On the stability of capillary jets of elasto-viscous liquids

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THE AUTHOR has discussed the problem of stability of capillary jets with special emphasis paid to the discrepancy between the known experimental data and some theoretical predictions. A formal analysis of hydrodynamic stability of the homogeneous jets in a quasi one-dimensional approximation was preceeded by previous examinations of the influence of the elastic tensile stresses on the stability of jet. The problem of stability at small disturbances was examined for two models of visco-elastic liquids. Assuming the process to be sufficiently "rapid" it was confirmed that the quickly increasing disturbances can not exist so that the visco-elastic properties strengthen the stability of jet. Another destabilizing factor i.e. an interaction between fluid and air circumfluent a jet was also consireded.

Autor przedyskutował zagadnienie stabilności kapilarnej strugi cieczy, zwracając szczególną uwagę na rozbieżności między znanymi wynikami eksperymentów i niektórymi przewidywaniami teoretycznymi. Formalną analizę hydrodynamicznej stateczności strugi jednorodnej w przybliżeniu quasi-jednowymiarowym poprzedzono rozważaniami nad wpływem rozciągających naprężeń sprężystych na stateczność strugi. Zagadnienie stateczności przy małych zaburzeniach zbadano dla dwóch modeli cieczy lepkosprężystych. W założeniu, że proces jest wystarczająco „szybki" stwierdzono, że szybko wzrastające zaburzenia nie mogą istnieć, a więc lepkosprężyste własności sprzyjają stabilizacji strugi. Rozpatrzono też inny czynnik destabilizujący: oddziaływanie między cieczą i powietrzem otaczającym strugę.

Автор обсудил вопрос устойчивости капиллярной струи жидкости, обращая особенное внимание на расходимости между известными экспериментальными результатами и некоторыми теоретическими предвидениями. Формальному анализу гидродинамической устойчивости однородной струи в квазиодномерном приближении предшествует обсуждение влияния растягивающих упругих напряжений на устойчивость струи. Вопрос устойчивости при малых возмущениях исследован для двух моделей вязкоупругих жидкостей. В предположении, что процесс является достаточно „быстрым" обнаружено, что быстро возрастающие возмущения не могут существовать, т. е. вязкоупругие свойства способствуют стабилизации струи. Рассмотрен тоже другой дестабилизирующий фактор: взаимодействие между жидкостью и воздухом окружающим струю.

IT IS SHOWN by numerous experiments that capillary jets of elasto-viscous liquids are much more stable (i.e. have a considerably greater break-up length) than jets of Newtonian viscous liquids of comparable viscosity [1–3].

A theoretical treatment of the jet stability problem within the framework of the theory of small perturbations leads to the opposite conclusion. The growth rate of perturbations is shown to be greater for an elastic liquid than for a Newtonian one of the same zero viscosity [1]. To elucidate the cause of such discrepance between the theory and experiments note that in experiments [1] it was not the decrease in the growth rate of small disturbances on the initial jet that was observed but the striking stability of thin filaments arising after deformation of the initially homogeneous jet (cf. Fig. 9 of [1]).

The overstability of elastic jets manifests itself in the ability of elastic liquids to form long and durable filaments, the so-called "spinnability phenomenon" [4]. It is quite obvious that for a jet to be pulled up from the free surface of a liquid as in [4] there must exist considerable longitudinal tension along the jet.

It is easy to present a simple experiment which not only demonstrates the existence of longitudinal tension along the jet axis but allows for an immediate evaluation of this tension. To this end it is sufficient to shift aside carefully the glass in which the jet of an elastic liquid falls. The jet catches the wall of glass and follows it for some distance. Therefore with some skill one is able to get a stationary curved jet (Figs. 1 and 2).
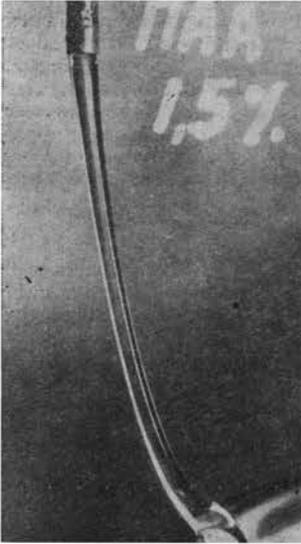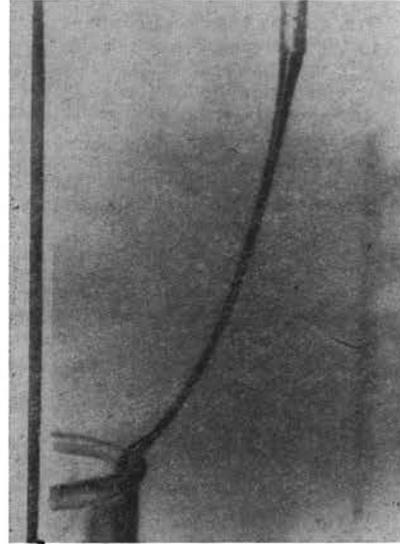

