Archives of Mechanics • Archiwum Mechaniki Stosowanej • 30, 6, pp. 801-826, Warszawa 1978

Prandtl-Reuss plastic material with scalar and tensor internal variables

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PRANDTL-REUSS plastic material with general work-hardening is investigated theoretically. A scalar internal variable and a symmetric tensor internal variable are introduced. Von Mises's plastic potential is assumed to be a function of the translated stress and the scalar internal variable. Thus the general constitutive equations of the rate type plastic material are derived. Four constitutive assumptions, i.e. isotropy, pressure insensitivity, no generalized Bauschinger effect and grade two, are supposed. Then, a generalized Prandtl-Reuss plastic material is defined. From its stress rate constitutive equation, the yield condition and the flow rule are derived. They correspond, respectively, to a generalization of the Huber-Von Mises yield condition and a generalization of the Lévy-St. Venant flow rule. A fracture condition is also defined. It is assumed that the fracture occurs when a scalar function, called the fracture function, of two internal variables reaches a critical value. General solution for the steady simple extension is obtained. The behaviour of the incompressible material for the uniaxial stress extension is also investigated. The stress-strain-internal variables relations are depicted in the figure for three material functions and several values of the material constants in the loading-unloading-reloading processes and in the loading-unloading cycles. The calculated results show both isotropic work-hardening and translation workhardening, as well as the rounding phenomenon and the Bauschinger effect.

Przeprowadzono badania teoretyczne materiału plastycznego Prandtla-Reussa ze wzmocnieniem ogólnej postaci. Wprowadzono skalarową i tensorową zmienną wewnętrzną. Założono, że potencjał plastyczny Misesa jest funkcją naprężenia translacyjnego i skalarowej zmiennej wewnętrznej. Następnie wyprowadzono ogólne równania konstytutywne dla materiałów plastycznych typu prędkościowego. Przyjęto cztery założenia konstytutywne, mianowicie: izotropię, niezależność od ciśnienia hydrostatycznego, niewystępowanie uogólnionego efektu Bauschingera, materiał stopnia drugiego. Z kolei zdefiniowano uogólniony materiał plastyczny Prandtla-Reussa. Z otrzymanych czułych na prędkość naprężenia równań konstytutywnych wyprowadzono warunek plastyczności i prawo płynięcia. Odpowiadają one odpowiednio uogólnionemu warunkowi plastyczności Hubera-Misesa i uogólnionemu prawu płynięcia Levy'ego-St. Venanta. Określono również warunek zniszczenia. Założono, że zniszczenie zachodzi, gdy funkcja skalarowa zależna od dwóch zmiennych wewnętrznych, zwana funkcją zniszczenia, osiąga wartość krytyczną. Otrzymano rozwiązanie ogólne dla ustalonego prostego rozciągania. Zbadano również zachowanie się materiału nieściśliwego dla jednoosiowego naprężenia rozciągającego. Związki naprężeniowo-odkształceniowe dla zmiennych wewnętrznych są pokazane graficznie na wykresach dla trzech funkcji materiałowych w procesie obciążenie-odciążenie i dla cyklu obciążenie-odciążenie. Wyniki obliczeń wskazują zarówno na wzmocnienie izotropowe jak i wzmocnienie translacyjne kinematyczne. Wykazują również zjawisko "zaokrąglania" i efekt Bauschingera.

Проведены теоретические исследования пластического материала Прандтля-Рейсса с упрочнением общего вида. Введены скалярная и тензорные внутренние переменные. Предположено, что пластический потенциал Мизеса является функцией трансляционного напряжения и скалярной внутренней переменной. Затем выведены общие определяющие уравнения для пластических материалов скоростного типа. Приняты четыре определяющие предположения, а именно: изотропия, независимость от гидростатического давления, невыступающие обобщенного эффекта Баушингера, материал второй степени. В свою очередь определен обобщенный пластический материал Прандтля-Рейсса. Из полученных определяющих уравнений чувствительных на скорость напряжения, выведены условие пластичности и закон течения. Они отвечают соответственно обобщенному условию пластичности Губера-Мизеса и обобщенному закону течения Леви-Сан-Венана. Определено тоже условие разрушения. Предположено, что разрушение наступает, когда скалярная функция зависящая от двух внутренних переменных, называемая функцией разрушения, достигает критического значения. Получено общее решение для установившегося простого растяжения. Исследовано также поведение несжимаемого материала для одноосного растягивающего напряжения. Соотношения напряжение–деформация для внутренних переменных показаны графически на диаграммах для трех материальных функций в процессе нагрузка–разгрузка–повторная нагрузка и для цикла нагрузка–разгрузка. Результаты расчетов указывают как на изотропное упрочнение, так и на трансляционное (кинематическое) упрочнение, а также на явление „округления'' и эффект Баушингера.

1. Introduction

In general the typical behaviour of plasticity are *yield*, *flow* and *work-hardening*. Therefore the constitutive equations, which define a plastic material, must contain and represent the above three properties. The *yield condition*, which is expressed as a stress relation, can be expressed geometrically by a surface, i.e. the *yield surface* in the stress space. The *plastic flow* can be represented by the vector of the strain increment in the space. When the plastic flow proceeds, the yield condition changes in general and the change shows work-hardening which can be expressed by the deformation and the motion of the yield surface. When the centre of the surface remains at the origin and the surface expands similarly with respect to the origin, we say that the material has *isotropic work-hardening*. When the surface translates rigidly in the space, we say that it has *translation* (or *kinematic*) *work-hardening*. In general, the plastic material may have both types of work-hardening. When the strain increment at a point on the yield surface has external, tangential, and internal direction, we say that the process is *loading*, *neutral* and *unloading*, respectively. If the yield stress for a loading direction and that for the opposite direction have different magnitudes, we say that the *Bauschinger effect* exists. Then, isotropic work-hardening has no such effect but translation work-hardening has it.

In general, plastic deformation is accompanied by an irreversible change of the internal state of the material. For a method which includes this change into the constitutive equation of continuum mechanics, there is the *theory of internal variable* [1–8]. The external variables such as the deformation and the stress, which are explicitly observable, are distributed continuously in a continuum and so are the *internal variables* which are implicitly observable as a result of the observed external variables. These internal variables may be of scalar, vector or tensor character. Thus the theory of internal variable has two types of *constitutive equation*; one is the relations between the external variables and these relations depend upon the internal variable and are called the constitutive equation in a narrow sense, and the other is the *evolutional equations*, which prescribe the time evolution of the internal variables and depend upon the external variables.

The plasticity and the viscosity can be distinguished by the dependence on the time scale. The plastic stress-deformation relation does not depend upon the time scale but the viscous relation does. The *rate type constitutive equation*, which is a linear relation between the stress rate and the deformation rate, does not depend upon the time scale. The *Prandtl-Reuss plastic material* has a special case of the rate type constitutive equation. The disintegration of the strain into the elastic and the plastic part can be taken with some ambiguous consideration [9]. On the other hand, the disintegration of the deformation rate, i.e. the stretching, has a definite meaning. Further, the rate type equations can express collectively many variaties of responses to the initial values.

Two types of construction of the rate type plastic equations can be considered. One is the *hypo-elasticity* by TRUESDELL [10] and another is the method derived from the *Von Mises' plastic potential* and the *rate type elastic equation*. By the former method the author introduced the scalar, vector and tensor internal variables into the hypo-elastic equation and obtained as special cases the *Prandtl-Reuss plastic material* and $\mathcal{T}$ material [11, 12]. Also by the latter method the author introduced the scalar and the tensor internal variables into the plastic potential and obtained the Prandtl-Reuss plastic material with isotropic and kinematic work-hardening [13–15].

In this paper we introduce the scalar and the tensor internal variables into the second method and propose the general constitutive equations of the *Prandtl-Reuss plastic material with general work-hardening*. The contents of this paper is a development of the results given in [15]. Further, the general behaviour of the material is analysed for the steady simple extension. Specially, the behaviour of the incompressible material for the steady uniaxial stress extension is analysed and depicted in the figure for the loading-unloading-reloading processes and for the cyclic loading-unloading processes.

2. Rate type constitutive equations

The positions of a material particle at the reference configuration and the current configuration are denoted by $\mathbf{X}$ and $\mathbf{x}$, respectively. The deformation gradient is $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$, and the left Cauchy-Green tensor is $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. The Cauchy stress $\mathbf{T}$ of the isotropic elastic material is a function of $\mathbf{B}$, i.e.

$$\mathbf{T} = \mathbf{K}(\mathbf{B}). \tag{2.1}$$

For the basic concepts of continuum mechanics refer, for example, to TRUESDELL and NOLL [16]. From the *principle of material frame-indifference* the material function $\mathbf{K}$ cannot be arbitrarily taken, but it must satisfy the identity

$$\mathbf{K}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T) = \mathbf{Q}\mathbf{K}(\mathbf{B})\mathbf{Q}^T, \tag{2.2}$$

where $\mathbf{Q}$ is any orthogonal tensor.

Differentiating the relation (2.1) with respect to time and we have

$$\dot{\mathbf{T}} = \frac{\partial\mathbf{K}}{\partial B_{mn}}(\mathbf{D}\mathbf{B}+\mathbf{B}\mathbf{D}+\mathbf{W}\mathbf{B}-\mathbf{B}\mathbf{W})_{nm}, \tag{2.3}$$

where the stretching $\mathbf{D}$ and the spin tensor $\mathbf{W}$ are the symmetric part and the skew symmetric part of the velocity gradient $\partial\dot{\mathbf{x}}/\partial\mathbf{x}$, respectively. Let us consider that $\mathbf{Q}$ is a function of a parameter a and that $\mathbf{Q} = \mathbf{1}$ and $d\mathbf{Q}/da = \mathbf{W}$ at $a = 0$. Then, differentiating the relation (2.2) with respect to a and setting $a = 0$, we have

$$\frac{\partial\mathbf{K}}{\partial B_{mn}}(\mathbf{W}\mathbf{B}-\mathbf{B}\mathbf{W})_{nm} = \mathbf{W}\mathbf{K}-\mathbf{K}\mathbf{W}. \tag{2.4}$$

Substituting Eq. (2.4) into Eq. (2.3) we can obtain

$$\overset{\circ}{\mathbf{T}} = \mathscr{E}[\mathbf{D}], \quad \overset{\circ}{T}_{ij} = \mathscr{E}_{ijkl} D_{kl}, \tag{2.5}$$

where

$$\overset{\circ}{\mathbf{T}} \equiv \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W} \tag{2.6}$$

is the co-rotational stress rate and the bracket denotes the linear dependence.

