilled automatically.

Let us formulate the problem (1.1)–(1.4) in Lagrangian coordinates, for the case $\nu = 0$, $\sigma = 0$ (and ideal fluid with a zero surface tension). It is necessary to find the vector $x(\xi, t)$ and the function $p(\xi, t)$ in the region $\Omega \times (0, T)$ so that the following equations, initial and boundary conditions

$$\mathcal{U}^* \cdot x_{tt} + \nabla_\xi p = 0, \quad \det \mathcal{U} = 1, \tag{1.9}$$

$$p = 0 \quad \text{if} \quad \xi \in \Gamma, \tag{1.10}$$

$$x = \xi, \quad x_t = v_0(\xi) \quad \text{if} \quad t = 0 \tag{1.11}$$

are satisfied ($\nabla_\xi \cdot v_0 = 0$). Here ∇_ξ is the gradient with respect to the variables (ξ_1, ξ_2, ξ_3), $\mathcal{U}$ is the Jacobi matrix of the mapping (1.7) at fixed t, $\mathcal{U}_{ij} = \partial x_j / \partial \xi_i$, $i, j = 1, 2, 3$.

This paper presents a review of works devoted to the investigation of the problem (1.1)–(1.4) in a strict formulation as well as to some of its models. The results of the investigations obtained at the Institute of Hydrodynamics of the Siberian Branch of the USSR Academy of Sciences are used as a basis of this work. These investigations were carried out upon the initiative and under the guidance of L. V. OVSJANNIKOV.

2. Existence theorems

The first result on the solvability of the problem (1.9)–(1.11) was obtained by L. V. OVSJANNIKOV [1] and is based on the theory of the nonlinear Cauchy problem in the scale of Banach spaces [2] developed by the same author. Let Ω be a simply-connected plane region with an analytical boundary, and v_0 be a potential plane vector field which is analytical in some region $\overline{\Omega}' \supset \Omega$. Under these assumptions the exintence and uniqueness of the solution of a plane analogy of the problem (1.9)–(1.11) in the scale of Banach spaces of analytical functions are proved.

Generalization of this result for a spatial case is of particular interest. In connection with this, it should be noted that V. I. NALIMOV [3] proved the solvability of the three-dimensional Cauchy-Poisson problem of the water waves in the classes of analytical functions which is close to Eqs (1.9)–(1.11). The proof of the solvability of the problem (1.9)–(1.11) in the class of finite smoothness functions gives rise to many difficulties which are explained below (see Sect. 7).

The problem (1.1)–(1.4) for $\nu > 0$ and $\sigma = 0$ was considered by V. A. SOLONNIKOV [4]. He obtained the following results. Let $\Gamma \in C^{2+\alpha}$, $v_0 \in C^{2+\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, and the conditions of agreement (1.5) and (1.6) be valid. Let $|v_0|_{\Omega}^{2+\alpha} = R$. For any $R > 0$ there exists such $T > 0$ that the problem (1.1)–(1.4) with $\nu > 0$ and $\sigma = 0$ has a unique classical solution if $t \in [0, T]$ and v and p belong to some Hölder classes. Besides, it is possible to find such R, on the basis of any T, that the problem (1.1)–(1.4) should be identically solvable within the interval $[0, T]$.

It should be noted that all the above-mentioned theorems of the solvability of the problem of unsteady, liquid free boundary flow in a strict statement have a local character. This restriction is connected with the subject matter. Let us assume that in the process of motion of two points of a free surface being at the initial moment of time at a finite distance, and then one part of the fluid impacts with the other part. The mathematical nature of singularities existing in this phenomenon is very complicated and has not been examined so far. The question of the possible spoiling in time of the smoothness of a free surface has not been investigated, either. After all, there are no exact results of a general character concerning the solvability of the problem (1.1)–(1.4) for $\sigma \neq 0$.

3. Finite-dimensional models

Below, examples of the solutions of the problems (1.1)–(1.4) are presented. To find them, we must integrate the system of ordinary equations. The widest class of such solutions is admissible in the case when $\nu = 0$, $\sigma = 0$. The motions with a linear velocity field found by DIRICHLET [5] and investigated in detail by L. V. OVSJANNIKOV [6] (cf. also [7, 8]) belong to this class. Here the mapping (1.7) is given by the formula

$$x = \mathscr{A}(t)\xi$$

thus $\mathscr{A} = \mathscr{U}$. Due to Eqs. (1.9)–(1.11) the matrix $\mathscr{A}$ is the solution of the Cauchy problem

$$\mathscr{A}'' = q\mathscr{A}^{*-1}\mathscr{N}, \quad \mathscr{A}(0) = \mathscr{E}, \quad \mathscr{A}'(0) = \mathscr{A}'_0, \tag{3.1}$$

where

$$q = \frac{\mathscr{S}p(\mathscr{A}^{-1}\mathscr{A}')^2}{\mathscr{S}p(\mathscr{A}^{-1}\mathscr{A}^{*-1}\mathscr{N})},$$

$\mathscr{A}'_0$ is the arbitrary matrix with $\mathscr{S}p\,\mathscr{A}'_0 = 0$, $\mathscr{N} = \mathrm{diag}\{n_1, n_2, n_3\}$ and n_1, n_2, n_3 are the arbitrary positive numbers. The pressure formula is in the form

$$p = \frac{q}{2}(c^2 - \xi\cdot\mathscr{N}\cdot\xi),$$

where $c > 0$ is constant. A free surface Γ is the ellipsoid with the equation

$$c^2 - \xi\cdot\mathscr{N}\cdot\xi = 0.$$

The Cauchy problem (3.1) has a unique solution at all t [7]. The system (3.1) has eight integrals which express the conservation of mass ($\det\mathscr{A} = 1$), energy, circulation and moment of momentum of a deformable liquid ellipsoid [5]. The problem (3.1) has a number of exact solutions. One of them will be considered below [6].

