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A variational principle and a finite element method for compressible flow with free boundaries

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BASED on the stream function formulation, a variational principle is presented for steady subsonic two-dimensional flows with free streamlines. We begin, for fixed boundaries, by stating two variational principles for a rotational flow of perfect gas, one with the stream function and the other with the density as an independent variable. A numerical method is devised by a finite element approximation of these variational principles and then applied to the problem of exit flow from two stream nozzles.

Wykorzystując sformułowanie rozważanego zagadnienia przez funkcję prądu, przedstawiono zasadę wariacyjną dla ustalonych dwuwymiarowych przepływów poddźwiękowych ze swobodnymi liniami prądu. Najpierw, dla brzegów zamocowanych, sformułowano dwie zasady wariacyjne dla wirowego przepływu gazu doskonałego, przyjmując w pierwszej funkcję prądu, a w drugiej gęstość jako zmienne niezależne. Jako metodę numeryczną wybrano metodę ele mentów skończonych, a następnie zastosowano ją do zagadnienia wypływu gazu z dwustrumieniowych dysz.

Используя формулировку рассматриваемой задачи через функцию тока, представлен вариационный принцип для установившихся двумерных дозвуковых течений со свободными линиями тока. Сначала, для закрепленных границ, сформулированы два вариационных принципа для вихревого течения идеального газа, принимая в первом функцию тока, а во втором плотность как независимые переменные. Как численный метод избран метод конечных элементов, а затем он применен для задачи истечения газа из двухструйных выходных отверстий.

Introduction

VARIATIONAL principles for problems in classical fluid dynamics have been studied for a long time. Apart from giving an elegant derivation for some equations of fluid dynamics, these variational principles lead to convincing proofs for existence and uniqueness. The fact that they can also generate efficient numerical methods received during the last years some illustrations in finite element methods.

The present paper deals with such a constructive aspect of variational principles in the case of a steady subsonic two-dimensional flow of a perfect gas with free boundaries. First, we formulate three variational principles: for stream function, density and free boundaries; then we derive a numerical method based on these principles and the finite element approximation to a model problem, namely the steady subsonic irrotational and axisymmetric exit flow from two-stream nozzles.

1.

1.1. Notations and equations

Through this paper, Ω is a bounded domain for plane or axisymmetric meridianal flow. The boundaries are fixed or partly unknown streamlines (Γ_D), and inlet or outlet sections (Γ_N).

$$\Gamma_D = \Gamma_{DL} \cup \Gamma_{DU}, \quad \Gamma_N = \Gamma_{NI} \cup \Gamma_{NO}.$$

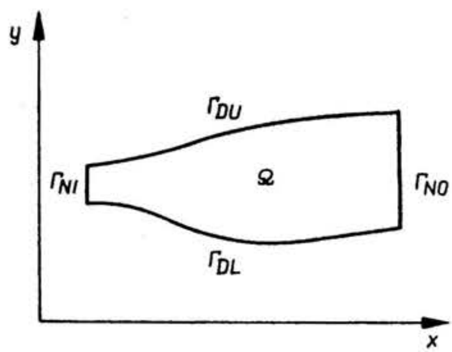


FIG. 1.

After assuming the case of a perfect gas with constant specific heats, we have the following thermodynamical relations:

(1.1) $$p = \varrho RT, \quad R = c_p - c_v;$$

(1.2) $$p = \varrho^{\gamma} e^{(S-S_0)/c_v}, \quad \gamma = c_p/c_v,$$

where p, ϱ, T, S are pressure, density, temperature and specific entropy.

The dynamical equations are expressed in terms of the stream function ψ for the plane ($\varepsilon = 0$) or the axisymmetric flow ($\varepsilon = 1$) with the following notations:

$$\vec{V} = \begin{vmatrix} u \\ v \end{vmatrix} = \begin{vmatrix} \dfrac{1}{\varrho y^{\varepsilon}} \dfrac{\partial \psi}{\partial y} \\ -\dfrac{1}{\varrho y^{\varepsilon}} \dfrac{\partial \psi}{\partial x} \end{vmatrix}, \quad \overrightarrow{\nabla \psi} = \begin{vmatrix} \dfrac{\partial \psi}{\partial x} \\ \dfrac{\partial \psi}{\partial y} \end{vmatrix},$$

$$q^2 = |\vec{V}|^2, \quad \lambda = \varrho^2 q^2 = \frac{1}{y^{2\varepsilon}} |\overrightarrow{\nabla \psi}|^2.$$

Using the property that the entropy S and the total enthalpy H remain constant on each streamline, we assume that they are given functions of ψ

$$S = S(\psi), \quad H = H(\psi).$$

The Bernoulli's equation provides us with an expression for λ in terms of ϱ, H, S:

(1.3) $$\lambda = \frac{|\nabla \psi|^2}{y^{2\varepsilon}} = 2\varrho^2 H - \frac{2\gamma}{\gamma - 1} \varrho^{\gamma+1} e^{(S-S_0)/c_v}.$$

The equation of motion and boundary conditions are

$$(1.4)\qquad \begin{gathered} \nabla\cdot\left(\frac{1}{\varrho y^{\varepsilon}}\overrightarrow{\nabla}\psi\right) = \varrho y^{\varepsilon}\big(H'(\psi)-TS'(\psi)\big) \quad \text{in} \quad \Omega, \\ \psi|_{\Gamma_{DL}} = 0, \quad \psi|_{\Gamma_{DU}} = Q, \quad \left.\frac{\partial\psi}{\partial n}\right|_{\Gamma_N} = 0, \end{gathered}$$

where Q is the value of the total mass flow in Ω.

