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A

TREATISE

ON

THE DIFFERENTIAL CALCULUS,

AND

ITS APPLICATION TO GEOMETRY:

FOUNDED CHIEFLY ON THE METHOD OF INFINITESIMALS.

BY

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PREFACE.

The following Treatise consists of two parts, in the former of which are discussed the general principles of the Differential Calculus, and theorems which arise out of them; in the latter, the principles and deductions are applied to Geometry, and to the discussion of properties of geometrical quantities, both in plane and in space. Were the writer's system carried out to the full, it would be necessary to add a third part, containing mechanical applications.

It has been the author's object to exhibit the principles in a form less repulsive than is usual with English writers. With this view, many illustrations have been introduced, which may, at first sight, appear extraneous to the matter in hand; as, for instance, in Chap. III., the Theory of Successive Differentiation and the Independent Variable is illustrated by what is in Mechanics the foundation of our means of determining velocity and accelerating force. But it is thought that whatever tends to present to the student the principles in a sensible form, and thereby enables him the better to grasp the matter, is not foreign to the purpose.

The treatise is essentially one of Differentials; it is not a Calculus of derived functions, and wherever the latter have been introduced, it has been only for the purpose of determining differentials, of which they are modified forms. Neither have (with a single exception, where the convergency of the series has been proved) series been introduced, because we have no general means of determining convergency.

As the greater part of the treatise has been delivered, from time to time, in the form of lectures, a colloquial style has been adopted. The writer is not aware that he is indebted, exclusively, to any living English author, for methods which have been inserted; whatever has been extracted from English books has been known and published for so long a time as to have become public property. But to foreign writers he is under many obligations, and especially to M. Cauchy, for almost the whole method of treating one of the most abstruse parts of the science, viz. the theorems of Chap. IV.: and he cannot but acknowledge his debt to Professor De Morgan for much valuable information, obtained from his large treatise; also, to his friend W. Spottiswoode, Esq. B.A., of Balliol College, Oxford, the author is indebted for the latter part of Chap. XVI., on Curvature of Surfaces. He would add a few words on the *method* which has been pursued. It has been his object so to frame the definitions of the technical terms, that they should contain, in germ, the contents of the articles dependent on them; and that the object of the discussion which follows should be to evolve and develope the principles and facts which the definitions import. Such seems, at least to the writer, to be the true and logical method of treating such subjects; and the following brief sketch of the course pursued at the commencement will best explain what is meant.

We begin by defining the subject-matter of the Calculus, and deducing from it such properties as are specially applicable; for on a just conception of these will depend whether we work with mere symbols, or whether our symbols are σημεῖα of real philosophical ideas, which we understand. Thus we are led to consider continuous variables, viz. variables *insensibly growing*, which are combined with other symbols, and form continuous functions. It is of these continuous functions of continuous variables that we are about to treat; our means of doing so is the Differential Calculus, which is "a general method, or system of rules, by which are determined corresponding changes of the variables and functions, when the variations of the variables are small, and the code of laws to which such small quantities are subject." Thus our definition leads us to investigate these small quantities; and the method by which we arrive at them is, to consider the difference between two quantities under two successive states. This requires a research into the theory of such limiting differences; and as they often assume indeterminate forms, involving quantities infinitesimally small, or zeros, we are obliged to investigate the nature of them; which problem presents itself under the form of a question, Are all 0's equal? It will appear that they are not, that they are of relative magnitude; that there are different orders of them; that those of the same order may have a finite ratio to one another; that those of different orders cannot; and that such infinitesimals are subject to the two following laws:

Two finite quantities, which differ from one another by an infinitesimal, may be considered equal.

purpose of determining differentials, of which they are modified forms. Neither have (with a single exception, where the convergency of the series has been proved) series been introduced, because we have no general means of determining convergency.

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Two finite quantities, which differ from one another by an infinitesimal, may be considered equal.

Two infinitesimals of the same order, which differ from one another by an infinitesimal of a higher order, may be considered equal.

With these laws before us, we proceed to solve our problem, which is the determination of the infinitesimal variation of the function due to the infinitesimal variation of the variable; and the form which it most conveniently assumes is that of determining a ratio, which is called the derived function, and which enables us to determine the absolute change of the function due to the change of the variable. Thus, derivation is an operation subservient to that of differentiation: we wish to differentiate, and derivation is a simpler method of enabling us to do so. Thus we find small quantities or infinitesimals; on these we proceed to operate, and to deduce such properties of them as are applicable to questions of Geometry and Physics.

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$$\frac{\mathrm{F}(x_0+h)}{f(x_0+h)}=\frac{\mathrm{F}^n(x_0+\theta h)}{f^n(x_0+\theta h)},$$

$$\frac{\mathrm{F}(x_0+h)-\mathrm{F}(x_0)}{f(x_0+h)-f(x_0)}=\frac{\mathrm{F}^n(x_0+\theta h)}{f^n(x_0+\theta h)},$$

$$\frac{\mathrm{F}(x)}{f(x)}=\frac{\mathrm{F}^n(\theta x)}{f^n(\theta x)},\qquad \frac{\mathrm{F}(x)-\mathrm{F}(0)}{f(x)-f(0)}=\frac{\mathrm{F}^n(\theta x)}{f^n(\theta x)}$$ - - 78

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DIFFERENTIAL CALCULUS.

PRELIMINARY PROPOSITIONS.

I.

If $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots \frac{a_n}{b_n}$ be a series of fractions the numerators of which are of either sign, and *the denominators all of the same sign*, then $\frac{a_1 + a_2 + a_3 + \ldots + a_n}{b_1 + b_2 + b_3 + \ldots + b_n}$ is equal to some quantity less than the greatest, and greater than the least, of the given fractions, that is, to some fraction which is a mean between the greatest and least of the given ones.

The proof of this proposition depends on the fact, if both terms of an inequality are multiplied or divided by a positive quantity, the sign of inequality remains the same, that is, the one that was greater before the multiplication is the greater after; but, if the terms are multiplied by a negative quantity, the sign of the inequality is changed. This is easily shown by an example: e. g. $5 > 2$. If we multiply by $+4$, the sign of the inequality is unchanged, $20 > 8$; but, if we multiply both by a negative quantity, as e. g. -2, the $>$ is changed into a $<$, $-10 < -4$, because -10 is less than -4.

First, let all the denominators in the above fractions be positive; let L be the least and G the greatest of the fractions; then

$$\frac{a_1}{b_1} > \text{L}, < \text{G}$$

$$\frac{a_2}{b_2} > \text{L}, < \text{G}$$

$$\frac{a_3}{b_3} > \text{L}, \ < \text{G}$$

. . . . . .

$$\frac{a_n}{b_n} > \text{L}, \ < \text{G}.$$

Multiply these several inequalities by the positive quantities $b_1, b_2, .. b_n$, the signs are not changed:

$$a_1 > \text{L}b_1, \ < \text{G}b_1$$

$$a_2 > \text{L}b_2, \ < \text{G}b_2$$

. . . .

$$a_n > \text{L}b_n, \ < \text{G}b_n:$$

therefore $(a_1+a_2+ \ldots +a_n) > \text{L} \times (b_1+b_2+b_3+ \ldots +b_n)$,
$< \text{G} \times (b_1+b_2+b_3+ \ldots +b_n)$;

and $$\frac{a_1+a_2+a_3+ \ldots +a_n}{b_1+b_2+b_3+ \ldots +b_n} > \text{L}, \ < \text{G}.$$

Secondly, let $b_1, b_2, b_3, \ldots b_n$ be negative:

$$\frac{a_1}{b_1} > \text{L}, \ < \text{G}$$

$$\frac{a_2}{b_2} > \text{L}, \ < \text{G}$$

. . . . . .

$$\frac{a_n}{b_n} > \text{L}, \ < \text{G}.$$

Multiplying by the negative quantities $b_1, b_2, \ldots b_n$, the signs of inequality are changed:

$$a_1 < \text{L}b_1, \ > \text{G}b_1$$

$$a_2 < \text{L}b_2, \ > \text{G}b_2$$

. . . .

$$a_n < \text{L}b_n, \ > \text{G}b_n;$$

$$\therefore \quad (a_1 + a_2 + \dots + a_n), < \text{L} \times (b_1 + b_2 + \dots + b_n),$$
$$> \text{G} \times (b_1 + b_2 + \dots + b_n):$$

and again, dividing by a negative quantity, the signs of inequality are changed:

$$\text{and} \quad \frac{a_1 + a_2 + a_3 + \dots + a_n}{b_1 + b_2 + b_3 + \dots + b_n} > \text{L}, < \text{G};$$

whence the proposition is proved.

II.

If $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$ are quantities of the same sign, then $\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \dots + \alpha_n a_n$ is equal to $\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n$ multiplied by some quantity less than the greatest and greater than the least of the quantities $a_1, a_2, a_3, \dots a_n$.

Let L be the least and G the greatest of the quantities $a_1, a_2, a_3, \dots a_n$.

$$\therefore \; a_1 \text{ is } > \text{L}, < \text{G}$$
$$a_2 \text{ is } > \text{L}, < \text{G}$$
$$\dots\dots$$
$$a_n \text{ is } > \text{L}, < \text{G}.$$

First, let the quantities $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$ be all positive, then the signs of the above inequalities will not be altered, when the several terms are multiplied as under:

$$\alpha_1 a_1 \text{ is } > \text{L}\alpha_1, < \text{G}\alpha_1$$
$$\alpha_2 a_2 \text{ is } > \text{L}\alpha_2, < \text{G}\alpha_2$$
$$\dots\dots$$
$$\alpha_n a_n \text{ is } > \text{L}\alpha_n, < \text{G}\alpha_n;$$

$$\therefore \quad \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n \text{ is } > \text{L}(\alpha_1 + \alpha_2 + \dots + \alpha_n),$$
$$< \text{G}(\alpha_1 + \alpha_2 + \dots + \alpha_n);$$

$$\therefore \quad \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n = (\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n) \times \text{some mean value of the } a\text{s}.$$

Similarly, if $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$ be all negative, may the same result be arrived at: and therefore the proposition is proved.

III.

If there be a series of *equal* fractions, $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$, ... $\frac{a_n}{b_n}$, then each of these is equal to $\frac{a_1+a_2+a_3+\ \dots\ +a_n}{b_1+b_2+b_3+\ \dots\ +b_n}$; and, if m_1, m_2, m_3,...m_n be any multipliers, to $\frac{m_1a_1+m_2a_2+m_3a_3+\dots+m_na_n}{m_1b_1+m_2b_2+m_3b_3+\dots+m_nb_n}$, and to $\frac{\sqrt{(a_1^2+a_2^2+\ \dots\ +a_n^2)}}{\sqrt{(b_1^2+b_2^2+\ \dots\ +b_n^2)}}$.

For let each fraction $= r$

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\dots\dots=\frac{a_n}{b_n}=r.$$

$$\begin{array}{lll} \therefore\ a_1=b_1r & m_1a_1=m_1b_1r & a_1^2=b_1^2r^2 \\ \quad a_2=b_2r & m_2a_2=m_2b_2r & a_2^2=b_2^2r^2 \\ \quad a_3=b_3r & m_3a_3=m_3b_3r & a_3^2=b_3^2r^2 \\ \quad \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \quad a_n=b_nr & m_na_n=m_nb_nr & a_n^2=b_n^2r^2 \end{array}$$

therefore, by addition and division,

$$r=\frac{a_1+a_2+a_3+\dots\dots+a_n}{b_1+b_2+b_3+\dots\dots+b_n}=\frac{m_1a_1+m_2a_2+\dots\dots+m_na_n}{m_1b_1+m_2b_2+\dots\dots+m_nb_n}$$
$$=\frac{\sqrt{(a_1^2+a_2^2+\dots\dots+a_n^2)}}{\sqrt{(b_1^2+b_2^2+\dots\dots+b_n^2)}}=\frac{a_1}{b_1}=\frac{a_2}{b_2}=\dots\dots=\frac{a_n}{b_n};$$

and the proposition is proved.