FIG. 1.

FIG. 2.

The projection of momentum balance on the normal to jet axis gives readily

(0.1) $T = \varrho Q^2/f + \varrho g f \cos\varphi/(d\varphi/ds)$

the continuity being taken into account.

Here T denotes the force of axial tension in the jet, f — the area of the jet cross-section, φ — the angle between the horizontal line and the jet axis, q — the flow rate. Equation (0.1) expresses jet tension in terms of flow rate and jet geometry. For the experiment shown in Fig. 2 we have the capillary inner radius $a = 0.1$ cm, $Q = 0.30$ cm^3 sec^{-1} and Eq. (0.1) gives for the axial tension $T = 30$—40 dynes, the liquid being a water solution of 1.5 per cent PAA of zero viscosity 5.4 Po.

One can express the tension T as the sum of contributions of capillary (T_c), elastic (T_e) and viscous (T_v) forces, $T_c = T_e + T_v$. Simple estimates give $T_c \simeq 11$ dynes, $T_v \simeq 1$ dyne for the case shown(*), so we have $T_e = 20 - 30$ dynes and, with the jet cross-section area $f \simeq 7.10^{-3}$ cm^2, elastic stresses are of the order of

$$\sigma_e \sim T_e/f \simeq 3.10^3 \text{ dynes} \cdot \text{cm}^{-2}.$$

(*) Experimental work was carried out by S. Makhkamov and K. Mukuk.

In other words, in the case considered the contribution of elastic stresses is of the order of the capillary force.

Note that the static equilibrium of the sufficiently thin filaments might be well supported by the capillary forces only. So it is natural to suppose that the main role of elastic stresses consists in stabilizing the jet, i.e. in preventing the development of local constrictions on the jet. This supposition will be supported by the detailed analysis at the end of the paper. Now we shall discuss the role of elastic stresses in jet qualitatively.

Note that the local construction on the jet will not develop further if the "static part of tension"

(0.2) $$T_s = T_c + T_e$$

increases as the diameter of the jet decreases.

Indeed, be the contrary true the total tension T being constant along the jet,

(0.3) $$T = T_s + T_v = \text{const}$$

the decrease in T_s causes an increase in T_v and a corresponding increase of the local rate of stretching and rapid local thinning of the jet. This is, in a few words, the mechanism of jet break-up if inertial forces can be neglected.

The static tension T_s is of the order of

(0.4) $$T_s = \pi \alpha r + 2G\pi\lambda^2 r^2,$$

α being the surface tension, G — the shear modulus of the liquid, λ — the "elastic" (reversible) part of the stretch ratio of a liquid element. The jet tension is assumed to be sufficiently great, so that $\lambda \gg 1$. Let $r = r_0$, $\lambda = \lambda_0$ correspond to the unperturbed state of the jet and the disturbance in question is fast (the time of the disturbance development τ is less than the liquid relaxation time θ) so that the relaxation can be neglected. Then $\lambda = \lambda_0 r_0^2/r^2$ and

(0.5) $$T_s = \pi \alpha r + \pi\sigma_0 r_0^4/r^4, \qquad \sigma_0 = 2G\lambda_0^2,$$

T_s has a maximum as the function of r at

(0.6) $$r = r_* = \left(\frac{2\sigma_0 r_0}{\alpha}\right)^{1/3} r_0$$
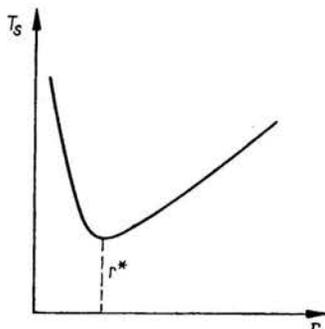

FIG. 3.