If we define the function space $\vartheta(\Omega)$ and the subset $\vartheta_\alpha(\Omega)$,

$$\vartheta(\Omega) = \left\{\psi|\psi, \frac{1}{y^{\varepsilon}}\frac{\partial\psi}{\partial x}, \frac{1}{y^{\varepsilon}}\frac{\partial\psi}{\partial y} \in L^2(\Omega)\right\},$$

$$\vartheta_\alpha(\Omega) = \{\psi\in\vartheta(\Omega)|\, \psi|_{\Gamma_{DL}} = 0;\, \psi|_{\Gamma_{DU}} = \alpha\}, \quad \alpha \geqslant 0,$$

we can give a weak formulation for the equation of motion: Find $\psi\in\vartheta_Q(\Omega)$ such that for all φ in $\vartheta_0(\Omega)$

$$(1.5)\qquad \int_\Omega \left\{\frac{1}{\varrho y^{2\varepsilon}}\overrightarrow{\nabla}\psi\cdot\overrightarrow{\nabla}\varphi + \varrho\big(H'(\psi)-TS'(\psi)\big)\varphi\right\} y^{\varepsilon}dx\,dy = 0,$$

where ϱ and T implicitly depend on ψ and $|\overrightarrow{\nabla}\psi|^2$ by Eqs. (1.1), (1.2), (1.3).

1.2. A variational principle for the stream function

We have seen that λ depends explicitly on ϱ, H, S, then for $\frac{\partial\lambda}{\partial\varrho} < 0$ we can define ϱ implicitly in terms of λ, H, S and therefore $\varrho = \varrho(\psi)$. In the same way $p+\varrho q^2$ expressed in terms of ϱ, H, S can be considered as a function of ψ:

$$(1.6)\qquad P(\psi) = p+\varrho q^2 = 2\varrho H - \frac{\gamma+1}{\gamma-1}\varrho^{\gamma}e^{(S-S_0)/c_v}.$$

The condition $\frac{\partial\lambda}{\partial\varrho} < 0$ is nothing else than the subsonic condition

$$(1.7)\qquad (\gamma+1)c_pT-2H > 0 \quad \text{where} \quad T = \frac{1}{c_v(\gamma-1)}\varrho^{\gamma-1}e^{(S-S_0)/c_v}.$$

Let us consider the functional

$$(1.8)\qquad I(\Omega,\psi) = \int_\Omega P(\psi)y^{\varepsilon}dx\,dy \quad \text{for} \quad \psi\in\vartheta_Q(\Omega).$$

Assuming a sufficiently small value for a given Q and a sufficiently smooth $\partial\Omega$ without a reentrant corner for a given Ω, we have the following variational principle for the stream function ψ^* solution of the equation of motion:

The stream function ψ^* satisfying the equation of motion (1.4) gives a stationary value to $I(\Omega,\psi)$ over $\vartheta_Q(\Omega)$.

Moreover, this stationary value is a local minimum under the two sufficient conditions:

$$(1.9)\qquad (\gamma+1)c_pT-2H > 0$$

$$(1.10)\qquad H''(\psi)-T(S''(\psi)+S'(\psi)^2/c_v) \geqslant 0 \quad \text{a.e.on} \quad \Omega.$$

We only sketch the proof in giving the first and second Gateaux-derivatives of $I(\Omega, \psi)$ with respect to ψ. First, it is quite easy to obtain

$$dP = \frac{1}{2\varrho} d\lambda + \varrho(dH - TdS).$$

Therefore, denoting $\left(\frac{\partial I}{\partial \psi}, \varphi\right) = \lim_{\theta \to 0} [I(\Omega, \psi+\theta\varphi) - I(\Omega, \psi)]/\theta$ we have

$$\left(\frac{\partial I}{\partial \psi}, \varphi\right) = \int_{\Omega} \left\{\frac{\nabla\psi \cdot \nabla\varphi}{\varrho y^{2\varepsilon}} + \varrho\big(H'(\psi) - TS'(\psi)\big)\varphi\right\} y^{\varepsilon} dx\, dy$$

and the stationarity of $I(\Omega, \psi)$ for $\psi^* \in \vartheta_Q(\Omega)$ follows from Eq. (1.5)

$$\left(\frac{\partial I}{\partial \psi}, \varphi\right)\Big|_{\psi=\psi^*} = 0, \quad \forall \varphi \in \vartheta_0(\Omega).$$

We can show that the second Gateaux-derivative of $I(\Omega, \psi)$ is

$$\left(\frac{\partial^2 I}{\partial \psi^2}, \varphi, \varphi\right)\Big|_{\psi=\psi^*} = \int \left\{\frac{|\overrightarrow{\nabla\varphi}|^2}{\varrho y^{2\varepsilon}} + \varrho B\varphi^2 + \frac{C^2}{\varrho A}\right\} y^{\varepsilon} dx\, dy,$$

where

$$A = (\gamma+1)c_p T - 2H,$$
$$B = H'' - T(S'' + S'^2/c_v),$$
$$C = \frac{1}{\varrho y^{2\varepsilon}} \overrightarrow{\nabla\psi^*} \cdot \overrightarrow{\nabla\varphi} - \varrho(H' - \gamma TS')\varphi,$$

so that $A > 0$ and $B \geqslant 0$ suffice to insure the local convexity of $I(\Omega, \psi)$.

This variational principle is not truly new. The functional $I(\Omega, \psi)$ is one of the two well-known Bateman integrals. (The other one is the pressure integral, the maximization of which leads to the velocity potential equation).