Let $\mathscr{N} = \mathscr{E}$, and the matrix $\mathscr{A}'_0$ have the following non-zero elements: $a_{22} = a_{33} = -2a_{11} = -2b$, $a_{31} = -a_{23} = \omega$. The matrix $\mathscr{A}$ has non-zero elements $a_{11} = m$, $a_{22} = a_{33} = k$, $a_{31} = -a_{23} = n$ for

$$k = \frac{1}{\sqrt{m}}\cos\left(\omega\int\limits_0^t m(s)ds\right), \quad n = \frac{1}{\sqrt{m}}\sin\left(\omega\int\limits_0^t m(s)ds\right)$$

and the function m is found as a quadrature from the equation

$$m'^2 = 4m^3\,\frac{3b^2+\omega^2(1-m)}{1+2m^3}$$

for $m(0) = 1$. The equation of the free boundary Γ_t in Eulerian coordinates is

$$\frac{x_1^2}{m^2} + m(x_2^2+x_3^2) = c^2.$$

The interpretation of this solution is as follows. At the initial moment of time the liquid fills the sphere Γ and is in the state of a uniform rotation which is imposed upon a potential linear velocity field. Let us assume, for definiteness, that $\omega \neq 0$, $b > 0$. Then for $0 < t < t_*$ the sphere Γ is extended taking the form of the rotation ellipsoid Γ_t with the axis x_1, as long as its large hemi-axis takes the largest value of $cm_* = c[3(b/\omega)^2+1]^{1/2}$. After it the ellipsoid is contracted; at the moment $t = 2t_*$ it takes the form of the sphere Γ and then shrinks to the plane $x_1 = 0$ merging with it when $t \to \infty$.

Now let us consider the case when $\omega = 0$, i.e. the matrix is diagonal and the motion is potential. If $b > 0$ then $m > 1$ for $t > 0$ and $m \to \infty$ for $t \to \infty$. Thus, if $b > 0$ and $\omega = 0$, the ellipsoid Γ_t is expanded in the direction of the x_1 axis when t increases, and is contracted to this axis when $t \to \infty$. From the viewpoint of the motion stability, this result means that the potential motion pre-determined by the $\mathscr{A}$ matrix is unstable with respect to any small vortex disturbances for $\omega = 0$.

The question of the behaviour of the Cauchy problem (3.1) in the case when $t \to \infty$ has not been studied so far. It seems to be likely that when $\mathscr{A}'_0 \neq 0$, this problem has no limited solutions.

Now let us consider plane motions with a linear velocity field. In this case ξ and x denote two-dimensional vectors, and $\mathscr{A}$, $\mathscr{A}'_0$ and $\mathscr{N}$ are second-order matrices. The solution of the problem (3.1) here describes the rotational deformable ellipse motion. Using the integrals of motion, this problem can be integrated in quadratures. It turns out that if an initial motion is not a rest, the following alternative takes place: either one of hemi-axes of an ellipsoid increases infinitely when $t \to \infty$, or the motion is a uniform rotation of a liquid circle around its centre.

Now let us consider the plane problem (3.1) for

$$\mathscr{A}'_0 = \begin{pmatrix} b & \omega \\ -\omega & b \end{pmatrix}$$

and $\mathscr{N} = \mathscr{E}$ (Γ is the circle of the radius c). In this case the semi-axes of the ellipse $a_1(t)$, $a_2(t)$ are connected by the relations

(3.2)
$$\frac{1}{2}(a_1'^2 + a_2'^2) + \frac{4c^4\omega^2}{(a_1+a_2)^2} = (b^2+\omega^2)c^2,$$
$$a_1 a_2 = c^2, \quad a_1(0) = a_2(0) = c$$

and the angular velocity of the ellipse rotation is equal to $4c^2\omega(a_1+a_2)^{-2}$. The case $b = 0$ corresponds to the rotation of a circle as a solid. From Eq. (3.2) it is seen that any small initial deformation of the velocity field ($b \neq 0$) destroys the above-mentioned stationary motion.

The number of exact solutions of Eqs. (1.1)–(1.4) in the case $\nu \neq 0$, $\sigma \neq 0$ is extremely poor. The only non-trivial example of a solution describies the radial inertial motion of a spherical layer [9–12]. A plane analogy of this solution describes a radial motion of a circular ring [10, 13]. The plane problem of a rotationally symmetric motion of a rotating ring is more general. Section 4 is devoted to the consideration of this problem.

