Archives of Mechanics • Archiwum Mechaniki Stosowanej • 30, 6, pp. 853–857, Warszawa 1978

Acceleration waves in hyperelastic Cosserat bodies

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THE PAPER is concerned with the general theory of acceleration waves in nonlinear micropolar elasticity. The condition is established which suffices to ensure that the speed of propagation and the amplitude vector of the wave are real. The solution of the growth equation is given for plane waves.

1. Basic equations and formulae

THROUGHOUT this paper a rectangular coordinate system is employed. The motion of a Cosserat medium is described by the functions

$$y_j = y_j(x_k, t), \quad R_{kj} = R_{kj}(x_p, t), \tag{1.1}$$

where R_{kj} are the components of a proper orthogonal tensor which determine the position of the directors at time t.

We consider as strain tensors [1]

$$c_{kj} = y_{p,k} R_{pj}, \quad \gamma_{kj} = \frac{1}{2} \varepsilon_{jmn} R_{pn} R_{pm,k}, \tag{1.2}$$

and the angular velocity vector

$$\nu_k = \frac{1}{2} \varepsilon_{kpq} R_{nq} \dot{R}_{np}. \tag{1.3}$$

The deformation tensor γ_{ij} and the angular velocity vector ν_i may be represented in the form [2]

$$\gamma_{kj} = H'_{jp} \varphi_{p,k}, \quad \nu_k = H_{kp} \dot{\varphi}_p, \tag{1.4}$$

where

$$H_{jp} = \frac{2\delta_{jp} + \varepsilon_{jpn} \varphi_n}{2\left(1 + \frac{1}{4} \varphi_k \varphi_k\right)},$$

and φ_k are the components of the micro-rotation vector.

The equations of motion are [1]

$$\begin{gathered} (R_{pj} t_{kj})_{,k} + \varrho F_p = \varrho \ddot{u}_p, \\ (R_{pj} m_{kj})_{,k} + \varepsilon_{pjk} y_{j,n} R_{kq} t_{nq} + \varrho G_p = \varrho \dot{\sigma}_p, \end{gathered} \tag{1.5}$$

where

$$t_{kj} = \frac{\partial}{\partial c_{kj}} W(c_{nj}, \gamma_{nj}), \quad m_{kj} = \frac{\partial}{\partial \gamma_{kj}} W(c_{pq}, \gamma_{pj}), \tag{1.6}$$
$$\sigma_j = R_{jk} J_{kp} \nu_p,$$

J_{kp} are the components of the material tensor of inertia density.

A propagating singular surface $\Sigma(t)$ is said to be an acceleration wave if the functions y_j, φ_j and their first partial derivatives are continuous everywhere, while all second derivatives of y_j, φ_j suffer jump discontinuities across $\Sigma(t)$, but are continuous everywhere else.

The geometrical and kinematical conditions of compatibility for acceleration waves are ([3], p. 496, 505)

$$[z_{p,kj}] = \frac{s_p}{V^2} N_k N_j, \quad [\dot{z}_{p,j}] = -\frac{s_p}{V} N_j,$$

$$[\dddot{z}_p] = b_p V^2 + 2\frac{\delta s_p}{\delta t} - \frac{s_p}{V}\frac{\delta V}{\delta t}, \tag{1.7}$$

$$[\dot{z}_{k,pj}] = b_k N_p N_j - 2N_{(p} x_{j);}{}^{\alpha}\left(\frac{s_k}{V}\right)_{;\alpha} + \frac{s_k}{V} x_{(p;}{}^{\alpha} x_{j);}{}^{\beta} b_{\alpha\beta},$$

where

$$s_p = [\ddot{z}_p], \quad b_p = [z_{p,kj}] N_k N_j,$$

$b_{\alpha\beta}$ are the coefficients of the second fundamental form of the surface $\Sigma(t)$, $\delta/\delta t$ is called the displacement derivative with respect to the surface $\Sigma(t)$, and V is the speed of propagation of the wave.

From Eqs. (1.7) we obtain

$$[c_{pj,k}] = \frac{1}{V^2} a_j N_p N_k, \quad [\gamma_{pj,k}] = \frac{1}{V^2} \tilde{a}_j N_p N_k, \quad [\dot{\nu}_p] = -\tilde{a}_p, \tag{1.8}$$

where

$$a_j = R_{kj}[\ddot{y}_k], \quad \tilde{a}_j = H_{jp}[\ddot{\varphi}]. \tag{1.9}$$

The vector $A = a_k, \tilde{a}_k$ is called the amplitude of the wave.

2. Wave propagation

The differential equations (1.5) must hold on each side of the surface. Substracting the limit values of these two equations as the surface is approached from one side or the other, and assuming that the external body force, body couple and the second-order derivatives of the function W are continuous across the wave, we obtain from Eqs. (1.8) the relations of the form

$$\frac{\partial^2 W}{\partial c_{pq}\partial c_{kj}} N_p N_k a_q + \frac{\partial^2 W}{\partial \gamma_{pq}\partial c_{kj}} N_p N_k \tilde{a}_q = \varrho V^2 a_j, \tag{2.1}$$
$$\frac{\partial^2 W}{\partial c_{pq}\partial \gamma_{kj}} N_p N_k a_q + \frac{\partial^2 W}{\partial \gamma_{pq}\partial \gamma_{kj}} N_p N_k \tilde{a}_q = \varrho V^2 J_{jn} \tilde{a}_n.$$

We shall refer to Eqs. (2.1) as the propagation condition. Let us introduce the matrix $Q = ||Q_{jk}||_{6\times 6}$, $B = ||B_{kj}||_{6\times 6}$, where

$$Q_{jk} = \frac{\partial^2 W}{\partial c_{qk}\partial c_{pj}} N_p N_q, \qquad Q_{j;k+3} = \frac{\partial^2 W}{\partial c_{qj}\partial \gamma_{pk}} N_q N_p,$$

$$Q_{j+3;k} = \frac{\partial^2 W}{\partial \gamma_{qj}\partial c_{pk}} N_p N_q, \quad Q_{j+3;k+3} = \frac{\partial^2 W}{\partial \gamma_{qj}\partial \gamma_{pk}} N_p N_q,$$

$$B_{kj} = \delta_{kj} \quad B_{k,j+3} = 0, \quad B_{k+3;j} = 0, \quad B_{k+3;p+3} = J^{-1}_{kp},$$

J^{-1}_{kp} is the inverse to J_{kp} which is symmetric and positive definite.