We refer to the illuminating paper of SEWELL [1] which also recalls the contributions of LIN [2], LUSH and CHERRY [3], SERRIN [4] for the stationarity of the integral of $P = p + \varrho q^2$.

With regard to the local convexity of $I(\Omega, \psi)$, it seems that our second condition is new. This sufficient condition concerns only rotational flow and simplifies for

a) isoenergetic flow ($H' \equiv 0$): $S'' + S'^2/c_v \leqslant 0$,

b) isentropic flow ($S' \equiv 0$): $H'' \geqslant 0$.

We have as far not further studied this convexity condition which could be connected with the stability of the flow.

Finally, the fact that for given values of S and $H > 0$ there exists an upper bound for λ (precisely at a sonic point) gives us a somewhat qualitative argument for justifying our assertion about the necessity of boundedness of Q for the existence of a solution.

1.3. A variational principle for the density

Let us now consider the following functional:

$$(1.11) \qquad \mathcal{J}(\Omega, \varrho, \psi) = \int_{\Omega} \left\{\frac{1}{2} \frac{|\overrightarrow{\nabla\psi}|^2}{\varrho y^{2\varepsilon}} + \varrho H(\psi) - \frac{\varrho^{\gamma}}{\gamma-1} e^{(S(\psi)-S_0)/c_v}\right\} y^{\varepsilon} dx\, dy,$$

defined for each $\psi \in \vartheta_Q(\Omega)$ and each ϱ a positive and bounded function over Ω. This integral was already introduced by LIN and RUBINOV [5] in the case of iso-energetic flow $(H(\psi) \equiv H_0)$.

Then we define $\psi(\varrho)$ as the solution in $\vartheta_Q(\Omega)$ of

$$(1.12)\qquad \int_\Omega \left\{ \frac{\overrightarrow{\nabla\psi}\cdot\overrightarrow{\nabla\varphi}}{\varrho y^{2\varepsilon}} + \varrho\left(H'(\psi) - \frac{\varrho^{\gamma-1}}{c_v(\gamma-1)} e^{(S(\psi)-S_0)/c_v} S'(\psi)\right)\varphi\right\} y^\varepsilon dx\,dy = 0,$$

$$\forall \varphi \in \vartheta_0(\Omega).$$

This is the weak form (1.5) of the equation of motion but for ϱ not necessarily connected with ψ by the Bernouilli equation.

Assuming for a moment the existence and uniqueness of the solution $\psi(\varrho)$, we set

$$J(\Omega, \varrho) = \mathscr{J}(\Omega, \varrho, \psi(\varrho)),$$

and with the same restrictions as before about the given Q and Ω we can state the following:

The Bernouilli equation connecting ϱ and $\psi(\varrho)$ is the stationarity condition for $J(\Omega, \varrho)$ with respect to ϱ.

Moreover, the stationary value is a maximum under the two sufficient conditions:

$$(1.13)\qquad \gamma\frac{\gamma+1}{\gamma-1}\varrho^{\gamma-1}e^{(S-S_0)/c_v} - 2H > 0,$$

$$(1.14)\qquad H'' - \frac{\varrho^{\gamma-1}}{c_v(\gamma-1)} e^{(S-S_0)/c_v}(S'' + S'^2/c_v) \geqslant 0 \quad \text{a.e.in} \quad \Omega.$$

Proof:

With the choice of $\psi(\varrho)$ according to Eq. (1.12) we get

$$\left(\frac{\partial\mathscr{J}}{\partial\psi}, \varphi\right)\bigg|_{\psi=\psi(\rho)} = 0, \quad \forall\varphi \in \vartheta_0(\Omega),$$

and therefore,

$$\left(\frac{\partial J}{\partial\varrho}; \tau\right) = \left(\frac{\partial\mathscr{J}}{\partial\varrho}, \tau\right)\bigg|_{\psi=\psi(\rho)} = \int_\Omega \left\{ -\frac{|\overrightarrow{\nabla\psi}|^2}{y^{2\varepsilon}} + 2\varrho^2 H - \frac{2\gamma\varrho^{\gamma+1}}{\gamma-1} e^{\frac{S-S_0}{c_v}}\right\}_{2\rho^2} \tau y^\varepsilon dx\,dy$$

giving the proof of the first assertion.

We then have to use a Lagrange multiplier technique for the evaluation of the second Gateaux-derivative $\left(\frac{\partial^2 J}{\partial\varrho}, \tau_1, \tau_2\right)$ and after the definition of χ as the solution of

$$-\nabla\left(\frac{1}{\varrho y^\varepsilon}\overrightarrow{\nabla\chi}\right) + \varrho y^\varepsilon B\chi = \nabla\left(\frac{\tau}{\varrho^2 y^\varepsilon}\overrightarrow{\nabla\psi}\right) + y^\varepsilon\tau\left(H' - \frac{\gamma\varrho^{\gamma-1}}{c_v(\gamma-1)} e^{\frac{S-S_0}{c_v}} S'\right),$$

$$\chi\bigg|_{\Gamma_D} = 0, \quad \frac{\partial\chi}{\partial n}\bigg|_{\Gamma_N} = 0 \quad \text{with} \quad \varrho = \varrho^*, \quad \psi = \psi(\varrho)^*,$$

we get

$$\left(\frac{\partial^2 J}{\partial \varrho^2}, \tau, \tau\right)\Big|_{\rho=\rho^*} = -\int_{\Omega} \left\{\frac{A\tau^2}{\varrho^*} + \frac{|\overrightarrow{\nabla\chi}|^2}{\varrho^* y^{2\varepsilon}} + \varrho^* B\chi^2\right\} y^\varepsilon dx\,dy,$$

$$A = \gamma\frac{\gamma+1}{\gamma-1}\varrho^{\gamma-1}e^{(S-S_0)/c_v} - 2H,$$

$$B = H'' - \frac{\varrho^{\gamma-1}}{c_v(\gamma-1)}e^{(S-S_0)/c_v}(S'' + S'^2/c_v).$$