4. Rotating ring

Now let us consider the plane problem (1.1)–(1.4) with special initial data: Ω is the circle with $r_{10} < r = |x| < r_{20}$

(4.1)
$$v_r = \chi_0 r^{-1}, \quad v_\theta = v_0(r) \quad \text{if} \quad t = 0, \quad r_{10} < r < r_{20}.$$

Here v_r and v_θ are the radial and circumferential velocity components respectively, in the polar coordinate system (r, θ), χ_0 the given constant; v_0 the given function. The solution of this problem has the form

$$v_r = \chi(t) r^{-1}, \quad v_\theta = v_\theta(r, t), \quad p = p(r, t).$$

The free boundary equations are $r = r_i(t)$, $i = 1, 2$. This solution is interpreted as a liquid rotating ring motion due to inertia, viscosity and surface tension.

If $\nu > 0$, the problem (1.1)–(1.3) is reduced to the solution of the connected system of one parabolic and three ordinary equations for the functions v_θ, χ, r_i; and to the quadrature for finding p. This problem was studied by V. O. BYTEV [14] for the case $\sigma = 0$, and by O. M. LAVRENT'EVA [15]. The obtained results are presented below.

Let the moment of momentum of a ring be denoted by L, and its square by Σ. The values of Σ and L are the integrals of motion. Let the dimensionless parameter be introduced:

$$\beta = \frac{L^2}{\varrho\sigma\Sigma^{5/2}}$$

(ϱ is the density of the liquid). Suppose that $\sigma > 0$. Then if the inequality $\beta > \beta_* \approx 5.17$ is valid, the problem has two stationary solutions describing the rotation of a ring as a solid. If $\beta < \beta_*$, the stationary solutions do not exist.

Now suppose that $v_0 \in C^{2+\alpha}[r_{10}, r_{20}]$ and the agreement conditions $v_0'(r_{10}) = v_0'(r_{20}) = 0$ are satisfied. If $\beta < \beta_*$, then either the solution of the problem with the initial data (1.1)–(1.3), (4.1) exists at any $t > 0$, and then $r_1 = 0(t)$ at $t \to \infty$, or such t_0 is found (finite or infinite) that $r_1(t) \to 0$ if $t \to t_0$.

If $\beta \geqslant \beta_*$, the existence of the two regimes of motion is possible: vanishing of an inner radius and stabilization of motion to the ring rotation as a solid. In the paper [15] sufficient conditions to realize each of these regimes are presented. For example, under the condition

$$8r_{20}^2 \Sigma \varrho E_0 < L^2,$$

where E_0 is the total liquid energy at moment $t = 0$, the inner radius cannot turn into zero.

In the case $\sigma = 0$, the qualitative picture of motion significantly changes. If $L \neq 0$ and in the condition (4.1) $\chi_0 < 0$, at first the rotating ring converges to the centre, until the inner radius reaches a positive minimum. Then the ring diverges, thus $r_1 \to \infty$ if $t \to \infty$. At $\chi_0 \geqslant 0$ divergence begins immediately. Two different divergence regimes exists. If the inequalities

$$\frac{\chi_0}{\nu} < 4, \quad \frac{r_{10}^2}{\Sigma \nu^2} \int\limits_{r_{10}}^{r_{20}} r v_0^2(r)\,dr < 2 \tag{4.2}$$

are satisfied, then $r_1 = 0(\sqrt{t})$ when $t \to \infty$. If one of these inequalities is changed by the opposite one, $r_1 = 0(t)$ when $t \to \infty$ [14].

The case $v_0 = 0$ in Eq. (4.1) corresponds to a purely radial ring motion. If in this case $\sigma \neq 0$, either the ring diverges up to infinity or its inner radius turns into zero at some moment. If $\sigma = 0$ and the first of the inequalities (4.2) is valid, then $\lim r_1(t) = r_{1\infty} > 0$ exists when $t \to \infty$. In the opposite case r_1 tends to infinity when $t \to \infty$.

So far we have considered the motion of a viscous fluid ring. In the case $\nu = 0$, the problem is simplified and the integration in quadratures [7] is admissible. Here the different regimes of motion dependent on initial data are realized. In particular, when $\sigma \neq 0$ and $L \neq 0$, radial free-oscillations of a rotating ideal ring are possible.

5. Stationary motions

The stationary motions of an isolated volume of a capillary viscous fluid admit a simple description: the fluid rotates as a solid around its axis which is parallel to the given vector of momentum, and a free surface is unmovable in a rotating coordinate system. It is determined as a closed instantaneous surface in the field of centrifugal which restricts the given volume.

Existence, stability and branching of equilibrium forms of a rotating fluid have been considered in detail in [16] and here they will not be discussed. Among the problems unsolved, the existence problem of the equilibrium forms of a rotating drop which are not the surface of revolution, is of particular interest.

It should be noted that for a rotating cylindrical column such forms exist [17].

If the fluid has no surface tension, then for $\nu \neq 0$ the stationary motion of an isolated volume can be only a translational motion. In a plane case the rotation of a circle and a ring as a solid is also admissible. If, simultaneously, $\sigma = 0$ and $\nu = 0$, non-trivial stationary motions of an isolated liquid volume are possible. Below, the example of such a motion will be presented where an ideal flow is axi-symmetric with respect to the z axis, and its vorticity is proportional to the r distance to this axis.