IV.

When three unknown quantities are involved in three equations of the following forms, a convenient method of obtaining the value of any one of them in terms of the constants is that given below:

$$a_1x+b_1y+c_1z=d_1 \qquad (1)$$
$$a_2x+b_2y+c_2z=d_2 \qquad (2)$$
$$a_3x+b_3y+c_3z=d_3. \qquad (3)$$

Multiply (1) by λ_1, (2) by λ_2, (3) by λ_3, and add:

$$(a_1 \lambda_1 + a_2 \lambda_2 + a_3 \lambda_3)x + (b_1 \lambda_1 + b_2 \lambda_2 + b_3 \lambda_3) y$$
$$+ (c_1 \lambda_1 + c_2 \lambda_2 + c_3 \lambda_3)z = d_1 \lambda_1 + d_2 \lambda_2 + d_3 \lambda_3.$$

As we have introduced three undetermined quantities, λ_1, λ_2, λ_3, we may make three suppositions respecting them. Leaving one to be determined afterwards, let two be, that the coefficients of y and z shall in the above equation be equal to zero; viz.

$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = 0$$
$$c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 = 0;$$

whence by elimination,

$$\frac{\lambda_1}{b_2 c_3 - c_2 b_3} = \frac{\lambda_2}{b_3 c_1 - c_3 b_1} = \frac{\lambda_3}{b_1 c_2 - c_1 b_2},$$

the last following from the symmetry of the formulæ. Thus it appears that only the ratio of the multipliers has been determined, and therefore there are an indefinite number satisfying the conditions we have made. To put them into the most simple form, let us introduce the third supposition, that each of these ratios be equal to unity; therefore

$$\lambda_1 = b_2 c_3 - c_2 b_3$$
$$\lambda_2 = b_3 c_1 - c_3 b_1$$
$$\lambda_3 = b_1 c_2 - c_1 b_2;$$

and therefore

$$x = \frac{d_1(b_2 c_3 - c_2 b_3) + d_2(b_3 c_1 - c_3 b_1) + d_3(b_1 c_2 - c_1 b_2)}{a_1(b_2 c_3 - c_2 b_3) + a_2(b_3 c_1 - c_3 b_1) + a_3(b_1 c_2 - c_1 b_2)},$$

and similar values for y and z.

This method of elimination is generally known by the name of Lagrange's rule of cross multiplication; the mode by which the forms of the multipliers have been determined is called the method of indeterminate multipliers, on which subject more will be said in Chapter VII.

PART I.

Analytical Investigations.

CHAPTER I.

GENERAL PRINCIPLES, AND EXPLANATION OF TERMS.

1.] THE quantities which are used in the following treatise, and which are the subject matter of it, are of two kinds, variable and constant.

A *constant* quantity is one which we suppose to have the same determinate value during a given operation, although in another operation, or under another mode of considering it, it may be supposed to vary.

A *variable* quantity is one which we suppose capable of receiving values different from one another; such a quantity may vary in two ways, either continuously or discontinuously.

A quantity varies *continuously* when it passes from one value to another only by going through all intermediate values; that is, it changes gradually, not "per saltus," to use the language of the *Principia;* and it may consequently receive any intermediate value at pleasure: but a quantity varies *discontinuously*, when it passes abruptly from some one value to another, without going through all the intermediate values.

Thus, for instance, if we consider a circle, and pass along the curve, we do so continuously; for we cannot go from one point to another without going through all the intermediate points. So again, if a body has moved from one position to another, it has been in all intermediate positions; it has not passed abruptly from one place to another, but it has occupied a series of places between them, and has moved along a certain determinate and continuous line: but if we consider a variable quantity of such a nature as to admit only of values differing one from another by certain determinate quantities, and not admitting of values intermediate to these, then such a variable is discontinuous; as, for instance, if a variable quantity admitted only of values corresponding to the integral numbers, 1, 2, 3, 4, &c., it is discontinuous.

This may be thus illustrated. If we consider only the parts of the paths which meet the surface of the earth, the line generated by the crawling motion of a worm is a continuous variable, and that marked by the hopping motion of a frog is a discontinuous one.

It is of continuous variables only that we shall speak in the following treatise, and it is plain that such variables may increase or decrease by very small quantities: in this remark lies the germ of the calculus.

2.] When one or more of such variables and constants are combined in an analytical expression, then that expression is said to be a *function* of such variables: if one variable quantity is involved in the expression, such expression is said to be a function of one variable; if two variable quantities are involved, a function of two variables; and so on. Thus, e^{ax}, $\log(a+bx)$, $\surd(a^2-x^2)$, are functions of one variable, x; e^{ax+by}, $\sin(ax+by)$, are functions of two variables, x and y; $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$ is a function of three variables, x, y, z, and so on. Functions are designated by the symbols F, f, ϕ, ψ. Thus $f(x)$ means a function of one variable, x, combined or not, as the case may be, with constants; $F(x^2)$ means a function of x^2; $\phi(x, y)$ symbolises a function of two variables, $\psi(x, y, z)$ a function of three variables, and so on. These letters are the general types or symbols of the various forms in which the variables can be combined; they are the analytical symbols of the laws by which the variables are related; thus $\surd(a^2-x^2)$, $\sin x$, e^{ax}, would all be symbolised by $f(x)$; $\log(x+y)$ by $f(x, y)$.

3.] Functions are divided into two classes, *explicit* and *implicit*. If by any artifice or operation, as, for instance, by an algebraical resolution of the expression, one variable can be expressed in terms of all the others, then this is said to be an explicit function of them; but if the equation be not solved, and the variables remain involved in one expression, then the function is said to be implicit. Thus, for instance, $y=\surd(a^2-x^2)$, $y=\sin x$, each of which is of the form $y=f(x)$, are explicit functions of x; but $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is of the form $\text{F}(x,y)=c$ (c being a constant), is an implicit function of two variables: so again, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is an implicit function of three variables of the form $\text{F}(x,y,z)=c$; whereas $z=c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$, which is of the form $z=f(x,y)$, is an explicit function of two variables. Implicit functions are often expressed in the form $u=\text{F}(x,y,z,)=c$ or 0, as the case may be.

Functions have again been divided into two classes, *algebraical* and *transcendental:* the former being those functions which involve the operations of addition, subtraction, multiplication, division, involution, and evolution; the latter where the operations symbolised are such as e^x, $\log_e x$, $\sin x$, $\sec x$. This, however, is a division not necessary to our present purpose.

Functions, again, may be *simple* or *compound*, that is, according as one or several operations (the results of which are the functions in question) are involved. Thus, $y=\sin x$, $y=\log_a x$ are simple functions of x; but $y=\log\sin x$, $y=e^{\tan ax}$ are compound functions.

It is necessary to observe, that, if two functions are represented by the same functional symbol, they are formed in the same manner by means of the variables which they involve. Thus, if $f(x)=\sin x$, $f(y)=\sin y$; if $f(x)=e^{bx}$, $f(y)=e^{by}$.

4.] Functions may be either *continuous* or *discontinuous*. A continuous function is subject to the two following conditions:

1st. As the variable gradually changes, the function must gradually change.

2d. The law symbolised by the functional character must not abruptly change.

When these two conditions are not satisfied, the function is discontinuous.

Thus, for instance, both conditions are fulfilled in the functions

$$\left.\begin{array}{l} y=ax+b \\ y=\sin x \end{array}\right\}$$; in which, as the variable x changes, the value of

the function also changes, but it changes gradually, and there is no abrupt passage from one value to another; neither does the law symbolised by the functional character change, it remains the same: but, if the function were such as to express a curve of the form in the diagram fig. 1., so that BA should be a continuous curve drawn after some determinate law, but at A the law suddenly should change, and the curve, from being, say, a circle, become a straight line, then neither of the above conditions is satisfied, and the function is discontinuous. A is called a point of discontinuity. As an instance of a function of this description the following may be mentioned. It may easily be proved that

$$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots \text{ad infin.} = -\frac{\sin\left(\alpha - \frac{\beta}{2}\right)}{2 \sin \frac{\beta}{2}}$$

Suppose that $\alpha = \frac{\beta}{2}$, then the series becomes

$$\cos \alpha + \cos 3\alpha + \cos 5\alpha + \dots \text{ad infin.} = -\frac{0}{2 \sin \alpha}:$$

but if α = any multiple of π, then the sum of the series assumes the indeterminate form $\frac{0}{0}$.* Hence we have this remarkable result, each term of the series varies continuously with α, but the sum of the series varies discontinuously, being always zero, except when α passes through some multiple of π, when the sum of the series suddenly and abruptly becomes $\frac{0}{0}$; i. e. some unknown or indeterminate quantity. On this series see some more remarks in page 18. of Mr. O'Brien's *Mathematical Tracts*, to whom I am indebted for the above illustration.

* That $\frac{0}{0}$ is indeterminate, as far as the expression shows, is hence apparent; it is that quantity which is equal to 0, after it has been multiplied by 0; and since every quantity multiplied by 0 is equal to 0; therefore $\frac{0}{0}$ may, *a priori*, be *any* quantity. It does not, however, follow that any quantity will satisfy the conditions of the particular problem under consideration; hence the actual value of $\frac{0}{0}$ may possibly be determined: but of this more hereafter, when we come to treat generally of such indeterminate expressions.

We proceed, then, to consider the properties of continuous functions of continuous variables, of which only we shall treat; and, if discontinuous functions are introduced, we shall consider them only for those values of the variables for which they are continuous. Let us first consider the simple case of an explicit function of one variable, viz. $y=f(x)$.

The law of continuity expressed in the sentence "non agit per saltus" implies that x may vary by quantities as small as we please; and the law of the continuity of the function implies that the variation in the function due to the small variation of the variable is continuous. As, for instance, if $y=f(x)$ were the equation to a plane curve, we might pass from one point to the next consecutive point on it, and such two points would be those through which a tangent would pass; or, if we consider a curve to be generated by a moving point, the motion being regulated by some law, and being carried on during a finite time, and the time to be resolved into very short instants, then the space passed over during one of these instants is the small increase in the length of the curve, and the projection of this small quantity on the axis of x is the quantity by which we consider x to have changed: it is plain, then, that the variation of y or $f(x)$ due to the small variation in the value of x will, in general, be continuous. The same will be true of functions of any number of variables.

5.] The Differential Calculus is a general method or system of rules by which are determined the corresponding changes in the variables and functions, when the variations of the variables are small; and the code of laws to which such small quantities are subject, and conformably to which they may be applied to questions of geometry and physics.

The symbolisation we employ is as follows:

Δ is the character which indicates a finite augment of the variable of the function; thus, Δx symbolises the finite augment or increment that x receives, and Δy the corresponding finite change in the value of y or $f(x)$, viz.

$$\Delta y=f(x+\Delta x)-f(x):$$

this will in general be finite also.* And, when these augments

* Δx may be negative; in which case it might, perhaps, be more properly called a decrement; but in the following treatise we shall use the words augments and increments to express the variations in the values of the variables, whether such variations cause them to increase or decrease.

become very small, let d be the character to symbolise them, so that dx and dy are the corresponding small variations in x and y; hence we have

$$dy = f(x+dx) - f(x):$$

Δ being the symbol for difference, and d for differential or small difference. Similarly, if we are considering a function of several variables, as e. g.

$$u = \mathrm{F}(x\, y\, z\, \ldots),$$

we shall use the characters Δu, Δx, Δy, Δz ..., to symbolise the several changes in the functions and the variables when they are finite; and du, dx, dy, dz ..., when the changes are very small.

