Geometrical and dynamical nonlocality(*)

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THE PRINCIPAL aim of the paper is to clarify the mathematical nature of nonlocality in physical theories and to elucidate the relation between the notions of continua and pseudo-continua. The problem of operators of infinite singular order is also explained on the basis of non-classical continua.

Podstawowym celem pracy jest wyjaśnienie sensu matematycznego nielokalności w teoriach fizycznych oraz związku między pojęciami kontinuum i pseudokontinuum. Wyjaśniono również pojęcie operatorów osobliwych nieskończonego rzędu na podstawie teorii kontinuów niekla-sycznych.

Основной целью работы является определение математического смысла нелокаль-ности в физических как и отношений между континуом и цсевдоконтинуом. На основе теории некласических континуумов дано выяснение понятия особых опе-раторов бесконечного порядка.

1. Introduction

1.1.

CONSIDER a physical theory based on a fundamental equation of the form

$$(1.1) \qquad\qquad Ay = f,$$

where $y = y(x)$ and $f = f(x)$ are functions of $x \in \Omega$, the domain Ω being a certain region in (Euclidean) space. The symbol A denotes an operator which can be nonlinear.

Equation (1.1) can play the role of a governing equation of the theory, or basic consti-tutive relation. As examples we can consider elasticity with y — displacements, f — external forces, or y — deformations, f — stresses, or electrostatics with y — electric field, f — free charge, or y — electric field, f — electric induction.

1.2.

In classical theories the operator A is usually local. It means that Eq. (1.1) states no direct relation between $y(x)$ and $f(x')$ for $x \neq x'$. This colloquial definition of locality can be made more precise in the following way.

Let Y be the domain of A, i.e. the set of functions y for which Ay is defined in the consid-ered theory. To make our statement clear, we shall formulate our definition in two steps.

DEFINITION 1.1. *A linear operator A is local if*

(1.2) $$\text{supp}\, Ay \subset \text{supp}\, y$$

for every $y \in Y$.

DEFINITION 1.2. *An arbitrary (nonlinear) operator A is local if Eq. (1.2) is satisfied and*

(1.3) $$\text{supp}\,[A(y+\varphi) - A(y)] \subset \text{supp}\, \varphi$$

for every $y, \varphi \in Y$.

1.3.

The need to describe the physical reality in a more adequate way gives rise to various modifications of the classical theories. The modified operators do not necessarily fulfill the locality conditions.

1.4.

To define an operator A means (i) to define the domain Y of A and (ii) to specify the action of A in Y.

Hence, to modify an operator A means either (i) to modify the domain Y, or (ii) to modify the action of A, or still both.

This opens two general ways of modifying the classical theories.

The second of them, which can be called *dynamical*, consists in modifying the form of the operator A without making any substantial changes of the domain. In terms of material continua it corresponds to a modification of interactions of material particles.

The first way, the *geometrical* (or kinematical) one, consists in making a substantial modification of the domain. Without changing the physical laws of interaction of particles, the set of admissible configurations (or motions) is controlled.

1.5.

Although the dynamical approach is nowadays most popular, it does not mean that the geometrical one is not interesting. The impression that the physical contents of a theory can not be seriously changed by modifying the set of admissible functions is due to considering not too significant modifications of Y. The pseudo-continuum theory [1, 2] furnishes an example of the contrary. Generally, from the atomistic point of view one can argue that the set of functions admissible in the classical continuum theory is definitely too large. In fact, many unphysical results of this theory are directly due to excessive freedom of the classical continuum in forming singularities, vibrations of very high frequency, etc. Therefore, the idea of forming the domain Y by a substantial *restriction* of the classical resources of functions seems to be promising.

1.6.

The Definitions 1.1 and 1.2 express the intuitive idea of locality of an operator provided that Y contains funtions of arbitrarily small compact support, concentrated around an

arbitrary point $x \in \Omega$. If Y does not contain functions of compact support at all, then there is no difference between local and nonlocal operators, although the conditions (1.2) and (1.3) are trivially satisfied.