If the stream function is denoted by ψ, and the radial and axial velocities are denoted by v_r and v_z, respectively, we have $v_r = -r^{-1}\, \partial\psi/\partial z$, $v_z = r^{-1}\, \partial\psi/\partial r$; and ψ satisfied the equation

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)=kr, \tag{5.1}$$

where k is constant. The angular velocity v is equal to Cq/r, where Cq is constant. For the flows of the above-mentioned type, the Eulerian equations admit the integral

$$p+\frac{1}{2}\left(v_r^2+v_z^2+C^2q^2/r^2\right)+k\psi = \frac{1}{2}C^2 \equiv \text{const}. \tag{5.2}$$

Now we will find free surfaces Γ close to a torus. Let the meridional section Γ be denoted by γ, and a plane domain bounded by the curve γ, by ω. The conditions at a free boundary $p = 0$, $\psi = 0$ and the equality (5.2) lead to the relations

$$\psi\Big|_{\gamma} = 0, \qquad \frac{1}{r}\frac{\partial\psi}{\partial n}\Big|_{\gamma} = C\sqrt{1-q^2/r^2}, \tag{5.3}$$

where $\partial/\partial n$ denotes the operator of differentiation in the direction of an outer normal γ. By virtue of Eqs. (5.1) and (5.3), the constants k and C are connected by the equality

$$k = C\int_{\gamma}\sqrt{1-q^2/r^2}\,d\gamma \Big/ \int_{\omega} r\,d\omega.$$

In Eqs. (5.1) and (5.3) let us make a substitute of the variables

$$r = a+bx, \qquad z = by, \qquad \psi(r, z) = abC\sqrt{1-\mu^2}\,w(x, y),$$

where a and b are some constant dimensions of length, $\mu = q/a$, and let $\varepsilon = b/a$. The images of the curve γ and domains ω on the x, y plane are denoted by γ_0 and ω_0, respectively. In the new variables the problem (5.1) and (5.3) takes the form

$$\Delta w - \frac{\varepsilon}{1+\varepsilon x}\frac{\partial w}{\partial x} = \frac{(1+\varepsilon x)^2}{\sqrt{1-\mu^2}}\int_{\gamma_0}\frac{\sqrt{1-\mu^2/(1+\varepsilon x)^2}\,d\gamma_0}{\int_{\omega_0}(1+\varepsilon x)\,d\omega_0},$$

$$w\Big|_{\gamma_0} = 0, \qquad \frac{\partial w}{\partial n_0}\Big|_{\gamma_0} = \frac{(1+\varepsilon x)}{\sqrt{1-\mu^2}}\sqrt{1-\mu^2/(1+\varepsilon x)^2}$$

(Δ is the Laplacian operator with respect to x, y variables).

If $\varepsilon = 0$ the problem (5.4) has a one-parametric family of solutions where γ_0 is a circle $x^2+y^2 = c^2$ and $w = (x^2+y^2-c^2)/2c$. It is proved that for sufficiently small ε and the certain dependence $\mu = \mu(\varepsilon)$, this problem has a three-parametric family of solutions. A four-parametric family of solutions of the problem (5.1), (5.3) corresponds to it. A kinetic energy of fluid, the inertia moment of the meridional free-surface section with respect to the line r = const containing the gravity centre of the section, the length of the meridional free-surface section and the distance between the gravity centre of the section and the rotation axis can be predetermined as physical parameters (the ratios of the two latter values must be sufficiently small).

6. Small disturbances

Let us consider the motion of an ideal isolated liquid volume with zero surface tension. Such a motion can be described in terms of the functions $x(\xi, t)$, $p(\xi, t)$ determined in a cylinder $Q_T = \Omega \times [0, T]$ and satisfying the relations (1.9)–(1.11). The solution x, p corresponds to the initial field $x_t(\xi, 0) = v_0(\xi)$ and will be called the basic solution.

Let us consider, in the cylinder Q_T, the solution of the problem (1.9)–(1.11) with the other initial velocity field $\tilde{v}_0(\xi) = v_0(\xi)+V_0(\xi)$, $\nabla_\xi \cdot V_0 = 0$. The solution $\tilde{x}, \tilde{p}$ corresponding to the initial function $\tilde{v}_0$ is called the disturbed solution, and the function V_0 the initial disturbance. Assume that $\tilde{x} = x+X$, $\tilde{p} = p+P$ and call the functions X, P the basic solution disturbances. Assuming the smallness of the initial disturbance, we can hope that the functions X, P are small within some interval of time. Substituting the expressions for $\tilde{x}, \tilde{p}$ into the relations (1.9)–(1.11) and neglecting the terms which are nonlinear with respect to disturbances, we will come to the linear problem for the functions X, P which is considered in this section.

The linear model in the theory of unsteady free boundary flow is of interest for two reasons. In the first place, linearization on the solution of the free boundary problem makes it possible to understand the mathematical nature of this problem. Secondly, if we have a certain solution determined for all $t > 0$, the analysis of the small disturbance behaviour for $t \to \infty$ allows to judge the stability of the given solution.