6.] The first object, then, of the Calculus is to determine

$$dy = f(x+dx) - f(x),$$

when dx is very small; and the most convenient method of doing this is to determine the ratio of the change in $f(x)$ to the change in x, when x varies by a very small quantity, that is, to determine $\frac{d\,f(x)}{dx}$ when dx approaches to zero, or, in other words, differs from zero by a quantity less than any assignable quantity.

Since $\frac{df(x)}{dx} = \frac{f(x+dx)-f(x)}{dx}$, it is plain that as dx becomes very small, this quantity approaches more and more nearly to the form $\frac{0}{0}$.

7.] The quantity towards which any expression converges for certain values of the variable or variables on which it depends is called its *limit*. Thus the limit of $\frac{1}{1+x}$ is 1, when $x = 0$; that is, although for every value of x greater than 0 it is less than 1, yet the nearer x approaches to 0, the less becomes the difference between $\frac{1}{1+x}$ and 1: so again, as x increases, the quantity becomes less and less; and finally, when x becomes very large, the quantity is very small; and when x is greater than any finite quantity, that is when x is infinite, the fraction is less than any finite quantity, and assumes the value zero or 0: so again, as x

approaches to -1, the difference between the fraction and $\frac{1}{0}$ is less than any assignable quantity: or again, to take another instance, the limit of $\tan x$ is $-\frac{1}{0}$ when $x=-90°$, and 0 when $x=0$, and $+\frac{1}{0}$ when $x=90°$. Sometimes the limiting value of an expression assumes an indeterminate form: as, e. g., the fraction $\frac{a^2-x^2}{a-x}$, when $x=a$, becomes $\frac{0}{0}$ (of which, however, the real value is $2a$, as might be found by actual division); so, again, the value of $\frac{f(x+dx)-f(x)}{dx}$ approaches to $\frac{0}{0}$, as dx approximates to its limit 0; another indeterminate form is $1^{\frac{1}{0}}$, as, for instance, the value of $(1+x)^{\frac{1}{x}}$, when x approaches to 0. A fraction, it appears then, assumes the form $\frac{0}{0}$ from its numerator and denominator becoming 0 simultaneously. A question then arises, are all 0s equal? because, if they are, the above fraction $=1$; or are they commensurable, so as to have a finite ratio? because, if so, $\frac{df(x)}{dx}$ may be a finite quantity: on this subject we speak as follows.

8.] If any quantity be divided into a number of parts, equal or not equal, as the case may be, the larger the number of parts, the smaller is each part; and if the number of them becomes infinitely great, then each part becomes infinitely small: and the nearer the number is to infinity, the nearer does each part approach to what we may call the zero of its kind. When a finite quantity is thus resolved, then each part is called an *infinitesimal;* the word infinitesimal implying a relative term which imports the number of these infinitesimals which are required to make the finite quantity, which relative term we call *infinity:* and the relation of the terms is, that *the* particular infinity of *the* particular infinitesimals must be added together to make up the finite quantity. Thus, then, we may say that a finite quantity is an infinitesimal infinitely quantupled; or that infinity is the number of infinitesimals into which a finite quantity has been divided. It is hence plain, also, that these terms are the reciprocals of each other; the greater the number of parts, the less is the infinitesimal; and the less the infinitesimal, the greater the infinity. Again, in the same manner as we conceive any magnitude to be thus resolved into a very large number of very small parts, may we conceive that each of these parts admits of a similar resolution: we may divide each into such a large number of such minute parts, that an infinite number of them would be required to be added together to make up the whole; so, again, may we conceive each of these to be subdivided. Performing similar processes on these parts successively, we arrive at different orders of infinitesimals, and, of course, of infinities. One of the great objects of the Calculus is to compare these, and on this subject more will be said hereafter. It is, however, plain, that under this conception all infinitesimals are not equal, nor are all infinities; neither have all infinitesimals a finite ratio to each other. To illustrate what has been said, let us consider the following convergent series in geometrical progression:

$$\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}\dots \text{ ad infin.}$$

We know by simple division that the sum of this series is $\frac{1}{9}$.

There are an infinite number of terms, and yet the sum of them is finite; why is this? It is because the terms are becoming less and less; that is, are becoming such small quantities, or infinitesimals, that an infinite number of them are required to be added together to make the finite sum. The last terms of the series are becoming inappreciably small; and, if we were to neglect a finite number of them, no sensible error would be caused in the sum of the series. This, then, illustrates what we have said, that a finite quantity may be resolved into an infinite number of small parts, each of which is called an infinitesimal, and is so small that nothing short of an infinite number will produce an error appreciable in reference to a finite quantity. In the same way, also, it is conceivable that a finite quantity may be but an infinitesimal part of some magnitude infinitely greater than itself; as, for instance, twice the earth's radius may be an infinitesimal compared with the distance of any fixed star of which the parallax has not been discovered. More, however, will be said on this subject, and on the geometrical conception of it, in a future chapter. Now, although the form which these quantities assume is 0, it is ever to be borne in mind that they are not absolute zeros, they are quantities subject to the laws of Algebra, as any other symbolised quantities are, not creatures of the mathematician's imagination (see Poisson, *Traité de Mécanique*, vol. i. edit. 2de), but real quantities, and to be dealt with as such. Whenever we consider the variations of a quantity or function subject to the law of continuity, and consider it in two immediately successive states, the difference between the two successive values corresponding to these states is an infinitesimal of the kind we are considering, and is called a differential; and these differentials are, of course, so small that it requires an infinite number of them to make a finite quantity: and successive orders of differentials may be obtained in the same way that orders of infinitesimals are obtained; we may resolve a differential, small though it be, into an infinite number of smaller quantities, and each of these again in a similar manner.

If, then, we have infinitesimals, such that one is twice or three times or half another, that is, such that they have a finite ratio to each other, then they would be of the same order: but if there are infinitesimals bearing such a relation to other infinitesimals that an infinity of the former must be summed to constitute a quantity bearing a finite ratio to the latter, these would be infinitesimals of different orders; and the object of a future chapter will be to determine the relative orders of infinitesimals. Hence it appears that there will be a scale of infinitesimals and infinities in regular order, such that an infinitesimal of the $(n-1)^{\text{th}}$ order must be infinitely subdivided to produce one of the n^{th} order, and infinitely quantupled to produce one of the $(n-2)^{\text{th}}$ order. Infinitesimals of each order, then, in the following scale, bear such a relation to those on either side of it, that they are infinitesimal parts of the one, and the aggregate of an infinity of the other. Assuming that what are commonly called finite quantities are infinitesimals of the order 0, as we shall by this means have a fixed standard to start from, and symbolising infinitesimals and infinities by the powers of 0, which is the form such quantities assume, we have the following scale:

$\ldots 0^n, 0^{n-1}, \ldots 0^3, 0^2, 0^1, 0^0, 0^{-1}, 0^{-2}, \ldots 0^{-n}, 0^{-(n+1)}, \ldots$

and which are subject to the following laws :

I. Two finite quantities, which differ from one another by an infinitesimal of any order, are considered as equal, because it will require an infinite number of infinitesimals to constitute a finite quantity (the infinity being of the same order as the infinitesimal); and, when this infinite number is taken, the quantity is no longer an infinitesimal, which is contrary to our hypothesis. Hence, then, *pari ratione*,

II. Two infinitesimals of the same order, which differ from one another by an infinitesimal of a higher order, are considered as equal.

9.] The form, then, which these two rules practically assume is this. In any expression involving finite quantities and infinitesimals (always bearing in mind that the coefficients of such infinitesimals are not infinite), the latter may be neglected, and the results, affecting only the finite quantities, will be rigorously exact; the same will, of course, be true of infinitesimals of any order. Now we are not required to determine of what order absolutely any infinitesimal is; only to assure ourselves what quantities may be neglected in the expressions without affecting the truth of the result: this we can do by dividing through by an infinitesimal of any order, and then discarding those terms which are of the form 0, as in the following example. Suppose β, α, i to be infinitesimals, and A to be a finite coefficient, and

$$\beta = \alpha(\mathrm{A} + i),$$

$$\therefore \quad \frac{\beta}{\alpha} = \mathrm{A} + i.$$

The last term may be neglected, as being an infinitesimal; A being a finite quantity, and $\frac{\beta}{\alpha}$ being of the indeterminate form $\frac{0}{0}$, which may be a finite quantity, and therefore is not to be neglected; whence we have

$$\frac{\beta}{\alpha} = \mathrm{A}.$$

It is plain, then, that infinitesimals of a higher order may be neglected without affecting the truth of the results we arrive at; and it is plain, also, that all infinitesimals of the same order need not be equal; they may have a finite ratio: and therefore, when the limit of any expression is of the form $\frac{0}{0}$, if the infinitesimal in the numerator is of a higher order than that in the denominator, the result is an infinitesimal; if of the same order, the result is a finite quantity; if of a lower order, the result is infinity.

In considering, then, the corresponding small variations of the function and the variable, in the case of the simple explicit continuous function

$$y=f(x),$$

although $\frac{f(x+dx)-f(x)}{dx}$, which is equal to $\frac{d\,.f(x)}{dx}$, when $dx=0$, assumes the indeterminate form $\frac{0}{0}$, it will in general be a finite quantity, which if we can determine, we shall know the absolute change in the function due to the change in the variable.

Let us symbolise this ratio, when dx is very small, by $f'(x)$, so that we have

$$\Delta y=f(x+\Delta x)-f(x),$$

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)+\text{R};^{*}$$

R being a small quantity which $=0$, when Δx becomes dx, whence

$$\frac{dy}{dx}=f'(x),$$

$$\text{and} \qquad \therefore \quad dy=f'(x)\,.\,dx.$$

$f'(x)$ is called the *derived function*, and is so related to x and $f(x)$, that, when we know it, we have determined the absolute change in $f(x)$ due to the change of x. It is also called the *differential coefficient*, being the coefficient of dx in the equation $d\,.f(x)=f'(x)\,.\,dx$.

* When Δx is not small, R has generally some definite value; the value of which we shall determine in a subsequent chapter.

10.] The operation by which this derived function is determined is called *derivation*, and that by which the differential of the function is determined is called *differentiation*; the immediate work of the Differential Calculus is to carry out these operations. In the following pages we shall determine whichever of the two is most convenient, but for the most part we shall differentiate; because differentials are in themselves more tangible, and because the determination of the variation of a function of several variables depends on them.

11.] The problems which are proposed for solution are such as the following:

1st. Suppose we had given $y=x^2$; it is required to determine the change of value of y or x^2, due to a small change in the value of x.