Bearing that in mind, we shall introduce the following definitions.

DEFINITION 1.3. *A space Y of functions over a domain Ω is called local if for any $x \in \Omega$ and for any neighbourhood N such that $\Omega \supset N \ni x$, there exists $y \in Y$ which satisfies*

$$(1.4) \qquad\qquad\qquad x \in \mathrm{supp}\, y \subset N^{(1)}.$$

DEFINITION 1.4. *A theory based on a fundamental equation of the form (1.1) is local if Y is a local space and A is a local operator.*

A theory which is not local will be considered to be nonlocal. From Definition 1.4 it follows that there are two basic reasons for the nonlocality of a theory: either nonlocality of the space Y or nonlocality of the operator A. These two cases will be referred to as geometrical and dynamical nonlocality, respectively.

2. The spaces Y

2.1.

To avoid considering too poor or too irregular spaces Y let us make the following assumptions.

Let Ω be an open region in a Euclidean space E, $\overline{\Omega}$ — its closure. Let C^∞ denote the space of functions on E which have continuous derivatives of arbitrary order. In the case of Ω with a boundary, by C^∞ we shall understand the space of restrictions of these functions from E to $\overline{\Omega}$. If it is necessary to indicate the region, we shall write explicitly $C^\infty(\overline{\Omega})$ or $C^\infty(E)$. The same applies to other function spaces, such as C_0^∞ which consists of infinitely differentiable functions of compact support, or C_b^∞ which is composed of bounded C^∞-functions.

2.2.

The spaces Y will be considered as configuration spaces of certain physical systems, although other interpretations are also possible. In this paper we shall not profit much from the conceptual difference between a physical system and its configuration space. Thus we shall treat the corresponding expressions as synonyms.

2.3.

The spaces Y will be assumed to be linear, and endowed with an appropriate topology when necessary.

DEFINITION 2.1. *A system is called classical continuum iff*

$$(2.1) \qquad\qquad\qquad C_0^\infty \subset Y.$$

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(1) The concept of local space introduced here differs from that given by HÖRMANDER [3].

A classical continuum is always geometrically local.

DEFINITION 2.2. *A system is called non-classical continuum if (i) there exists a function* $\varphi \in C_0^\infty$ *which does not belong to Y, and (ii) Y contains all the functions of the form*

(2.2) $e^{ikx} \cdot \varphi(\mathbf{x})$

(or their real counterparts) with arbitrary real k and $\varphi(\mathbf{x}) \in Y$.

As to geometrical locality or nonlocality, both occur in non-classical continua.

DEFINITION 2.3. *A system is called pseudo-continuum if there exists a compact set D in the k-space such that for any* $y \in Y$

(2.3) $\operatorname{supp}\hat{y} \subset D$,

where $\hat{y}$ *denotes the Fourier transform of y (or, more generally, of a certain extension of y from* $\overline{\Omega}$ *to E).*

Pseudo-continua are always nonlocal.

2.4.

We shall not be interested in functions which grow up very fast when $|x| \to \infty$. We shall consider (i) functions of *tempered growth*, i.e. such that there exist C and N (dependent on φ and μ) satisfying the inequality

(2.4) $|\partial^\mu \varphi(\mathbf{x})| \leqslant C(1+|\mathbf{x}|^N)$,

(ii) functions which are bounded together with all their derivatives,

(2.5) $|\partial^\mu \varphi(\mathbf{x})| \leqslant \text{const}$,

where the constant can depend on μ, and (iii) functions of fast decrease, i.e. such that for any μ and N there exists C satisfying the inequality

(2.6) $|\partial^\mu \varphi(\mathbf{x})| \leqslant C(1+|\mathbf{x}|^{-N})$.

3. Construction of non-classical continua

3.1.

In this section we shall describe a spectrum of non-classical continua. In order to concentrate our attention on the basic facts, we shall avoid discussing multidimensional cases, restricting ourselves to one-dimensional E. The q-th order derivative of φ will be denoted by $\varphi^{(q)}$.