There is a great number of works dealing with small disturbances at rest or at a uniform liquid rotation (the works on the linear wave theory belong to the same number). However, up to present time there have not been any works devoted to the investigation of small disturbances of arbitrary solutions of Eulerian equations in all — or partly free boundary region. The formulation of this problem and the first results for the case of a potential flow are attributed to L. V. OVSJANNIKOV [10]. A general problem of small disturbances of an ideal free boundary flow in a potential field of mass forces was investigated by V. K. ANDREEV [18]. When the whole boundary of flow is free, this problem reduces to the question of finding the function $\Phi(\xi, t)$ adhering to the following relation:

$$\begin{aligned}
&\nabla[\mathcal{U}^{-1}\mathcal{U}^{*-1}(\nabla\Phi+V_0)] = -\nabla\Big[(\mathcal{U}^{-1}W)_t \\
&\qquad\qquad \times \int_0^t W^{-1}\mathcal{U}^{*-1}(\nabla\Phi+V_0)dt\Big], \quad \xi\in\Omega, \quad 0<t<T,\\
(6.1)\qquad &(a\Phi_t)_t+n\mathcal{U}^{-1}\mathcal{U}^{*-1}(\nabla\Phi+V_0)\\
&\qquad\qquad = -n(\mathcal{U}^{-1}W)_t\int_0^t W^{-1}\mathcal{U}^{*-1}(\nabla\Phi+V_0)dt, \quad \xi\in\Gamma, \quad 0<t<T,\\
&\Phi=0, \quad \Phi_t=0 \quad \text{if} \quad t=0, \quad \xi\in\Omega.
\end{aligned}$$

Here $a = -(\partial p/\partial n)^{-1}$, n is the unit vector of an outer normal to the boundary Γ of the region Ω; $\partial p/\partial n$ is the p derivative with respect to the normal n to Γ. The matrix W is the solution of the Cauchy problem

$$W_t = \mathcal{U}^{*-1}\mathcal{U}_t^* W, \quad W|_{t=0} = \mathcal{E}$$

and $\mathcal{U}$ is the Jacobi matrix of the mapping (1.7). The operator ∇ in this section denotes the gradient in the Lagrangian variables ξ_1, ξ_2, ξ_3. The unknown function in the problem (6.1) is connected with the pressure disturbance by the relation $\Phi_t = -P$. After solving the problem (6.1), the vector X is restored according to the formula

$$X = W\int_0^t W^{-1}\mathcal{U}^{*-1}(\nabla\Phi + V_0)\,dt.$$

The relations (6.1) are strongly simplified if basic and disturbed flows are potential. In this case $V_0 = \nabla\Phi_0$ $(\Delta\Phi_0 = 0)$, the matrices $\mathcal{U}$ and W coincide, and for the function $\hat{\Phi} = \Phi + \Phi_0$ the following problem is obtained:

$$\begin{aligned}
&\nabla(\mathcal{U}^{-1}\mathcal{U}^{*-1}\nabla\hat{\Phi}) = 0 \quad \text{if} \quad \xi\in\Omega, \quad 0<t<T,\\
&(a\hat{\Phi}_t)_t + n\mathcal{U}^{-1}\mathcal{U}^{*-1}\nabla\hat{\Phi} = 0 \quad \text{if} \quad \xi\in\Gamma, \quad 0<t<T,\\
&\hat{\Phi} = \Phi_0(\xi), \quad \hat{\Phi}_t = 0 \quad \text{if} \quad \xi\in\Omega, \quad t=0.
\end{aligned} \tag{6.2}$$

In turn this problem can be reduced to the initial value problem for the differential equation in Hilbert space

$$\begin{aligned}
&(a\varphi_t)_t + \mathcal{K}(t)\varphi = 0,\\
&\varphi = \varphi_0, \quad \varphi_t = 0 \quad \text{if} \quad t = 0
\end{aligned} \tag{6.3}$$

with the unknown function $\varphi = \hat{\Phi}|_\Gamma$. The operator $\mathcal{K}$ associates the element $\varphi\in W_2^{1/2}(\Gamma)$ to the function $\mathcal{K}_\varphi \in W_2^{-1/2}(\Gamma)$ by the following rule: The solution of the Direchlet problem $\hat{\phi}|_\Gamma = \varphi$ for Eq. (6.2) is found by the function φ, and then the derivative of the solution is calculated by the co-normal to Γ, $\mathcal{K}\varphi = n\mathcal{U}^{-1}\mathcal{U}^{*-1}\,\nabla\hat{\phi}$. The writing $\mathcal{K}(t)$ underlines the dependence of the operator $\mathcal{K}$ on t (it is explained by the fact that $\mathcal{U}$ depends on t). In the initial condition (6.3), φ_0 is a trace of the function $\Phi_0(\xi)\in W_2^1(\Omega)$ on the surface Γ.

The operator $\mathcal{K}$ is symmetric and positively determined on the sub-space of the space $W_2^{1/2}(\Gamma)$ formed by the functions φ with a zero mean value on Γ. Let us assume that for all $\xi\in\Gamma$, $t\in[0,T]$ the condition

$$\left(-\frac{\partial p}{\partial n}\right)^{-1} \equiv a(\xi,t) \geqslant a_0 > 0 \tag{6.4}$$

is valid. In this case Eq. (6.3) can be considered as a hyperbolic pseudo-differential equation in the manifold Γ. If the basic solution is such that $\Gamma\in C^2$, $x_i, p\in C^3(Q_t)$, $i = 1,2,3$ and the condition (6.4) is satisfied, the solution of the problem (6.2) admits the estimate

$$\int_\Gamma \hat{\Phi}_t^2\,d\Gamma + \int_\Omega |\nabla\hat{\Phi}|^2 d\Omega \leqslant C(T)\int_\Omega |\nabla\Phi_0|^2 d\Omega \tag{6.5}$$

for $t\in[0,T]$. Basing on this estimate, V. K. ANDREEV proved the solvability of the problem (6.2) at the condition (6.4) [19]. Under the same conditions, he proved the existence and uniqueness theorem for the solution to the more general problem (6.1) [18] within the space $\mathcal{L}_2 \equiv (0, T; W_2^1(\Omega))$.

