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Some further investigations on non-unique solutions of the Navier-Stokes equations for the Kármán swirling flow

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THE ROTATING disc problem is considered in terms of von Kármán similarity variables. Recently the authors have solved the problem occuring near a ratio between the angular velocity of the fluid at infinity and the angular velocity of the disc of -0.16 when other numerical methods seemed to fail. We have found, by applying special numerical solution techniques that the solution branches at the critical value $s = -0.160538613$, at that a second branch occurs which ranges backwards to positive values of s. The structure of the solutions obtained is discussed.

Problem wirującej tarczy rozważono w zmiennych podobieństwa Kármána. Ostatnio autorzy rozwiązali problem dla przypadku stosunku prędkości kątowej cieczy w nieskończoności do prędkości kątowej tarczy bliskiego 0.16, tj. dla przypadku, gdy różne metody numeryczne zawodzą. Stosując specjalną technikę numeryczną wykazano, że rozwiązanie rozgałęzia się przy wartości krytycznej $s = -0.160538613$, przy której pojawia się druga gałąź skierowana do tyłu w stronę dodatnich wartości s. Przedyskutowano strukturę otrzymanych rozwiązań.

Задача вращающегося диска рассмотрена в переменных подобия Кармана. В последнее время авторы решили задачу имеющую место для отношения угловой скорости жидкости в бесконечности к угловой скорости диска близкого 0,16, т. е. для случая, когда разные численные методы непригодны. Применяя специальную численную технику, показано, что решение расщепляется при критическом значении $s = -0{,}160538613$, при котором появляется вторая ветвь, направленная назад в сторону положительных значений s. Обсуждена структура полученных решений.

1. Introduction

THE PROBLEM which we want to consider is that of the flow of a rotating fluid above an infinite disk which is itself rotating. An important quantity in the analysis is the ratio s of the angular velocity of the fluid and of the disk.

Solutions to this problem can be found by solving a set of two ordinary nonlinear differential equations to which the Navier-Stokes equations can be reduced in this case together with appropriate boundary conditions.

There is, however, a range of s values for which no solutions can be obtained: $-0.16054 > s > -1.4355$. For $s = -1.4355$ it has been proved by BODONYI [1] that the solution of the equations becomes singular. Quite recently the present authors have clarified the situation for $s = -0.16054$ [2]. By means of carefully designed computational methods, we have shown that at that point branching of the solution occurs. In this way it turns out that a second solution branch can be constructed which ranges back to positive values of s. In the meantime we have found that at $s = 0.074526$ another branching point occurs and that it is possible to construct a third branch, which again passes $s = 0$.

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We will start by giving a brief outline of these results and the way in which they were obtained. It will become clear then that it is becoming more and more difficult to obtain further computational results, due to the rather singular behaviour of the solutions. So as to gain further insight into the nature of the solutions we used analytical methods. It turned out that most of our needs were already covered in literature, especially in the papers by KUIKEN [3] and OCKENDON [4]. Taking this into consideration we were able to derive some interesting results for our present case.

We will not give full details here, but rather give an outline of some of the essential steps taken.

2. The construction of the solution branches

In a cylindrical coordinate system (r, ϕ, z) the disk is the plane $z = 0$ and the corresponding velocities are (see Fig. 1)

$$u = r\Omega f'(x), \quad v = r\Omega g(x), \quad w = -2(\nu\Omega)^{1/2} f(x). \tag{2.1}$$

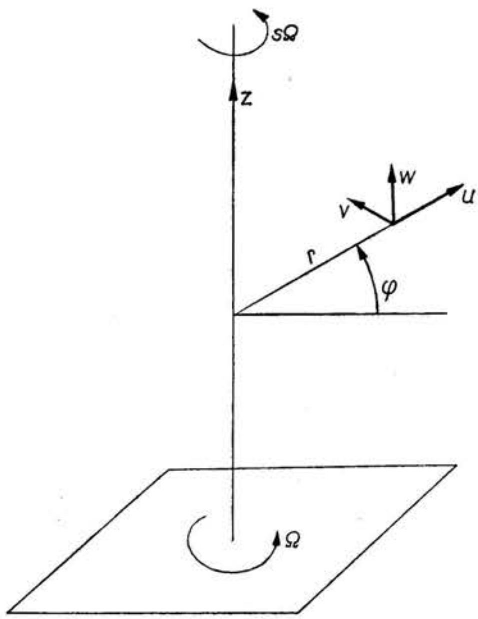

FIG. 1.