3.2.

Let us start from the following observation. Let $\{b_q\}$ be an arbitrary positive sequence,

(3.1) $b_q > 0$ for $q = 0, 1, \ldots$,

and x — an arbitrary point in E. We shall say that a function $\varphi \in C^\infty$ is majorized at x by the sequence $\{b_q\}$ if there exists a constant C such that

(3.2) $|\varphi^{(q)}(x)| \leqslant C b_q$

for any $q \geqslant 0$. With this definition one can assert that for any $\{b_q\}$ there exists a function $\varphi \in C_0^\infty$ which is not majorized by $\{b_q\}$. A proof of this assertion can be given by constructing such a function [5].

Informally speaking the sequence of consecutive derivatives of an infinitely differentiable function (C_0^∞ and, a fortiori, C^∞) can grow up arbitrarily fast.

3.3.

The above observation suggests the idea of constructing non-classical continua by making use of the inequalities (3.2) with appropriate classes of sequences $\{b_q\}$. In general, the constant C in the inequalities (3.2) can be dependent on x. This depednence, although insignificant on compact $\overline{\Omega}$'s, can be made use of for defining the behaviour of admissible functions at infinity.

We shall be interested in functions which grow up not faster than polynomials when $|x| \to \infty$.

3.4.

Which sequences $\{b_q\}$ are interesting from the point of view of constructing non-classical continua?

Let us consider first the $\{b_q\}$'s which grow up more slowly than any power sequence, i.e. for any $B > 0$ there exists C such that

$$(3.3) \qquad\qquad B \leqslant CB^q.$$

Then φ is an entire analytic function such that for any z and B

$$(3.4) \qquad\qquad |\varphi(x+z)| \leqslant Ce^{|B||z|},$$

which means that it is either of the fractional order of growth or of the minimal exponential type [6]. In both cases it follows that all the φ's which are not polynomials have to grow up faster than any polynomial when $x \to \infty$ or $x \to -\infty$ (i.e. at least in one real direction).

Therefore we shall restrict ourselves to sequences $\{b_q\}$ which grow up like power sequences or faster. Any such sequence can be represented in the form

$$(3.5) \qquad\qquad b_q = d_q B^q,$$

where d_q is a non-decreasing sequence.

3.5.

Consider the sequences which satisfy the inequality (3.3) with a finite B. Then, from the inequality (3.4) it follows that φ is an entire analytic function of a finite exponential type [6]. There exist functions of this type and of moderate growth for $|x| \to \infty$. By the Paley-Wiener-Schwartz theorem any such function has the Fourier transform of compact support [3].

Hence sequences $\{b_q\}$ satisfying the inequality (3.3) with a finite B define pseudo-continua in the sense of the Definition 2.3.

3.6.

Consider now the sequences $\{b_q\}$ of faster growth, i.e. of the form (3.5) with increasing d_q,

(3.6)
$$\limsup d_q = \infty.$$

Let

(3.7)
$$\sum_{q=0}^{\infty} c_q z^q$$

where

(3.8)
$$c_q = \frac{1}{q!}\,\varphi^{(q)}$$

is the Taylor expansion of φ at a certain fixed point x. According to basic theorems on analytic functions, the radius of convergence of this series equals

(3.9)
$$R = \liminf |c_q|^{-1/q}.$$

If $R = \infty$, φ is an entire analytic function of the order given by

(3.10)
$$\varrho = \limsup \frac{q\ln q}{\ln \dfrac{1}{|c_q|}}.$$

Appropriate calculations show that if φ satisfies the inequality (3.2) then

(3.11)
$$\ln\frac{1}{R} \leqslant 1 + \limsup\left(\frac{1}{q}\ln b_q - \ln q\right),$$

(3.12)
$$\frac{1}{\varrho} \geqslant 1 - \limsup \frac{\ln b_q}{q\ln q}.$$

By substituting Eq. (3.5) into the inequality (3.12) we obtain

(3.13)
$$\ln\frac{1}{R} \leqslant 1 + B + \limsup\left(\frac{1}{q}\ln d_q - \ln q\right),$$