The *elasticity* $\mathscr{E}$ obtained above is a function of **B** and given by the material function **K(B)**; however, we consider that Eq. (2.5) with any fourth-order tensor $\mathscr{E}(\mathbf{B})$ define a rate type constitutive equation of the isotropic elastic material [17].

In an elastic-plastic deformation the stretching is assumed to be the sum of the elastic stretching ${}_E\mathbf{D}$ and the plastic stretching ${}_P\mathbf{D}$, i.e.

$$\mathbf{D} = {}_E\mathbf{D} + {}_P\mathbf{D}. \tag{2.7}$$

Further, we assume that the elastic equation (2.5) holds for an elastic stretching in an elastic-plastic deformation and

$$\overset{\circ}{\mathbf{T}} = \mathscr{E}[{}_E\mathbf{D}] = \mathscr{E}[\mathbf{D}] - \mathscr{E}[{}_P\mathbf{D}]. \tag{2.8}$$

It will be shown that the above equation is reduced to the plastic equation.

3. General rate type plastic equations

In order to include the change of the internal state in the theory, we introduce the scalar internal variable α and the symmetric tensor internal variable $\boldsymbol{\beta}$, which is assumed to have stress dimension without no generality. Their physical meanings depend upon a particular material. They may be the dislocation density for metal of the crystallization density for plastics. Here we do not discuss their physical interpretation.

Let us introduce Von Mises' plastic potential g and assume that it is a function of the stress and the internal variables. According to PRAGER [18], who introduced the translated stress

$$\tilde{\mathbf{T}} \equiv \mathbf{T} - \boldsymbol{\beta}, \tag{3.1}$$

we assume that

$$g = g(\tilde{\mathbf{T}}, \alpha). \tag{3.2}$$

The internal parameter $\boldsymbol{\beta}$ is called the translation tensor.

In the elastic state

$$g(\tilde{\mathbf{T}}, \alpha) < 0 \tag{3.3}$$

holds. In the yield state the yield condition

$$g(\tilde{\mathbf{T}}, \alpha) = 0 \tag{3.4}$$

holds and the flow rule

$${}_P\mathbf{D} = \varepsilon \frac{\partial g}{\partial \tilde{\mathbf{T}}} \tag{3.5}$$

is assumed, where ε is a proportionality factor.

The combination of Eqs. (2.8) and (3.5) gives the constitutive equation of a plastic equation in a narrow sense. We assume that the behaviour of the internal variables are independent of the time scale. The evolutional equations are assumed to be

(3.6) $$\dot{\alpha} = \boldsymbol{\Phi}(\tilde{\mathbf{T}}, \alpha)[{}_{\mathrm{P}}\mathbf{D}], \quad \dot{\alpha} = \Phi_{kl\,\mathrm{P}} D_{kl},$$

(3.7) $$\mathring{\boldsymbol{\beta}} = \boldsymbol{\Psi}(\tilde{\mathbf{T}}, \alpha)[{}_{\mathrm{P}}\mathbf{D}], \quad \mathring{\beta}_{ij} = \boldsymbol{\Psi}_{ijkl\,\mathrm{P}} D_{kl},$$

where

(3.8) $$\mathring{\boldsymbol{\beta}} \equiv \dot{\boldsymbol{\beta}} - \mathbf{W}\boldsymbol{\beta} + \boldsymbol{\beta}\mathbf{W}$$

is the co-rotational translation rate. Then we have

(3.9) $$\mathring{\tilde{\mathbf{T}}} = \mathscr{E}[\mathbf{D}] - (\mathscr{E} + \boldsymbol{\Psi})[{}_{\mathrm{P}}\mathbf{D}],$$

where

$$\mathring{\tilde{\mathbf{T}}} \equiv \dot{\tilde{\mathbf{T}}} - \mathbf{W}\tilde{\mathbf{T}} + \tilde{\mathbf{T}}\mathbf{W} = \mathring{\mathbf{T}} - \mathring{\boldsymbol{\beta}}$$

is the co-rotational translated stress rate.

From the principle of material frame-indifference, the identity

(3.10) $$g(\mathbf{Q}\tilde{\mathbf{T}}\mathbf{Q}^T, \alpha) = g(\tilde{\mathbf{T}}, \alpha)$$

holds for every orthogonal tensor **Q**. By the same process for Eq. (2.4) we have

(3.11) $$\mathrm{tr}\left(\frac{\partial g}{\partial \tilde{\mathbf{T}}}(\mathbf{W}\tilde{\mathbf{T}} - \tilde{\mathbf{T}}\mathbf{W})\right) = 0.$$

For the yield state Eq. (3.4) holds. Differentiating it with respect to time and referring to Eq. (3.11), we have

(3.12) $$\mathrm{tr}\left(\frac{\partial g}{\partial \tilde{\mathbf{T}}}\mathring{\tilde{\mathbf{T}}}\right) + \frac{\partial g}{\partial \alpha}\dot{\alpha} = 0.$$

Let us obtain the proportional factor ε. Substituting Eqs. (3.9), (3.6) and (3.5) into Eq. (3.12), we can obtain

(3.13) $$\varepsilon = \mathrm{tr}(\mathbf{G}\mathbf{D}),$$

where

(3.14) $$\mathbf{G} \equiv \frac{\left[\dfrac{\partial g}{\partial \tilde{\mathbf{T}}}\right]\mathscr{E}}{\left[\dfrac{\partial g}{\partial \tilde{\mathbf{T}}}\right](\mathscr{E} + \boldsymbol{\Psi})\left[\dfrac{\partial g}{\partial \tilde{\mathbf{T}}}\right] - \boldsymbol{\Phi}\left[\dfrac{\partial g}{\partial \tilde{\mathbf{T}}}\right]\dfrac{\partial g}{\partial \alpha}}$$

and $([\partial g/\partial \tilde{T}]\mathscr{E})_{kl} = (\partial g/\partial \tilde{T}_{mn})\mathscr{E}_{mnkl}$. Therefore, in the yield state the plastic stretching is proportional to the total stretching and

(3.15) $${}_{\mathrm{P}}\mathbf{D} = \frac{\partial g}{\partial \tilde{\mathbf{T}}}\,\mathrm{tr}(\mathbf{G}\mathbf{D}).$$

Also we can obtain the general rate type plastic equations:

(3.16) $$\mathring{\tilde{\mathbf{T}}} = \tilde{\mathscr{P}}[\mathbf{D}],$$

(3.17) $$\dot{\alpha} = \boldsymbol{\Pi}[\mathbf{D}],$$

(3.18) $$\mathring{\boldsymbol{\beta}} = \boldsymbol{\Pi}[\mathbf{D}],$$

where

$$\tilde{\mathscr{P}} \equiv \mathscr{E} - (\mathscr{E} + \mathbf{\Psi})\left[\frac{\partial g}{\partial \tilde{\mathbf{T}}}\right] \otimes \mathbf{G}, \tag{3.19}$$

$$\Pi \equiv \mathbf{\Phi}\left[\frac{\partial g}{\partial \tilde{\mathbf{T}}}\right]\mathbf{G}, \tag{3.20}$$

$$\mathbf{\Pi} \equiv \mathbf{\Psi}\left[\frac{\partial g}{\partial \tilde{\mathbf{T}}}\right] \otimes \mathbf{G} \tag{3.21}$$

and $\otimes$ denotes the tensor product.

4. Constitutive assumptions

The general plastic constitutive equations given in the last section are too general, so we assume here the following four constitutive assumptions:

(i) $\mathscr{E}$ and $\mathbf{\Psi}$ are isotropic constant tensors;

(ii) g and $\mathbf{\Phi}$ are pressure-insensitive;

(iii) there is no generalized Bauschinger effect with resepct to the translated stress;

(iv) the plastic potential and the evolutional equations are second-order polynomials for the translated stress.

The assumption (i) denotes that

$$\mathscr{E}_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{4.1}$$

$$\Psi_{ijkl} = l\delta_{ij}\delta_{kl} + m(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{4.2}$$

where λ, μ, l and m are material constants and δ_{ij} is the Kronecker delta.

The assumption (ii) gives

$$g = g(\tilde{\mathbf{T}}^*, \alpha), \quad \mathbf{\Phi} = \mathbf{\Phi}(\tilde{\mathbf{T}}^*, \alpha), \tag{4.3}$$

where $\tilde{\mathbf{T}}^* = \mathbf{T} - (1/3)\mathrm{tr}(\tilde{T})\,\mathbf{1}$ is the deviatoric translated stress. The principle of material frame-indifference demands that the identities

$$g(\mathbf{Q}\tilde{\mathbf{T}}^*\mathbf{Q}^T, \alpha) = g(\tilde{\mathbf{T}}^*, \alpha), \tag{4.4}$$

$$\mathbf{\Phi}(\mathbf{Q}\tilde{\mathbf{T}}^*\mathbf{Q}^T, \alpha) = \mathbf{Q}\mathbf{\Phi}(\tilde{\mathbf{T}}^*, \alpha)\mathbf{Q}^T \tag{4.5}$$

hold for every orthogonal tensor $\mathbf{Q}$. Then, by the representation theorem we have the expressions

$$g = g(\tilde{II}^*, \tilde{III}^*, \alpha), \tag{4.6}$$

$$\mathbf{\Phi} = \phi_0\mathbf{1} + \phi_1\tilde{\mathbf{T}}^* + \phi_2\tilde{\mathbf{T}}^{*2}, \tag{4.7}$$

where

$$\tilde{II}^* \equiv \mathrm{tr}(\tilde{\mathbf{T}}^{*2}), \quad \tilde{III}^* \equiv \mathrm{tr}(\tilde{\mathbf{T}}^{*3}) \tag{4.8}$$

are invariants of the deviatoric translated stress and

$$\phi_\Gamma = \phi_\Gamma\,(\tilde{II}^*, \tilde{III}^*, \alpha) \quad (\Gamma = 0, 1, 2) \tag{4.9}$$

are material functions.