The angular velocity of the disk is Ω and $x = z(\Omega|\nu)^{\frac{1}{2}}$. The Navier-Stokes equations in this case reduce to

$$f''' + 2ff'' = f'^2 + s^2 - g^2, \tag{2.2}$$

$$g'' + 2fg' = 2f'g. \tag{2.3}$$

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The boundary conditions are

$$(2.4)\qquad \begin{aligned} x = 0:&\quad f = 0,\quad f' = 0,\quad g = 1,\\ x = \infty:&\quad f' = 0,\quad g = s. \end{aligned}$$

As it has been proved by ROGERS and LANCE [5] and MC LEOD [6], the asymptotic solution for $x \to \infty$ is given by

$$(2.5)\qquad \begin{aligned} f &\sim a+e^{px}\left\{\frac{bp+cq}{p^2+q^2}\sin qx+\frac{cp-bq}{p^2+q^2}\cos qx\right\},\\ f' &\sim e^{px}\{b\sin qx+c\cos qx\},\\ g &\sim s+e^{px}\{c\sin qx-b\cos qx\}. \end{aligned}$$

The relations between a, p, q and s are given by $p^2-q^2 = -2\,ap$, $pq = -aq+s$.

This means that the asymptotic character is determined by the three quantities a, b and c.

For the construction of the solution various methods can be employed. Originally Rogers and Lance used a shooting method from $x = 0$ onwards. It turned out this is a very troublesome method, because of the instabilities occurring in the method.

A second method consists in using a finite difference technique on a finite interval by applying the boundary conditions at $x = \infty$ at a suitable chosen value $x = x_0$.

A third method makes use of a shooting technique from essentially $x = \infty$ towards $x = 0$. This can be done by using the asymptotic formulae (2.5), asuming values for the quantities a, b and c and trying to fulfill the boundary values at $x = 0$.

A combination of the second and the third method proved to be very successful. The finite difference technique was used as a kind of scout, whereas very accurate calculations could then be made with the last method, which made use of a stabilized highly accurate solution method.

Using these methods we discovered that a branching of the solutions occurs for $s = -0.1605387613$.

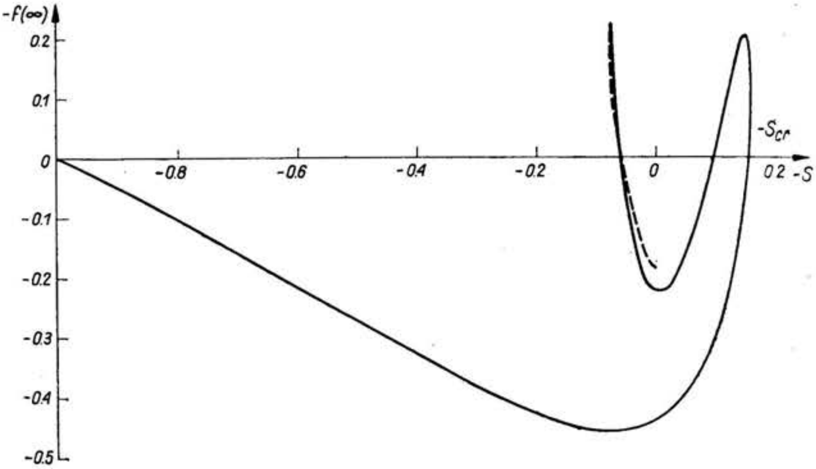

FIG. 2.

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This is illustrated in Fig. 2 where we give the value of $-f(\infty)$ as a function of the quantity s.

For precise details as to how the solution was obtained in the vicinity of the critical value of s we should like to refer to our paper [2].

Quite naturally the question arises how this graph would proceed, giving thereby insight in further possible solutions of the Navier-Stokes equations. Now, in the meantime, we have produced further solutions for the third branch, which are also given in Fig. 2. It is perhaps interesting also to give the quantities $f''(0)$ and $g'(0)$ as a function of s for the various branches (Fig. 3). An important remark to be made here is that these quantities have become practically constant for the third solution branch. To gain an insight in the