It should be noted that the problem (6.1) can also be reduced to an operator equation of the (6.3) type. This equation is nonhomogeneous and the operator $\mathcal{K}(t)$ is nonlocal with respect to t and nonsymmetric. However, its main part is a symmetric and positively determined operator; thus the above mentioned operator equation conserves the properties of a hyperbolic equation at the condition (6.4).

It should be emphasized that it is precisely the condition $\partial p/\partial n < 0$ which provides the correctness of the problem of small disturbances. The importance of the latter condition was noted in the works by G. TAYLOR [20], L. V. OVSJANNIKOV [10], V. I. NALIMOV [21]. V. I. Nalimov proved the solvability of the plane Cauchy-Poisson problem in an exact formulation in the class of finite smoothness functions. He established a remarkable peculiarity of this problem. The Cauchy problem for linearized equations turned out to be correct only when linearization was based on an exact solution of nonlinear equations. Probably it is characteristic of the problem of an ideal isolated liquid motion without a surface tension.

Now let us come back to the problem of small disturbances (6.1). It is rather difficult to characterize the class of basic motions for which the inequality (6.4) is valid providing the correctness of the problem (6.1). However, it should be noted that for potential motions of an isolated volume differing from a constant flow, this inequality is always valid [10, 7].

If a basic motion is such that for all $\xi \in \Gamma$ and $t \in [0, T]$, the inequality

$$\frac{\partial p}{\partial n} > 0 \tag{6.6}$$

takes place, Eqs. (6.3) is an elliptic one, and the Cauchy problem is not well posed in the sense of Hadamard. The condition (6.6) can be satisfied even for potential motions if the boundary of the region Ω is not wholly free, or if external forces act onto the fluid. Examples of the uncorrectness of the problem of small disturbances of a plane free boundary flow were presented by G. TAYLOR [20]. As for an isolated flow, the uncorrectness of the problem of disturbances can be explained by a heavy vorticity of flow. In particular, if the solution from Sect. 3, which describes the rotating ellipsoid motion, is taken as a basic one, the inequality (6.6) is valid when t is close to t_*.

Since the linearized problem (6.3) or more general problem (6.1) when $\partial p/\partial n > 0$ are solvable only in the class of analytical functions, it would not be expected that the initial nonlinear problem (1.9) – (1.11) is solvable at arbitrary initial data in the class of finite smoothness functions.

It is interesting to note that in the case when a basic flow is a uniform rotation of fluid as a solid, the problem of small disturbances is correct [22] though the inequality (6.6) is satisfied. This case is an exclusive one, because in the rotating coordinate system the basic motion is a rest.

In these considerations we did not take into account surface tension. As was mentioned by G. Taylor, the surface tension turned out to be the factor which stabilizes short wave disturbances and makes the mathematical problem of small disturbances well posed. The existence and uniqueness theorem in the problem of small disturbances of an ideal isolated capillary flow was proved due to V. K. ANDREEV [23] (without any restrictions to the sign $\partial p/\partial n$).

The problem of small disturbances of a viscous capillary flow was studied only in the case when a basic motion is a rest or uniform rotation [16]. The disturbances of an arbitrary basic motion for $\nu \neq 0$, $\sigma = 0$ were considered in [4].

In conclusion, it should be noted that the other formulation of the problem of the free boundary flow disturbances is possible: that is, at the invariable initial velocity field, the domain of the mapping (1.7) is changed. For the potential ideal flow, the problem of the domain disturbances was investigated in [10].

7. The problem of stability

Assume that a certain solution of the problem (1.1) – (1.4) definite for all $t \geqslant 0$ is known. Then we can consider the stability of this solution with respect to the change of initial data. If the disturbances which are caused by this change are small, the problem of stability can be considered within the framework of a linear theory.

It should be noted that so far no results on the stability of the problem (1.1) – (1.4) within an infinite interval of time have been obtained. For this reason, the problem of the proof of a linear approximation within the theory of stability of a free boundary flow is far from being solved.

If the basic solution is non-stationary, the coefficients of linearized equations depend on time. In connection with this it is rather difficult to obtain sufficient stability conditions in a linear approximation in the general case. All the results on the stability of an unsteady flow of a finite liquid volume were obtained by considering some special examples. In these examples the fluid was assumed to be ideal, and its motion to be potential (except for the work by V. M. MENSHCHIKOV [24] where the stability of a rotating ring of an ideal fluid was considered).

The problem of potential disturbances of a potential ideal flow in the case when $\sigma = 0$, is reduced to finding the function $\hat{\Phi}(\xi, t)$ from the relation (6.2). It is assumed that the elements of the matrix $\mathcal{U}$ (det $\mathcal{U} = 1$) and the coefficient a are definte and sufficiently smooth functions of ξ, t in the cylinder $Q_T = \bar{\Omega} \times [0, T]$ for any $T > 0$. The boundary Γ of the region Ω is also assumed to be smooth. Now suppose that for any $(\xi, t) \in \Gamma \times [0, T]$ the "hyperbolicity condition" (6.4) is satisfied for some $a_0(T) > 0$ (it is permissible that $a_0 \to 0$ when $T \to \infty$). This condition assures the correctness of the problem under consideration.