Let x represent a straight line, then x^2 or y represents a square, of which $x=$ AP is a side; see fig. 2. Suppose the side x to be increased by a quantity $\Delta x=$ PQ, then from the figure it is plain that the square is increased by the rectangles DB, BQ, and the square BR, the values of which are $x \times \Delta x$, $x \times \Delta x$, $(\Delta x)^2$, whence

$$\Delta y = 2x\,\Delta x + (\Delta x)^2.$$

Suppose that PQ is very small, so that Δy, Δx become respectively dy, dx, then

$$dy = 2x\,dx + dx^2,$$

$$\therefore \; \frac{dy}{dx} = 2x + dx;$$

and as dx is very small in comparison with x, i. e. as dx is an infinitesimal when x is a finite quantity, dx may, in accordance with the laws at the end of Article 8., be neglected, and we have

$$\frac{dy}{dx} = 2x,$$

$$\therefore \; dy = d\,.\,x^2 = 2x\,dx,$$

the geometrical interpretation of which process is this: $x\,dx$ symbolises approximately a straight line, of which the length is x, and the breadth, if one may so speak, is dx; but dx^2 represents a square whose side is dx, and as this is an infinitesimal, it represents a point, and as it will require an infinity of such to make a straight line, and as the coefficient of dx is not infinity, we may neglect it; that is, in calculating the enlargement of the square due to the enlargement of a side, we take account of the infinitely narrow rectangles which adjoin the sides, but neglect the small point which is required to complete the square, and which is situated at one of the angles, as at B, and no appreciable error is committed by our so doing. Or if we introduce the idea of motion, the enlargement of the square is due to the moving forwards of the two sides PB and CB, and the rectangles by which the square is increased are the several spaces passed over by the sides, which are the spaces contained between the lines before and after the motion; and as the spaces through which the lines have passed are very small, the lines being considered to be in two immediately *successive* positions, the small element at B becomes a point, and, as we have not an infinity of such points, the accuracy of our result is not affected if we neglect this small quantity; and therefore, again, the increase of the square due to the increase of the side is $2x\,dx$. This is algebraically solved as follows:

$$y = x^2, \quad \therefore \quad y + \Delta y = (x + \Delta x)^2,$$

$$\therefore \text{ by subtraction} \quad \Delta y = (x + \Delta x)^2 - x^2$$

$$= 2x\,\Delta x + \Delta x^2,$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = 2x + \Delta x;$$

and taking differentials instead of differences, we have

$$\frac{dy}{dx} = 2x,$$

$$\therefore \quad dy = d\,.\,x^2 = 2x\,dx.$$

2d. As a second example, let the given function be

$$y = \sin x,$$

x being the arc of a circle whose radius $= 1$; it is required to find the small variation in the value of the sine due to a small variation of the arc. See fig. 3.

Let arc $\text{AP} = x$, $\quad \therefore \quad \text{MP} = \sin x = y$,

arc $\text{PQ} = \Delta x$, $\quad \text{NQ} = \sin(x + \Delta x) = y + \Delta y$;

$\therefore$ by subtraction $\quad \Delta y = \Delta\,.\,\sin x = \text{NQ} - \text{MP} = \text{QR}.$

Now, when the increment of the arc PQ is very small, PQ is a straight line, becoming, in that case, the line drawn between two points in the circle which are indefinitely near to each other, and then Δx becomes dx, and the angle QPC is a right angle.

$$\therefore \text{ QR} = \text{PQ} \sin \text{QPR} = \text{PQ} \cos \text{RPC} = \text{PQ} \cos \text{PCA};$$

$$\therefore \text{ QR} = \Delta x \cos x;$$

$$\therefore \Delta . \sin x = \Delta x \cos x,$$

$$\text{and} \qquad d . \sin x = \cos x \, dx;$$

$$\therefore \frac{d . \sin x}{dx} = \cos x;$$

whence it appears that the differential of $\sin x$ is $\cos x \, dx$, and the derived function is $\cos x$.

3d. As a third example, let us take the following. The tangent to a curve being defined to be the straight line which passes through two points in the curve which are indefinitely near to each other, it is required to find the equation to a tangent to a circle.

Let the equation to the circle be

$$y^2 + x^2 = a^2;$$

and let η and ξ be the current co-ordinates to the tangent, and $(x \, y)$ $(x' \, y')$ the co-ordinates to the two points through which the line passes; then the equation to the line is

$$\eta - y = \frac{y' - y}{x' - x} (\xi - x):$$

but when the two points are indefinitely near to each other, $y' - y$ and $x' - x$ become infinitesimals, and their ratio assumes the indeterminate form $\frac{0}{0}$, the value of which we can thus determine:

$$y'^2 + x'^2 = a^2,$$

$$y^2 + x^2 = a^2,$$

$$\therefore \quad y'^2 - y^2 + x'^2 - x^2 = 0;$$

$$(y' - y)(y' + y) + (x' - x)(x' + x) = 0,$$

$$\therefore \quad \frac{y' - y}{x' - x} = - \frac{x' + x}{y' + y}:$$

and therefore, when $y'=y$ and $x'=x$,

$$\frac{y'-y}{x'-x} = -\frac{x}{y};$$

and the equation to the tangent becomes

$$\eta - y = -\frac{x}{y}(\xi - x),$$

which, after reduction, becomes

$$\eta\, y + \xi\, x = a^2.$$

Thus it appears that although the two points in the circle through which the tangent passes are indefinitely near to one another, the distance between them being an infinitesimal, and therefore dy and dx the projections of this distance on the axes of co-ordinates being generally infinitesimals also, yet these have a finite ratio, and this ratio being determined we are enabled to draw a tangent to the circle.

12.] We proceed, then, to the construction of rules for the determination of such quantities from general forms of functions, but before doing so it is necessary to evaluate algebraically the two following functions of x, which, when $x=0$, assume indeterminate forms.

LEMMA I. The Evaluation of $(1+x)^{\frac{1}{x}}$ when $x = 0$.

By the Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3.}x^3 + \&c.$$

Let $n = \dfrac{1}{x}$

$$(1+x)^{\frac{1}{x}} = 1 + \frac{1}{x}x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{1.2}x^2 + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{1.2.3}x^3 + \&c.$$

$$= 1 + 1 + \frac{1-x}{1.2} + \frac{(1-x)(1-2x)}{1.2.3} + \&c.$$

Suppose that x were some small positive fractional number, it is plain that each factor in the numerators of the several terms of the series is less than 1, and therefore no term being negative, the whole series is greater than its first two terms, that is, is greater than 2; and since

$$3 = 1 + \frac{1}{1 - \frac{1}{2}} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \&c.$$

each term of which after the second being greater than the corresponding term in the series above, the whole series is greater; and therefore, when x is a small positive fractional number, $(1+x)^{\frac{1}{x}}$ is equal to some number greater than 2 and less than 3.

Let $x = 0$, and we have

$$(1+0)^{\frac{1}{0}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \&c.$$

which must be summed arithmetically as follows:

$$\begin{array}{rcl}
1 & = & 1\cdot \\
1 & = & 1\cdot0000000 \\
\frac{1}{1.2} & = & \cdot5000000 \\
\frac{1}{1.2.3} & = & \cdot1666666 \\
\frac{1}{1.2 \ldots 3.4} & = & \cdot0416666 \\
\frac{1}{1.2 \ldots 4.5} & = & \cdot0083333 \\
\frac{1}{1.2 \ldots 5.6} & = & \cdot0013888 \\
\frac{1}{1.2 \ldots 6.7} & = & \cdot0001984 \\
\frac{1}{1.2 \ldots 7.8} & = & \cdot0000248 \\
\frac{1}{1.2 \ldots 8.9} & = & \cdot0000027 \\
\frac{1}{1.2 .. 9.10} & = & \cdot0000002 \\
\hline
1 + 1 + \frac{1}{1.2} + \&c. & = & 2.7182818 \&c.
\end{array}$$

the correct value to seven places of decimals. This arithmetical quantity, which is not commensurable with any digit of the common scale of notation, and which is the base of the Napierian logarithms, is symbolised by e; so that we have

$$(1+0)^{\frac{1}{0}} = 2 \cdot 7182818$$
$$= e:$$

and, whenever we meet with e, it is to be borne in mind that it is the symbol for this numerical quantity.

If x were a small negative fractional quantity, then $1+x$ is a positive quantity less than 1; let $1+x = \frac{1}{1+z}$, where z is a small positive quantity less than 1, and becoming 0 at the same time with x; then

$$(1+x)^{\frac{1}{x}} = \left(\frac{1}{1+z}\right)^{-\frac{1+z}{z}} = (1+z)^{\frac{1+z}{z}} = \left\{(1+z)^{\frac{1}{z}}\right\}^{1+z}$$

which, when $z=0$, becomes e.

Hence we conclude that when x is a small positive or negative quantity, approximating to 0, $(1+x)^{\frac{1}{x}}$ approximates to the value e, that is, differs from e by a quantity less than any assignable quantity, when x diminishes without limit.

Lemma II. To determine the relation between tan x, x, and sin. x, when x is less than any assignable quantity.

Let AP (see fig. 4.) be the arc of a circle whose radius = 1, and let the arc AP $= x$, x being the circular measure of the angle PCA; AT is the tangent and PM the sine of the arc. At P draw a tangent to the circle, viz. PT′, and draw the other lines as in the figure; then, since TPT′ is a right angle, TT′ is greater than PT′,

$$\therefore \quad \text{AT} > \text{AT}' + \text{T}'\text{P};$$

and because two sides of a triangle are greater than the third,

$$\text{RT}' + \text{T}'\text{S} > \text{RS};$$

$$\therefore \quad \textit{à fortiori} \quad \text{AT} > \text{AR} + \text{RS} + \text{SP}:$$

and similarly, if tangents be drawn to the arc at points between A and Q and Q and P, it may be shown that the sum of all the lines similar to SR is less than AT; but the limit of all such lines is the circular arc, therefore AT is greater than the arc.

Again; the chord AP is greater than PM, which is the sine of x; and PQ + QA is greater than AP; therefore

$$\text{PQ} + \text{QA} > \text{PM}:$$

and, drawing other chords from A and P to intermediate points on the arcs, it may be shown that the sum of such chords is greater than the chord AP; and therefore, *à fortiori*, than PM; and, the arc itself being the limit of such chords, the arc AP is greater than the sine PM. Therefore the arc is greater than the sine of the arc, and less than the tangent.

Again; we have

$$\frac{\sin x}{\tan x} = \cos x = 1, \text{ when } x = 0.$$

$\therefore$ $\sin x = \tan x$, and therefore $= x$, since x is intermediate to $\sin x$ and $\tan x$; therefore when $x = 0$, the ratio of the sine the arc and the tangent is one of equality, and we have

$$\sin x = \tan x = x,$$

$$\frac{\sin x}{x} = \frac{\tan x}{x} = \frac{\sin x}{\tan x} = 1.$$

CHAP. II.

THE CONSTRUCTION OF RULES FOR THE DIFFERENTIATION AND DERIVATION OF FUNCTIONS.

In the following processes particular attention must be paid to what has been said in Art. 9. of the former chapter, as the accuracy of the results entirely depends on the fact, that no error is committed by neglecting infinitesimals such as R in that Article.

For examples illustrative of the rules which are here deduced, the reader is referred to *Examples of Processes in Differential Calculus*, by Mr. Gregory, Cambridge, 1841, and to *Examples of the Application of the Differential Calculus*, by the Rev. J. Hind, Cambridge, 1832.

13.] The Differentiation of a Constant connected with a Function of a Variable by the symbol of Addition or Subtraction.

Let $$y = f(x) \pm c,$$

$$y + \Delta y = f(x + \Delta x) \pm c;$$

by subtraction $$\Delta y = \Delta\{f(x) \pm c\} = f(x + \Delta x) - f(x):$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

$$\therefore \quad \frac{dy}{dx} = f'(x),$$

$$dy = d\{f(x) \pm c\} = f'(x)dx.$$

Therefore, if a constant be connected by the symbols of addition or subtraction with a function whose variation is to be calculated, it disappears in the process of differentiation.