The assumption (iii) means the following two conditions:

(a) if $(\tilde{\mathbf{T}}^*, \alpha)$ satisfies the yield condition, $(-\tilde{\mathbf{T}}^*, \alpha)$ does the same;

(b) $(\tilde{\mathbf{T}}^*, \alpha, {}_P\mathbf{D})$ and $(-\tilde{\mathbf{T}}^*, \alpha, -{}_P\mathbf{D})$ give the same rates of the internal variables.

These conditions show that g and ϕ_1 are even functions of $\tilde{\mathbf{T}}^*$, and ϕ_0 and ϕ_2 are its odd functions.

We assume that g and ϕ_Γ $(\Gamma = 0, 1, 2)$ are analytic functions of the invariants $\tilde{II}^*$ and $\tilde{III}^*$. The assumption (iv) and the parity of the functions depicted above give the expressions

$$g = g_0(\alpha) + g_1(\alpha)\tilde{II}^*, \tag{4.10}$$

$$\mathbf{\Phi} = \phi(\alpha)\tilde{\mathbf{T}}^*, \tag{4.11}$$

where $g_0(\alpha)$, $g_1(\alpha)$ and $\phi(\alpha)$ are material functions of α.

From Eq. (4.10) the yield condition (3.4) and the flow rule (3.5) reduce, respectively, to

$$\tilde{II}^* = -\frac{g_0(\alpha)}{g_1(\alpha)} \equiv g_2(\alpha), \tag{4.12}$$

$${}_P\mathbf{D} = 2\varepsilon g_1(\alpha)\tilde{\mathbf{T}}^*. \tag{4.13}$$

Equation (4.13) gives ZIEGLER's rule [19] when $g_1(\alpha)$ takes a constant value.

Let us obtain special forms of the constitutive equations (3.16)–(3.18) by the constitutive assumptions. Substituting the expressions (4.1), (4.2), (4.10) and (4.11) into the constitutive funcitons (3.19)-(3.21) and referring to the condition (4.12), we can obtain

$$\tilde{\mathscr{P}} = \tilde{\mathscr{E}} - \frac{2\mu}{k(\alpha)^2}\tilde{\mathbf{T}}^* \otimes \tilde{\mathbf{T}}^*, \tag{4.14}$$

$$\Pi = \frac{\mu}{\mu+m}\,\frac{g_2(\alpha)\phi(\alpha)}{k(\alpha)^2}\tilde{\mathbf{T}}^*, \tag{4.15}$$

$$\mathbf{\Pi} = \frac{2\mu m}{(\mu+m)k(\alpha)^2}\tilde{\mathbf{T}}^* \otimes \tilde{\mathbf{T}}^*, \tag{4.16}$$

where $\mathscr{E}$ is given by Eq. (4.1) and

$$k(\alpha) = \left[g_2(\alpha)\left\{1 + \frac{(\partial g_2(\alpha)/\partial\alpha)\phi(\alpha)}{4(\mu+m)}\right\}\right]^{1/2}. \tag{4.17}$$

Here we introduce a new parameter

$$\bar{\alpha} = \int_0^\alpha \frac{\mu+m}{2\mu^2}\,\frac{k(\alpha)^2}{g_2(\alpha)\phi(\alpha)}\,d\alpha, \tag{4.18}$$

which is defined by the material constants μ and m and the material functions $g_2(\alpha)$ and $\phi(\alpha)$. Therefore the relation between α and $\bar{\alpha}$ is specified by a given material. Then the parameter $\bar{\alpha}$ may be interpreted as a new internal variable, and henceforth $\bar{\alpha}$ is rewritten as α. By this transformation the expressions (4.14) and (4.16) are formally unchangeable and Eq. (4.15) reduces to

$$\Pi = \frac{1}{2\mu}\tilde{\mathbf{T}}^*. \tag{4.19}$$

The constitutive equations (3.16)–(3.18) are, then, expressed as

$$\mathring{\mathbf{T}} = \mathscr{E} - \frac{2\mu}{k(\alpha)^2} w\tilde{\mathbf{T}}^*, \tag{4.20}$$

$$\dot{\alpha} = \frac{1}{2\mu} w, \tag{4.21}$$

$$\mathring{\boldsymbol{\beta}} = \frac{2\mu m}{(\mu+m)k(\alpha)^2} w\tilde{\mathbf{T}}^*, \tag{4.22}$$

where

$$w = \operatorname{tr}(\tilde{\mathbf{T}}^*\mathbf{D}) \tag{4.23}$$

denotes the work done on the material per unit volume and per unit time by the deviatoric translated stress; it is called the substantial stress power.

We must remark here that the above equations have been derived under the yield condition (3.4) and $k(\alpha)$ is not an arbitrary function of α but is given by Eq. (4.17).

5. Prandtl-Reuss plastic material

A typical property of the plasticity is the loading-unloading phenomenon. The stress-strain diagram has two different paths for the loading and for the unloading, and two paths crossing at a point which is the starting point of the unloading. A theory of ordinary differential equation assures that they have unique solution for a given initial condition. The rate type constitutive equations may be regarded as the ordinary differential equations when the time is regarded as the independent variable. Then a single set of rate-type constitutive equations cannot express the loading-unloading phenomenon. Therefore we must adopt two sets of equations, one is for the loading state and the other is for the unloading state.

In the unloading state, which is not defined now, we assume that the elastic equation (2.5) holds and the internal variables remain their initial values, i.e.,

$$\dot{\alpha} = 0, \tag{5.1}$$

$$\mathring{\boldsymbol{\beta}} = \mathbf{0}. \tag{5.2}$$

In the loading state we assume that Eqs. (4.20)–(4.22) hold, where the material function $k(\alpha)$ is not given by Eq. (4.17) but it must be regarded as any given function of α. This is a drastic change of our standpoint; here, Eqs. (4.20)–(4.22) must not be regarded as the equations under the yield condition.

When the substantial stress power w is positive, zero and negative, the material receives the mechanical work, is adiabatic with it, and gives it to the exterior, respectively. So we may define that the three states

$$w > 0, \quad w = 0, \quad w < 0 \tag{5.3}$$

are the loading, the neutral and the unloading state, respectively. By this definition the two sets of equations have the expressions

(5.4) $$\dot{p} = -(\lambda+\frac{2}{3}\mu)\mathrm{tr}\mathbf{D},$$

(5.5) $$\mathring{\tilde{\mathbf{T}}}^* = 2\mu\mathbf{D}^* - \frac{2\mu}{k(\alpha)^2}\langle w\rangle\tilde{\mathbf{T}}^*,$$

(5.6) $$\dot{\alpha} = \frac{1}{2\mu}\langle w\rangle,$$

(5.7) $$\mathring{\boldsymbol{\beta}} = \frac{2\mu m}{(\mu+m)k(\alpha)^2}\langle w\rangle\tilde{\mathbf{T}}^*,$$

where the symbol $\langle\ \rangle$ means

(5.8) $$\langle x\rangle = \begin{cases} x, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

These consist of the constitutive equations of the compressible Prandtl-Reuss plastic material.

If the material is incompressible, the pressure cannot be determined by the constitutive relations but it must be specified by the initial and the boundary condition. The constitutive equations of the incompressible Prandtl-Reuss plastic material are given by Eqs. (5.5)–(5.7) without Eq. (5.4). The incompressible material must be isochoric, then there are

(5.9) $$\mathrm{tr}\mathbf{D} = 0, \quad \mathbf{D}^* = \mathbf{D}.$$

Until now we have derived the constitutive equations (4.20)–(4.22) from the heuristic method with the aid of Von Mises's plastic potential. However, the author obtained them through the method of hypo-elasticity. Equations (7.2), (6.5) and (7.3) depicted in the reference [12] reduce to Eqs. (4.20)–(4.22) depicted in the last section if we put $K_M(\alpha) = k(\alpha)$, $b(\alpha) = m/(\mu+m)$.

For the sake of the numerical calculations, let us take the dimensionless expressions of the constitutive equations.

(5.10) $$\mathbf{S} \equiv \frac{\mathbf{T}}{2\mu}, \quad q \equiv \frac{p}{2\mu}, \quad \boldsymbol{\Upsilon} \equiv \frac{\boldsymbol{\beta}}{2\mu}, \quad M(\alpha) \equiv \frac{k(\alpha)}{2\mu}, \quad c \equiv \frac{m}{\mu+m}$$

are respective dimensionless quantities, function and constant. Equations (5.4)–(5.7) reduce to

(5.11) $$\dot{q} = -\left(\frac{\lambda}{2\mu}+\frac{1}{3}\right)\mathrm{tr}\mathbf{D},$$

(5.12) $$\mathring{\tilde{\mathbf{S}}}^* = \mathbf{D}^* - \frac{1}{M(\alpha)^2}\langle v\rangle\tilde{\mathbf{S}}^*,$$

(5.13) $$\dot{\alpha} = \langle v\rangle,$$

(5.14) $$\mathring{\boldsymbol{\Upsilon}} = \frac{c}{M(\alpha)^2}\langle v\rangle\tilde{\mathbf{S}}^*,$$

where

$$v \equiv \mathrm{tr}(\tilde{\mathbf{S}}^*\mathbf{D}). \tag{5.15}$$

We can say that the compressible Prandtl-Reuss plastic material is characterized by two constants λ/μ and c and a function $M(\alpha)$, and the incompressible Prandtl-Reuss plastic material by a constant c and a function $M(\alpha)$.

6. Yield condition and flow rule

The Prandtl-Reuss constitutive equations (5.4)-(5.7) are characterized by the material constants μ and m and the material function $k(\alpha)$, which does not have the form (4.17) but is a given function. Then the yield condition and the flow rule must be newly defined by Eq. (5.5).