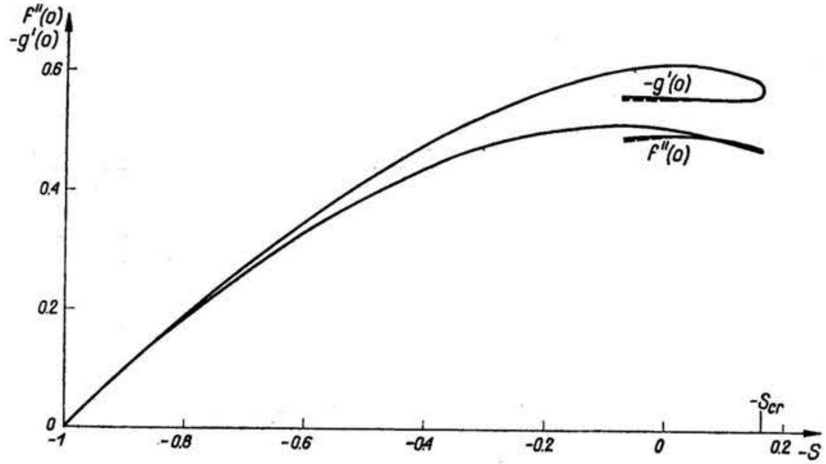

FIG. 3.

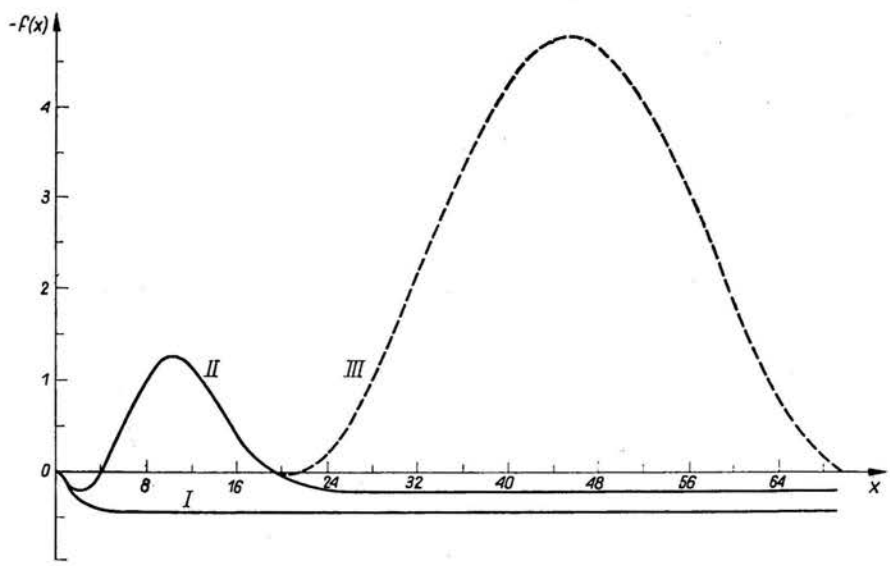

FIG. 4.

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difference between the various solutions we will consider the graph for f as a function of x for the cases which have been calculated for $s = 0$.

As it follows from Fig. 4 the second solution differs from the first by the addition of a large hill with strong upward velocities, whereas the third solution can be said to add a second much larger hill to the already existing first one.

Now it becomes clear at once why it is practically impossible to proceed with the solutions. We must expect that each branch will add every time a still larger hill to the ones already existing. In this way the ranges over which the calculations have to be made increase dramatically.

In order to obtain a deeper knowledge of this rather singular behaviour, it seems that we ought to study at least three different phenomena analytically:

the behaviour of a large hill,

the behaviour of the region between two hills,

the connection between two hills.

Of course this will not solve the whole problem quantitatively, but it may be expected that qualitatively, at least, a fair picture of the situation can be obtained.

In the rest of this paper we will therefore study these three problems and conclude with some final remarks. We will only give a rough outline of the results.

3. Large hills

In fact already Kuiken [3] studied the case of large hills, but we will use here a somewhat different approach to introduce the subject. If we use the system (2.2), (2.3) assuming that $f'(0) = 0$ and introducing

$$\int_0^\lambda f d\xi = F(\lambda), \tag{3.1}$$

we can derive the following formulae:

$$f(x) = e^{-F(x)} \int_0^x e^{F(\lambda)} \Big[\int_0^\lambda (\lambda-\mu)(3f'^2+s^2-g^2)d\mu + f''(0)\lambda + f(0) \Big] d\lambda + f(0)e^{-F(x)}, \tag{3.2}$$

$$g(x) = e^{-2F(x)} \int_0^x e^{2F(\lambda)} \Big[\int_0^\lambda 4f'g d\mu + g'(0) + 2f(0)g(0) \Big] d\lambda + g(0)e^{-2F(x)}. \tag{3.3}$$

Using these formulae we now consider the following situation for $s = 0$:

$$f(0) = A, \quad f'(0) = 0, \quad f''(0) = C, \quad g(0) = 1, \quad g'(0) = \varepsilon,$$

where A is large and $g(0)$ has been set equal to unity without loss of generality. The origin $x = 0$ has been taken at the top of a hill so that $f'(0) = 0$.