It is time to notice that the condition (1.14): $B \geqslant 0$ in $\Omega \forall \varrho > 0$ is a sufficient condition for

$$\left(\frac{\partial^2 \mathscr{J}}{\partial \psi^2}, \varphi, \varphi\right) > 0, \quad \forall \varphi \in \vartheta_0(\Omega),$$

therefore, in this case (with the supplementary assumption $A > 0$ in Ω);

(1.15) $$\max_{\varrho} J(\Omega, \varrho) \Leftrightarrow \max_{\varrho}[\min_{\psi} \mathscr{J}(\Omega, \varrho, \psi)].$$

For the solution fields ϱ^* and $\psi^* = \psi(\varrho^*)$ we have

(1.16) $$J(\Omega, \varrho^*) = \mathscr{J}(\Omega, \varrho^*, \psi^*) = I(\Omega, \psi^*) = \int_{\Omega} (p + \varrho q^2) y^\varepsilon dx\,dy.$$

This variational principle can also be seen as an optimal control problem (for fixed Q and Ω) where the density function ϱ is the distributed control, the stream function $\psi(\varrho)$ is the state with its nonlinear partial differential equation as "state equation" and $\mathscr{J}(\Omega, \varrho, \psi(\varrho))$ is the criterion to be maximized.

It is possible to extend this variational principle to other boundary conditions (non--homogeneous Neuman, periodic) for bounded Ω and even to exterior problems. Finally, as noticed by SEWELL [1] in a similar situation, the concavity of $J(\Omega, \varrho)$ with respect to ϱ can occur even for flow where the conditions $A \geqslant 0$, $B \geqslant 0$ should be slightly exceeded in some small part of Ω. Unfortunately, it seems difficult to give quantitative results about this situation.

1.4. A variational principle for free boundaries

It is convenient to denote by $\psi(\Omega)$ and $\varrho(\Omega)$ instead of ψ^*, ϱ^* the solution fields in Ω and to introduce for given Q

(1.17) $$K(\Omega) = J(\Omega, \varrho(\Omega)) = I(\Omega, \psi(\Omega)) = \int_{\Omega} (p + \varrho q^2) y^\varepsilon dx\,dy.$$

We intend to restrict the variations of Ω so that the boundaries remain smooth except perhaps in some convex corner. On the unknown portions $\Gamma_v \subset \Gamma_D$ of the limiting streamlines the Dirichlet conditions $\psi|_{\Gamma_{DL}} = 0$; $\psi|_{\Gamma_{DU}} = Q$ must be imposed.

Then, defining on Γ_v a function δn as a sufficiently small and smooth normal displacement of Γ_v, we can show that

(1.18) $$\delta K(\Omega) = \int_{\Gamma_v} p y^\varepsilon \delta n\,dy + O(\delta n^2).$$

A formal proof is easily derived using a Lagrangian

$$\mathcal{L}(\Omega, \psi, \mathfrak{z}) = I(\Omega, \psi) - \int_{\Gamma_D} \mathfrak{z}(\psi - \psi_D)\, d\gamma .$$

If we choose

$$\mathfrak{z}(\Omega)\Big|_{\Gamma_D} = \frac{1}{\varrho y^\varepsilon} \frac{\partial \psi}{\partial n}\Big|_{\Gamma_D} \quad \text{so that} \quad \left(\frac{\partial \mathcal{L}}{\partial \psi}, \varphi\right) = 0, \quad \forall \varphi \in \vartheta_0(\Omega).$$

Then

$$\delta \mathcal{L}\Big|_{\substack{\psi = \psi(\Omega) \\ \mathfrak{z} = \mathfrak{z}(\Omega)}} = \delta K(\Omega) = \int_{\Gamma_v} \left[(p + \varrho q^2) y^\varepsilon - \mathfrak{z}(\Omega) \frac{\partial \psi(\Omega)}{\partial n}\right] \delta n\, d\gamma = \int_{\Gamma_v} p y^\varepsilon \delta n\, d\gamma .$$

By means of this "Hadamard's formula" (1.18) we can build variational principles for several kinds of pressure condition on free streamlines.

For the model problem of the next part (see Fig. 2) of two perfect gases separated by an unknown free streamline Γ_{v_1} on which $p_1 = p_2$, the second free streamline Γ_{v_2}

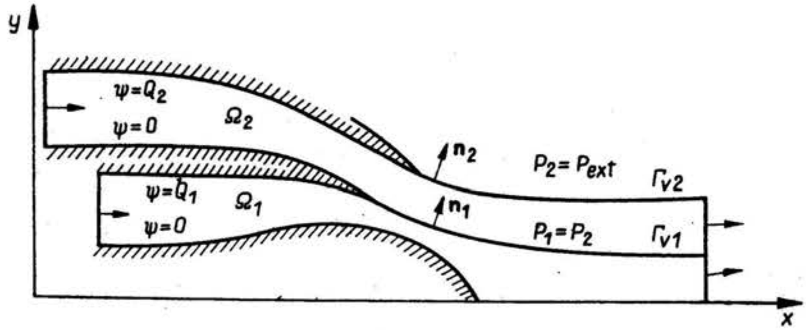


FIG. 2.

