Ad corrigendum et agendas concerning the paper
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Due to an unfortunate set of circumstances, caused inadvertently by a most regrettable lapse of communication between the co-authors associated with prolonged delays in mail delivery, a preliminary version of the manuscript for this paper has been published without the knowledge of the senior author. When the paper was presented at the Thirteenth Biennial Fluid Dynamics Symposium last year by one of the junior authors, the work was still in progress. In the meantime a number of conceptual errors, involving primarily the interpretation of the analysis, have been identified thanks especially to the comments provided by Dr. V. I. Korobeinikov and Dr. P. I. Chushkin. The senior author wishes to express his appreciation for the kind help they have rendered and at the same time submit apologies for the regrettable oversight that led to the premature publication of the manuscript. Its correct version with new results is now in preparation for eventual publication in a thoroughly modified form.

A. K. Oppenheim

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On the method of phase space for blast waves (*)

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PHASE space is a generalization of the concept of phase plane which has become well established as the most convenient means for the analysis of self-similar blast waves. Its extension to three dimensions provides a useful tool for the treatment of non-self-similar problems. In this manner, the solution for such problems is reduced to the task of determining an integral surface, or a family of integral curves, in a space defined in terms of appropriate reduced coordinates that, in effect, render the governing equations essentially independent of the physical coordinates of space and time. The paper provides the formulation of the equations and describes the technique used for their solution. Ot particular importance in this respect is the fact that conditions at the inner boundary of the blast wave correspond, in most cases, to a saddle point singularity in the phase space. Hence special precautions have to be taken to assure the stability of the solution in their vicinity. Salient features of the method of phase space are illustrated here by its application to the classical problem of point explosion propagating in a uniform atmosphere of finite (non-zero) pressure.

Przestrzeń fazowa jest uogólnieniem koncepcji płaszczyzny fazowej, która została uznana jako jeden z najbardziej dogodnych sposobów analizy samopodobnych fal wybuchowych. Jej rozszerzenie na przypadek trójwymiarowy stanowi pożyteczne narzędzie do analizy zagadnień niesamopodobnych. Rozwiązanie tych zagadnień sprowadza się do określenia powierzchni całkowej lub rodziny krzywych całkowych w przestrzeni odpowiednich współrzędnych sprowadzonych, co w efekcie prowadzi do równań wyjściowych zasadniczo niezależnych od współrzędnych fizycznych — czasu i przestrzeni. W niniejszej pracy przedstawiono sformułowanie problemu, wyprowadzono równania wyjściowe i opisano technikę ich rozwiązywania. Szczególne znaczenie ma tu fakt, że warunki na wewnętrznym brzegu fali wybuchowej odpowiadają w większości przypadków osobliwemu punktowi siodłowemu w przestrzeni fazowej. Dlatego też należy zachować szczególną ostrożność na zapewnienie stabilności rozwiązania w otoczeniu tego punktu. Charakterystyczne metody przestrzeni fazowej zostały tu zilustrowane na przykładzie klasycznego problemu punktowego wybuchu rozchodzącego się w jednorodnej atmosferze o skończonym (niezerowym) ciśnieniu.

Фазовое пространстно является обобщением концепции фазовой плоскости, которая признана как один из наиболее пригодных способов анализа автомодельных взрывных волн. Ее расширение на трехмерный случай составляет полезное орудие для анализа неавтомодельных задач. Решение этих задач сводится к определению интегральной поверхности иии семейства интегральных кривых в пространстве соответствующих приведенных координат, что в эффекте приводит к исходному уравнению в принципе независящему от физических координат — времени и пространства. В настоящей работе представлена формулировка проблемы, выведены исходные уравнения и описана техника их решения. Особенное значение имеет здесь факт, что условия на внутренней границе взрывной волны отвечают, в большинстве случаев, особой точке типа седла в фазовом пространстве. Поэтому следует сохранять особенную осторожность для обеспечения стабильности решения в окрестности этой точки. Характеристические методы фазового пространства иллюстрированы здесь на примере классической задачи точечного взрыва распространяющегося в атмосфере с равномерным, конечным (ненулевым) давлением.

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1. Introduction

BLAST waves are, essentially, geometrically symmetrical, non-steady flow fields of a compressible medium that are bounded by gasdynamic discontinuities. Generally they are formed by explosions. The process is governed by spatially one-dimensional time dependent equations expressing the conservation of mass, momentum, and energy, subject to appropriate boundary conditions at the center and at the front for the particular problem under consideration.

Base on the pioneering work of von NEUMANN (1947), SEDOV (1946) and TAYLOR (1941), self-similarity variables have been formulated that transform the governing equations of certain classes of problems into ordinary differential equations, thus making them amenable to simple analysis. Such self-similar problems are, as a rule, characterized either by a constant front velocity, or by a negligible (essentially zero) counter-pressure that causes the Mach number of the wave to remain infinite, irrespective of its actual velocity. A parametric study of such solutions has been presented by OPPENHEIM *et al.*, (1972).

In most physical situations, however, especially when one is interested in the interpretation of experimental records, self-similarity conditions are inapplicable. One has to take into account then the dependence of the gasdynamic parameters of the problem on the change in the conditions at the front, as well as in their distribution within the field. It thus becomes necessary to deal with a nonlinear, coupled non-homogeneous set of partial differential equations (viz. OPPENHEIM *et al.*, 1971).

Numerical solutions for some particularly simple cases, such as the problem of constant energy blast wave in a perfect gas with finite ambient pressure, are available in the literature. VON NEUMANN and GOLDSTINE (1955), for example, obtained one of the earliest solutions for this problem by using first-order difference approximations to the gasdynamic equations expressed in Lagrangian form, for which the position and strength of the shock front at each time step was determined by iteration. OKHOTSIMSKII *et al.* (1957) solved the same problem by using the method of characteristics in the Lagrangian frame of reference. BRODE (1955 and 1969) and WILKINS (1969), on the other hand, with the former including radiation effects, utilized Lagrangian difference schemes employing artificial viscosity. This concept, originally introduced by von NEUMANN and RICHTMYER (1950), assures the numerical stability of the solution. In effect, it spreads the shock discontinuity over several mesh steps whose number does not increase with time, while flow variables remain continuous as one integrates "through" the shock front. The penalty paid for this convenience is the loss of accuracy in determining the shock position, as well as difficulties that arise in multiple front blast waves.

In contrast to the above, problems formulated in the Eulerian frame of reference include explicit information on the evolution of the shock front with time. In Eulerian coordinates, KOROBEINIKOV and CHUSHKIN (1966; with SHAROVATOVA, 1969) solved the non-self-similar point explosion problem by the method of integral relations which was first proposed by DORODNITSYN (1956a and 1956b) and later developed by BELOTSERKOVSKII and CHUSHKIN (1965). Although this method yields good results, especially near the self-similar limit and away from the origin, the large number of ordinary differential equations that it generates (27 being a typical number) presents a definite drawback. This could possibly have been the reason why it was never utilized for the solution of more involved problems.