The author derived the yield condition and the flow rule of the material with the hypo-elastic equation [20]. Equation (4.20) or Eq. (5.5) is a linear relation between the stress rate and the stretching. By a given stretching the stress rate can be determined uniquely. The inverse correspondence may be singular. This singularity relation is called the yield condition and the null space of the stretching gives the flow rule. For a more simple method with matrix representation, see TOKUOKA [21]. From the equation we can obtain the yield condition

$$\tilde{II}^* = \mathrm{tr}(\tilde{\mathbf{T}}^{*2}) = k(\alpha)^2 \tag{6.1}$$

and the flow rule

$$\mathbf{D} = \varepsilon\tilde{\mathbf{T}}^*, \tag{6.2}$$

where ε is a new proportionality factor. Compare them with Eqs. (4.12) and (4.13).

The condition (6.1) represents a circular cylinder in the stress space with the axis crossing at the translation $\boldsymbol{\beta}$ and the radius $\sqrt{2}k(\alpha)$. The axis is parallel to the pressure-axis which spans an equal angle with three coordinate axes. Therefore, the change of $k(\alpha)$ with α denotes the isotropic work-hardening and the change of $\boldsymbol{\beta}$ denotes the translation work-hardening. The condition (6.1) is a generalization of the Huber-Von-Mises yield condition. The rule (6.2) denotes that the stretching is proportional to the deviatoric translated stress. This stretching is called the flow stretching. For this we have $\mathrm{tr}\mathbf{D} \equiv 0$, then we can say that the flow stretching is isochoric. The rule (6.2) is a generalization of the Lévy-St. Venant flow rule.

The flow cannot continue with indefinite magnitude. Because if the stretching changes by the manner of Eq. (6.2), the evolutional equations (5.6) and (5.7) for the loading state yield the changes of the internal variables which, in general, disorder the yield condition (6.1).

The evolutional equation (5.7) shows

$$\mathrm{tr}\overset{\circ}{\boldsymbol{\beta}} \equiv \mathrm{tr}\dot{\boldsymbol{\beta}} \equiv 0, \tag{6.3}$$

which gives $I_{\boldsymbol{\beta}} \equiv \mathrm{tr}\,\boldsymbol{\beta} = \text{constant}$. Then the translation point $\boldsymbol{\beta}$ in the stress space is on a plane that is perpendicular to the pressure-axis.

7. Fracture rule

Every material fractures when it is deformed largely or when the repeated loading-unloading cycles are applied on it. In general, we can say that the fracture occurs when the internal change is cumulated and the internal state reaches a critical state. The internal variables were introduced to express some changes of the internal state, so it is natural to assume that a material function of the internal variables exists and the fracture occurs when it has a critical value.

We introduce a scalar material function called the fracture function:

$$h = h(\alpha, \boldsymbol{\beta}), \tag{7.1}$$

and we assume that the fracture occurs when

$$h(\alpha, \boldsymbol{\beta}) = 0, \tag{7.2}$$

which is called the fracture condition.

The fracture function must be frame-indifferent, so

$$h(\alpha, \mathbf{Q}\boldsymbol{\beta}\mathbf{Q}^T) = h(\alpha, \boldsymbol{\beta}) \tag{7.3}$$

holds identically for every orthogonal tensor Q. Then the representation theorem gives

$$h = h(\alpha, \mathrm{II}_{\boldsymbol{\beta}}, \mathrm{III}_{\boldsymbol{\beta}}), \tag{7.4}$$

where

$$\mathrm{II}_{\boldsymbol{\beta}} \equiv \mathrm{tr}(\boldsymbol{\beta}^2), \quad \mathrm{III}_{\boldsymbol{\beta}} \equiv \mathrm{tr}(\boldsymbol{\beta}^3) \tag{7.5}$$

are the invariants of the translation tensor $\boldsymbol{\beta}$ and where the dependence on $I_{\boldsymbol{\beta}}$ is deleted because, from Eq. (6.3), it has a constant value.

8. Cauchy's laws of motion

Every material is subjected to the fundamental laws of motion, i.e. Cauchy's first and second law of motion:

$$\mathrm{div}\mathbf{T} + \varrho\mathbf{b} = \varrho\ddot{\mathbf{x}}, \tag{8.1}$$

$$\mathbf{T} = \mathbf{T}^T, \tag{8.2}$$

where ϱ is the mass density, $\mathbf{b}$ is the body force per unit mass and $\mathbf{T}^T$ denotes the transpose of $\mathbf{T}$.

The second law holds when the stress is assumed to be a symmetric tensor. Usually, the body force is neglected and then, the spatially homogeneous stress exists only when the motion is accelerationless. Even if the motion is not accelerationless, the acceleration reduces to a diminishingly small quantity if the rate of deformation tends to zero. Our rate type constitutive equations are independent of the time scale, therefore we can consider the law (8.1) to hold for the homogeneous stress, zero body force and sufficiently slow deformation. In this way we can now focus our attention on the constitutive equations.

9. Steady simple extension

A steady simple extension has a constant stretch tensor

$$D_{ij} = D_i \delta_{ij}, \tag{9.1}$$

where D_i are constant principal stretches with respect to a rectangular Cartesian coordinates and the summation convention is not applied in this section. From the definition of the stretching we have

$$\dot{x}_i = D_i x_i \tag{9.2}$$

and by integration we have

$$x_i = X_i \exp(D_i t), \tag{9.3}$$

$$E_i(t) \equiv \log \frac{dx_i}{dX_i} = D_i t, \tag{9.4}$$

where the material configuration at $t = 0$ is referred to the reference configuration and E is called the logarithmic strain.

Now let us analyse the behaviour of the Prandtl-Reuss plastic material for the steady simple extension, where the zero initial conditions

$$\mathbf{S} = \mathbf{0}, \quad \alpha = 0, \quad \boldsymbol{\gamma} = \mathbf{0} \quad \text{at } t = 0 \tag{9.5}$$

are assumed.

From Eq. (5.11) we have

$$q = q_0 - \left(\frac{\lambda}{2\mu} + \frac{1}{3}\right) \sum_{i=1}^{3} (E_i - E_{0i}), \tag{9.6}$$

where the subscript zero indicates the value when a steady simple extension starts. From Eq. (9.2) the spin tensor vanishes and the co-rotational time rate reduces to the usual time rate. From Eqs. (5.12) and (9.1) we have

$$\dot{\tilde{S}}^*_{ij} = -\frac{1}{M(\alpha)^2} \langle v \rangle \tilde{S}^*_{ij}, \quad (i \neq j), \tag{9.7}$$

and we can conclude with the initial condition $(9.5)_1$ that the shear components of $\tilde{S}^*$ vanish identically. We then have

$$\tilde{S}^*_{ij} = \tilde{S}^*_i \delta_{ij}. \tag{9.8}$$

Therefore, from Eqs. (5.14) and $(9.5)_3$, we have

$$\gamma_{ij} = \gamma_i \delta_{ij}. \tag{9.9}$$

Then Eqs. (5.12)–(5.14) can be written as

$$\frac{d\tilde{S}^*_i}{dt} = D^*_i - \frac{1}{M(\alpha)^2} \langle v \rangle \tilde{S}^*_i, \tag{9.10}$$

$$\frac{d\alpha}{dt} = \langle v \rangle, \tag{9.11}$$

$$\frac{d\gamma_i}{dt} = \frac{c}{M(\alpha)^2} \langle v \rangle \tilde{S}^*_i, \tag{9.12}$$

where

(9.13) $$\langle v\rangle = \sum_{i=1}^{3} \tilde{S}_i^* D_i = \sum_{i=1}^{3} \tilde{S}_i^* D_i^*.$$

For an unloading process we can easily obtain

(9.14) $$\tilde{S}_i^* = \tilde{S}_{0i}^* + E_i^* - E_{0i}^*,$$

(9.15) $$\alpha = \alpha_0,$$

(9.16) $$\gamma_i = \gamma_{0i}.$$

Then from $S_i = \tilde{S}_i^* + \beta_i - q$, we have

(9.17) $$S_i = S_{0i} + (E_i - E_{0i}) + \frac{\lambda}{2\mu} \sum_{i=1}^{3} (E_i - E_{0i}),$$

where we put $S_{0i} = \tilde{S}_{0i}^* + \beta_{0i} - q_0$.

For the loading process, multiplying Eq. (9.10) by D_i^*, summing in i and replacing the independent variable from t to α by Eq. (9.11) gives us

(9.18) $$\frac{dv^2}{d\alpha} - \frac{2v^2}{M(\alpha)^2} = 2 \sum_{i=1}^{3} D_i^{*2}.$$

Now we define the function

(9.19) $$f(\alpha, \alpha_0) \equiv \exp\left(\int_{\alpha_0}^{\alpha} \frac{d\xi}{M(\xi)^2}\right),$$

and by similar manipulations we can obtain the following:

(9.20) $$\tilde{S}_i^*(\alpha, \alpha_0) = \frac{1}{f(\alpha, \alpha_0)} \left[\tilde{S}_{0i}^* + D_i^* \int_{\alpha_0}^{\alpha} \frac{f(\xi, \alpha_0)}{v(\xi, \alpha_0)} d\xi\right],$$

(9.21) $$\gamma_i(\alpha, \alpha_0) = \gamma_{0i} + c\big(E_i^*(\alpha, \alpha_0) - E_{0i}^*\big) + c\big(\tilde{S}_i^*(\alpha, \alpha_0) - \tilde{S}_{0i}^*\big),$$

(9.22) $$t = t_0 + \int_{\alpha_0}^{\alpha} \frac{d\xi}{v(\xi, \alpha_0)},$$

(9.23) $$E_i = E_{0i} + D_i \int_{\alpha_0}^{\alpha} \frac{d\xi}{v(\xi, \alpha_0)}.$$

Then we have

(9.24) $$S_i(\alpha, \alpha_0) = S_{0i} + (1-c)\left(\frac{1}{f(\alpha, \alpha_0)} - 1\right) \tilde{S}_{0i}^* + \frac{(1-c)D_i^*}{f(\alpha, \alpha_0)} \int_{\alpha_0}^{\alpha} \frac{f(\xi, \alpha_0)}{v(\xi, \alpha_0)} d\xi$$
$$+ c\big(E_i(\alpha, \alpha_0) - E_{0i}\big) + \left(\frac{\lambda}{2\mu} + \frac{1-c}{3}\right) \sum_{i=1}^{3} (E_i(\alpha, \alpha_0) - E_{0i}).$$

These equations (9.20)–(9.24) are the relations of the stress, the strain, the translation and the time which are correlated with each other by the scalar internal variable α as a parameter.