being the external boundary on which the pressure p_2 has a constant value p_{ext}, we define the function $\mathcal{F}$ in the following way:

$$(1.19) \qquad \mathcal{F}(\Omega_1, \Omega_2) = \int_{\Omega_1} (p + \varrho q^2 - p_{ext}) y^\varepsilon dx\, dy + \int_{\Omega_2} (p + \varrho q^2 - p_{ext}) y^\varepsilon dx\, dy;$$

then, for given Q_1 and Q_2 and compatible variations of Ω_1 and Ω_2 we have

$$(1.20) \qquad \delta \mathcal{F}(\Omega_1, \Omega_2) \simeq \int_{\Gamma_{v1}} (p_1 - p_2) y^\varepsilon \delta n_1 d\gamma + \int_{\Gamma_{v2}} (p_2 - p_{ext}) y^\varepsilon \delta n_2 d\gamma .$$

We can state the following variational principle:

If, for the given mass flow values Q_1 and Q_2, there exists in the class of admissible domains for Ω_1 and Ω_2 a pair (Ω_1^*, Ω_2^*) which makes $\mathcal{F}$ stationary, then the actual flow in Ω_1^*, Ω_2^* satisfies the pressure equilibrium conditions $p_1 = p_2$ on Γ_{v1} and $p_2 = p_{ext}$ on Γ_{v2}.

This result generalizes those of RIABOUCHINSKI [6], GARABEDIAN and al. [7, 8], LUKE [9], O'CARROLL and HARRISSON [10].

Unfortunately, we were not able to find a variational principle which includes variations of Q_1 and Q_2 so that the Kutta-Joukowski conditions would appear as a stationarity condition of some functional.

It is important to notice here that the present formulation is valid for any number of juxtaposed fluids and not only for two of them.

2.

2.1. The model problem

We henceforth consider the problem of steady subsonic irrotational axisymmetric exit flow from a two-stream nozzle with constant external pressure.

With the irrotationality assumption, the stagnation pressure and density p_{0l}, ϱ_{0l} are constant in $\Omega_l (l = 1, 2)$. We can set

$$H(\psi)\Big|_{\Omega_l} \equiv H_{0l} = \frac{\gamma_l}{\gamma_l - 1}\frac{p_{0l}}{\varrho_{0l}}, \quad e^{\frac{(S(\psi)-S_0)}{c_v}}\Big|_{\Omega_l} = \frac{p_{0l}}{(\varrho_{0l})^{\gamma_l}},$$

and then the main formulae of the first part (with the subscript l omitted) become,

$$\mathcal{J}(\Omega, \varrho, \psi) = \int_\Omega \left\{\frac{1}{2}\frac{|\overrightarrow{\nabla\psi}|^2}{\varrho y^2} + \varrho_0 H_0\left[\frac{\varrho}{\varrho_0} - \frac{1}{\gamma}\left(\frac{\varrho}{\varrho_0}\right)^\gamma\right]\right\} y\,dx\,dy,$$

$$\left(\frac{\partial \mathcal{J}}{\partial \psi}, \varphi\right) = \int_\Omega \frac{\overrightarrow{\nabla\psi}\cdot\overrightarrow{\nabla\varphi}}{\varrho y}\,dx\,dy,$$

$$\left(\frac{\partial \mathcal{J}}{\partial \varrho}, \tau\right) = \int_\Omega \left\{-\frac{|\overrightarrow{\nabla\psi}|^2}{y^2} + 2\varrho_0^2 H_0\left[\left(\frac{\varrho}{\varrho_0}\right)^2 - \left(\frac{\varrho}{\varrho_0}\right)^{\gamma+1}\right]\frac{y\tau}{2\varrho^2}\right\} dx\,dy.$$

It is quite easy to verify that for

$$\frac{|\overrightarrow{\nabla\psi}|^2}{y^2} < \lambda_{\max} = \varrho_0^2 H_0(\gamma-1)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}$$

we can express $p+\varrho q^2$ in terms of $\frac{|\overrightarrow{\nabla\psi}|^2}{y^2}$ according to

$$P(\psi) = p_0 + \frac{1}{2}\int_0^{\frac{|\overrightarrow{\Delta\psi}|^2}{y^2}} \frac{d\lambda}{\varrho_{\text{sub}}(\lambda)}.$$

Assuming that the quantities $p_{\text{ext}}, (p_{0l}, \varrho_{0l}, \gamma_l)\,|l = 1, 2$ are given, the physical problem consists in finding the shape of the streamlines Γ_{v1}, Γ_{v2} and the mass flow values Q_1, Q_2 so that, for the corresponding flow in the domains Ω_1, Ω_2, the pressure equilibrium conditons $p_1 = p_2$ on Γ_{v1} and $p_2 = p_{\text{ext}}$ on Γ_{v2} should be satisfied up to the trailing edges.

The satisfaction of the Kutta-Joukowski conditions means to impose for each trailing edge the initial tangent of the curve Γ_v.

It is assumed that for the given data there exists a solution flow which is subsonic everywhere.

Due to the variational principles of previous sections we are now in the position to give the following mathematical formulation for the model problem:

Find (Q_1^*, Q_2^*) and an "admissible pair $(\Gamma_{v1}^*, \Gamma_{v2}^*)$" such that, for the corresponding (Ω_1^*, Ω_2^*), the function $\mathscr{F}(\Omega_1, \Omega_2)$ has a stationary value. In the evaluation of $\mathscr{F}(\Omega_1, \Omega_2)$, the functions $\psi(\Omega_l)$, $\varrho(\Omega_l)$ are defined as solutions of

$$\max_{\varrho} \left[\min_{\psi \in \vartheta_{Q_l^*}(\Omega_l)} \mathscr{I}(\Omega, \varrho, \psi)\right], \quad l = 1, 2.$$

The numerical method we have devised is based essentially on a discrete version of this variational formulation by the way of a finite element approximation.