For more complicated flow fields, however, or when the cost of numerical solutions becomes prohibitive, one should resort to analytical methods since they are capable of providing not only the desired results, but may also yield additional insight into the problem.

The first, and most popular, approximate analysis is that involving the expansion of the dependent variables in terms of the front coordinate, for which the self-similar solution provides the zeroth-order step (SAKURAI, 1965 and KOROBEINIKOV *et al.*, 1963). This technique is useful only for the study of flows near the self-similar limit.

Another approximate solution is that utilizing the so-called "quasi-similar" method developed by OSHIMA (1960 and 1962) where all the terms containing the front coordinate are taken to be equal to their values at the front. By virtue of its construction, this method gives exact results at the self-similar limit and just behind the front, with the accuracy deteriorating fast as one proceeds towards the center. In a third approximate method the density is represented by a power law of the field coordinate (MEL'NIKOVA, 1966 and BACH and LEE, 1970). Although this method gives good qualitative results for the problem of adiabatic point explosion, it cannot be utilized to solve many problems, especially those associated with energy addition in the course of the process since it disposes of the energy equation from the outset.

Useful closed form approximate solutions, especially valid when the front is decoupled from the source of explosion, have been developed by CHESTER (1954), CHISNELL (1957), WHITHAM (1958) and FRIEDMAN (1961); these solutions became known as the Whitham role. This solution is based on the assumption that the differential relations which must be satisfied by the flow field variables along one set of characteristic lines, are satisfied by the gasdynamic parameters of the state immediately behind the front. An older approximate solution, valid especially for the weak non-self-similar region of the point explosion problem, has been proposed by BRINKLEY and KIRKWOOD (1947). It was developed by seeking a self-consistent set of ordinary differential equations to specify the problem withou, having to treat explicitly the partial differential equations expressing the conservation principles.

It thus appears from the above survey that each of the existing analytical methods has quite a limited range of validity that precludes its universal applicability to non-self-similar blast wave problems.

In this paper, on the other hand, we exploit a novel analytical approach that holds promise over a very wide range of validity, providing means for the analysis of a great variety of blast wave problems of current interest. In contrast to numerical methods, this approach is basically analytical. It actually takes advantage of the physical properties of the flow field of a particular class of problems, as manifested by singularities specified in the appropriate mathematical phase space of the solution. Thus, our method derives its impetus from the same property that limits the success of other methods: the existence of singularities.

The fundamental concept on which our analysis is based is that of a phase space, a generalization of the well-known concept of phase plane used for the solution of self-similar blast waves (SEDOV, 1957 and COURANT and FRIEDRICHS, 1948). Its extension to third dimension turns it into a useful tool for the treatment of non-self-similar problems. By its use, getting a solution to such problems is reduced to the task of determining an integral surface, or a family of integral lines, in a space defined in terms of appropriate reduced coordinates that render, in effect, the governing equations essentially independent of the physical coordinates of space and time.

To test the utility of the phase space method, it is used here to obtain a solution to the classical case of non-self-similar point explosion. The results are compared to those of the most recently published detailed numerical solution, obtained for this problem by KOROBEINIKOV and CHUSHKIN (1966; with SHAROVATOVA, 1969).

2. Formulation

Problems under consideration here are concerned with the determination of the Eulerian space profiles in a sourceless flow field of a blast wave propagating into an atmosphere of uniform thermodynamic state. Following our systematic method of approach (OPPENHEIM *et al.* 1971), they are formulated in terms of the following variables:

1. Physical space coordinates, the independent variables of the problem:

$$X \equiv \frac{r}{r_n}, \quad \xi \equiv \frac{r_n}{r_0}$$

the first referred to as the *field* and the second as the *front* coordinate. The symbol r denotes the space coordinate of a point in the flow field, while r_n is the radius of the front at the same instant of time, subscript 0 specifying its reference value. It should be noted that when the front trajectory is known ξ becomes a measure of time, t.

2. Front parameters, i.e. variables pertaining uniquely to the front motion and therefore functions of only the front coordinate, ξ:

$$y \equiv \frac{a_a^2}{w^2}, \quad \mu \equiv \frac{d\ln r_n}{d\ln t} = \frac{wt}{r_n}, \quad \lambda \equiv -2\frac{d\ln w}{d\ln r_n} = \frac{d\ln y}{d\ln \xi},$$

where $w = dr_n/dt$ is the front propagation velocity, a is the velocity of sound, while subscript a denotes conditions of the ambient atmosphere into which the front of the blast wave propagates.

3. Gasdynamic parameters of the flow field, the dependent variables describing the structure of the flow field:

$$f \equiv \frac{u}{W}, \quad h \equiv \frac{\varrho}{\varrho_a}, \quad g \equiv \frac{p}{\varrho_a w^2},$$

where u is the particle velocity, ϱ the density and p the pressure.

4. Reduced variables; coordinates of the phase plane:

$$F \equiv \frac{f}{x} = \frac{tu}{r\mu}, \quad Z = \frac{\Gamma g}{x^2 h} = \left(\frac{ta}{r\mu}\right)^2,$$

where

$$\Gamma \equiv \frac{a^2\varrho}{p}.$$

As shown by OPPENHEIM *et al.* (1971), in terms of these variables the conservation equations can be expressed by the following set:

(2.1) $$\left(\frac{\partial Z}{\partial F}\right)_y = \frac{Z}{1-F}\,\frac{\mathbf{P}(F, Z; \Phi^g, \Phi^F, \Phi^Z)}{\mathbf{Q}(F, Z; \Phi^g, \Phi^F)},$$

(2.2) $$\left(\frac{\partial \ln x}{\partial F}\right)_y = -\frac{\mathbf{D}(F, Z)}{\mathbf{Q}(F, Z; \Phi^g, \Phi^F)},$$

(2.3) $$\left(\frac{\partial \ln h}{\partial F}\right)_y = \frac{\mathbf{H}(F, Z; \Phi^h, \Phi^g, \Phi^F)}{\mathbf{Q}(F, Z; \Phi^g, \Phi^F)}$$

and

(2.4) $$\left(\frac{\partial \ln h}{\partial F}\right)_y = \frac{\mathbf{G}(F, Z; \Phi^g, \Phi^F)}{\mathbf{Q}(F , Z; \Phi^g, \Phi^F)}.$$

In the above

$$\Phi^k \equiv \frac{\partial \ln k}{\partial \ln \xi} = \lambda \frac{\partial \ln k}{\partial \ln y},$$

where $k = h, g, F$ or Z, are the logarithmic cross-derivatives, taken with respect to the front coordinate, while

(2.5) $$\mathbf{D}(F, Z) \equiv Z-(1-F)^2,$$

(2.6) $$\mathbf{Q}(F, Z; \Phi^g, \Phi^F) \equiv (j+1)(F-\hat{F}_D)Z-F(1-\hat{F})(F_F-F),$$

(2.7) $$\mathbf{P}(F, Z; \Phi^g, \Phi^F, \Phi^Z) \equiv \delta(\hat{F}_B-F)\mathbf{D}+(\Gamma-1)\mathbf{Q},$$