From the conditions above it follows that

$$f \sim A + \frac{1}{2} Cx^2 - \frac{1+2AC}{6} x^3, \quad g \sim 1 + \varepsilon x - A\varepsilon x^2. \tag{3.4}$$

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We now substitute these expressions into the integral expressions (3.2) and (3.3). Thereby we obtain better aproximations for the functions f and g. However, it turns out that this expression contains exponential increasing functions. So f contains a term

$$f(x) = \left\{2\frac{C}{A^2}+\frac{1}{A^3}+\dots\right\}e^{-F(x)}\sim e^{-Ax}\left\{2\frac{C}{A^2}+\frac{1}{A^3}+\dots\right\}.$$

Now we have assumed that A is large and hence such a term should not occur. Therefore the coefficient has to be zero which means that there should hold

$$C \approx -\frac{1}{2A}. \tag{3.5}$$

A similar analysis applied to the function g reveals that

$$\varepsilon \approx \frac{1}{4A^3}. \tag{3.6}$$

It will be clear that by inserting more terms better asymptotic formulae can be derived for C and ε and, in addition, also a better representation for f and g.

It can be shown, see also KUIKEN [3], that

$$f \sim A\cos^2\frac{x}{2A}, \quad g \sim \cos^2\frac{x}{2A}. \tag{3.7}$$

It is evident from the relations (3.5) and (3.4) that the third derivative of f is zero and since this term represents the viscous term, we can therefore speak of large inviscid hills which are apparently due in these solutions. Another remark to be made is that once given A (with $g(0) = 1$) the solution is completely determined locally. For instance, the last hill of the third solution branch is already given accurately (4 digits) by this analysis.

For reasons of later use we also give the formulae, when $g(0)$ is left free, say equal to B. Then

$$f \sim A\cos^2\frac{Bx}{2A} \quad \text{and} \quad g \sim B\cos^2\frac{Bx}{2A}. \tag{3.8}$$

Now in order to see how two hills can be connected we first want to analyze the region between two hills.

For obvious reasons it is to be expected that this is a viscous region.

4. The viscous interlayer

The viscous interlayer is a region where f is small, while we can assume the origin in $f' = 0$ and again may set $g = 1$. It then follows from the equations that f'' should be large. We therefore set

$$f(0) = \mu, \quad f'(0) = 0, \quad f''(0) = L, \quad g(0) = 1 \quad \text{and} \quad g'(0) = \delta.$$

From these conditions it follows that

$$f = \mu+\frac{L}{2}x^2-\frac{1+2\mu L}{6}x^3, \quad g = 1+\delta x-\delta\mu x^2,$$

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while in this case the quantity F is equal to

$$F = \mu x + \frac{L}{6} x^3.$$

It is obvious that the leading term is $\frac{L}{6} x^3$.

Therefore f will contain an asymptotically increasing term of the form

$$f = e^{-F(x)} \int_0^\infty e^{\frac{L}{6}x^3} \{\text{Polynomial in} x\} dx.$$

By using the fact that $\int_0^\infty e^{\frac{L}{6}x^3} x^n dx = \frac{1}{3}\left(-\frac{6}{L}\right)^{\frac{n+1}{3}} \Gamma\left(\frac{n+1}{3}\right)$ the coefficient of the function $e^{-F(x)}$ can be analyzed.

It turns out that this coefficient contains terms of the form either $1/L$ or μ which means that L is of the order $1/\mu$ since, naturally, this coefficient should vanish.

A similar calculation for g reveals that there the coefficient is either constant or $\frac{\delta}{L^{1/3}}$ which shows that δ is of the order $L^{1/3}$.

Now in this case a complication arises because it turns out that in order to find the correct coefficients all terms in g induced by the term $\frac{L}{2} x^2$ in f are needed.

This can be done following the approach given here, but the procedure becomes laborious. It is preferable to consider the differential equation for g, as has already been done by Ockendon. Following OCKENDON [4], calculation reveals that in this case there holds

$$L = -\frac{1.03038}{\mu}, \quad \delta = -1.06291 \ (-L)^{1/3},$$

where μ is a small positive constant. Of course it is possible to furnish further terms in these expressions.