(see Fig. 3). Hence the static axial tension in jet decreases as r grows if $2\sigma_0 > \alpha/r_0$, so that the jet is stable. Be the contrary true there would develop a local constriction of the radius r_1 determined by $T(r_1) = T(r_0)$

(0.7) $$r_1 = [\sigma_0 r_0^2 + (\sigma_0^2 r_0^4 + 4\sigma_0 \alpha r_0^3)^{1/2}]/2\alpha.$$

If the jet is initially under zero tension ($\lambda_0 = 1$), Eq. (0.7) changes into

(0.8) $$r_1 = [Gr_0^2 + (G^2 r_0^4 + 2Gr_0^3 \alpha)^{1/2}]/2\alpha.$$

For $G \ll \alpha/r_0$, the inequality being valid for thin filaments and (or) dilute solutions, Eq. (0.8) simplifies to

(0.9) $$r_1 = (Gr_0^3/2\alpha)^{1/2}, \quad r_1/r_0 = (Gr_0/2\alpha)^{1/2} \ll 1.$$

The above argument, elementary as it is, gives some insight into the significance of axial tension in the stabilizing of capillary jets or thin filaments of elastic liquids and corresponds qualitatively to the observations [1].

Below we give a formal stability analysis of the capillary jet of an elastic liquid.

1.

The stability of a cylindrical jet is considered in a long-wave (quasi-unidimensional) approximation.

The balances of mass and momentum are

(1.1) $$\frac{\partial \varrho f}{\partial t} + \frac{\partial \varrho f v}{\partial x} = 0,$$

(1.2) $$\frac{\partial \varrho f v}{\partial t} + \frac{\partial \varrho f v^2}{\partial x} = \frac{\partial \sigma f}{\partial x} + \frac{\partial \Pi \alpha}{\partial x}$$

(ϱ — the density of liquid, $f = \pi a^2$, $\Pi = 2\pi a$, a — the radius of the jet, v — the axial velocity, σ — the axial stress, x — the axis coincides with the jet axis).

Two different types of liquid constitutive equations were used. Assume first the constitutive equation of the form used in [4]

(1.3) $$\theta \Delta s/\Delta t - \varepsilon\theta[se + es - 2/3\delta \cdot \text{trace (se)}] + s = 2\eta e,$$

$$\sigma = -p\delta + s.$$

Here θ — the relaxation time, s — the stress deviator, $\Delta/\Delta t$ — the symbol of Jaumann's derivative, e — the strain rate tensor, σ — the stress tensor, δ — the unit tensor.

Under the assumptions mentioned we have

(1.4) $$\mathbf{e} = \begin{pmatrix} e & & 0 \\ & -1/2e & \\ 0 & & -1/2e \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} s & & 0 \\ & -1/2s & \\ 0 & & -1/2s \end{pmatrix}.$$

As the jet surface is stress-free, we get

(1.5) $$-p - 1/2s = -q_\alpha, \quad p = q_\alpha - 1/2s$$

(1.6)
$$\sigma = \begin{pmatrix} {}^3/_2 s - q_\alpha & & 0 \\ & -q_\alpha & \\ 0 & & -q_\alpha \end{pmatrix},$$

q_α being the capillary pressure.

Therefore Eq. (1.3) reduces to an equation for s of the form

(1.7)
$$\theta\left(\frac{\partial s}{\partial t} + v\,\frac{\partial s}{\partial x}\right) - \varepsilon\theta s\,\frac{\partial v}{\partial x} + s = 2\eta\,\frac{\partial v}{\partial x},$$

(1.8)
$$\sigma = {}^3/_2 s - q_\alpha.$$

Equations (1.1), (1.2), (1.7) and (1.8) describe the jet motion in a long-wave approximation. Consider small perturbations (denoted below by primes) of an initial state which represents the state of a long relaxing cylinder, so that

(1.9)
$$v = v_0 \equiv 0, \quad s = s_0 = S^0 \exp(-t/\theta), \quad f = f_0.$$

The perturbations are described by equations of the form

(1.10)
$$\frac{\partial f'}{\partial t} + f_0\,\frac{\partial v'}{\partial x} = 0,$$