(2.8) $$\mathbf{H}(F, Z; \Phi^h, \Phi^g, \Phi^F) \equiv \frac{1}{1-F}[\mathbf{Q}-\{(j+1)F+\Phi^h\}\mathbf{D}]$$

and

(2.9) $$\mathbf{G}(F, Z, \Phi^g, \Phi^F) \equiv -\Gamma[F(\hat{F}_F-F)-(j+1)(F-\hat{F}_D)(1-F)].$$

Here $j = 0, 1, 2$ for flow fields of plane, line or point symmetry, respectively, while

(2.10) $$\hat{F}_F \equiv 1+\frac{\lambda}{2}-\Phi^F,$$

(2.11) $$\hat{F}_D \equiv \frac{1}{(j+1)\Gamma}[\lambda-\Phi^g],$$

(2.12) $$\hat{F}_B \equiv \frac{1}{\delta}[\lambda+2-\Lambda^Z]$$

and

$$\delta \equiv (j+1)(\Gamma-1)+2.$$

In the above set Eq. (2.1) defines the problem in the phase space. Integral curves corresponding to fixed values of y computed on the basis of this equation, subject to appropriate boundary conditions at the front, provide essentially the solution to the problem. When such integral curves are determined, profiles of the gasdynamic parameters describing the structure of the flow field are obtained from the quadratures of Eqs. (2.2), (2.3), and (2.4), while the relationship between the front decay parameter, λ, and y, evaluated at the same time, provide all the necessary information that is required for the determination of the front trajectory in the time-space domain.

It is worth noting at this point that Eq. (2.1) has the same form as the governing equation in the self-similar phase plane analysis (OPPENHEIM *et al.*, 1972) except that the terms $\hat{F}_F$, $\hat{F}_D$ and $\hat{F}_B$ vary along the integral curves, rather than remain constant.

3. Method of solution

The main objective if the analysis is to determine the integral surface of the solution in the phase space. This is specified in terms of integral curves that correspond to fixed values of the front parameter, y. It should first be noted, however, that, in general, the flow fields under study are each bounded by a front and an inner boundary. The front is usually comprised of a gasdynamic discontinuity that is described by the appropriate jump conditions. The state immediately behind the front represents the outer boundary and is denoted here by subscript n, while the inner boundary, referred to by subscript i, expresses the conditions at the center of symmetry on a piston face or at any internal surface along which all the conditions are prescribed. The problem is thus basically one of a double boundary value.

To develop an analytic solution, an appropriate expression for the variation of the front derivatives must be adopted *a priori*. For a given value of y these derivatives are, in general, functions of F or Z, as well as Φ_i^k and Φ_n^k. Since the form of these functions has a definite bearing on the solution, a judicious choice must be made. For example, they must satisfy conditions at both boundaries, as well as comply with any other constraints of the problem. However once these conditions of constraint are satisfied if one starts to evaluate integral curves on the basis of the self-similar solution, which is exact, the particular form of these functions becomes practically immaterial. The reason for this is that for self-similar solutions, where $y = 0$, all $\Phi^k = 0$, whereas for y close to $y = 0$, $\Phi^k{}_{,s}$ are relatively small. Consequently, the proper form of these functions may, in principle, be developed step by step, as y is varied from 0 to 1, to approach the exact solution.

As a consequence of the above argument one may define, without much loss in generality, a progress variable, ε, as follows

$$\Phi^k = \Phi_i^k + \varepsilon(\Phi_n^k - \Phi_i^k). \tag{3.1}$$

The value of ε must be equal to 1 at the front and 0 at the inner boundary while, within the flow field, ε may be taken as a function of either Z or F. Specifically, in general, one may have

$$\varepsilon = \frac{a_1\zeta + a_2\zeta^2 + a_2\zeta^3 + \dots}{a_1 + a_2 + a_3 + \dots}, \tag{3.2}$$

where

$$\zeta \equiv Z_n/Z.$$

The gasdynamic profiles obtained by the use of a given form of $\varepsilon(F)$ or $\varepsilon(Z)$ must of course satisfy the global conservation equations. The extent to which this is achieved serves as a check of the appropriateness of the form used for this function.

For example, if one takes $\varepsilon = 1$ throughout, Eq. (3.1) reduces to $\Phi^k = \Phi^k_n$, corresponding to the postulate of the so-called quasi-self-similar theory (OSHIMA, 1962). In this case one loses the constraints imposed by the inner boundary conditions of the center and, consequently, the mass integral cannot be satisfied. On the other hand, as will be demonstrated later, if one takes in Eq. (3.2) $a_2 = a_3 = \ldots = 0$ so that $\varepsilon = \zeta$ one can achieve for the classical case of point explosion in an atmosphere of finite pressure as accurate an agreement with the mass integral as one wishes.

4. Application

As an illustration of the method of phase space, a solution obtained by its use is described for the classical case of an adiabatic point explosion in uniform atmosphere of finite (non-zero) pressure. The medium is assumed to behave essentially as a perfect gas with constant specific heats, so that $\Gamma = \gamma$, the specific heat ratio.

Under such circumstances the state immediately behind the front is given by the Rankine-Hugoniot relations, yielding

(4.1) $$F_n = \frac{2}{\gamma+1}(1-y),$$

(4.2) $$h_n = \frac{\gamma+1}{\gamma-1+2y},$$

(4.3) $$g_n = \frac{\gamma-1}{\gamma(\gamma+1)}\left(\frac{2\gamma}{\gamma-1}-y\right)$$

and

(4.4) $$Z_n = \frac{\gamma-1}{2}\left(\frac{2}{\gamma-1}+F_n\right)(1-F_n).$$

For a given value of y, the first and last equations specify the front boundary condition for the integral curves in the phase space.

For this particular problem the inner boundary that specifies the integral curve at the other limit is singularity D (OPPENHEIM *et al.*, 1972) which is a saddle point representing conditions at the center of symmetry. In the phase space this singularity is located at $Z = \infty$ and $F = F_D$. The latter is specified by Eq. (2.11) corresponding to Φ^g evaluated at $Z = \infty$.

The front derivatives at this singularity are as follows:

(4.5) $$\Phi^F_D = \lambda y\left(\frac{\partial \ln F}{\partial y}\right)_D,$$

(4.6) $$\Phi^h_D = (1-F_D)\alpha_D-(j+1)F_D,$$

(4.7) $$\Phi^g_D = (j+1)\gamma\left[\frac{\lambda}{(j+1)\gamma}-F_D\right]$$

and

(4.8) $$\Phi^z_D = \Phi^g_D-\Phi^h_D,$$

where

$$\alpha_D \equiv \left(\frac{\partial \ln h}{\partial \ln x}\right)_D.$$

The first equation in the above set is obtained directly from the definition of λ; the second — by dividing Eq. (2.3) by Eq. (2.2) and expressing the result at point D with the aid of Eqs. (2.5) and (2.8); the third — from Eqs. (2.6) and (2.11), corresponding to point D, while the fourth — from the definition of Z.