10. Uniaxial stress extension

10.1. Constitutive relations

Here let us analyse the uniaxial stress extension of the incompressible Prandtl-Reuss plastic material with the initial conditions (9.5). The loading direction is taken as the x_1-axis and we can put

(10.1) $$S_1 \equiv S, \qquad S_2 = S_3 = 0,$$

(10.2) $$D_1 \equiv D(t), \quad D_2 = D_3 = -\frac{1}{2} D(t),$$

where $D(t)$ is a given function of time. From the initial condition $(9.5)_3$ and Eq. (6.3) we have $\mathrm{tr}\boldsymbol{\gamma} \equiv 0$. Then we can put

(10.3) $$\gamma_1 \equiv \frac{2}{3}\gamma, \quad \gamma_2 = \gamma_3 = -\frac{1}{3}\gamma.$$

The stress power in Eq. (5.15) is given by

(10.4) $$v = \tilde{S}D,$$

where

(10.5) $$\tilde{S} \equiv S - \gamma.$$

We can say that for the loading state

(i) $\tilde{S} > 0, \quad S > \gamma \quad \text{and} \quad D > 0,$

(ii) $\tilde{S} < 0, \quad S < \gamma \quad \text{and} \quad D < 0,$

and for the unloading state

(iii) $\tilde{S} > 0, \quad S > \gamma \quad \text{and} \quad D < 0,$

(iv) $\tilde{S} < 0, \quad S < \gamma \quad \text{and} \quad D > 0.$

The pressure is indefinite and the constitutive equations (5.5)–(5.7) in the uniaxial stress extension are given by the following: For the unloading state,

(10.6) $$\frac{d\tilde{S}}{dE} = \frac{3}{2}, \quad \frac{d\alpha}{dE} = 0, \quad \frac{d\gamma}{dE} = 0,$$

and for the loading state,

(10.7) $$\frac{d\tilde{S}}{dE} = \frac{3}{2} - \frac{\tilde{S}^2}{M(\alpha)^2}, \quad \frac{d\alpha}{dE} = \tilde{S}, \quad \frac{d\gamma}{dE} = \frac{c\tilde{S}^2}{M(\alpha)^2},$$

where

(10.8) $$\frac{dE}{dt} = D$$

and E is the logarithmic strain along the x_1-axis.

Equations (10.6) can easily be integrated and we thus have the relations for the unloading state

(10.9) $$S = S_0 + \frac{3}{2}(E - E_0), \quad \alpha = \alpha_0, \quad \gamma = \gamma_0.$$

After some manipulations we can integrate Eqs. (10.7) and we have the relations for the loading state

(10.10) $$\tilde{S}(\alpha, \alpha_0) = \pm \frac{1}{f(\alpha, \alpha_0)} \left[\tilde{S}_0^2 + 3 \int_{\alpha_0}^{\alpha} f(\xi, \alpha_0)^2 d\xi\right]^{1/2},$$

(10.11) $$E(\alpha, \alpha_0) = E_0 + \int_{\alpha_0}^{\alpha} \frac{d\xi}{\tilde{S}(\xi, \alpha_0)},$$

(10.12) $$\gamma(\alpha, \alpha_0) = \gamma_0 + c \int_{\alpha_0}^{\alpha} \frac{\tilde{S}(\xi, \alpha_0)^2}{M(\xi)^2} d\xi$$

$$= \gamma_0 + \frac{3}{2} c\big(E(\alpha, \alpha_0) - E_0\big) - c\big(\tilde{S}(\alpha, \alpha_0) - \tilde{S}_0\big).$$

Then, from Eq. (10.5) we have

(10.13) $$S(\alpha, \alpha_0) = S_0 + (1-c)\big(\tilde{S}(\alpha, \alpha_0) - \tilde{S}_0\big) + \frac{3}{2} c\big(E(\alpha, \alpha_0) - E_0\big),$$

where $f(\alpha, \alpha_0)$ was defined by Eq. (9.19) and the subscript zero denotes as before a value at a starting instance.

10.2. Work-hardening

Let us estimate the magnitude of the work-hardening. At the zero-th order approximation we can put $d\tilde{S}/dE = 0$ in Eq. $(10.7)_1$ and we have the yield condition

(10.14) $$\tilde{S} = \pm\sqrt{\frac{3}{2}}\, M(\alpha),$$

which is equivalent to (6.1).

From Eqs. $(10.7)_2$ and $(10.7)_3$ we have the increment relations:

(10.15) $$\Delta\alpha = \pm\sqrt{\frac{3}{2}}\, M(\alpha)\Delta E,$$

(10.16) $$\Delta\gamma = \frac{3}{2} c\Delta E.$$

By the yield condition (10.14) and the relation (10.15) we have

(10.17) $$\Delta\tilde{S} = \frac{3}{4} \frac{dM(\alpha)^2}{d\alpha} \Delta E.$$

The increment relations (10.16) and (10.17) give, respectively, the translation and the isotropic work-hardening. Then the material constant c expresses the magnitude of the translation work-hardening and the material function $M(\alpha)$ characterizes the isotropic work-hardening. From Eq. (10.5) we have

$$\Delta S = \frac{3}{4}\left(\frac{dM(\alpha)^2}{d\alpha}+2c\right)\Delta E, \tag{10.18}$$

which indicates that the total work-hardening is a sum of two hardenings.

10.3. Perfect plastic material

In the case where

$$c = 0, \quad M(\alpha) = M_0 = \text{constant}, \tag{10.19}$$

there is no work-hardening and the material is reduced to a perfect plastic material. The translation always vanishes, i.e., $\gamma \equiv 0$ and then $\tilde{S} \equiv S$. The scalar internal variable disappears in Eq. $(10.7)_1$ and its sinificance vanishes naturally.

Equation $(10.7)_1$ may be integrated and we have

$$S = \sqrt{\frac{3}{2}}\, M_0 \, \frac{S_0+\sqrt{\frac{3}{2}}\, M_0 \tanh\left(\sqrt{\frac{3}{2}}\,\frac{E-E_0}{M_0}\right)}{\sqrt{\frac{3}{2}}\, M_0+S_0 \tanh\left(\sqrt{\frac{3}{2}}\,\frac{E-E_0}{M_0}\right)}. \tag{10.20}$$

When $|E-E_0| \gg M_0$, we have the yield conditions $S = \pm\sqrt{3/2}\, M_0$, and when $|E-E_0| \ll M_0$, we have the unloading relation $(10.9)_1$

10.4. Plastic material with translation work-hardening

In the case where

$$c > 0, \quad M(\alpha) = M_0 = \text{constant}, \tag{10.21}$$

there is the translation work-hardening. The relation of the translated stress magnitude $\tilde{S}$ and the strain is given by Eq. (10.20) by the replacements of S by $\tilde{S}$ and S_0 by $\tilde{S}_0$. The magnitudes of the translation and the stress are obtained from Eqs. (10.12) and (10.13).

10.5. Plastic material with isotropic work-hardening

In the case where

$$c = 0, \quad \frac{dM(\alpha)}{d\alpha} \neq 0, \tag{10.22}$$

there is isotropic work-hardening. Here $\gamma \equiv 0$ and $\tilde{S} \equiv S$. The stress-strain relation is given by Eqs. (10.10) and (10.11) by replacement of $\tilde{S}$ by S and $\tilde{S}_0$ by S_0.

10.6. Fracture function

In Sect. 7 we introduced the fracture function. In the uniaxial stress extension the translation tensor is represented by the magnitude γ. Then we can put

$$h = \hat{h}(\alpha, \gamma), \tag{10.23}$$

and then the fracture condition is given by

$$\hat{h}(\alpha, \gamma) = 0. \tag{10.24}$$

11. Numerical calculations

In order to execute the numerical calculations we must adopt a concrete form of the material function. Now we assume

$$M(\alpha) = M_0(1+a\alpha)^n, \tag{11.1}$$

where M_0, a and n are material constants. When $a = 0$, there is no isotropic work-hardening.

Let us consider a parameter transformation:

$$M_0 \to \zeta M_0, \quad a \to \zeta^{-2}a, \quad n \to n, \quad c \to c, \tag{11.2}$$

and

$$S \to \zeta S, \quad E \to \zeta E, \quad \alpha \to \zeta^2\alpha, \quad \gamma \to \zeta\gamma. \tag{11.2'}$$

By this transformation the constitutive relations (10.6) and (10.7), the stress-strain-translation relations (10.9)–(10.12) and (10.20), the yield condition (10.14) and the work-hardening relations (10.15)–(10.17) are invariant. Therefore, one of the two material constants M_0 and a may assume any value without any loss of generality.

The function defined by the relation (9.19) has the forms

$$f(\alpha, \alpha_0) = \exp\left[\frac{(1+a\alpha)^{1-2n}-(1+a\alpha_0)^{1-2n}}{(1-2n)M_0^2a}\right], \quad n \neq \frac{1}{2}, \tag{11.3}$$

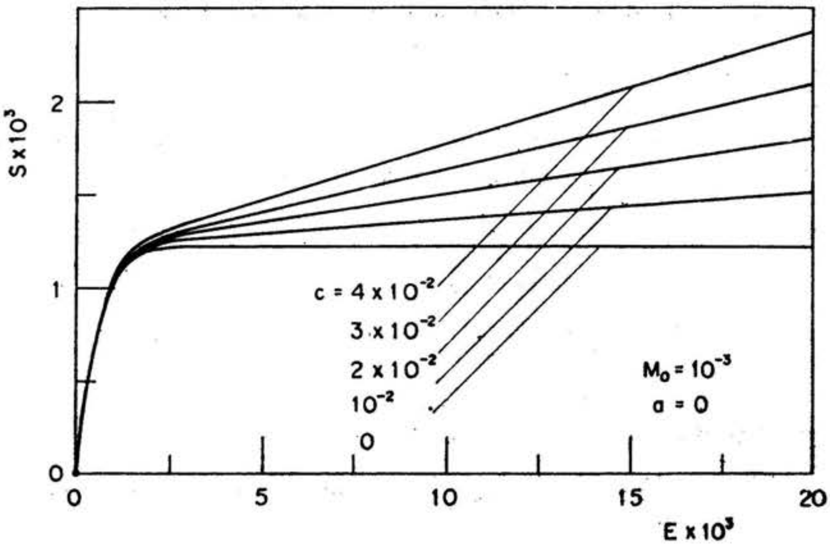
$$f(\alpha, \alpha_0) = \left(\frac{1+a\alpha}{1+a\alpha_0}\right)^{1/M_0^2a}, \quad n = \frac{1}{2}, \tag{11.3'}$$

which are invariant with respect to the transformation (11.2).