(1.11)
$$\frac{\partial v'}{\partial t} = \frac{1}{\varrho}\,\frac{\partial \sigma'}{\partial x} + \frac{\alpha}{\varrho f_0}\,\frac{\partial \Pi'}{\partial x} + \frac{\sigma_0}{\varrho f_0}\,\frac{\partial f'}{\partial x},$$

(1.12)
$$\sigma' = {}^3/_2 s' - q'_\alpha,$$

(1.13)
$$\theta\,\frac{\partial s'}{\partial t} + s' = 2\eta^*\,\frac{\partial v'}{\partial x}, \quad \eta^* = \eta + {}^1/_2\varepsilon\theta s_0.$$

In what follows we consider "fast" processes characterized by a time constant $\tau \sim \dfrac{1}{\mu} \ll \theta$. So it is possible to neglect the time variation of s and to treat s_0 and η^* as constants. Let

(1.14)
$$f' = f_0 F e^{\mu t}\cos kx, \quad v' = V e^{\mu t}\sin kx, \quad q'_\alpha = Q e^{\mu t}\cos kx,$$
$$\sigma' = \Sigma e^{\mu t}\cos kx, \quad s' = S e^{\mu t}\cos kx, \quad \Pi' = \Pi e^{\mu t}\cos kx.$$

Substitution of Eqs. (1.14) in Eqs. (1.10) — (1.13) results in

$$\mu F = -\frac{\Sigma}{\varrho}\,k - \frac{\alpha\Pi}{\varrho f_0}\,k - \frac{\sigma_0 k}{\varrho}\,F = -\frac{\Sigma k}{\varrho} - \frac{\alpha^*}{\varrho a}\,kF,$$

(1.15)
$$\mu F + kV = 0, \quad \Sigma = {}^3/_2 S - Q,$$

$$(1 + \mu\theta)S = 2\eta^* kV, \quad \alpha^* = \alpha + a\sigma_0 = {}^3/_2 aS_0;$$

(1.16)
$$\Pi = 2\pi a, \quad f = \pi a^2, \quad q'_\alpha = -\alpha\left(\frac{1}{a^2} + \frac{\partial^2}{\partial x^2}\right)a';$$

(1.17)
$$Q = -\frac{1}{2}\,\alpha\left(\frac{1}{a^2} - k^2\right)F.$$

Equations $(1.15)-(1.17)$ have non-trivial solutions if

(1.18)
$$\mu^2 + \frac{3\eta^* k^2 \mu}{\varrho(1+\theta\mu)} = \frac{\alpha^* k^2}{\varrho a} + \frac{\alpha k^2}{2a\varrho}(1-k^2 a^2).$$

Equation (1.18) is valid for fast processes only, so $\theta\mu \gg 1$ and Eq. (1.18) simplifies to

(1.19)
$$\mu^2 = -\frac{3}{2}\frac{(\varepsilon-1)s_0 + 2\eta/\theta}{\varrho} + \frac{\alpha k^2}{2a\varrho}(1-k^2 a^2).$$

The r.h.s. of Eq. (1.19) is negative provided that

(1.20)
$$3[(\varepsilon-1)s_0 + 2\eta/\theta] > \alpha/a.$$

Thus rapidly growing disturbances $\left(\text{with positive Re } \mu \gg \dfrac{1}{\theta}\right)$ are absent provided that the inequality (1.20) holds true. In other words the initial axial tension s_0 stabilizes the jet with respect to axisymmetrical disturbances. Here we assumed $\varepsilon > 1$, which, according to [4], is characteristic of the spinnability of a liquid.

2.

Consider the same problem in the framework of constitutive equations of an elastic liquid proposed by A. I. LEONOV [5]. In the simplest form Leonov's equations are

(2.1)
$$\Delta\mathbf{C}/\Delta t - \mathbf{Ce} - \mathbf{eC} = -2\mathbf{Ce}_p,$$

(2.2)
$$\boldsymbol{\sigma} + p\boldsymbol{\delta} = 2G\mathbf{C},$$

(2.3)
$$\mathbf{e}_p = -\frac{1}{2\theta}\left[\mathbf{C} + \mathbf{C}^{-1} - \frac{1}{3}\boldsymbol{\delta}(I_1 + I_2)\right].$$