As pointed out earlier, in order to obtain a satisfactory solution for the adiabatic point explosion, it is found that taking $\varepsilon \equiv Z_n/Z$ in Eq. (3.1) is quite sufficient. The cross-deriv, atives at the singularity are obtained from Eqs. (4.5)—(4.8), while those behind the front are evaluated by differentiating directly Eqs. (4.1)—(4.4) with respect to y, keeping in mind that $\Phi^k = \lambda y \dfrac{\partial \ln k}{\partial y}$, while the position of singularity D corresponds to $\varepsilon = 0$ and $y = 0$.

For fixed values of y or ξ, Eqs. (2.1)–(2.4) are reduced to a set of ordinary differential equations. Their solution yields the integral curves delineating the phase space, as well as the gasdynamic profiles of the flow field. Each integral curve, however, is still coupled to three parameters that are not known *a priori*: the decay coefficient $\lambda(y)$, specifying the front trajectory; the singularity location, $F_D(y)$, which is related to λ through Eq. (4.7); and the parameter α_D. One has thus, at each step corresponding to a fixed value of y, a double boundary value problem whose solution requires an iterative procedure to determine the unknown parameters λ, F_D and α_D. This is accomplished in the following manner.

Preliminary values for the unknown parameters, as well as for $(\partial F/\partial y)_D$, are based upon an extrapolation from a solution corresponding to a smaller value of y. Since singularity D is in general a saddle point, one cannot integrate into it. Consequently, asymptotic analysis is used to specify the solution in the vicinity of the singularity, i.e. for $100 \leqslant Z \leqslant \infty$ Then Eqs. (2.1)–(2.4) are integrated from $Z = 100$ to the shock front. At the same time the mass integral I_1 and the energy integral J_3 defined as

$$J_1 \equiv \int_0^1 hx^j dx \tag{4.9}$$

and

$$J_3 \equiv \int_0^1 \left(\frac{Z}{(\gamma-1)\gamma} - \frac{F^2}{2}\right) hx^{j+2} dx \tag{4.10}$$

are evaluated, while the decay coefficient $\lambda(y)$ is calculated on the basis of the energy equation expressed in the following integral form (OPPENHEIM et al., 1971):

$$\left(\frac{1}{\xi}\right)^{j+1} = \frac{\gamma}{y} J_3 - \frac{1}{(j+1)(\gamma-1)}. \tag{4.11}$$

Noting that λ is only a function of y, the above may be differentiated with respect to y yielding thus

$$\lambda = \frac{(j+1)J_3 - y/(\gamma-1)\gamma}{J_3 - y dJ_3/dy}. \tag{4.12}$$

The logarithmic density gradient at the center α_D is then varied until the solution conserves mass globally, i.e.

$$J_1 = \frac{1}{j+1} \tag{4.13}$$

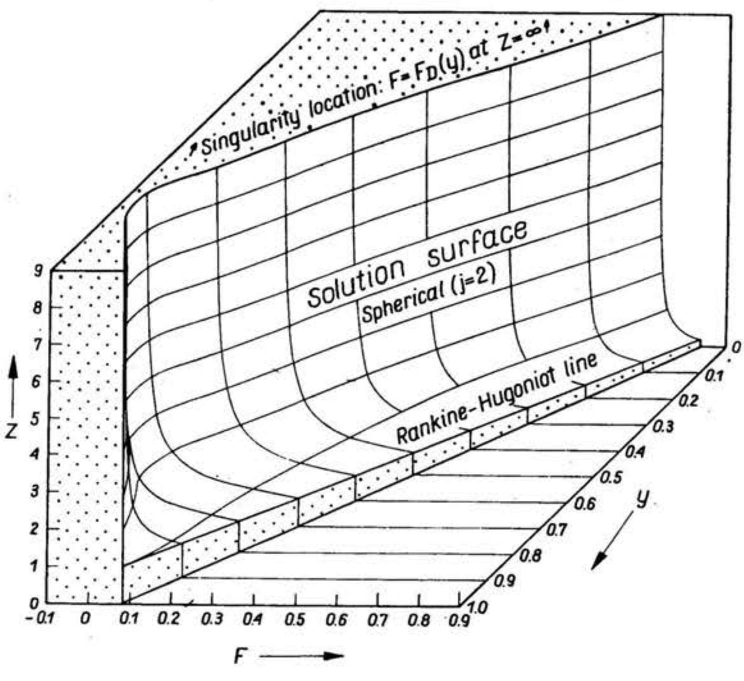
and the decay coefficient λ is varied until it agrees with the value given by Eq. (4.12). Finally, the singularity location is adjusted so that the integral curve terminates at the assumed shock front boundary conditions F_n and Z_n given by the specific value of y for which the solution at a given step is sought.

5. Results

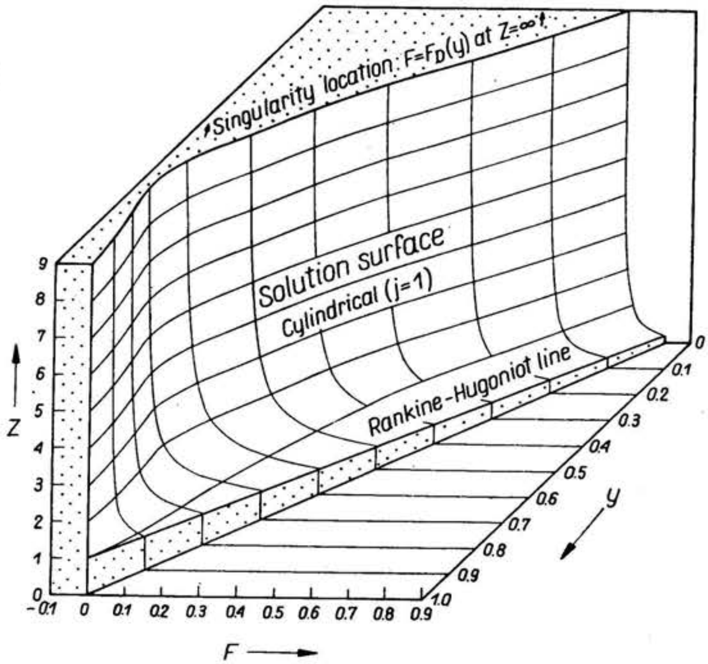
The method of phase space was applied to the classical adiabatic point explosion problem in spherical, cylindrical and planar geometries, the blast wave propagating into a uniform quiescent atmosphere of finite pressure. The medium was assumed to behave as a perfect gas with $\gamma = 1.4$.