14.] The Differentiation of the Product of a Constant and Function of a Variable.

$$y = cf(x),$$

$$y + \Delta y = cf(x + \Delta x);$$

by subtraction $\Delta y = \Delta \,.\, cf(x) = c\{f(x+\Delta x) - f(x)\}$:

$$\therefore \quad \frac{\Delta y}{\Delta x} = c\,\frac{f(x+\Delta x)-f(x)}{\Delta x},$$

$$\therefore \quad \frac{dy}{dx} = \frac{d.cf(x)}{dx} = cf'(x),$$

$$\therefore \quad dy = d.cf(x) = cf'(x)\,dx.$$

Similarly, if $y = \dfrac{f(x)}{c} = \dfrac{1}{c}f(x)$,

$$dy = d\left\{\frac{1}{c}f(x)\right\} = \frac{1}{c}f'(x)\,dx,$$

$$\frac{dy}{dx} = \frac{d\left\{\frac{1}{c}f(x)\right\}}{dx} = \frac{1}{c}f'(x).$$

Therefore a constant connected with a function of a variable by the processes of multiplication or division is not affected by differentiation or derivation.

Ex. Suppose $c = -1$,

$$\therefore \quad y = -f(x),$$

$$\frac{dy}{dx} = \frac{d\{-f(x)\}}{dx} = -\frac{d.f(x)}{dx} = -f'(x);$$

$$\therefore \quad dy = d\{-f(x)\} = -d.f(x) = -f'(x)\,dx.$$

15.] Differentiation of an algebraical Sum of Functions of Variables.

$$y = f(x) \pm \mathrm{F}(x) \pm \phi(x) \pm \text{\&c.}$$

$$y + \Delta y = f(x+\Delta x) \pm \mathrm{F}(x+\Delta x) \pm \phi(x+\Delta x) \pm \text{\&c.}$$

$$\therefore \quad \Delta y = f(x+\Delta x) - f(x) \pm \{\mathrm{F}(x+\Delta x) - \mathrm{F}(x)\}$$
$$\pm \{\phi(x+\Delta x) - \phi(x)\} \pm \text{\&c.}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} \pm \frac{\mathrm{F}(x+\Delta x)-\mathrm{F}(x)}{\Delta x}$$
$$\pm \frac{\phi(x+\Delta x)-\phi(x)}{\Delta x} \pm \text{\&c.}$$

$$\therefore \quad \frac{dy}{dx} = f'(x) \pm \text{F}'(x) \pm \phi'(x) \pm \text{\&c.}$$

$$dy = f'(x)dx \pm \text{F}'(x)dx \pm \phi'(x)dx \pm \text{\&c.}$$

$$= d.f(x) \pm d.\text{F}(x) \pm d.\phi(x) \pm \text{\&c.}$$

Hence the differential of a sum of functions is equal to the sum of the differentials of the functions, and the derived function is equal to the sum of the several derived functions.

From the last two Articles it is plain that if the function to be differentiated be of the form

$$y = f(x) + \sqrt{(-1)}\,\phi(x),$$

one of the functions being what is commonly called impossible,

$$dy = d.f(x) + \sqrt{(-1)}\,d.\phi(x)$$

$$= f'(x)\,dx + \sqrt{(-1)}\,\phi'(x)\,dx;$$

$$\therefore \quad \frac{dy}{dx} = f'(x) + \sqrt{(-1)}\,\phi'(x).$$

Whence it appears that, in the differentiation of impossible quantities, we may treat the symbol of impossibility in the same way as we treat an ordinary constant or symbol of affection.

16.] Differentiation of a Product of Two Functions.

$$y = f(x) \times \phi(x),$$

$$y + \Delta y = f(x + \Delta x) \times \phi(x + \Delta x),$$

$$\Delta y = f(x + \Delta x) \times \phi(x + \Delta x) - f(x) \times \phi(x),$$

$$= \{f(x + \Delta x) - f(x)\}\,\phi(x + \Delta x) + \{\phi(x + \Delta x) - \phi(x)\}\,f(x),$$

by the addition and subtraction of $\phi(x + \Delta x) \times f(x)$,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}\,\phi(x + \Delta x) + \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}\,f(x),$$

$$\frac{dy}{dx} = f'(x) \times \phi(x) + \phi'(x) \times f(x),$$

$$dy = f'(x)\,dx \times \phi(x) + \phi'(x)\,dx \times f(x);$$

$$\therefore \quad dy = d\{f(x) \times \phi(x)\} = d.f(x) \times \phi(x) + d.\phi(x) \times f(x).$$

Hence it appears that the differential of the product of two functions is the sum of the product of each function and the differential of the other.

17.] Differentiation of the Quotient of Two Functions.

$$y = \frac{f(x)}{\phi(x)},$$

$$y + \Delta y = \frac{f(x+\Delta x)}{\phi(x+\Delta x)};$$

$$\therefore \quad \Delta y = \frac{f(x+\Delta x)}{\phi(x+\Delta x)} - \frac{f(x)}{\phi(x)}$$

$$= \frac{f(x+\Delta x)\,\phi(x) - f(x)\,\phi(x+\Delta x)}{\phi(x)\,\phi(x+\Delta x)}$$

$$= \frac{\{f(x+\Delta x) - f(x)\}\,\phi(x) - \{\phi(x+\Delta x) - \phi(x)\}\,f(x)}{\phi(x)\,\phi(x+\Delta x)};$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\dfrac{f(x+\Delta x) - f(x)}{\Delta x}\,\phi(x) - \dfrac{\phi(x+\Delta x) - \phi(x)}{\Delta x}\,f(x)}{\phi(x)\,\phi(x+\Delta x)},$$

$$\frac{dy}{dx} = \frac{f'(x)\,\phi(x) - \phi'(x)\,f(x)}{\{\phi(x)\}^2};$$

$$\therefore \quad dy = d\left\{\frac{f(x)}{\phi(x)}\right\} = \frac{f'(x)\,dx\,\phi(x) - \phi'(x)\,dx\,f(x)}{\{\phi(x)\}^2},$$

$$= \frac{d.f(x)\,\phi(x) - d.\phi(x)\,f(x)}{\{\phi(x)\}^2}.$$

18.] Differentiation of a Compound Function.

It is to be observed that we have been calculating the variation of a function of x due to a small variation of x, and the only condition to which x is subject is, that it is a continuous and finite variable for the values which are assigned to it. Now this condition may be fulfilled if x be replaced by a continuous and finite function of x or of any other variable, in which case y will vary in consequence of a variation of this other function; and this other function may of course vary in consequence of the variation of the variable of which it is a function; and so on: when this is the case y becomes a function of a function of x, or a function of several functions of x, and the differential of y is calculated as before; but the differential of the quantity under the first functional symbol must be replaced by its equivalent under this new supposition.

Thus, if
$$y = f(x),$$
$$dy = f'(x)\,dx;$$
but if x is replaced by $\phi(x)$, then dx must by replaced by $d.\phi(x)$, i. e. by $\phi'(x)\,dx$; and therefore, if
$$y = f\{\phi(x)\},$$
$$dy = f'\{\phi(x)\}\,\phi'(x)\,dx.$$
And so again, if $\phi(x)$ were a function of some other function of x, as for instance, if $\phi(x)$ were replaced by $\phi\{\psi(x)\}$, then $d.\phi(x)$ must be replaced by $d.\phi\{\psi(x)\}$, i. e. by $\phi'\{\psi(x)\}\,\psi'(x)dx$, and similarly if we had to differentiate a function of a function, &c. (to n functions) of x. Suppose, then, we adopt the following symbolisation:
$$y = f(z) \qquad \text{where} \quad z = \phi(x),$$
$$dy = f'(z)dz \qquad\qquad dz = \phi'(x)\,dx;$$
$$dy = f'(z)\,\phi'(x)\,dx,$$
$$= f'\{\phi(x)\}\,\phi'(x)\,dx,$$
which we may write under the form
$$dy = \frac{dy}{dz}\,\frac{dz}{dx}\,dx.$$

Ex. $y = f(cx)$, $\quad dy = f'(cx)\,d(cx) = cf'(cx)\,dx.$

$y = f(-x)$, $\quad dy = -f'(x)\,dx.$

$y = f(a+x)$, $\quad dy = f'(a+x)\,d(a+x) = f'(a+x)dx.$

$y = f(x^n)$, $\quad dy = f'(x^n)\,d.x^n.$

19.] Differentiation of a^x.
$$y = a^x,$$
$$y + \Delta y = a^{x+\Delta x},$$

$$\Delta y = a^{x+\Delta x} - a^x = a^x (a^{\Delta x} - 1),$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = a^x \frac{a^{\Delta x} - 1}{\Delta x}, \text{ which} = \frac{0}{0} \text{ when } \Delta x \text{ is very small.}$$

To evaluate this, let $a^{\Delta x} - 1 = z$; $\therefore$ $z = 0$, when $\Delta x = 0$,

$$a^{\Delta x} = 1 + z,$$

$$\Delta x \log_e a = \log_e (1 + z);$$

$$\therefore \quad \Delta x = \frac{\log_e (1 + z)}{\log_e a},$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = a^x \log_e a \,.\, \frac{z}{\log_e (1+z)} = \log_e a \,.\, a^x \frac{1}{\log_e (1 + z)^{\frac{1}{z}}}:$$

but when $\Delta x = 0$, that is, when $z = 0$, $(1 + z)^{\frac{1}{z}} = e$ by Lemma I. Art. 12., in which case $\log_e (1 + z)^{\frac{1}{z}} = 1$,

$$\therefore \quad \frac{dy}{dx} = \log_e a \,.\, a^x;$$

$$\therefore \quad \text{if } f(x) = a^x,$$

$$f'(x) = \log_e a \,.\, a^x,$$

$$d \,.\, a^x = \log_e a \,.\, a^x \, dx.$$

Hence it appears that if the function to be differentiated be e^x, since $\log_e e = 1$, we have

$$d \,.\, e^x = e^x \, dx:$$

that is, the differential of a^x is the product of the quantity, the Napierian logarithm of a, and the differential of the exponent; and the differential of e^x is the product of the quantity and the differential of the exponent.

Ex. $y = a^{cx}$, $dy = \log_e a \,.\, a^{cx} d \,.\, cx = c \log_e a \,.\, a^{cx} \, dx.$

$y = e^{f(x)}$, $dy = e^{f(x)} \, d \,.\, f(x) = e^{f(x)} f'(x) \, dx.$

$y = e^{-x^2}$, $dy = e^{-x^2} d(-x^2) = -e^{-x^2} d(x^2).$

$y = e^{x\sqrt{(-1)}} + e^{-x\sqrt{(-1)}}$,

$$dy = \sqrt{(-1)} \, e^{x\sqrt{(-1)}} \, dx - \sqrt{(-1)} \, e^{-x\sqrt{(-1)}} \, dx,$$
$$= \sqrt{(-1)} \, \{e^{x\sqrt{(-1)}} - e^{-x\sqrt{(-1)}}\} \, dx.$$

20.] Differentiation of $\log_a x$.

$$y = \log_a x,$$

$$y + \Delta y = \log_a (x + \Delta x);$$

$$\therefore \quad \Delta y = \log_a (x + \Delta x) - \log_a x,$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\log_a (x + \Delta x) - \log_a x}{\Delta x};$$

an expression which assumes the usual indeterminate form $\frac{0}{0}$ when Δx is an infinitesimal, i. e. when $\Delta x = 0$.