(3.14)
$$\frac{1}{\varrho} \geqslant 1 - \limsup \frac{\ln d_q}{q\ln q}.$$

These formulae suggest the following choice of $\{d_q\}$'s:

(3.15)
$$\frac{\ln d_q}{q\ln q} = \beta$$

which gives

(3.16)
$$d_q = q^{\beta q}.$$

In order to obtain increasing sequences $\{d_q\}$, we must assume $\beta > 0$.

3.7.

Now we can define the following family of function spaces.

DEFINITION 3.1. *The space $Q^{\beta,B}(E)$ consists of all the C^∞-functions of moderate growth such that*

(3.17)
$$|\varphi^{(q)}(x)| \leqslant Cq^{\beta q}\bar{B}^q$$

for a certain $C = C(x)$ of moderate growth and any $\bar{B} > B$ (i.e. $\wedge \bar{B} \vee C \wedge q$).

Extending the above definition by admitting $\beta = 0$ allows us to include also quasi-continuum spaces $Q^{0,B}$.

By substituting into Definition 3.1 phrases like "of bounded derivatives" or "of fast decrease" in place of the phrase "of moderate growth", one can obtain valid definitions for the coresponding spaces with different behaviour of the functions at infinity.

DEFINITION 3.2. *The space $Q^\beta(E)$ is the union of all spaces $Q^{\beta,B}(E)$.*

DEFINITION 3.3. *The spaces $Q^\beta(\overline{\Omega})$ and $Q^{\beta,B}(\overline{\Omega})$ are composed of corresponding restrictions of functions from $Q^\beta(E)$ and $Q^{\beta,B}(E)$, respectively.*

The spaces $Q^{\beta,B}$ and Q^β will be briefly referred to as Q-spaces.

4. Basic properties of Q-spaces

4.1.

If either $\beta' < \beta$ or $\beta' = \beta$ and $B' < B$, then the following inclusion

$$(4.1) \qquad Q^{\beta',B'} \subset Q^{\beta,B}$$

holds and is proper. It follows from the fact that the space $Q^{\beta,B}$ contains $S^{\beta,B}_{\alpha,A}$, where $S^{\beta,B}_{\alpha,A}$ are the spaces [4] of C^∞-functions satisfying the inequalities

$$(4.2) \qquad |x^k \varphi^{(q)}(x)| \leqslant Cq^{\beta q} \overline{B}^q k^{\alpha k} \overline{A}^k$$

for a certain C and any $\overline{B} > B$, $\overline{A} > A$. The parameters α and A in the relation (4.1) satisfy $\alpha \geqslant 0$, $A > 0$, and otherwise are arbitrary. As the spaces $S^{\beta,B}_{\alpha,A}$ with $\alpha + \beta > 1$ are non-empty, so are all the spaces $Q^{\beta,B}$.

This conclusion can be strengthened a little by taking into account the fact that $S^{\beta,B}_{\alpha,A}$ contain only functions of fast decrease.

4.2.

By putting Eq. (3.16) into Eqs. (3.13) and (3.14) one obtains

$$R = 0 \qquad \text{for} \qquad \beta > 1,$$

$$(4.3) \qquad R \geqslant \frac{1}{eB} \qquad \text{for} \qquad \beta = 1,$$

$$R = \infty \qquad \text{for} \qquad \beta < 1$$

and

$$(4.4) \qquad \varrho \leqslant \frac{1}{1-\beta}.$$

Hence:

if $\beta > 1$, then $Q^{\beta,B}$ — functions are not analytic;

if $\beta = 1$, then $Q^{\beta,B}$ — functions are either analytic with a finite radius of convergence satisfying Eq. (4.3)$_2$, or entire analytic functions of infinite order;

if $\beta < 1$, then $Q^{\beta,B}$ — functions are entire analytic of finite order given by Eq. (4.4). When β varies from 1 to 0, the order ϱ varies from ∞ to 1.