Figure 1 depicts the solution surface in the phase space. As pointed out at the outset, the surface consists of integral curves, each for a fixed value of y. Each integral curve starts from an appropriate point on the Rankine-Hugoniot curve, representing conditions at the front, and ends at singularity $D(Z = \infty;\ F = \hat{F}_D)$, representing conditions at the center. The surface is bounded on one side by the integral curve $y = 0$, the self-similar

a)



b)



c)

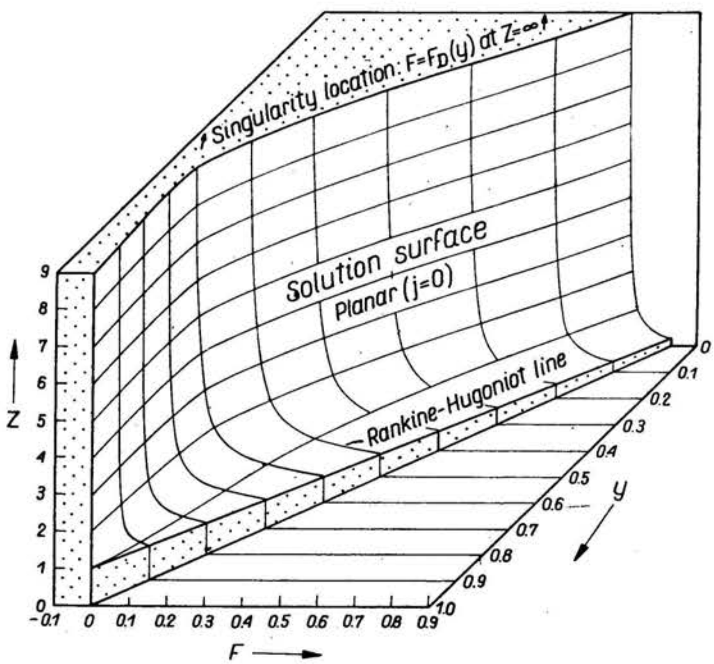


FIG. 1 (a, b, c). Phase space for point explosion in an inviscid and non-conductive atmosphere of finite pressure with $\gamma = 1.4$.

limit, while on the other side by the vertical line $y = 1$ corresponding to the acoustic limit. Curves of constant values of Z are included to facilitate the visualization of the integral surface. It should be noted that integral surfaces, within the range $0.6 < y < 1$, extend into negative values of F, indicating that the particles after their initial outward shift due to the passage of the front move towards the center of the flow field. One should also note that the integral surface approaches the acoustic solution smoothly, as one should expect.

In order to compare our solution with that obtained by KOROBEINIKOV *et al.* (1966) using the method of integral relations, the integral curves of both solutions are compared in Fig. 2 for some representative values of y. Solid lines represent solutions obtained by the phase space method. From this comparison it appears that for intermediate values of y the method of integral relations yields higher values for F at any given Z, thus slightly shifting the location of singularity D in the phase plane. This shift is most probably due to the fact that the central strip used in the method of integral relations, where the asymptotic solution is considered accurate, extends over a larger portion of the flow field than in the present solution. For example, for $y = 0.2$, in the case of $j = 2$ and $\gamma = 1.4$, the central strip extends to $x = 0.5$ which corresponds to $Z = 20.34$, while for the same conditions the asymptotic solution, taken along the axis of singularity D, extends only to $x = 0.45$

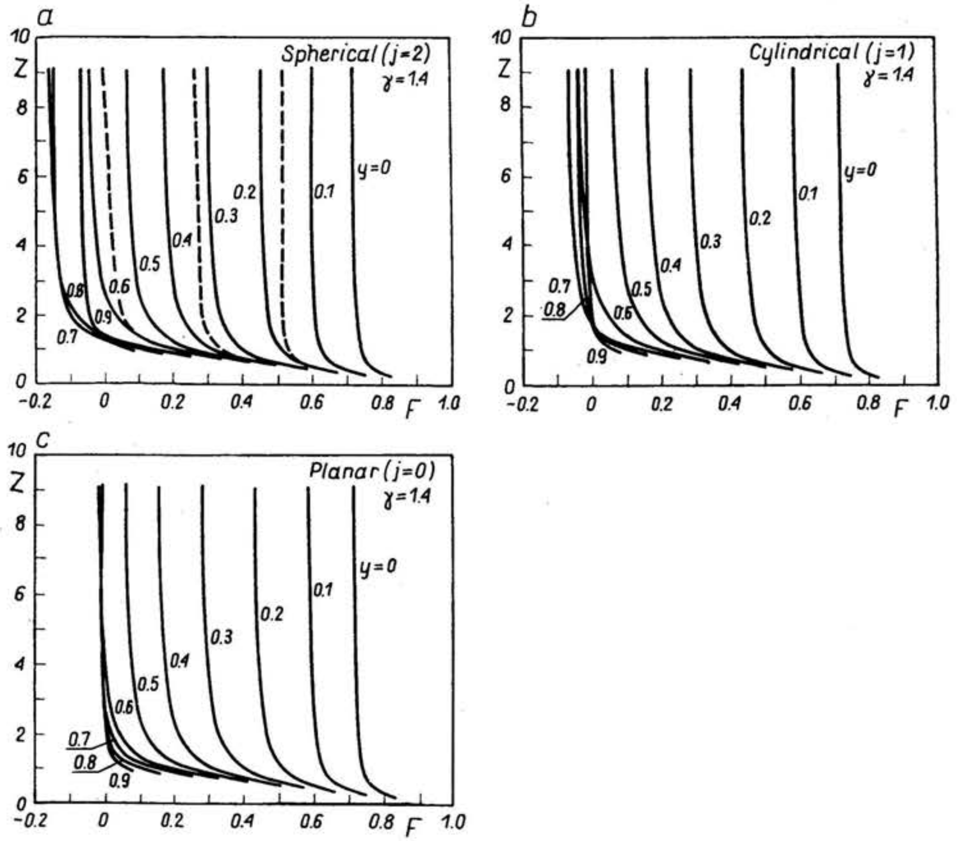


FIG. 2. Integral curves for representative values of y. The dashed lines are those of KOROBEINIKOV *et al.* (1969).

and $Z = 100$. In addition, the marching technique inherent in the method of integral relations tends to decrease the accuracy of the solution in the vicinity of singularity D especially if it is far from the self-similar limit. This point is especially apparent near the acoustic