To evaluate it, let $x + \Delta x = xz$, whence $z = 1$ when $\Delta x = 0$.

$$\Delta x = x(z-1),$$

$$\frac{\Delta y}{\Delta x} = \frac{\log_a (xz) - \log_a x}{x(z-1)},$$

$$= \frac{\log_a z}{x(z-1)} = \frac{1}{x} \log_a (z)^{\frac{1}{z-1}} :$$

but when $z = 1$, $\log_a (z)^{\frac{1}{z-1}} = \log_a e$ by Lemma I. Art. 12.,

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x} \log_a e = \frac{1}{\log_e a} \frac{1}{x}, \quad \because \log_a e = \frac{1}{\log_e a},$$

$$\therefore \quad dy = d . \log_a x = \frac{1}{\log_e a} \frac{dx}{x};$$

and, if the base be the Napierian base e, $\log_e e = 1$,

$$d . \log_e x = \frac{dx}{x},$$

that is, the differential of the Napierian logarithm of x is the differential of the quantity divided by the quantity itself.

Hence, also, this rule, in conjunction with the rule in Art. 18., shows that the differential of the Napierian logarithm of any function of x is equal to the differential of the function divided by the function itself.

Ex. $y = \log_a x^2$, $\qquad dy = \frac{1}{\log_e a} \frac{d(x^2)}{x^2}$.

$y = \log_e (a + bx)$, $\qquad dy = \frac{d(a+bx)}{a+bx} = \frac{b\,dx}{a+bx}$.

$$y = \log_a f(x), \qquad dy = \frac{1}{\log_e a} \frac{d.f(x)}{f(x)};$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\log_e a} \frac{f'(x)}{f(x)}.$$

This rule might also have been proved from the last Art. as follows:

$$y = \log_a x,$$

$$\therefore \quad x = a^y.$$

$$dx = \log_e a \; a^y dy,$$

$$= \log_e a \,.\, x\, dy,$$

$$\therefore \quad dy = d \,.\, \log_a x = \frac{1}{\log_e a} \frac{dx}{x}.$$

21.] The Differentiation of x^n.

Let $\qquad y = x^n,$

$$\therefore \quad \log_e y = n \log_e x,$$

$$\therefore \text{ by last Art. } \quad \frac{dy}{y} = n \frac{dx}{x};$$

$$dy = n \frac{y}{x} dx = n \frac{x^n}{x} dx = n x^{n-1} dx,$$

$$\therefore \quad d \,.\, x^n = n x^{n-1} dx.$$

that is, to differentiate x^n, multiply by the exponent, diminish the exponent by unity, and multiply the product by dx. In the case in which $y = -x^n$, in order to avoid the logarithms of negative quantities, which are impossible, it is necessary to square both sides of the equation; whence

$$y^2 = x^{2n},$$

$$\frac{d \,.\, y^2}{y^2} = \frac{d \,.\, x^{2n}}{x^{2n}};$$

$$\therefore \quad \frac{2y\, dy}{y^2} = \frac{2n x^{2n-1} dx}{x^{2n}},$$

$$dy = d(-x^n) = -n x^{n-1} dx:$$

and the same rule of differentiation is true for any function of x.

This differentiation might also have been performed as follows, by means of an expansion :

$$y = x^n,$$

$$y + \Delta y = (x + \Delta x)^n;$$

$$\therefore \quad \Delta y = x^n + n x^{n-1} \Delta x + \frac{n(n-1)}{1.2} x^{n-2} (\Delta x)^2 + \ldots - x^n,$$

$$= n x^{n-1} \Delta x + \frac{n(n-1)}{1.2} x^{n-2} (\Delta x)^2 + \&c.$$

$$\frac{\Delta y}{\Delta x} = n x^{n-1} + \frac{n(n-1)}{1.2} x^{n-2} \Delta x + \&c.$$

Let Δx become an infinitesimal, i. e. become 0, and we have

$$\frac{dy}{dx} = n x^{n-1},$$

$$dy = d \,.\, x^n = n x^{n-1} dx.$$

Ex.

$$y = a + bx^2, \qquad dy = 2bx\,dx.$$

$$y = \frac{a}{x} = a x^{-1}, \qquad dy = -a x^{-2} dx = -\frac{a\,dx}{x^2}.$$

$$y = c \sqrt{(x)} = c x^{\frac{1}{2}}, \qquad dy = \tfrac{1}{2} c x^{-\frac{1}{2}} dx = \frac{c\,dx}{2\sqrt{(x)}}.$$

$$y = \{\log_a (c + e x^2)\}^2, \quad dy = 2 \frac{1}{\log_e a} \log_a (c + e x^2) \frac{2 e x\,dx}{c + e x^2}.$$

22.] The Differentiation of Functions of x connected with one another by Multiplication or Division.

$$y = f(x) \times \mathrm{F}(x) \times \phi(x) \times \Phi(x) \times \&c.$$

If all the functions are positive, we have

$$\log y = \log f(x) + \log \mathrm{F}(x) + \log \phi(x) + \&c.$$

$$\therefore \quad \frac{dy}{y} = \frac{d.f(x)}{f(x)} + \frac{d.\mathrm{F}(x)}{\mathrm{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \&c.$$

$$dy = d\,\{f(x)\,\mathrm{F}(x)\,\phi(x) \ldots\},$$

$$= f(x)\,\mathrm{F}(x)\,\phi(x) \ldots \left\{\frac{d.f(x)}{f(x)} + \frac{d.\mathrm{F}(x)}{\mathrm{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \ldots\right\}.$$

or, if some of the functions be negative,

$$y^2 = [f(x)]^2\,[\mathrm{F}(x)]^2\,[\phi(x)]^2\,\&c.$$

$$\log(y^2) = \log[f(x)]^2 + \log[\mathrm{F}(x)]^2 + \log[\phi(x)]^2 + \&c.$$

$$\frac{dy}{y} = \frac{d.f(x)}{f(x)} + \frac{d.\mathrm{F}(x)}{\mathrm{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \&c.$$

$$\therefore\quad dy = f(x)\,\mathrm{F}(x)\,\phi(x)\ldots\left\{\frac{d.f(x)}{f(x)} + \frac{d.\mathrm{F}(x)}{\mathrm{F}(x)} + \frac{d.\phi(x)}{\phi(x)} + \ldots\right\}.$$

Therefore, in either case,

$$d\{f(x)\mathrm{F}(x)\phi(x)\ldots\} = d.f(x)\mathrm{F}(x)\phi(x)\ldots + d.\mathrm{F}(x)f(x)\phi(x)\ldots$$
$$+ d.\phi(x)f(x)\mathrm{F}(x)\ldots + \&c.$$

and, if $\qquad y = \dfrac{f(x)}{\phi(x)},$

$$\log(y^2) = \log[f(x)]^2 - [\log\phi(x)]^2;$$

$$\therefore\quad \frac{dy}{y} = \frac{d.f(x)}{f(x)} - \frac{d.\phi(x)}{\phi(x)},$$

$$\therefore\quad dy = d\left\{\frac{f(x)}{\phi(x)}\right\} = \frac{f(x)}{\phi(x)}\left\{\frac{d.f(x)}{f(x)} - \frac{d.\phi(x)}{\phi(x)}\right\}$$

$$= \frac{d.f(x)\,\phi(x) - d.\phi(x)\,f(x)}{[\phi(x)]^2},$$

the same result as that obtained in Art. 17.

Ex.

$y = f(x) \times \phi(x), \qquad d.\{f(x)\phi(x)\} = \phi(x)d.f(x) + f(x)d.\phi(x).$

$y = x^2 \log_e x, \qquad dy = 2x\log_e x\,dx + x\,dx.$

$y = x^n e^x, \qquad dy = e^x(nx^{n-1} + x^n)dx.$

$y = \{f(x)\}^{\phi(x)},$

$\therefore \log_e y = \phi(x)\log_e f(x),$

$$\frac{dy}{y} = d.\phi(x)\log_e f(x) + \phi(x)\frac{d.f(x)}{f(x)},$$

$$dy = f(x)\}^{\phi(x)}\left\{\log_e f(x)\,d.\phi(x) + \frac{\phi(x)}{f(x)}\,d.f(x)\right\}.$$

23.] Differentiation of (α) $\sin x$, (β) $\cos x$, (γ) $\tan x$, (δ) $\sec x$, (ϵ) $\text{versin}\, x$.

(α)
$$y = \sin x,$$
$$y + \Delta y = \sin (x + \Delta x);$$
$$\therefore \quad \Delta y = \sin (x + \Delta x) - \sin x$$
$$= 2 \cos \left(x + \frac{\Delta x}{2}\right) \sin \frac{\Delta x}{2};$$
$$\therefore \quad \frac{\Delta y}{\Delta x} = \cos \left(x + \frac{\Delta x}{2}\right) \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}:$$

$\therefore \quad \frac{dy}{dx} = \cos x$; since $\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1$ when Δx is small, as is shown in Lemma II. Article 12., and $\frac{\Delta x}{2}$ may be neglected, on account of the law of infinitesimals enunciated in Art. 8., since it is added to x, which is a finite quantity:
$$\therefore \quad dy = d.\sin x = \cos x\, dx.$$

(β)
$$y = \cos x,$$
$$y + \Delta y = \cos (x + \Delta x);$$
$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\cos (x + \Delta x) - \cos x}{\Delta x},$$
$$= -\sin\left(x + \frac{\Delta x}{2}\right) \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}};$$
$$\therefore \quad \frac{dy}{dx} = -\sin x,$$
$$dy = d.\cos x = -\sin x\, dx.$$

(γ)
$$y = \tan x,$$
$$y + \Delta y = \tan (x + \Delta x);$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\tan(x+\Delta x) - \tan x}{\Delta x}$$

$$= \frac{\tan x + \tan \Delta x - \tan x\,(1 - \tan x \tan \Delta x)}{(1 - \tan x \tan \Delta x)\,\Delta x}$$

$$= \frac{1 + \tan^2 x}{1 - \tan x \tan \Delta x}\,\frac{\tan \Delta x}{\Delta x};$$

$$\therefore \frac{dy}{dx} = 1 + \tan^2 x = \sec^2 x; \;\because \text{by Lem. II. Art. 12.}\; \frac{\tan \Delta x}{\Delta x} = 1$$

$$\therefore\; d.\tan x = \sec^2 x\, dx.$$

(δ) $$y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1};$$

$$\therefore\; dy = (\cos x)^{-2} \sin x\, dx, \text{ by Art. 21.};$$

$$\therefore\; dy = d.\sec x = \frac{\sin x\, dx}{\cos^2 x} = \sec x \;\tan x\, dx.$$

(ε) $$y = \text{versin}\, x = 1 - \cos x;$$

$$\therefore\; dy = d.(1 - \cos x) = \sin x\, dx.$$

It is also manifest from the geometry of the figure (see fig. 5.), that the increments of the trigonometrical functions due to the increments of the arc are such as have been deduced in the above formulæ. For let AP be the arc of a circle whose radius is unity, and PQ be any small arc added to it:

Let AP $= x$, PQ $= \Delta x$.

Then PM $= \sin x$, NQ $= \sin(x + \Delta x)$; $\therefore$ QR $= \Delta.\sin x$.

CM $= \cos x$, CN $= \cos(x + \Delta x)$; $\therefore$ NM $= -\Delta.\cos x$.

AT $= \tan x$, AT′ $= \tan(x + \Delta x)$; $\therefore$ TT′ $= \Delta.\tan x$.

CT $= \sec x$, CT′ $= \sec(x + \Delta x)$; $\therefore$ ST′ $= \Delta.\sec x$.

ST : QP :: CT : CP, i. e. ST : Δx :: $\sec x$: 1.

$\therefore$ ST $= \Delta x \sec x$.