Here $\mathbf{C}$ is the elastic strain tensor, I_1 and I_2 — its invariants, G — the shear modules, $\mathbf{e}_p$ — the irreversible part of the strain rate. For the quasi-undimensional motion in question we get

(2.4)
$$\frac{\partial C}{\partial t} + v\frac{\partial C}{\partial x} - 2C\frac{\partial v}{\partial x} = \frac{2}{3\theta}(C + C^{1/2} - C^{-1/2} - C)C, \quad C = C_{11},$$

(2.5)
$$\sigma = -q_\alpha + 2G(C - C^{-1/2}).$$

Linearization of the equations in the neighbourhood of undisturbed state $(C_0, \sigma_0, q_0, v = 0)$ results in

(2.6)
$$\frac{\partial C'}{\partial t} + \frac{2}{3\theta}\left(2C_0 + \frac{3}{2}C_0^{1/2} - \frac{1}{2}C_0^{-1/2}\right)C' = 2C_0\frac{\partial v'}{\partial x},$$

(2.7)
$$\sigma' = -q'_\alpha + 2GC'(1 + {}^1/_2 C_0^{-3/2}),$$

assuming

(2.8)
$$C' = C^+ e^{\mu t}\cos kx, \quad V' = Ve^{\mu t}\sin kx,$$

we get through Eqs. (2.4) and (2.7)

(2.9) $\qquad \mu C^+ + C^+/\theta^* = 2C_0 kV, \qquad \theta^* = 3\theta[4C_0 + 3C_0^{1/2} - C_0^{-1/2}]^{-1},$

(2.10) $\qquad \Sigma = -Q + 2GC^+(1 + {}^1/_2 C^{-3/2}).$

Equations (2.9) and (2.10) become identical with the last two in the set of equations (1.15) if we put θ^* for θ and $4/3\, C_0\theta^* G(1 + {}^1/_2 C_0^{-3/2})$ for η^*. So the characteristic equation for "fast" perturbations takes the following form:

$$\mu^2 = (\alpha^* - 3a\eta^*/\theta^*)k^2/\varrho + \frac{1}{2}\,\alpha k^2(1 - k^2 a^2)/(a\varrho).$$

Accounting for Eq. (2.5) and the formula for η^* and θ^*, we get

(2.11) $\qquad \mu^2 = -\dfrac{2GC_0(1 - 2C_0^{-3/2})ak^2}{\varrho} + \dfrac{\alpha k^2(1 - k^2 a^2)}{2a\varrho}.$

Thus for sufficiently great C_0,

(2.12) $\qquad 4C_0 G(1 - 2C_0^{-3/2})a > \alpha$

the r.h.s. of Eq. (2.11) is negative for all wave-numbers k so that the growth of disturbances is impossible.

3.

Up to now we have been considering the surface tension as the single destabilizing factor. For capillary jets moving through air with sufficiently large velocities, the dynamical action of air may be of importance.

Following the work of WEBER [6] it is easy to get a corresponding characteristic equation in long-wave approximation, the flow of air being considered as a potential one. The r.h.s. of the equation [see Eqs. (1.18) or (2.11)] has an additional term

(3.1) $\qquad \dfrac{\varrho_1}{2\varrho}\, ak^3 f_0(ka)U^2.$

Here ϱ_1 is the air density, U — the air velocity relative to the jet,

(3.2) $\qquad f_0 = -K_0(\xi)/K_0'(\xi),$

K_0 — the modified Bessel function of the second kind. In the long wave range, $ak < 1$, $f_0 < 1$. So the sufficient condition of jet "stability" takes the form

(3.3) $\qquad 3(\varepsilon - 1)S_0 + 6\eta/\theta > \varrho_1 U^2,$

or

(3.4) $\qquad 4C_0 G(1 - 2C_0^{-3/2})a > \varrho_1 U^2,$

for models considered in Parts 1 and 2, correspondingly, capillary forces being neglected. In other words, the initial elastic tension stabilizes the capillary jet in air if "elastic" stresses are of the order of the dynamical pressure of air.

The last argument must be considered in qualitative terms as the assumption of potential air flow results in a somewhat overestimated dynamical pressure of air (see [7] for details). It may be hoped nonetheless that the order of necessary stabilizing stresses is correct.

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