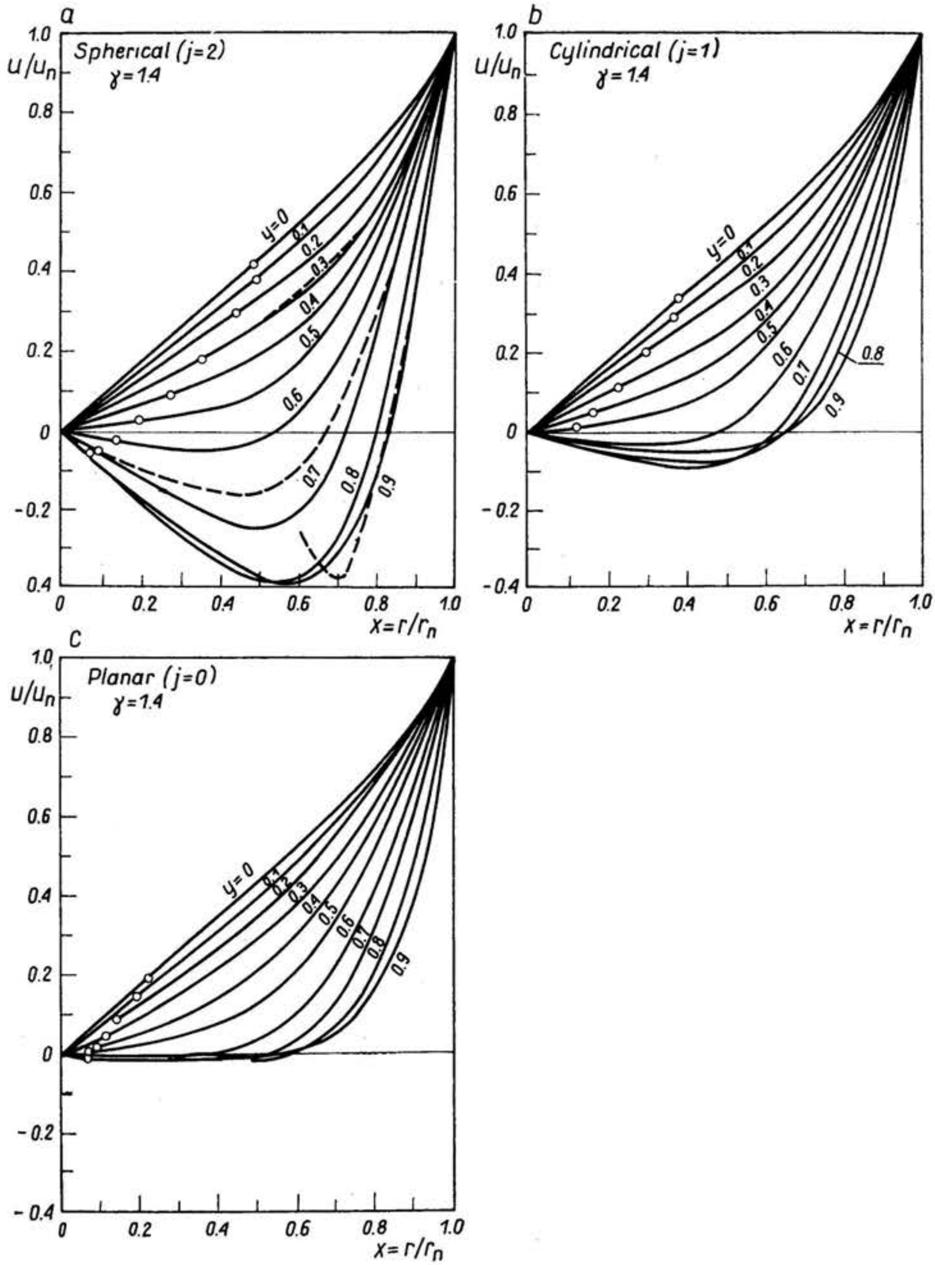


FIG. 3. Velocity profiles obtained by phase space method for point explosion without transport phenomena, ($\gamma = 1.4$). Broken lines represent solution of KOROBEINIKOV *et al.* (1969). Small circles mark the matching point with asymptotic solution.

limit. For example, while the curve for $y = 0.9$ in Fig. 2, evaluated by the phase space method, approaches smoothly its asymptotic value at the center, that obtained by the method of integral relations bulges pronouncedly in the negative F direction before returning to the value at the center, a feature which appears quite unnatural.

The profiles of the gasdynamic parameters, namely velocity, pressure and density, are presented respectively, in Figs. 3—5 for $j = 2$, 1 and 0. Broken lines show the solutions obtained by the computations of KOROBEINIKOW *et al.* (1969) using the BELOTSERKOVSKII's method of integral relations (1965). Circles on the curves indicate transition points from the asymptotic analysis to numerical integration.

As they appear in Fig. 3, the velocity profiles we obtained tend to fill out the void caused by the condition of zero velocity at the center more completely than they do according to the solution of KOROBEINIKOW *et al.* (1969).

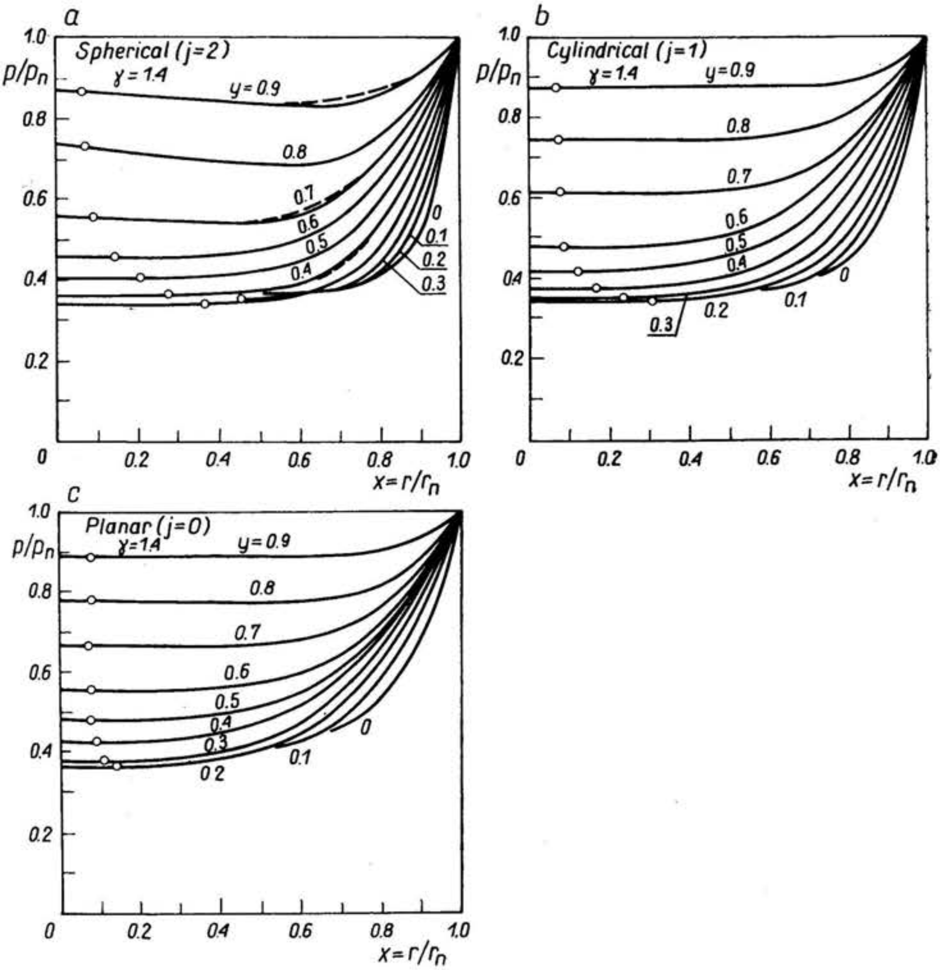


FIG. 4. Pressure profiles obtained by phase space method for point explosion without transport phenomena, ($\gamma = 1.4$). Broken lines represent solution of KOROBEINIKOV *et al.* (1969). Small circles mark the matching point with asymptotic solution.