In order to find the connection between two hills it is necessary to know the asymptotic behaviour of f and g as the solution leaves the viscous interlayer and merges with the large hill solution.

To leading order the function f is continuous and equal to $Lx^2/2$. Hence to leading order

$$f \sim \frac{1}{2} Lx^2 \quad \text{as} \quad |x| \to \infty.$$

As for g the results are

$$\begin{aligned} &g \sim \alpha x^2, && x \to \infty, \\ &g \sim -2\alpha x^2, && x \to \infty, \end{aligned} \tag{4.1}$$

where α is a constant.

By now we have sufficient material to consider the problem of the connection between two hills in the first order.

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5. The connection between two hills

In order to connect two hills together we make use of the formulae (3.8) and the remarks made about the asymptotic behaviour of the viscous interlayer solution.

If we call the relevant quantities of the first hill A_1 and B_1 and these of the second A_2 and B_2 we can then remark that we have to analyze f and g when $\frac{Bx}{2A} \sim \frac{\pi}{2}$. In that case we can put

$$f \sim A\cos^2 \frac{Bx}{2A} \sim A\cos^2\left(\frac{\pi}{2} - \frac{Bx'}{2A}\right) \sim A\sin^2 \frac{Bx'}{2A} \sim \frac{B^2}{4A} x'^2. \tag{5.1}$$

In the first order x' will be precisely the coordinate of the viscous interlayer. Hence the following will hold:

$$\frac{B_1^2}{4A_1} = \frac{L}{2} x'^2. \tag{5.2}$$

The same argument applied to the other side of the interlayer yields

$$\frac{B_2^2}{4A_1} x'^2 = \frac{L}{2} x'^2. \tag{5.3}$$

Hence one condition valid for the unknown A and B is

$$\frac{B_1^2}{A_1} = \frac{B_2^2}{A_2}. \tag{5.4}$$

Using Eq. (4.1) we obtain with a similar process for g

$$\frac{B_1^3}{A_1^2} = -2\frac{B_2^3}{A_2^2}. \tag{5.5}$$

From these two last equations the remarkable result follows:

$$\begin{aligned} B_2 &= -2B_1, \\ A_2 &= 4A_1. \end{aligned} \tag{5.6}$$

Hence, to leading order each large hill has an amplitude which is 4 times as large as the preceding one while the range over which the hill extends is two times as large as in the preceding hill. Already for the third hill (third branch) the ratio of A_2 and A_1 is 3.83 which means that already these first order approximations are fairly good even on the branches which we are considering now. It may be added that in a forthcoming publication a much more detailed analysis will be given, with second-order corrections included.

6. Final remarks

So far we have answered the three questions which we stated in this work. But evidently this does not put an end to our considerations. The next question which first comes to mind is: what occurs after the last hill on a certain branch, or putting it otherwise: how does the solution proceed towards infinity. But this is not a big problem since the question was

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already answered by KUIKEN [3] who matched the solution at infinity with a large hill. From his analysis it follows that the value of $f(\infty)$ expressed in the quantities A and B of the last hill should be to the first order

$$f(\infty) = 1.2096 \left|\frac{B^2}{16A}\right|^{1/3}. \tag{6.1}$$

It is interesting to compare the numerical value of $f(\infty)$ as calculated for the third branch ($s = 0$) with the value calculated from the relation (6.1) with the quantities A and B taken from the last hill in the solution. It is found that

$$f(\infty) \text{ as calculated} = 0.1937,$$
$$f(\infty) \text{ from Eq. (6.1)} = 0.2022.$$

We can therefore conclude that already in this case the first-order terms give a rather accurate picture of the phenomena. Summarizing, we can now state the following description of what occurs at $s = 0$.

The graph of $f(x)$ as given in Fig. 2 will be continued indefinitely, that is to say, there will be infinitely many solution branches, each branch adding a large hill to the solution, whereas these branches will tend to converge to a certain value of $f(\infty)$. As can be seen from Eqs. (5.4) and (6.1) this value will then be about $f(\infty) = 0.2$. Hence we may be in a position to state that by this analysis we have gained much more insight into the nature of possible solutions of the Navier-Stokes equations.

But of course also a number of interesting further questions arises. One of these questions is: what is the essential difference between a solution with n hills and one with $(n+1)$ hills, bearing in mind that there is no difference for the conditions at $x = 0$ or, putting it otherwise: is a large hill a kind of eigenfunction of the Navier-Stokes equations?

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