4.3.

Since the functions belonging to $Q^{\beta,B}$ with $\beta \leqslant 1$ are analytic (at least of a finite radius at any x), these spaces do not contain functions of compact support and, in consequence, are not local.

4.4.

Making use of the fact that $S_{0,A}^{\beta,B}$ consists of functions which vanish identically for $|x| > A$ [4], and taking $\alpha = 0$, $\beta > 1$ in Eq. (4.1), one concludes that for $\beta > 1$ any $Q^{\beta,B}$ contains C_0^∞-functions of arbitrarily small supports. Hence, according to Definition 1.1, the spaces $Q^{\beta,B}$ with $\beta > 1$ are local.

5. Basic operations in Q-spaces

5.1.

The spaces $Q^{\beta,B}$ are linear.

5.2.

Consider the following $\varphi \to \psi$ operations:
(i) translation by arbitrary real a

$$(5.1) \qquad\qquad \psi(x) = \varphi(x-a),$$

(ii) differentiation

$$(5.2) \qquad\qquad \psi(x) = \varphi'(x).$$

PROPOSITION 5.1. The operations of translation and differentiation are

$$(5.3) \qquad\qquad Q^{\beta,B} \to Q^{\beta,B}$$

(i.e. defined on $Q^{\beta,B}$ and having values in $Q^{\beta,B}$) for all β, B.

P r o o f. For translation evident, for differentiation given in the Appendix.

5.3.

A function f is called *multiplier* in $Q^{\beta,B}$ if the multiplication operation

$$(5.4) \qquad\qquad \varphi \to f\varphi$$

is the relation (5.3).

PROPOSITION 5.2. Let $\beta' < \beta$ and

$$(5.5) \qquad\qquad f \in Q^{\beta',B}$$

with an arbitrary B'. Then f is a multiplier in $Q^{\beta,B}$.

PROPOSITION 5.3. Let the relation (5.5) hold. Then the multiplication operation (5.4) is

$$(5.6) \qquad\qquad Q^{\beta',B} \to Q^{\beta',B+B'}$$

The proofs are given in the Appendix.

COROLLARY 5.1. Polynomials are multipliers in all spaces $Q^{\beta,B}$.

COROLLARY 5.2. The functions $\exp(ikx)$ with arbitrary real k are multipliers in any space $Q^{\beta,B}$ with $\beta > 0$.

P r o o f. Follows from Proposition 5.2 since the functions $\exp(ikx)$ belong to $Q^{0,k}$.

It follows from this Corollary that all the spaces $Q^{\beta,B}$ with $\beta > 0$ are non-classical continua.

5.4.

Consider a scale transformation operation defined by the equation

(5.7)
$$\psi(x) = \varphi(\lambda x), \quad \lambda > 0.$$

PROPOSITION 5.4. The scale transformation operation (5.7) is

(5.8)
$$Q^{\beta,B} \to Q^{\beta,\lambda B}.$$

COROLLARY 5.3. The scale transformation operation (5.7) is

(5.9)
$$Q^{\beta} \to Q^{\beta}.$$

Hence the spaces Q^{β} are invariant with respect to the scale transformations, while $Q^{\beta,B}$ are not.

6. Concluding remarks

A brief summary of Q-spaces and the corresponding terminology is given in Table 1. The last column indicates another interesting property of non-classical continua: while every differential operator in a classical continuum has to be of finite order, non-classical continua admit linear differential operators of infinite order. These conclusions follow from the following simple considerations.