Pressure profiles are given in Fig. 4. One notes that pressure at the center is always finite, first becoming smaller as the Mach number of the front decreases from infinity corresponding to the self-similar solution, and then increases towards unity as the blast wave decays to a sound wave.

Figure 5 displays the density profiles. It is evident that the density at the center is always zero, corresponding to infinite temperature, a characteristic property of the solutions obtained for an inviscid gas without taking into account the effects of transport properties.

The results of the phase space method turned out to be in a significantly better agreement with the mass conservation integral than those of the integral relations method. This is reflected especially in the differences between the density profiles near the center for weaker shocks. For example, for a spherical point explosion with $\gamma = 1.4$, the error in mass obtained by KOROBEINIKOV *et al.* (1969) reaches 4% while, by the use of the phase

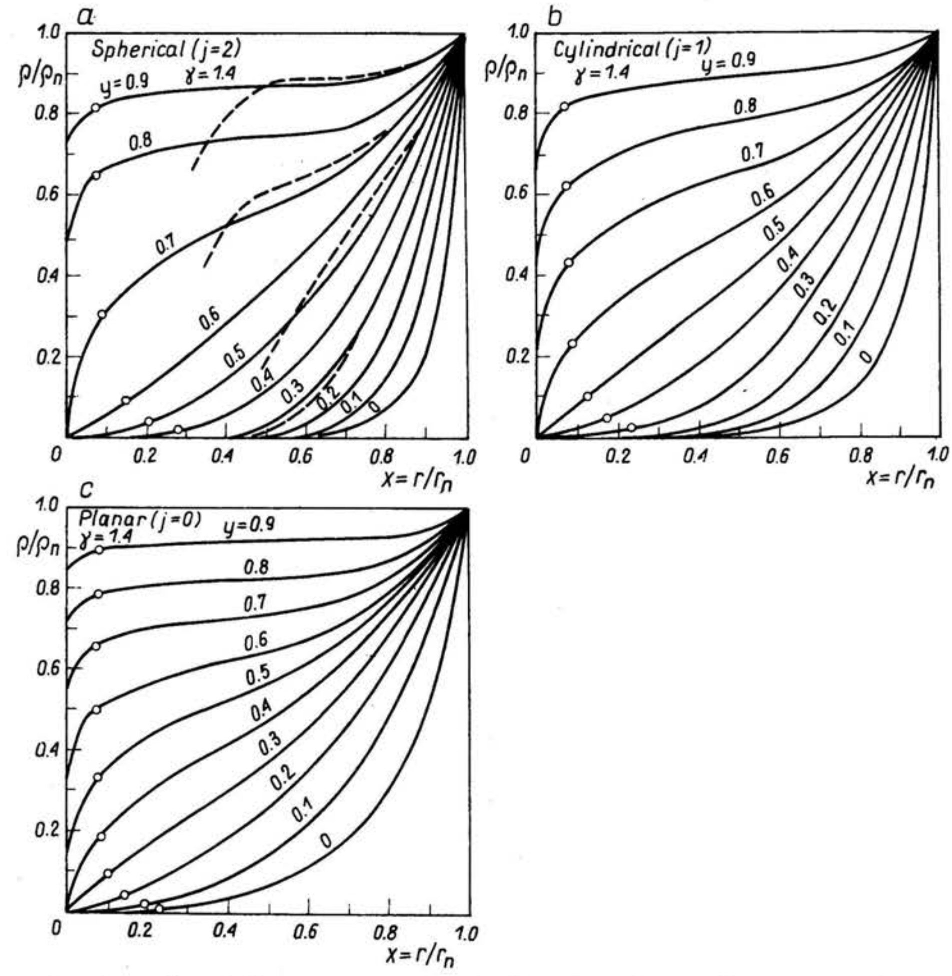


FIG. 5. Density profiles obtained by phase space method for point explosion without transport phenomena, ($\gamma = 1.4$). Broken lines represent solution of KOROBEINIKOV *et al.* (1969). Small circles mark the matching point with asymptotic solution.

space method, this error is maintained at the level of 0.01% for the same conditions, thus demonstrating the accuracy of the present technique.

The main parameters of our solution are presented in Figs. 6, 7 and 8. They were determined by iteration as described in the previous section.

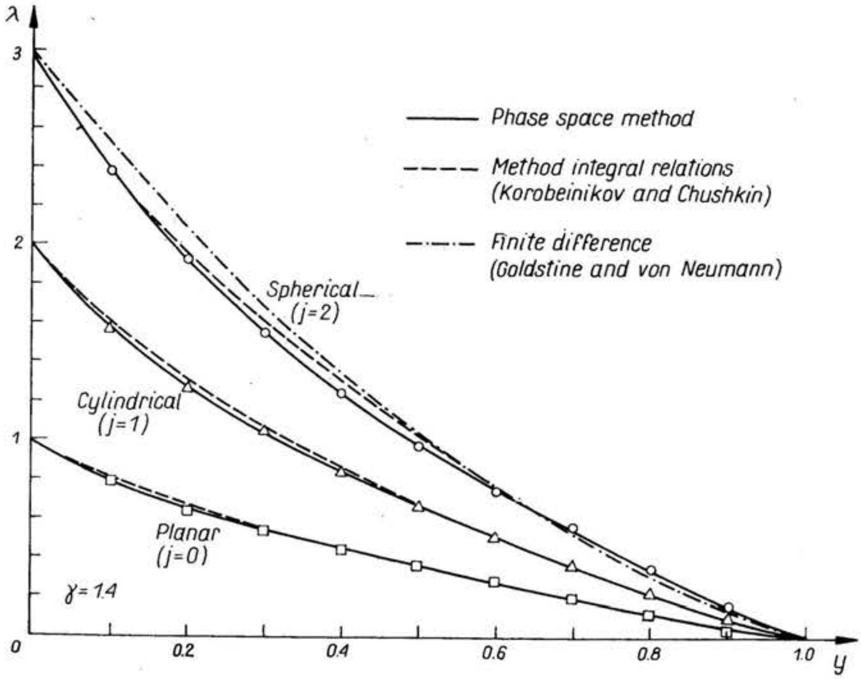


FIG. 6. Decay coefficient, λ, as a function of shock strength, y, for spherical, cylindrical and planar symmetries ($\gamma = 1.4$). Dashed lines are those obtained from the solution of KOROBEINIKOV *et al.* (1969), while the chain dotted line is based on the results of GOLDSTINE and VON NEUMANN (1951).

Figure 6 compares the front decay parameter, λ, with that obtained by the method of integral relations (KOROBEINIKOV and CHUSHKIN, 1966, 1969). They are in close agreement over the entire range of shock strengths, with the maximum difference of 7%, at $y \approx 0.7$. Also included for reference is the decay coefficient evaluated from the classical numerical solution of Goldstine and von Neumann.

Figure 7 provides information on the location of singularity D, $F_D(y)$ for point, line, and plane symmetrical explosions in comparison to $F_n(y)$, the front limit of the integral curve. For strong and moderate shock strengths ($0 \leqslant y \leqslant 0.5$) the singularity position is almost independent of the geometry. For weak shock strengths ($y \geqslant 0.55$), its location corresponds to negative values of F and becomes dependent on the geometry; this corresponds to the initiation of a negative velocity region in the space profiles (viz Fig. 3).