When Δx becomes very small, that is, becomes dx, PQ, ST become approximately straight lines, and perpendicular to CT, and ST becomes $dx \sec x$: whence

$$d.\sin x = \mathrm{QR} = \mathrm{PQ}\ \sin \mathrm{QPR} = \mathrm{PQ}\ \cos \mathrm{RPC} = \mathrm{PQ}\ \cos \mathrm{PCA} = dx \cos x.$$

$$d.\cos x = -\mathrm{NM} = -\mathrm{PR} = -\mathrm{PQ}\ \cos \mathrm{QPR} = -\mathrm{PQ}\ \sin \mathrm{PCA} = -dx \sin x.$$

$$d.\tan x = \mathrm{TT}' = \mathrm{ST}\ \sec \mathrm{T'TS} = \mathrm{ST}\ \mathrm{cosec}\, \mathrm{CTA} = \mathrm{ST}\ \sec \mathrm{PCA} = \sec^2 x\, dx.$$

$$d.\sec x = \mathrm{ST}' = \mathrm{ST}\ \tan \mathrm{T'TS} = \mathrm{ST} \tan \mathrm{PCA} = \sec x \tan x\, dx.$$

The differentials of tan x and sec x may also be determined from those of sin x and cos x as follows:

$$\tan x = \frac{\sin x}{\cos x}.$$

Therefore, by Art. 17.,

$$d.\tan x = \frac{\cos x\, d.\sin x - \sin x\, d.\cos x}{\cos^2 x},$$

$$d.\tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\, dx$$

$$= \frac{dx}{\cos^2 x} = \sec^2 x\, dx = (1 + \tan^2 x)\, dx;$$

$$\sec x = \frac{1}{\cos x} = (\cos x)^{-1};$$

$$\therefore\ d.\sec x = -(\cos x)^{-2}\, d.\cos x$$

$$= \frac{\sin x\, dx}{\cos^2 x} = \sec x \tan x\, dx.$$

Similarly may we differentiate cot x and cosec x:

$$\cot x = \frac{\cos x}{\sin x};$$

$$\therefore\ d.\cot x = \frac{\sin x\, d.\cos x - \cos x\, d.\sin x}{\sin^2 x}$$

$$= -\frac{dx}{\sin^2 x} = -\mathrm{cosec}^2\, x\, dx,$$

$$\text{cosec } x = \frac{1}{\sin x} = (\sin x)^{-1},$$

$$d.\text{cosec } x = -(\sin x)^{-2} d.\sin x$$

$$= -\frac{\cos x\, dx}{\sin^2 x} = -\text{cosec } x \cot x\, dx.$$

24.] Differentiation of (α) $\sin^{-1}\frac{x}{a}$, (β) $\cos^{-1}\frac{x}{a}$, (γ) $\tan^{-1}\frac{x}{a}$, (δ) $\sec^{-1}\frac{x}{a}$, (ϵ) $\text{versin}^{-1}\frac{x}{a}$.

(α) Let $y = \sin^{-1}\frac{x}{a}$; $\therefore$ $\sin y = \frac{x}{a}$,

$$\therefore \quad d.\sin y = \cos y\, dy = \frac{dx}{a},$$

$$\therefore dy = d.\sin^{-1}\frac{x}{a} = \frac{dx}{a\cos y};$$

but $\cos y = \sqrt{(1-\sin^2 y)} = \sqrt{\left(1 - \frac{x^2}{a^2}\right)}$,

$$\therefore \quad d.\sin^{-1}\frac{x}{a} = \frac{dx}{\sqrt{(a^2 - x^2)}}.$$

(β) $y = \cos^{-1}\frac{x}{a}$; $\therefore$ $\cos y = \frac{x}{a}$,

$$\therefore \quad d.\cos y = -\sin y\, dy = \frac{dx}{a};$$

but $\sin y = \sqrt{(1 - \cos^2 y)} = \sqrt{\left(1 - \frac{x^2}{a^2}\right)} = \frac{1}{a}\sqrt{(a^2 - x^2)}$,

$$dy = d.\cos^{-1}\frac{x}{a} = \frac{-dx}{\sqrt{(a^2 - x^2)}}.$$

(γ) $y = \tan^{-1}\frac{x}{a}$, $\therefore$ $\tan y = \frac{x}{a}$;

$$d.\tan y = (1 + \tan^2 y)\, dy = \frac{dx}{a},$$

$$\therefore \quad dy = d \,.\, \tan^{-1} \frac{x}{a} = \frac{dx}{a(1+\tan^2 y)} = \frac{a\,dx}{a^2+x^2}.$$

(δ) $$y = \sec^{-1} \frac{x}{a}, \qquad \therefore \quad \sec y = \frac{x}{a};$$

$$d \,.\, \sec\ y = \sec y \tan y \, dy = \frac{dx}{a};$$

$$\text{but } \tan y = \sqrt{(\sec^2 y - 1)} = \sqrt{\left(\frac{x^2}{a^2} - 1\right)},$$

$$\therefore \ dy = d \,.\, \sec^{-1} \frac{x}{a} = \frac{dx}{a \frac{x}{a} \sqrt{\left(\frac{x^2}{a^2} - 1\right)}} = \frac{a\,dx}{x\sqrt{(x^2-a^2)}}.$$

(ε) $$y = \text{versin}^{-1} \frac{x}{a}.$$

$$\therefore \quad \text{versin}\ y = 1 - \cos y = \frac{x}{a},$$

$$d \,.\, \text{versin}\, y = \sin y \, dy = \frac{dx}{a};$$

$$\text{but } \sin y = \sqrt{[(1-\cos y)\ (1+\cos y)]} = \sqrt{\frac{x}{a}\left(2 - \frac{x}{a}\right)}$$

$$= \frac{1}{a}\sqrt{(2\,ax - x^2)},$$

$$\therefore \quad dy = d \,.\, \text{versin}^{-1} \frac{x}{a} = \frac{dx}{\sqrt{(2\,ax - x^2)}}.$$

(ζ) Similarly, if $$y = \cot^{-1} \frac{x}{a},$$

$$d \,.\, \cot^{-1} \frac{x}{a} = -\frac{a\,dx}{a^2+x^2}:$$

(η) And if $$y = \text{cosec}^{-1} \frac{x}{a},$$

$$d \,.\, \text{cosec}^{-1} \frac{x}{a} = -\frac{a\,dx}{x\sqrt{(x^2-a^2)}}.$$

It is to be observed that if we add together the differentials of $\sin^{-1} \frac{x}{a}$ and $\cos^{-1} \frac{x}{a}$, the sum is zero, which is as it ought to be, because the sum of the arcs is $\frac{\pi}{2}$, which is constant, and the differential of a constant is zero; a similar remark applies to the differentials of $\tan^{-1}\frac{x}{a}$ and $\cot^{-1}\frac{x}{a}$, $\sec^{-1}\frac{x}{a}$ and $\operatorname{cosec}^{-1}\frac{x}{a}$. In the case in which a (= the radius) = 1, we have

$$d.\sin^{-1}x = \frac{dx}{\sqrt{(1-x^2)}},$$

$$d.\cos^{-1}x = -\frac{dx}{\sqrt{(1-x^2)}};$$

and so on in the other formulæ, giving to a the value 1.

These results, like the formulæ in the last Article, might also have been obtained without difficulty from geometry; as e. g. (see fig. 5.):

Let $\text{MP} = x$ $\quad\therefore\ \text{AP} = \sin^{-1}x = y.$

$\text{NQ} = x + \Delta x,\quad \therefore\ \text{AQ} = \sin^{-1}(x+\Delta x) = y + \Delta y.$

$\therefore\ \text{RQ} = \Delta x.$

$$\Delta y = \Delta\sin^{-1}x = \text{the arc PQ} = \text{RQ}\sec\text{PQR} = \text{RQ}\sec\text{PCA};$$

$$\therefore\ dy = d.\sin^{-1}x = dx\sec(\sin^{-1}x) = \frac{dx}{\sqrt{(1-x^2)}}.$$

By a similar method may the other differentials be obtained.

Ex. $y = \sin(x^2)$, $\quad dy = \cos(x^2)\,d.(x^2) = 2x\cos(x^2)\,dx.$

$y = \cos^2 x - \sin^2 x,$

$dy = -2\cos x\sin x\,dx - 2\cos x\sin x\,dx = -2\sin 2x\,dx.$

$y = \sin x\tan x\sin^{-1}x\tan^{-1}x.$

$$\therefore\ \log_e y = \log_e\sin x + \log_e\tan x + \log_e\sin^{-1}x + \log_e\tan^{-1}x,$$

$$\frac{dy}{y} = \frac{\cos x\,dx}{\sin x} + \frac{\sec^2 x\,dx}{\tan x} + \frac{dx}{\sqrt{(1-x^2)}\sin^{-1}x} + \frac{dx}{(1+x^2)\tan^{-1}x};$$

$$\therefore\ dy = \sin x\tan x\sin^{-1}x\tan^{-1}x\left\{\cot x\,dx + \frac{dx}{\sin x\cos x} + \frac{dx}{\sqrt{(1-x^2)}\sin^{-1}x} + \frac{dx}{(1+x^2)\tan^{-1}x}\right\}.$$

25.] Differentiation of a Function of several Variables.

To fix our thoughts let us first consider a function of two variables, x and y, which we will symbolise as follows,

$$u = \mathrm{F}(x, y);$$

and let x and y be two variables not connected by any other equation, and therefore such that they may vary independently of each other; that is, a change of one does not of necessity involve a change of the other. It is plain that u may vary in three ways; either on account of x varying when y does not vary, or on account of y varying when x is constant, or on account of x and y varying simultaneously. These changes in the value of u are called respectively the partial variations of u with respect to x and y, and the total variation of u; and the differences between the values of u, when these changes in the variables are small, are called respectively the partial and total differentials of u. An example will illustrate our meaning. Let us take $u = xy$, where u represents a plane rectangle, of which one side is x and another y. The area may vary owing to a change in the length of either side, or of both. If x changes and y does not, the change in u is $y\,dx$; and if y varies and x does not, the change in u is $x\,dy$; such are the partial variations; and, if both vary together, the variation of u is equal to $x\,dy + y\,dx + dy\,dx$: but when the variations of x and y are very small, dy and dx become infinitesimals, and $dy\,dx$ may be neglected without sensible error, and we have the total change in u equal to the sum of the partial changes. Similarly, if u is a function of many variables, changes may take place in its value owing to the variations of the variables, one, or more, or all, at a time, and similarly there may be partial and total differentials. The symbolisation we adopt is as follows:

$\mathrm{D}u$ represents the total change in u due to the variations of all the variables,

$d_x u$ . . . the partial change in u due to the variation of x,

$d_y u$ . . . . y,

$d_z u$ . . . . z,

and so on; and $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$, $\left(\frac{du}{dz}\right)$ represent the ratios of the several variations of the function due to the variation of the variables separately; that is, they are partial differentials, the brackets indicating that they are so, and the variable in the denominator of the fractions being the one due to the variation of which the partial variation of u is calculated. As before, let Δu, Δx, Δy represent finite changes of u, x, y, respectively; and $\mathrm{D}u$, du, dx, dy, infinitely small variations; we proceed then to the differentiation of such functions as these.