For the solutions of the problem (6.2) the estimate (6.5) is valid. This estimate allows the value $N(t) = ||\Phi_t||_\Gamma + ||\nabla\Phi||_\Omega$ to be taken as a measure of stability (the symbol $||\cdot||$ denotes the norm in $\mathcal{L}_2$) and the basic solution to be termed stable if $N(t)$ is bounded at all $t > 0$ for any $\Phi_0 \in W_2^1(\Omega)$, and unstable in the opposite case. Such a definition of stability is not the only possible one. L. V. OVSJANNIKOV [10] suggested to characterize stability in terms of the behaviour of the free boundary perturbation vector component normal to Γ_t and equal to

$$R = n_x \cdot X\Big|_\Gamma \equiv \left(\frac{\partial p}{\partial n}\right)^{-1} \Phi_t\Big|_\Gamma \frac{|\nabla F|}{|\mathcal{U}^{*-1}\nabla F|}, \tag{7.1}$$

when $t \to \infty$. Here $F(\xi) = 0$ is the equation of the free boundary Γ, X is the perturbation of the vector of the variable x; n_x is the unit vector of an external normal to the surface Γ_t at the point $x(\xi, t)$.

If $R(\xi, t) \to \infty$ when $t \to \infty$ for some $\xi \in \Gamma$, the local disturbance of a free surface from its undisturbed state in the space x infinitely grows. Thus the value $R(\xi, t)$ finally characterizes the motion stability. In the meantime, the function $N(t)$ is its rough integral characteristic. Some solution can be steady in an integral sense, but the function R infinitely grows for $t \to \infty$ at some point of the boundary. Such a situation really arises in the problem of the deformable ellipse stability [25]. The corresponding basic solution is described at the end of Sect. 3 (in the expressions for the matrix $\mathscr{A}_0'$ and hemi-axes a_1, a_2 it should be assumed that $\omega = 0$). The peculiarity of the problem is that the eingenfunctions of the operator $\mathscr{K}(t)$ do not depend on t, and the function a doesn't depend on ξ. It allows to reduce the problem (6.3), which is equivalent to the problem (6.2), to the Cauchy problem for desintegrating the system of ordinary second-order differential equations.

In the case under consideration Γ is the circle $|\xi| = c$, $\Phi_0|_\Gamma = \varphi_0(\theta)$ is the periodical function of Γ with a zero mean. Further, $\|\cdot\|$ denotes the norm in $\mathscr{L}_2(\Gamma)$. It turns out [25] that if $\varphi_0 \in \mathscr{L}_2(\Gamma)$, the value $\|\hat{\Phi}_{|\Gamma}\|$ is bounded for all $t > 0$. If $\varphi_0 \in W_2^{1/2}(\Gamma)$, the value $\|\hat{\Phi}_{t|\Gamma}\|$ is uniformly bounded. Increasing the smoothness φ_0, it is possible to obtain the estimates of the higher order derivarives $\hat{\Phi}$. Specifically, if $\varphi_0 \in W_2^1(\Gamma)$ then $\hat{\Phi}_{tt|\Gamma}$, $\hat{\Phi}_{\xi i|\Gamma}$, $i = 1, 2$, at the fixed t belong to $\mathscr{L}(\Gamma)$, their norms in $\mathscr{L}_2$ being bounded for all $t > 0$.

It should be noted that in the case when the initial function Φ_0 is even with respect ξ_1, the estimate $\|\hat{\Phi}_{|\Gamma}\| = 0(t^{-1})$ takes place if $t \to \infty$. The solution which is even with respect to ξ_1 describes the motion with the impermeable wall $\xi_1 = 0$.

The obtained results indicate that the basic motion is stable in a linear approximation if the norm in $\mathscr{L}_2$ of boundary values of the disturbances of the $\hat{\Phi}$ potential, or its derivatives, are considered as a measure of stability. However, if one will define stability from the deviation of a free surface from an undisturbed state, then the motion should be interpreted as an unstable one. In particular, if $\Phi_{0|\Gamma} = \sin n\theta$, $n = 1, 2, 3$, the function R determined from Eq. (7.1) has the form

$$R = \frac{\tau^3 \sin n\theta}{(\cos^2\theta + \tau^4 \sin^2\theta)^{1/2}} [\gamma_n + 0(t^{-4})] \tag{7.2}$$

if $t \to \infty$. Here γ_n is some constant, $\tau(t)$ is the function given by the equality

$$b\sqrt{2}t = \int_1^\tau \frac{\sqrt{s^4+1}}{s^2} ds,$$

$\tau = 0(t)$ when $t \to \infty$. From Eq. (7.2) it is seen that outside the zones $|\theta| < \varepsilon$, $|\pi - \theta| < \varepsilon$ ($\varepsilon > 0$ is fixed) R is $0(t)$ if $t \to \infty$. Thus, in this case a free boundary is unstable. In the meantime, $\|\hat{\Phi}_{|\Gamma}\| \to 0$ if $t \to \infty$.