Figure 8 shows the variation of the density modulus parameter α_D (logarithmic slope of the density profile at the center of symmetry) as a function of the shock strength. This parameter is not bounded by the self-similar value $\alpha_D(y = 0) = (j+1)F_D/(1-F_D)$, but reaches a maximum at a finite shock strength ($y \approx 0.10$). For weaker shock strengths α_D is less than one, which introduces an inflection point in the density profiles (viz. Fig. 5).

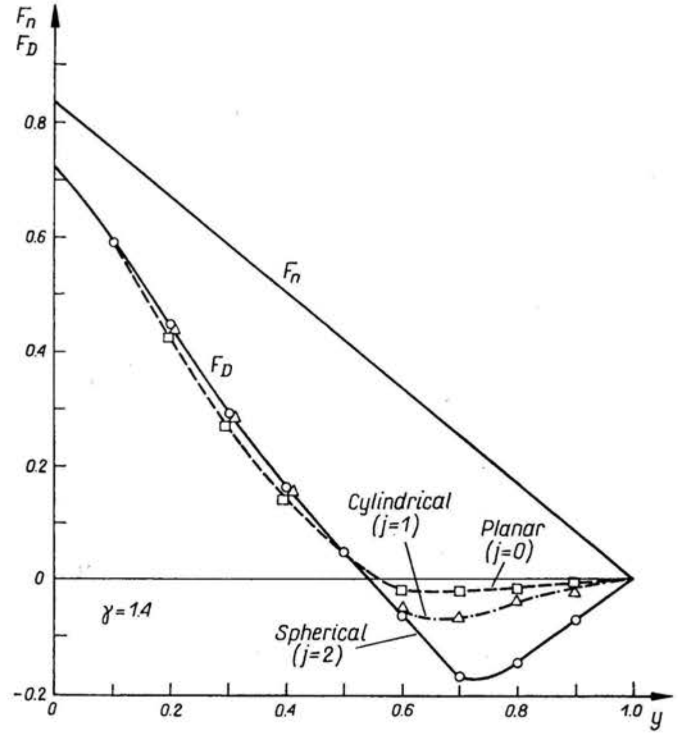


FIG. 7. Singularity location, F_D(at $Z_D = \infty$), and shock front location, F_n (at Z_n) as a function of shock strength, y, for $j = 0, 1$ and 2, with $\gamma = 1.4$.

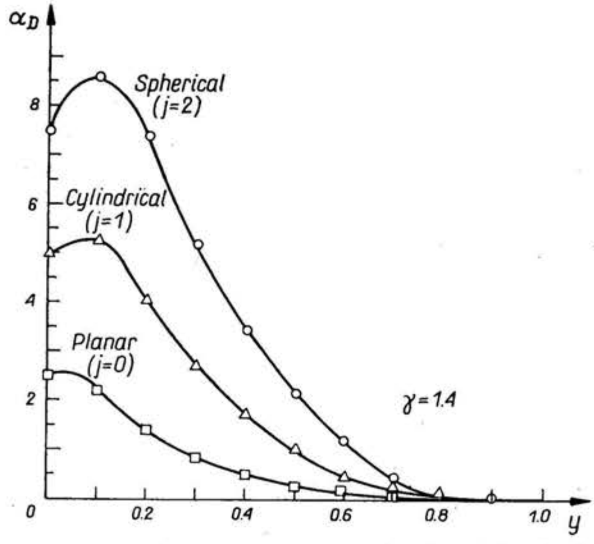


FIG. 8. Variation of the density modulus parameter, α_D, as a function of the shock strength y for $j = 0, 1$ and 2 with $\gamma = 1.4$.

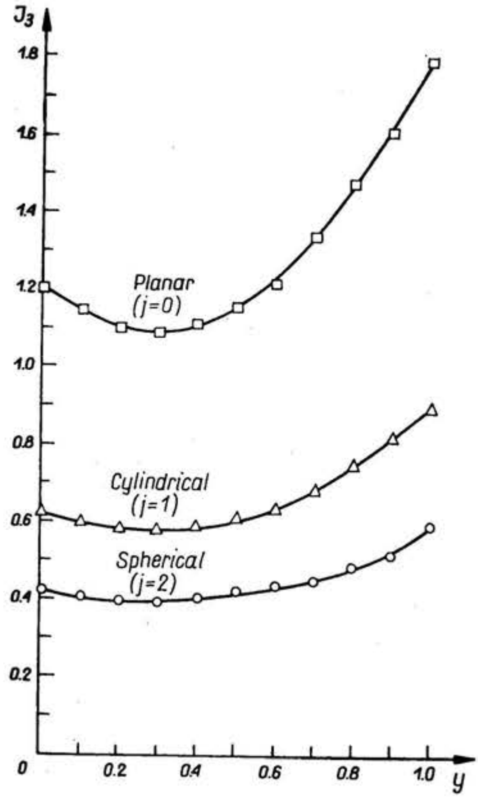


FIG. 9. Energy integral $J_3(y)$ as a function of shock strength y for $j = 0, 1$ and 2, with $\gamma = 1.4$.

Finally, the energy integral, $J_3(y)$, for spherical, cylindrical and planar explosions is presented in Fig. 9 as a function of the shock strength. The decrease of J_3 with y over the range $0 \leqslant y \leqslant 0.3$ is caused by the decrease of pressure at the center of the wave. The energy integral approaches the value of $1/\gamma(\gamma-1)(j+1)$ as the shock Mach number approaches unity.

6. Summary

A new technique, the phase space method, has been developed for solving non-steady flow fields with point, line or plane symmetry. On its basis conservation equations are reduced to a set of ordinary differential equations which have the same form as the phase plane equations for self-similar blast waves. For a fixed value of the shock strength the solution of this differential equation defines an integral curve. The full set of these curves forms a solution surface in phase space. Once this surface is determined, flow field profiles are obtained from quadratures. The singular behavior inherent in inviscid blast wave problems is merely a consequence of the singularities of the conservation equations; most other methods (finite difference, integral relations, etc.) ignore their effects. With the phase space method, however, this singular behavior is prominently taken into account in the integration procedure furnishing the solution.

The phase space method was applied to the classical problem of spherical, cylindrical, and planar explosion propagating in a uniform atmosphere of non-zero counter-pressure. The flow field profiles (pressure, density, and velocity) calculated by this method agree qualitatively over the entire range of shock strengths with results obtained by the method of integral relations. The mass integral, and, consequently, the density profiles calculated by the present method retain their accuracy even as the shock Mach number approaches unity.

In principle this method can be used to extend all existing self-similar solutions to cover the entire variation of shock Mach numbers from infinity to its asymptotic lower limit. Examples of other problems emenable to this approach are: explosions in a detonating gas, shock waves driven by accelerating and decelerating pistons or by deflagration waves, as well as implosions in a gas of finite (non-zero) pressure.

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