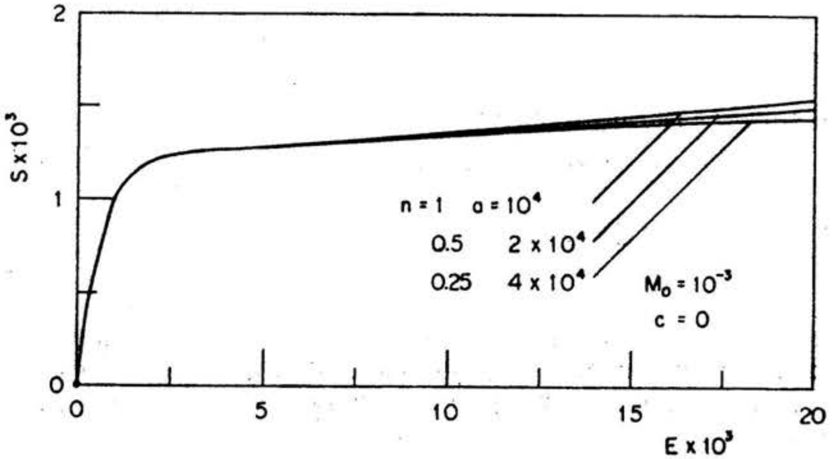
Figure 1 show the stress-strain relations for the simple loading process from the initial state (9.5). The diagrams depicted in Fig. 1 (a) refer to the material with translation work-hardening and the diagrams with $c = 0$ correspond to the perfect plastic material. The diagrams depicted in Fig. 1 (b)–(e) refer to the material with isotropic work-hardening. When $a\alpha \ll 1$, the material function (11.1) is approximated by

$$M(\alpha) = M_0(1+na\alpha). \tag{11.4}$$

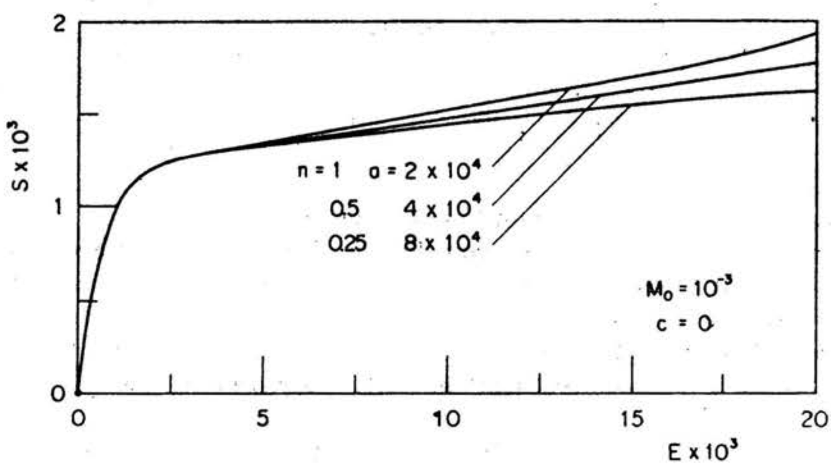
a)


S x 10³
0
1
2
c = 4 x 10⁻²
3 x 10⁻²
2 x 10⁻²
10⁻²
0
M₀ = 10⁻³
a = 0
0
5
10
15
20
E x 10³

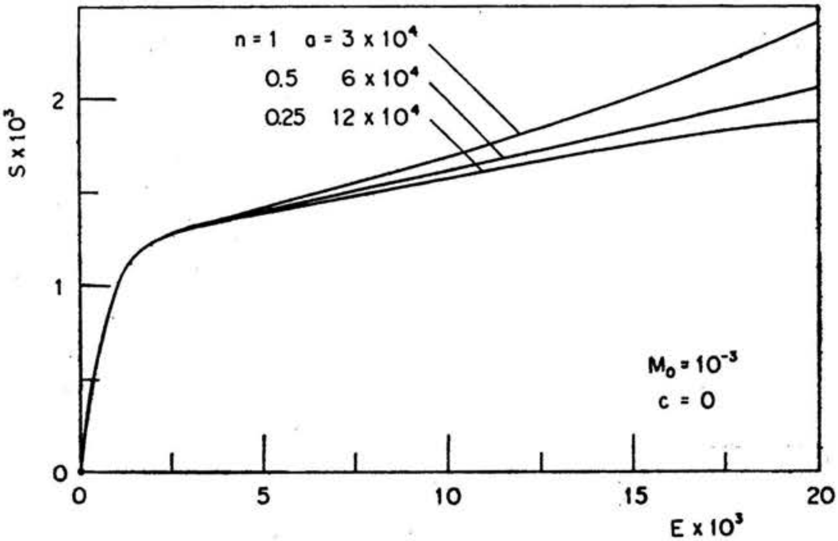
b)


S x 10³
0
1
2
n = 1   a = 10⁴
0.5   2 x 10⁴
0.25   4 x 10⁴
M₀ = 10⁻³
c = 0
0
5
10
15
20
E x 10³

c)


S x 10³
0
1
2
n = 1   a = 2 x 10⁴
0.5   4 x 10⁴
0.25   8 x 10⁴
M₀ = 10⁻³
c = 0
0
5
10
15
20
E x 10³

d)



e)

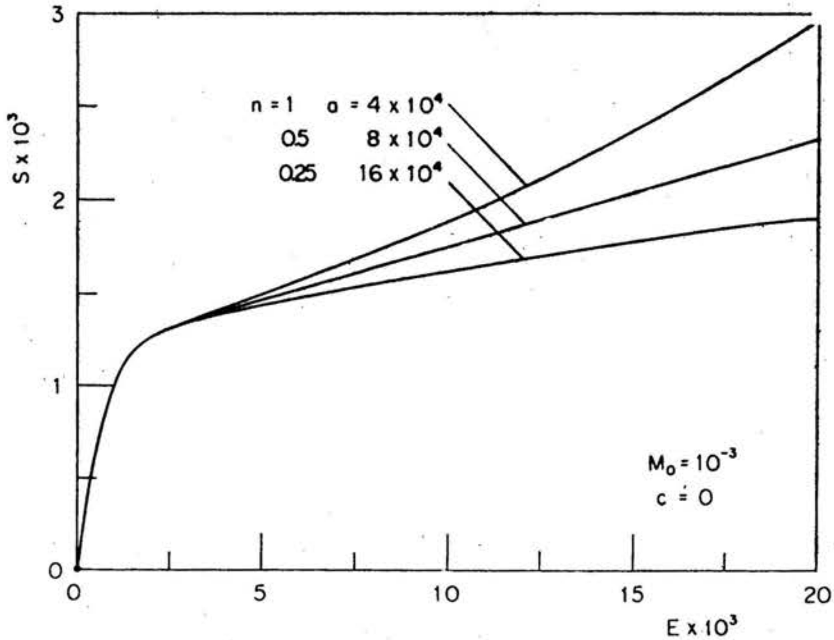


FIG. 1. Stress-strain relations for loading processes.