2.2. A moving grid system

We use quadrilateral element and consider that the curvilinear mesh over Ω results from the numerical computation of a transformation $\hat{F}$ which maps the reference unit square K onto Ω in a manner sketched in Fig. 3.

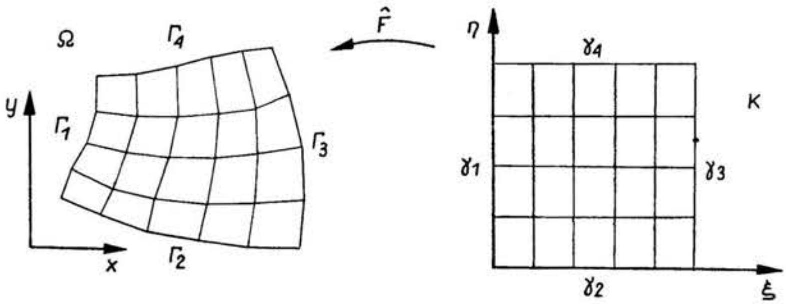


FIG. 3.

Instead of using an explicitly defined transformation, we want to characterize $\hat{F}(\xi, \eta) = \left(x(\xi, \eta), y(\xi, \eta)\right)$ as the pair of the functions x, y, solutions of the two elliptic partial differential equations in the square K:

$$\left\{\frac{\partial}{\partial \xi}\left(\frac{\mu}{\alpha(\xi)}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\alpha(\xi)}{\mu}\frac{\partial}{\partial \eta}\right)\right\}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \text{in} \quad K,$$

with the boundary conditions

$$x\Big|_{\gamma_l} = f_l, \quad \frac{\partial y}{\partial \xi}\Big|_{\gamma_1 \cup \gamma_3} = 0, \quad y\Big|_{\gamma_2} = g_2(\xi), \quad y\Big|_{\gamma_4} = g_4(\xi),$$

where

$$\mu = \int_0^1 (g_4 - g_2)\alpha d\xi \Big/ \int_0^1 (f_3 - f_1) d\eta$$

and

$$\alpha(\xi) > 0, \quad \int_0^1 \alpha(\xi) d\xi = 1.$$

Here we set

$$f_1(\eta) \equiv x_I, \quad f_3(\eta) \equiv x_0, \quad f_2(\xi) = f_4(\xi) = f(\xi),$$
$$f(0) = x_I, \quad f(1) = x_0, \quad \alpha(\xi) = f'(\xi)/(x_0 - x_1).$$

With this choice the shape of Γ_{DL} and Γ_{DU} are given by the parametric representation

$$x = f(\xi), \quad y = g_2(\xi); \quad x = f(\xi), \quad y = g_4(\xi)$$

and a variation of Ω is clearly defined by variations of the functions $g_2(\xi)$ and $g_4(\xi)$ with $f(\xi)$ kept fixed.

The simplest finite element approximation of the preceding equations leads to the linear systems

$$\bar{\bar{A}}\bar{x} = \bar{f}, \quad \bar{\bar{B}}\bar{y} = \bar{g}$$

giving the coordinates of the nodes of the grid.

These systems are solved by the fast Poisson's solver of BUNEMAN [11].

The two grids for Ω_1 and Ω_2 are separately computed by the same technique. The connection between the two systems is made on Γ_{v1} by choosing the same Dirichlet conditions for x and y on Γ_{DU1} and Γ_{DL2}.

We shall denote by u_1 and u_2 the fixed values of the ordinates of the trailing edges and by $\bar{v}_1, \bar{v}_2$ the vectors of the ordinates of the moving nodes of Γ_{v1} and Γ_{v2} as it can be seen in Fig. 4.

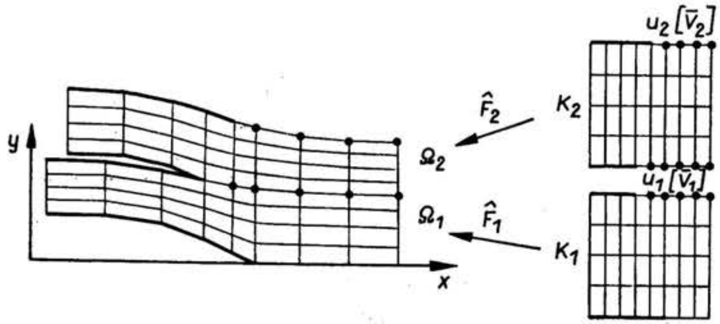


FIG. 4.

This method of automatic mesh generation (2.1) is a simple special case of more general techniques to be published elsewhere.

2.3. Finite element approximation

For approximating ψ over Ω we have taken the simplest isoparametric quadrilateral finite element for ψ_h, that is to say a continuous piecewise bilinear function for $\hat{\psi}_h = \hat{F}_h^{-1}$ o ψ_h on K. The approximate ϱ_h is a piecewise constant function. The approximation of $\mathcal{J}(\Omega, \varrho, \psi)$ was done with the aid of numerical integration formulae. For example, on a quadrilateral Ω_e(corresponding to an elementary rectangle $K_e = [0, H_1] \times [0, H_2]$) we set

$$\left\{ \int\limits_{\Omega_e} \frac{|\overrightarrow{\nabla\psi_h}|^2}{\varrho_h y} dx dy \right\}_h = \frac{1}{\varrho_e [y]_e} [(D\hat{F}_e)|\overrightarrow{\nabla\psi_h}|^2]_e H_1 H_2, \tag{2.1}$$

where $D\hat{F}_e$ is the Jacobian of the mapping $\hat{F}_e$ from K_e onto Ω_e; $[f]_e$ = arithmetic mean of the values of f at the four nodes of Ω_e.