If $\Phi_{0|\Gamma} = \cos n\theta$, $n = 1, 2, 3, \ldots$, then if $t \to \infty$

$$R = \frac{\tau \cos n\theta}{(\cos^2\theta + \tau^4 \sin^2\theta)^{1/2}} [\delta_n + 0(t^{-4})]$$

with the constant δ_n. It follows from this that for the given solution the free boundary instability is localized within the range of angles $|\theta| = 0(t^{-1})$, $|\pi-\theta| = 0(t^{-1})$. This conclusion is followed in part in the problem of deformable ellipse stability at the presence of an impermeable wall $\xi_2 = 0$. If $t \to \infty$, the liquid approaches the wall and stabilizes a free boundary.

We have considered above only potential disturbances. The solution which is stable with respect to potential disturbances can become an unstable one provided that the potentiality of a basic solution is retained, but the class of disturbances is extended without taking into account the potentiality condition. The corresponding examples are presented in [18]. Some exact solutions describing the motions of a rotating ellipse and rotating ellipsoid point to the same fact (see Sect. 3).

In addition to the case which was considered above, at the present time the stability in a linear approximation of the following potential motions of an ideal fluid has been investigated: the motions of a spherical layer [9], a circular ring [10, 13], an ellipsoid of revolution with a linear velocity field [26].

We did not consider here the equations of stability of equilibrium states of an isolated liquid volume. Here the sufficient conditions of stability and instability with respect to finite disturbances have been obtained. They are based on the results obtained by V. V. RUMYANTSEV [27] who established the analogy of the Lagrange theorem of stability for viscous capillary flow [16].

The question of limiting regimes of an isolated volume motion if $t \to \infty$ is closely connected with the problem of stability. The above-mentioned examples for an ideal fluid and viscous ring motion indicate the variety of possibilities existing in this field. In the general case this question is far from being solved. Only the following partial results takes place [7].

Assume that for $t > 0$ there exists the classical solution of the problem (1.9) – (1.11), and the mapping (1.7) defines the diffeomorphism of the regions Ω and Ω_t for any t. The diameter of the region Ω_t is denoted by $d(t)$,

$$d(t) = \max\{|x-y|: x, y \in \Omega_t\}.$$

Let $v_0(\xi) \not\equiv 0$ and $\operatorname{rot} v_0 = 0$ for $\xi \in \Omega$. Then $d(t) \to \infty$ when $t \to \infty$.

In other words, at the potential inertial motion of a finite ideal volume with non-costant velocity and zero surface, the volume diameter infinitely increases in time.

The proof of this result is based on the property of the pressure superharmonicity in a potential fluid motion and the identity of W. A. DAY [28]

$$\frac{d^2}{dt^2}\int\limits_{\Omega_t}|x|^2 d\Omega_t = 2\int\limits_{\Omega_t}|x_t|^2 d\Omega_t - 4\sigma\int\limits_{\Gamma_t} d\Gamma_t + 6\int\limits_{\Omega_t} p\, d\Omega_t,$$

which is valid for an arbitrary motion of an isolated volume of a viscous capillary fluid.

8. Boundary layers

Assume that the solution of the problem (1.1) – (1.4) is known for $\nu > 0$. How can its asymptotics be found if $\nu \to 0$? It is natural to expect that outside thin layers near a free boundary the motion will be close to that of an ideal fluid. Within the boundary layers, a sharp change of the velocity derivatives occurs. As a result of this, the tangential stresses on the free surface turn into zero. The formal asymptotics of the solution of this problem in the plane and axisymmetrical cases is presented by V. A. BATISHCHEV [29]. In particular, he investigated the boundary layer in the problem of motion of a non-rotating ellipsoid of revolution when $\sigma = 0$ (see Sect. 3).

The only example which affirms the validity of an asymptotical expansion is the problem of a rotating ring. The asymptotics is found in the form

$$v_\theta \sim v_\theta^{(0)} + \sqrt{\nu} v_\theta^{(1)} + \sqrt{\nu}(\zeta_1^{(1)} + \zeta_2^{(1)}) + \nu v_\theta^{(2)} + \nu(\zeta_1^{(2)} + \zeta_2^{(2)}) + \ldots,$$

$$\chi = \chi^{(0)} + \sqrt{\nu}\chi^{(1)} + \nu\chi^{(2)} + \ldots,$$

$$r_i \sim r_i^{(0)} + \sqrt{\nu} r_i^{(1)} + \nu r_i^{(2)} + \ldots, \quad i = 1, 2.$$

The notations v_θ, χ, r_i were introduced in Section 4. The functions $v_\theta^{(k)}$, $\chi^{(k)}$, $r_i^{(k)}$, $k = 0, 1, 2, \ldots$, were found by the Lusternik-Vishik first iterative process. If $k = 0$, we obtain the solution of the problem of an ideal fluid ring motion. The boundary layer type functions, $\zeta_i^{(k)}$, $i = 1, 2$, $k = 1, 2, \ldots$ are defined as a result of the second iterative process. They compensate discrepancy under the condition of the absence of the tangential stresses at the free boundary of a ring. The estimate of the asymptotic expansion error is given if $\nu \to 0$. It is valid within any finite interval of time if $\sigma = 0$, and within any interval $0 \leqslant t \leqslant T < \infty$ where $0 < \delta \leqslant r_1^{(0)}(t)$ if $\sigma \neq 0$ [30].

The question of the asymptotics construction for the solution of the problem (1.1) – (1.4) for $\nu \to 0$ remains open in the three-dimensional case.

The other interesting and unsolved problem is to find asymptotics of the solution of the problem of an isolated liquid volume motion if $\sigma \to 0$.

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