Table 1

Q-spaces	$\beta = 0$	$\infty > \beta > 1$	$\beta = 1$	$1 > \beta > 0$	$\beta = 0$
terminology	classical	non-classical			
	local		nonlocal		
	continuum				pseudo-continuum
functions admitted	C^{∞}	restricted C^{∞}	analytic	entire analytic	finite exponential type
linear differential operators	finite order	infinite order	—	—	—

Let

$$(6.1) \qquad f(\partial) = \sum_{q}^{\infty} f_q \partial^q$$

be a formal differential operator of (possibly) infinite order. Then the action of Eq. (6.1) on a C^∞-function should be given by

$$(6.2) \qquad f(\partial)\varphi = \sum^{\infty} f_q \varphi^{(q)}.$$

According to the observation formulated in the Subsection 3.2, for any $\{f_q\}$ there exists $\varphi \in C_0^\infty$ such that Eq. (6.2) is divergent, except the case of

$$(6.3) \qquad f_q = 0 \quad \text{for every} \quad q > q_0.$$

On the other hand, all φ's belonging to a non-classical Q-space satisfy Eq. (3.17). In consequence, for any Q-space there exist infinite $\{f_q\}$'s which make Eq. (6.2) convergent.

All admissible differential operators are local, and this locality is essential in local spaces. In nonlocal spaces every linear operator can be equivalently expressed in a differential form of order $\leqslant \infty$.

Besides the main aim, i.e. to understand better the mathematical nature of nonlocality, the results presented here elucidate the relation between continuum and pseudo--continuum [1]. They also allow for a deeper insight into operators of infinite singular order [7], by interpreting them in terms of non-classical continua.

Appendix

1. The p r o o f of Proposition 5.1

Let $\varphi \in Q^{\beta,B}$ and ψ be given by the relation (5.2). Then for any $\varepsilon > 0$

$$|\psi^{(q)}| \leqslant C(q+1)^{\beta(q+1)}(B+\varepsilon)^{q+1} = C'q^{\beta q}(B+\varepsilon)^q(q+1)^\beta \left(1+\frac{1}{q}\right)^{\beta q} \leqslant C''q^{\beta q}(B+2\varepsilon)^q,$$

where

$$C'' = C(B+\varepsilon)e^\beta \sup\left[\left(\frac{B+\varepsilon}{B+2\varepsilon}\right)^q (q+1)^\beta\right].$$

2. The p r o o f s of Propositions 5.2 and 5.3

Let $\varphi \in Q^{\beta,B}$, $\quad f \in Q^{\beta',B'}$. Then for any $\varepsilon, \varepsilon' > 0$

$$|(f\varphi)^{(q)}| \leqslant \sum_{p=0}^{q} \binom{q}{p}|f^{(p)}||\varphi^{(q-p)}| \leqslant CC' \sum_{p=0}^{q} \binom{q}{p} p^{\beta'p}(q-p)^{\beta(q-p)}(B'+\varepsilon')^p(B+\varepsilon)^{q-p}$$

$$\leqslant CC' \sum_{p=0}^{q} \binom{p}{q} q^{\beta'p}(q-p)^{\beta(q-p)}(B'+\varepsilon')^p(B+\varepsilon)^{q-p} \leqslant CC' \sum_{p=0}^{q} \binom{q}{p} q^{\beta q}q^{(\beta'-\beta)p}$$

$$\times (B'+\varepsilon')^p(B+\varepsilon)^{q-p} \leqslant CC'q^{\beta q} \sum_{p=0}^{q} \binom{q}{p}[(B'+\varepsilon')q^{\beta'-\beta}]^p(B+\varepsilon)^{q-p}$$

$$= CC'q^{\beta q}[(B'+\varepsilon')q^{\beta'-\beta}+B+\varepsilon]^q.$$

Now, if $\beta' < \beta$, then

$$|(f\varphi^{(q)})| \leqslant C''q^{\beta q}(B+\varepsilon)^q,$$

where

$$C'' = CC'\sup\left[1+\frac{B'+\varepsilon'}{B+\varepsilon}\,q^{\beta'-\beta}\right]^q,$$

which proves Proposition 5.2.

If $\beta = \beta'$, then

$$|(f\varphi)^{(q)}| \leqslant CC'q^{\beta q}[B+B'+\varepsilon+\varepsilon'],$$

which proves Proposition 5.3.

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