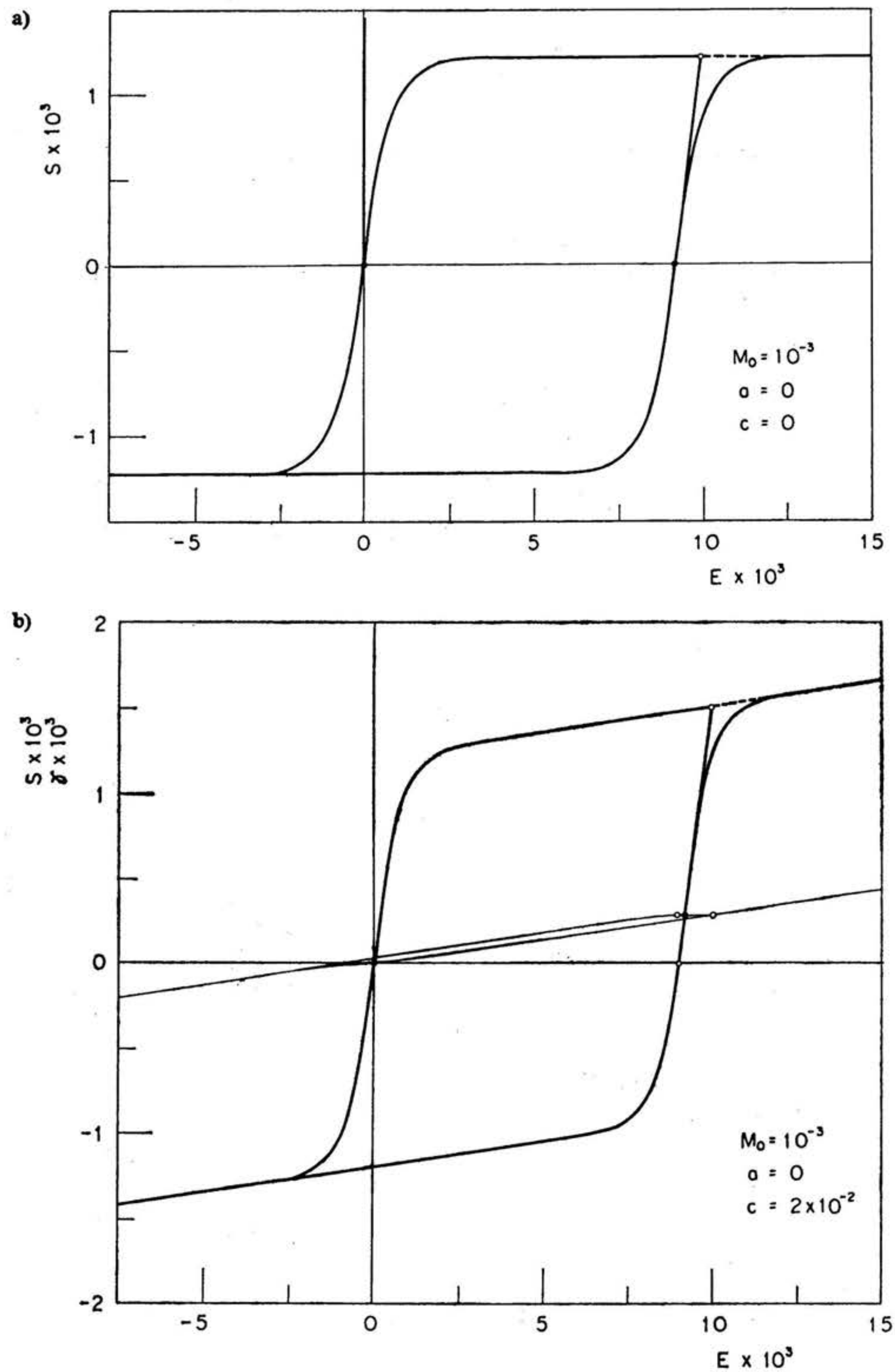
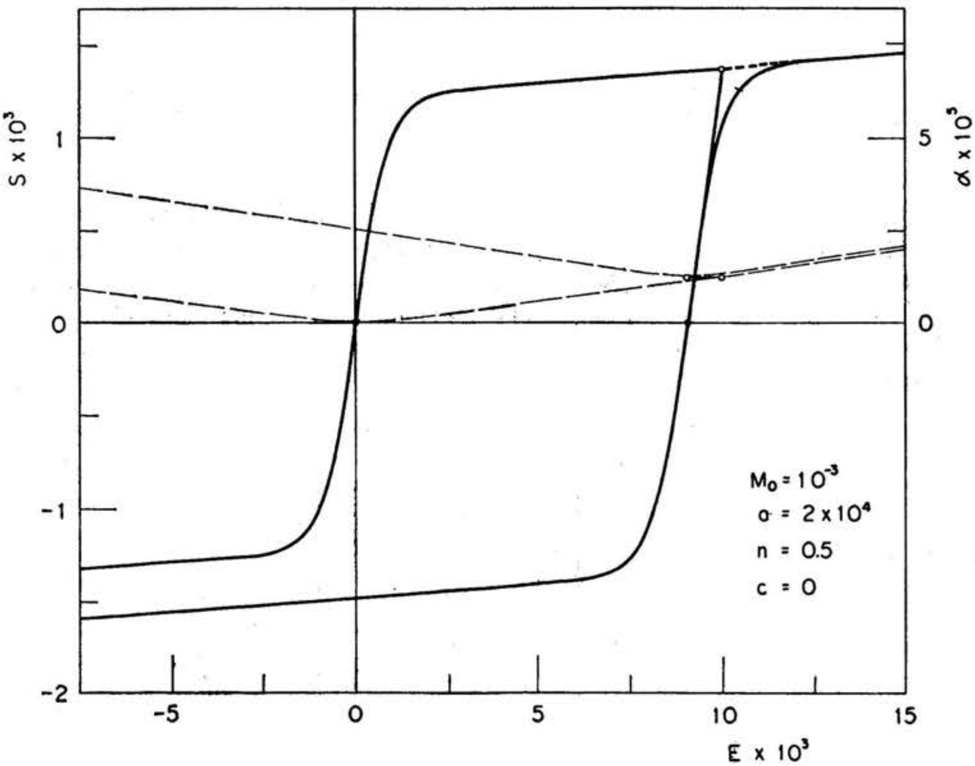
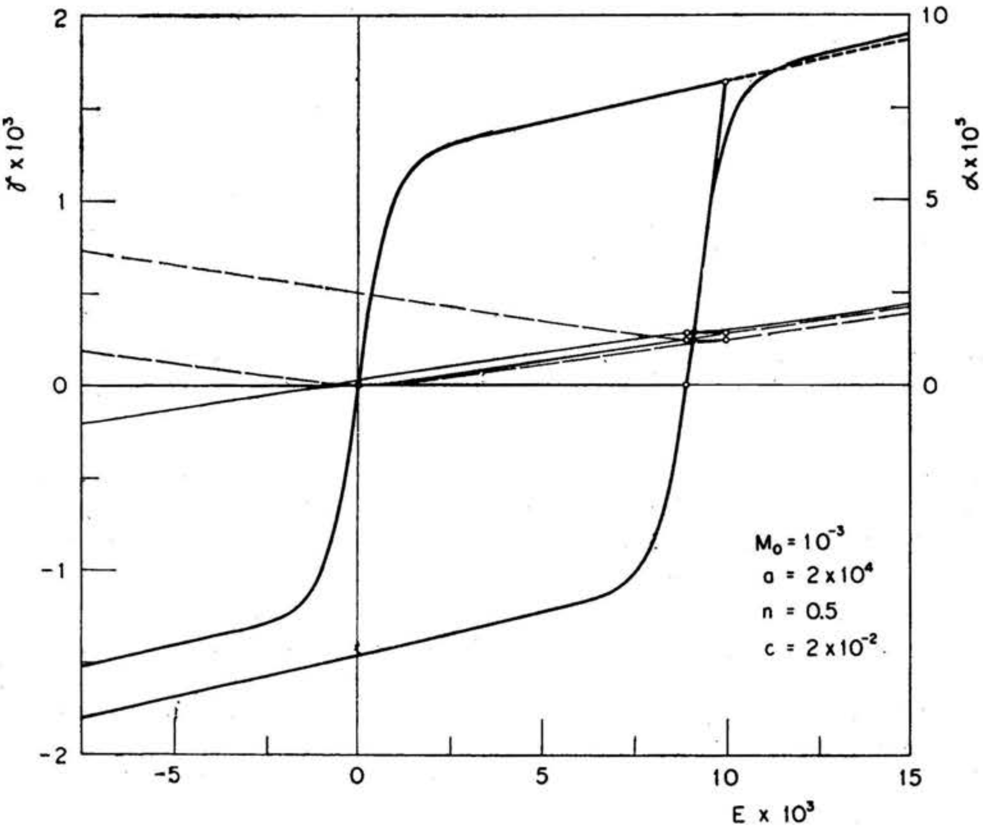


FIG. 2. Stress-strain-internal variables relations for loading-unloading-reloading processes, where the unloading starts at $E = 10^{-2}$.

c)


S x 10³
α x 10³
E x 10³
M₀ = 10⁻³
a = 2 x 10⁴
n = 0.5
c = 0
1
0
-1
-2
5
0
-5
0
5
10
15

d)


S x 10³
γ x 10³
α x 10³
E x 10³
M₀ = 10⁻³
a = 2 x 10⁴
n = 0.5
c = 2 x 10⁻²
2
1
0
-1
-2
10
5
0
-5
0
5
10
15

They correspond, respectively, to $na = 1, 2, 3$ and 4. We can say that the materials with $n > 1$, $n = 1$ and $n < 1$ have increasing, constant and decreasing work-hardening, respectively.

Figure 2 show the stress-strain-internal variables relations for the loading-unloading-reloading processes from the zero initial state. The diagrams in Figs. 2 (a), (b) (c) and (d) refer, respectively, to the perfect plastic material as well as to the materials with translation, isotropic and total work-hardening. The solid bold lines, the solid fine lines and the broken fine lines refer, respectively, to the stress-strain, the translation-strain and the scalar internal variable-strain diagrams. The unloadings are started at $E = 10^{-2}$ and the reloadings are