The system (2.1) can be written as

$$(\tilde{Q}_{kj} - \varrho V^2 \delta_{kj}) A_j = 0, \tag{2.2}$$

the matrix $\tilde{Q}$ is given by the relation

$$\tilde{Q} = BQ. \tag{2.3}$$

From Eq. (2.2) we obtain the following:

THEOREM 2.1. *The amplitude vector A must be a proper vector of $\tilde{Q}$; the speed V of propagation of the wave must be such that ϱV^2 is the corresponding proper number.*

REMARK 2.1. If the second-order derivatives of the function W are continuous, then the matrix Q is symmetric.

Taking into account Remark 2.1 and the relation (2.3), we have thus proved ([3], Appendix, Sect. 37).

LEMMA 2.1. The proper numbers of $\tilde{Q}$ are real.

The Theorem 2.1 asserts that any real right proper vector A of $\tilde{Q}$ is a possible amplitude vector provided that the corresponding proper value is real and positive. However, if it is known that A is a real proper vector of $\tilde{Q}$, the speed of propagation of a wave having the amplitude A is obtained more easily from the formula

$$\varrho V^2 (a_j a_j + J_{kp}\tilde{a}_k \tilde{a}_p) = A \cdot QA, \tag{2.4}$$

immediately from Eqs. (2.1). The quantity ϱV^2 is uniquely determined but need not be positive, thus V is not always real. If Q satisfies the condition

$$\lambda \cdot Q\lambda = Q_{ik}\lambda_i \lambda_k > 0, \tag{2.5}$$

for arbitrary non-vanishing vectors λ, then it is obvious that ϱV^2 is positive.

From Eq. (2.5) we have

THEOREM 2.2. *In a given Cosserat body whose internal energy satisfies the condition (2.5), for a wave of given amplitude and direction, there is at least one possible squared speed of propagation determined by the existing deformation from the reference state.*

3. Variation of the amplitude of plane waves

We assume that since the time $t = 0$, the plane wave has been propagating into a region which is at rest in a fixed homogeneous configuration. Let the constant tensors $y^0_{j,k}$, R^0_{jk} characterize the homogeneous configuration. Since the first partial derivatives of R_{jk} are continuous across an acceleration wave, from the relations (1.4), (1.6), (1.7) and (1.9) we obtain

$$
(3.1)\quad
\begin{aligned}
&[\dot{c}_{kj}] = -\frac{1}{V}a_j N_k, \quad [c_{kj,p}] = \frac{1}{V^2}a_j N_k N_p,\\
&[\dot{\gamma}_{kj}] = -\frac{1}{V}\tilde{a}_j N_k, \quad [\gamma_{kj,p}] = \frac{1}{V^2}\tilde{a}_j N_k N_p,\\
&[\dot{R}_{kq,p}] = -\frac{1}{V}R^0_{kn}\varepsilon_{qnj}\tilde{a}_j N_p,\\
&[\dot{c}_{kj,p}] = c_j N_k N_p - \frac{2}{V}a_{j,\alpha}N_{(k}x_{p),\alpha} - \frac{1}{V}\varepsilon_{jnq}c^0_{kn}\tilde{a}_q N_p,\\
&[\dot{\gamma}_{kj,p}] = \tilde{c}_j N_k N_p - \frac{2}{V}N_{(k}x_{p),\alpha}\tilde{a}_{j,\alpha},\\
&[\ddot{\sigma}_k] = R^0_{kq}J_{qp}\tilde{c}_p V^2 + 2R^0_{kq}J_{qp}\frac{\delta\tilde{a}_p}{\delta t},
\end{aligned}
$$

where

$$
(3.2)\qquad c_j = R_{kj}N_p N_q[y_{k,pq}], \quad \tilde{c}_j = H_{jk}N_p N_q[\varphi_{k,pq}].
$$

If the rank $(\tilde{Q}-\varrho V^2 I) = 5$, then the solution of the system (2.1) is of the form

$$
(3.3)\qquad a_j = ad_j, \quad \tilde{a}_j = a\tilde{d}_j,
$$

where a is an arbitrary function.

We assume that the discontinuity across the wave is uniform, so that the amplitude depends on t alone.

Following a procedure given in [4] we obtain the transport equation

$$
(3.4)\qquad p\frac{da}{dt} + qa^2 = 0,
$$

where p, q are constant quantities and $p > 0$.

The solution of Eq. (3.4) is readily found to be

$$
(3.5)\qquad a(t) = \frac{pa_0}{p+qa_0 t},
$$

where $a_0 = a(0)$.

Consequently we have

THEOREM 3.1.

1. *The vector amplitude A of the wave will decay to zero monotonically in infinite time if $qa_0 > 0$.*

2. *The vector amplitude* **A** *of the wave will become infinite monotonically within a finite time t given by*

$$t = -\frac{p}{qa_0}$$

if

$$qa_0 < 0.$$

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Received October 24, 1977.