Then, for fixed Ω_h and Q, the approximate ψ_h^*, ϱ_h^* are defined as the solutions of

$$\max_{\varrho_h} [\min_{\psi_h} \mathscr{J}_h(\Omega_h, \varrho_h, \psi_h)], \tag{2.2}$$

where $\psi_h \in V_h(Q, \Omega_h)$ finite dimension subspace of $\vartheta_Q(\Omega_h)$.

If we denote by $\overline{\psi}$; $^-$ the vectors of nodal values of ψ_h and element values of ϱ_h, we can express the Gateaux derivatives of $\mathscr{J}_h$ in vectorial notation:

$$\left(\frac{\partial \mathscr{J}_h}{\partial \psi_h}, \varphi_h\right) = \overline{\varphi}^t \cdot (\bar{\bar{C}}\overline{\psi} - \bar{h}); \quad \bar{\bar{C}}, \bar{h} \text{ depending on } \varrho_h, \Omega_h, Q.$$

$$\left(\frac{\partial \mathscr{J}_h}{\partial \varrho_h}, \tau_h\right) = \overline{\tau}^t \cdot \overline{G}; \quad \overline{G} = \overline{0} \text{ gives a discrete Bernouilli equation.}$$

Now, we call $\mathscr{F}_h(u_1, u_2, \bar{v}_1, \bar{v}_2, Q_1, Q_2)$ the approximation of $\mathscr{F}(\Omega_1, \Omega_2)$ consistent with the definition of $\mathscr{J}_h(\Omega_h, \varrho_h, \psi_h)$. Clearly, the variational principle for free boundaries can be interpreted in a discrete form according to

$$\left(\frac{\partial \mathscr{F}_h}{\partial \bar{v}_l}, \overline{\delta v_l}\right) = 0, \quad \forall \overline{\delta v_l} \in \mathrm{R}^{N_l}; \quad l = 1, 2 \tag{2.3}$$

however, we need two more equations if we want to adjust Q_1 and Q_2. For these two equations playing the role of the Kutta-Joukowski conditions we have chosen

$$\left(\frac{\partial \mathscr{F}_h}{\partial u_l}, \delta u_l\right) = 0, \quad \forall \delta u_l \in \mathrm{R}; \quad l = 1, 2 \tag{2.4}$$

We were led to this formulation by the somewhat heuristic argument of extending the variational equivalent of each pressure equilibrium condition up to the trailing edge node. The main benefit of this choice is the similarity of treatment of these N1+N2+2 equations (2.3) and (2.4). Due to the implicit character of the mesh generation it is necessary to use the optimal control theory (or the Lagrange multipliers technique) for evaluating the partial derivatives of $\mathscr{F}_h$.

Thus the discrete problem is now
Find $Q_1^*, Q_2^*, \bar{v}_1^*, \bar{v}_2^*$ such that

$$\frac{\partial \mathscr{F}_h}{\partial u_l} = 0, \quad \frac{\partial \mathscr{F}_h}{\partial \bar{v}_l} = 0, \quad l = 1, 2, \tag{2.5}$$

where $\psi_h^*(\Omega_l)$, $\varrho_h^*(\Omega_l)$ used for computing these derivatives are the solutions in Ω_l of

$$\max_{\varrho_h} \{\min_{\psi_h} \mathscr{J}_h(\Omega_h, \varrho_h, \psi_h)\}.$$

2.4. The method of solution

Since the discrete problem (2.5) has the structure of a three-level optimization process, the method of solution consists of three nested loops. In fact there are two nested loops in ϱ_h, ψ_h for each of Ω_1 and Ω_2 which are treated independently giving their contribution to the common external loop where $Q_1, Q_2, {}^-{}_1, {}^-{}_2$ are modified.

Internal loop: For fixed Q, Ω_h, ϱ_h,

$$\min_{\psi_h} \mathscr{J}_h(\Omega_h, \varrho_h, \psi_h) \to \psi_h(\varrho_h).$$

The quadratic functional $\mathscr{J}_h$ is minimized by a "conjugate directions" method with the mesh operator $\overline{\overline{B}}$ used as an auxiliary operator.

Middle loop: For fixed Q, Ω_h,

$$\max_{\varrho_h} J_h(\Omega_h, \varrho_h) = \mathscr{J}_h(\Omega_h, \varrho_h, \psi_h(\varrho_h)) \to \varrho_h(\Omega_h) \quad \text{and} \quad \psi_h(\Omega_h).$$

The non-quadratic functional J_h is maximized by a conjugate gradients method with a crude one-dimensional search.

External loop: The $N1+N2+2$ components of the gradients of $\mathscr{F}_h$ are computed providing information for modifying Q_l, $\overline{v}_l$ $(l = 1, 2)$ so that $\delta\mathscr{F}_h = 0$ hold. The $(N1+ +N2+2)$ nonlinear equations (2.3) and (2.4) of stationarity of $\mathscr{F}_h$ are solved by a least square minimization code using only the derivatives of $\mathscr{F}_h$[12].

As a preliminary step, the problem is solved with the assumption of a constant density $\varrho = \varrho_{0l}$ in Ω_h and for some reasonable guess of Γ_{vl}, Q_l.

This initialization phase furnishes good starting data $(\Gamma_{vl}, Q_l, \varrho, \psi)$ for the compressible case.