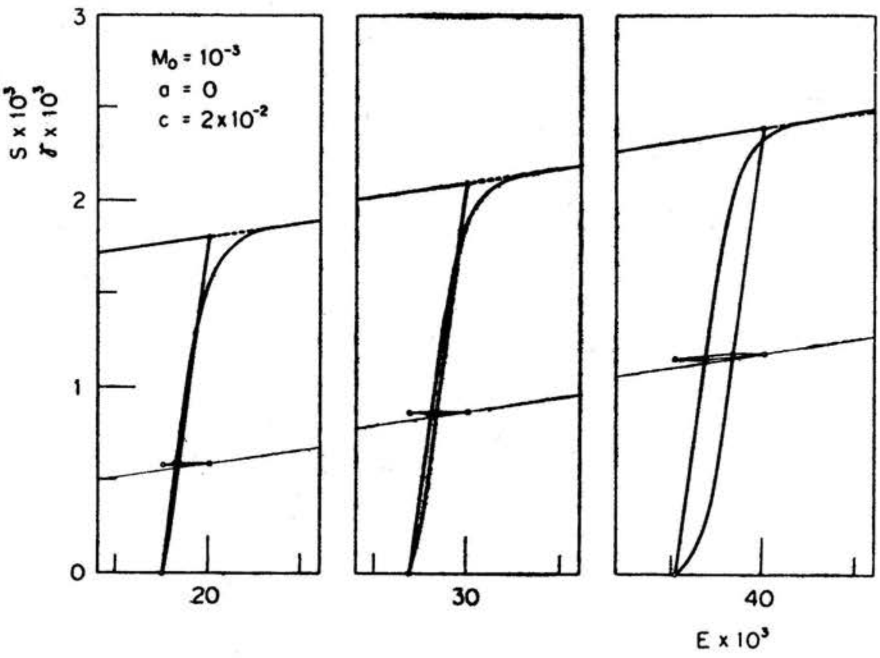
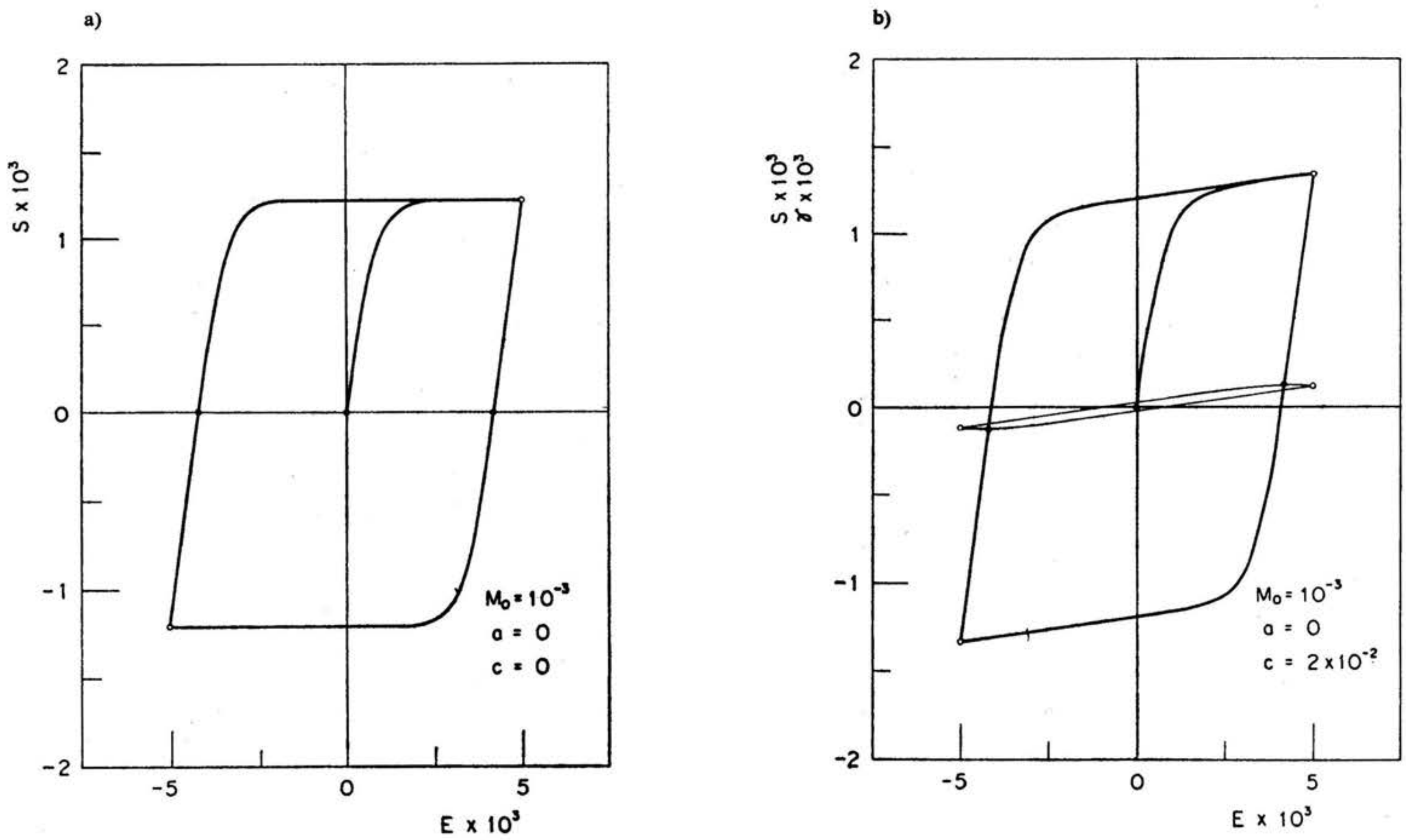


FIG. 3. Stress-strain-internal variables relations for the unloading-reloading processes, where the unloadings start at $E = 2\times10^{-2}$, 3×10^{-2} and 4×10^{-2}.

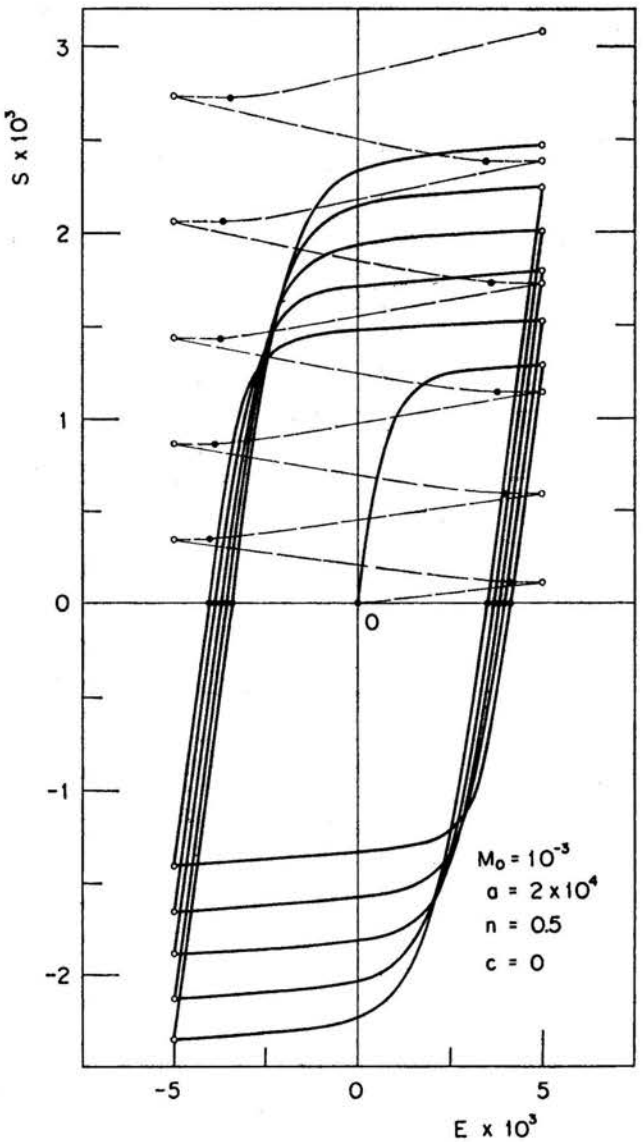
started at $S = 0$ for positive and negative directions. The reloading diagrams show the rounding phenomenon and Figs. 1 (b) and (d) show the remarkable Bauschinger effect.

Figure 3 show three unloading-reloading diagrams started at $E = 2\times10^{-2}$, 3×10^{-2} and 4×10^{-2} for the material with translation work-hardening. It is worth noting that the unloading and reloading paths make loops. Refer to TOKUOKA [22].

Figure 4 show the stress-strain-internal variables relations for the loading-unloading cycles in the limit of strain $\pm5\times10^{-3}$. We can say that the behaviour of the plastic material for this cycle depends largely upon the isotropic work-hardening and, a little, on the translation work-hardening.


a)
S x 10³
2
1
0
-1
-2
-5
0
5
E x 10³
M₀ = 10⁻³
a = 0
c = 0
b)
S x 10³
γ x 10³
2
1
0
-1
-2
-5
0
5
E x 10³
M₀ = 10⁻³
a = 0
c = 2 x 10⁻²

c)



d)

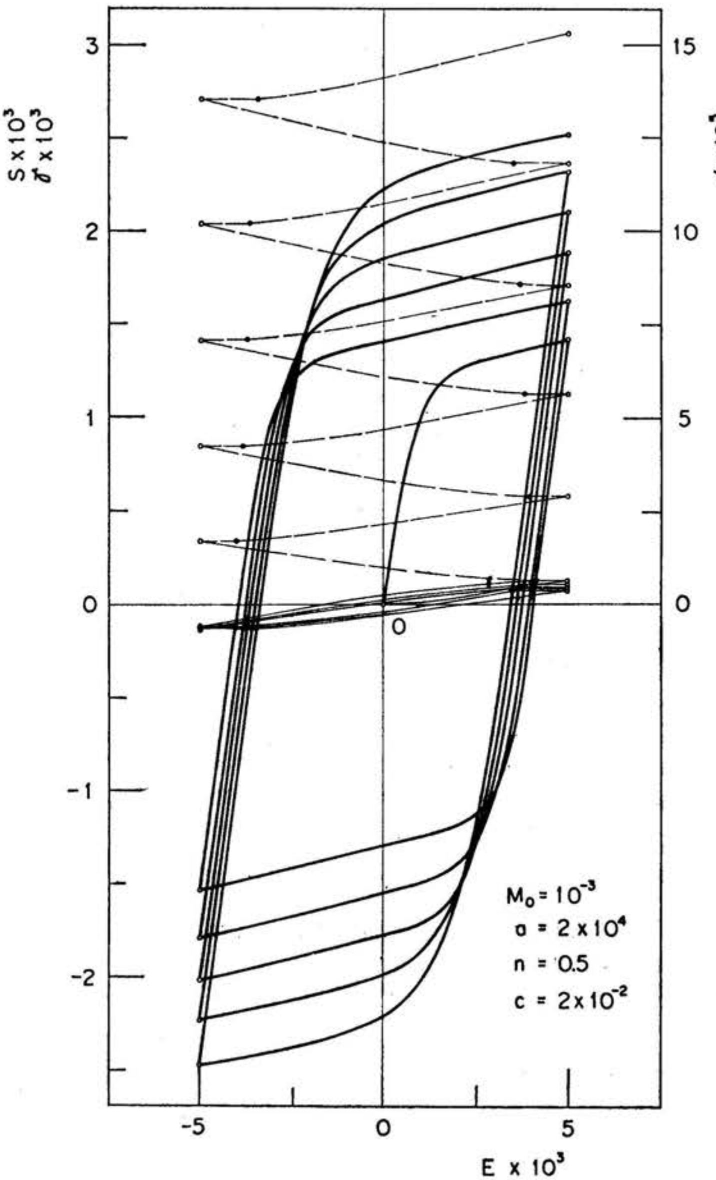


FIG. 4. Stress-strain-internal variables relations for the loading-unloading cycles in the limit of $E = \pm 5 \times 10^{-3}$.

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Received December 2, 1977.