2.5. Numerical results

We have studied various geometrical configurations for two-stream nozzles without encountering peculiar difficulties attached to any combination of solid walls.

We present here some results concerning two test cases. For each one, there are 320 quadrilateral elements. The number of control variables for Γ_{vl}, Q_l is 25 for case 1 and 18 for case 2 with respective total running times of 17 min and 15 min including the initialization phase taking nearly 2 min. Computation was run on a CII Iris 80 computer.

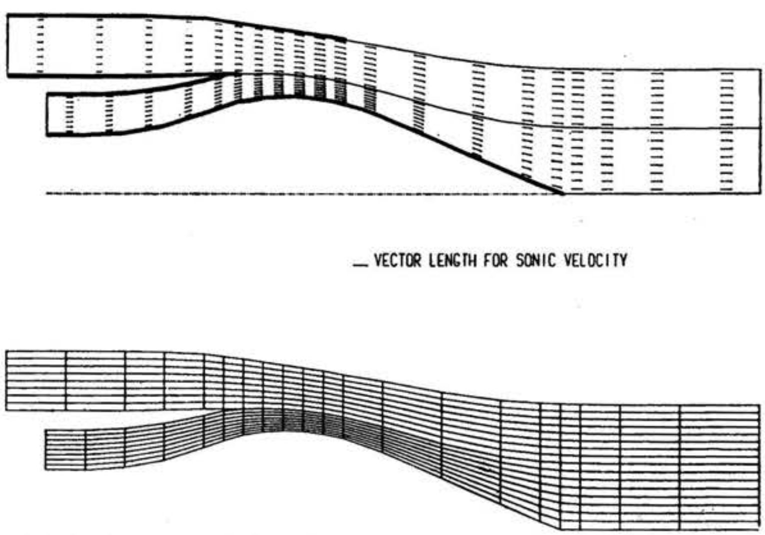
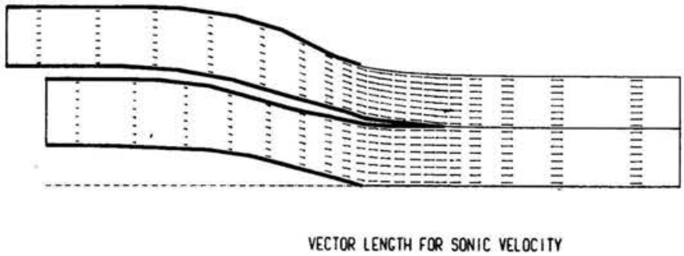


FIG. 5. Test case 1. Finite element grid for solution after 42 iterations.

The extrapolated pressure of free boundaries was found to satisfy the equilibrium conditions with a relative tolerance less than 1.10^{-2}.

Figures 5 and 6 represent for each configuration the final grid system and final values of velocity vectors the length of which is equal to the local Mach number. In Fig. 7 we have plotted for case 1 the boundary extrapolated pressures on fixed or free boundaries.



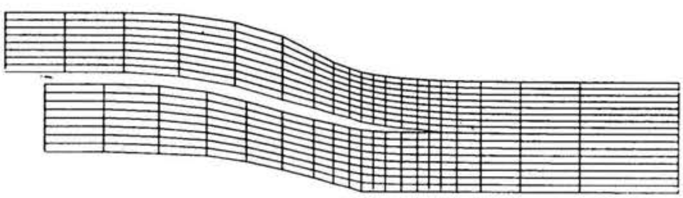


FIG. 6. Test case 2. Finite element grid for solution after 36 iteration.

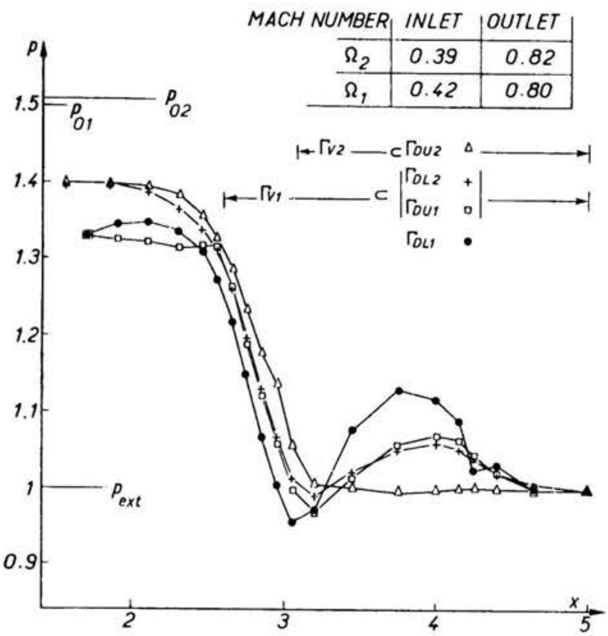


FIG. 7. Pressure distribution. Test case 1.

Conclusion

We have presented a variational principle for a two-dimensional steady compressible flow with free boundaries in the case of a perfect gas at subsonic velocity. This principle concerns rotational flow without being restricted to an insentropic or isoenergetic case. In the course of this derivation, we were led to formulate, for fixed boundaries, a variational principle for the stream function and a variational principle for the density. This last principle, seen as an optimal control problem, provides us with a tool for building a numerical method after a finite element approximation. We have chosen the lowest order for polynomial approximation on quadrilaterals but higher order choices are possible and could deserve attention. As concerns the free boundary problem to be solved, the methodology we have devised can easily be extended to other internal or external flows. Some work remains to be done in a theoretical and numerical way, but the safety and efficiency of the method of solution for the model problem are incitements for further applications.

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