26.] Differentiation of a Function of Two Variables.

$$u = f(x, y).$$

Let x and y receive the increments Δx and Δy, and let the corresponding increment of u be Δu; so that

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y);$$

$$\therefore \quad \Delta u = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)$$
$$+ f(x, y + \Delta y) - f(x, y).$$

Let the variations of x and y be very small, then

Δu becomes $\mathrm{D}u$, which is the total change in u;

$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)$ becomes $d_x u$;

$f(x, y + \Delta y) - f(x, y)$ becomes $d_y u$;

Since in the former of these last two y has the same value in both expressions, and in the latter x has the same value, therefore

$$\mathrm{D}u = d_x u + d_y u;$$

that is, the total differential of a function of two variables is equal to the sum of the partial differentials. Also, we have

$$\Delta u = \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} \Delta x$$
$$+ \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \Delta y;$$

$$\therefore \quad \mathrm{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy.$$

In which formula dx and dy are the small increments of the values of x and y; whence we can easily find the ratio of the total change of the function to the change of either variable: thus we have

$$\frac{\mathrm{D}u}{dx} = \left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right)\frac{dy}{dx},$$

$$\frac{\mathrm{D}u}{dy} = \left(\frac{du}{dx}\right)\frac{dx}{dy} + \left(\frac{du}{dy}\right).$$

But we must be careful to remember that the brackets indicate that we have estimated the variation of the function due to the change of the variable, which is in the denominator of the fraction, and due to that variable only.

Ex. $$u = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \quad \left(\frac{du}{dx}\right) = \frac{2x}{a^2}, \quad \left(\frac{du}{dy}\right) = \frac{2y}{b^2};$$

$$\therefore \quad \mathrm{D}u = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy.$$

$$u = x \tan y + y \tan x, \quad \left(\frac{du}{dx}\right) = \tan y + y \sec^2 x,$$

$$\left(\frac{du}{dy}\right) = x \sec^2 y + \tan x;$$

$$\therefore \quad \mathrm{D}u = (y \sec^2 x + \tan y)\,dx + (x \sec^2 y + \tan x)\,dy.$$

27.] Differentiation of an implicit Function of Two Variables.

Suppose that x and y are involved in an equation of the form

$$f(x, y) = c, \quad c \text{ being a constant};$$

since then $u = f(x, y) = c$, $\mathrm{D}u = 0$, and we have

$$\left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy = 0,$$

$$\frac{dy}{dx} = -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)},$$

which gives us the ratio of the increment of y to the increment of x, although the relation between y and x has not been expressed in the explicit form. On this process see the Note (A) at the end of the volume.

Ex. $$u = x^3 + y^3 = a^3,$$

$$\left(\frac{du}{dx}\right) = 3x^2, \qquad \left(\frac{du}{dy}\right) = 3y^2,$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)} = -\frac{x^2}{y^2}.$$

For other examples, see Gregory's *Examples*, p. 6.

28.] Differentiation of Functions of several Variables.

Let $$u = f(x, y, z \ldots),$$

$$u + \Delta u = f(x + \Delta x, y + \Delta y, z + \Delta z \ldots);$$

$$\begin{aligned} \therefore \Delta u &= f(x + \Delta x, y + \Delta y, z + \Delta z \ldots) - f(x, y, z \ldots) \\ &= f(x + \Delta x, y + \Delta y, z + \Delta z \ldots) - f(x, y + \Delta y, z + \Delta z \ldots) \\ &+ f(x, y + \Delta y, z + \Delta z \ldots) \quad - f(x, y, z + \Delta z \ldots) \\ &+ f(x, y, z + \Delta z \ldots\ldots) \quad - \; \ldots \\ &+ \ldots\ldots \quad - \; \ldots \\ &+ \ldots\ldots \quad - f(x, y, z \ldots), \end{aligned}$$

$$\therefore \quad \mathrm{D}u = d_x u + d_y u + d_z u + \&c.$$

that is, the total differential of a function of several variables is equal to the sum of the partial differentials.

And if we use partial derived functions, as in Art. 26.,

$$\mathrm{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz + \&c.$$

whence, as in that Art., we can find the ratio of the total differential to the increment of any variable.

And if the variables be so combined that

$$f(x, y, z \ldots) = c,$$

$$\mathrm{D}u = 0,$$

and $$\left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz + \ldots\ldots = 0;$$

and we can determine the total change any one of them, as e. g. x, has undergone, as follows:

$$dx = -\frac{\left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz + \ldots}{\left(\frac{du}{dx}\right)}.$$

Ex. $$u = \sin(ax + by + cz) + yz + zx + xy.$$

$$\left(\frac{du}{dx}\right) = a\cos(ax + by + cz) + (y + z).$$

$$\left(\frac{du}{dy}\right) = b\cos(ax + by + cz) + (z + x).$$

$$\left(\frac{du}{dx}\right) = c\cos(ax + by + cz) + (x + y).$$

dz

$$\therefore \text{D}u = \cos(ax + by + cz)\{a\,dx + b\,dy + c\,dx\} + (y + z)\,dx + (z + x)\,dy + (x + y)\,dx.$$

For examples, see Gregory's *Examples*, p. 7., and Hind's *Examples*, p. 116.

CHAP. III.

ON SUCCESSIVE DIFFERENTIATION AND THE THEORY OF THE INDEPENDENT VARIABLE.

A. *Of Functions of One Variable.*

29.] In the preceding chapter we have constructed a system of rules by which the variation of a function, due to the variation of the variable of which it is a function, may be calculated, and the ratio of the increment of the function to the increment of the variable, which we have designated by the symbol $f'(x)$, is in general a new function of x. Recurring to Art. 10., the process by which this new function is formed is called derivation, and the new function itself is called the derived function, and, for a reason which will shortly appear, the first derived; thus derivation may be considered an analytical artifice, by which this new function is deduced from the former one; and, as this new one is in general a function of x, it admits of having a similar process performed on itself, whence arises another new function, which is in general also a function of x, and from which another function may be derived in a similar manner. Thus we obtain a series of functions derived one from the other, which are called the derived functions of different orders, and we use the following symbolisation of them:

$$f(x), f'(x), f''(x), f'''(x), f^{\text{iv}}(x) \dots f^{n-1}(x), f^{n}(x);$$

where $f'(x)$ is the function derived from $f(x)$, $f''(x)$ that derived from $f'(x)$ and therefore called the second-derived, $f'''(x)$ that derived by a similar process from $f''(x)$ and called the third-derived, and so on, $f^{n}(x)$ being called the nth derived function of $f(x)$; so that we have the following series of equations:

$$y = f(x),$$
$$dy = d \, . \, f(x) = f'(x) \, dx, \qquad \therefore \quad f'(x) = \frac{dy}{dx}.$$
$$d \, . \, f'(x) = f''(x) \, dx,$$
$$d \, . \, f''(x) = f'''(x) \, dx,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$d \, . \, f^{n-1}(x) = f^n(x) \, dx.$$

Since then $dy = f'(x) \, dx$, considering dx to be constant, a supposition which will be explained hereafter,

$$d \, . \, dy = d \, . \, f'(x) \, dx = f''(x) \, dx^2,$$
$$d \, . \, d \, . \, dy = d \, . \, f''(x) \, dx^2 = f'''(x) \, dx^3;$$

and so on: and, since the operations symbolised by the character d are to be performed one on the back of another, we may, according to the index law, write for $d.dy$, d^2y; for $d.d.dy$, d^3y, and so on; whence we have

$$\frac{dy}{dx} = f'(x), \qquad dy = f'(x) \, dx,$$
$$\frac{d^2y}{dx^2} = f''(x), \qquad d^2y = f''(x) \, dx^2,$$
$$\frac{d^3y}{dx^3} = f'''(x), \qquad d^3y = f'''(x) \, dx^3,$$
$$. \;\; . = \; . \;, \qquad . \;\; . = \; . \;\; . \;\; . \;\; . \;,$$
$$\frac{d^ny}{dx^n} = f^n(x), \qquad d^ny = f^n(x) \, dx^n.$$

On account of these sets of equations, $\frac{d^2y}{dx^2} \; \{= f''(x)\}$, $\frac{d^3y}{dx^3}$ $\{= f'''(x)\}$, $\frac{d^ny}{dx^n} \; \{= f^n(x)\}$ are called the second, third, ... nth differential coefficients, because they are severally the coefficients of the second, third, ... nth powers of dx in the above equations; and dy, d^2y, d^3y ... d^ny are called the first, second, third ... nth differentials of y.

It is to be observed, that as $\frac{dy}{dx}$ represents the ratio of the variation of y or $f(x)$ to that of x, so $\frac{d^2y}{dx^2} = f''(x) = \frac{d \, . f'(x)}{dx}$ represents the ratio of the variation of $f'(x)$, that is of $\frac{dy}{dx}$, to the variation of x, and so $\frac{d^3y}{dx^3}$ represents the ratio of the variation of $\frac{d^2y}{dx^2}$ to that of x: and so on for the other differentials.

Subjoined are two examples in successive differentiation; for more, and some very elegant ones, see Gregory's *Examples*, chap. ii.

$$y = f(x) = a + x, \qquad y = x^r,$$

$$\frac{dy}{dx} = f'(x) = 1, \qquad \frac{dy}{dx} = f'(x). = rx^{r-1},$$

$$\frac{d^2y}{dx^2} = f''(x) = 0, \qquad \frac{d^2y}{dx^2} = f''(x) = r(r-1)x^{r-2},$$

$$\frac{d^ny}{dx^n} = f^n(x) = 0, \qquad \frac{d^ny}{d^nx} = f^n(x) = r(r-1)\ldots(r-n+1)x^{r-n}.$$

30.] First Application of the preceding theory.

Having given $f(x) = u \times v$, u and v being both functions of x, it is required to determine $f^n(x)$.

$$f(x) = u \times v, \qquad \therefore \text{ by the rule in Art. 16.,}$$

$$f'(x) = v\frac{du}{dx} + u\frac{dv}{dx},$$

$$f''(x) = v\frac{d^2u}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + \frac{d^2v}{dx^2}u,$$

$$f'''(x) = v\frac{d^3u}{dx^3} + 3\frac{dv}{dx}\frac{d^2u}{dx^2} + 3\frac{d^2v}{dx^2}\frac{du}{dx} + \frac{d^3v}{dx^3}u,$$

and so on, the law of the coefficients being that of the coefficients of $(1+x)^n$; in conformity with which let us assume

$$f^{n-1}(x) = v\frac{d^{n-1}u}{dx^{n-1}} + (n-1)\frac{dv}{dx}\frac{d^{n-2}u}{dx^{n-2}}$$
$$+ \frac{(n-1)(n-2)}{1 \cdot 2}\frac{d^2v}{dx^2}\frac{d^{n-3}u}{dx^{n-3}} + \&c.$$

$\therefore$ by actual derivation,

$$f^{n}(x) = v\frac{d^{n}u}{dx^{n}} + n\frac{dv}{dx}\frac{d^{n-1}u}{dx^{n-1}} + \frac{n(n-1)}{1 \cdot 2}\frac{d^2v}{dx^2}\frac{d^{n-2}u}{dx^{n-2}}$$
$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\frac{d^3v}{dx^3}\frac{d^{n-3}u}{dx^{n-3}} + \&c.$$

Therefore, if the formula be true for $n-1$, it is true for n; it is true when $n = 3$, $\therefore$ it is true when $n = 4$; and therefore for all positive integral values of n.

If we had calculated differentials instead of derived functions, the series would have been

$$d^n(u \times v) = vd^nu + ndvd^{n-1}u + \frac{n(n-1)}{1 \cdot 2}d^2vd^{n-2}u$$
$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}d^3vd^{n-3}u + \&c.$$

Several examples of this theorem are given in the second chapter of Gregory's *Examples*.

31.] Second Application.—To eliminate from a given expression Constants and determinate Functions by means of Derivation.

Suppose an equation to be given involving x and y, and m arbitrary constants; let n successively derived equations be formed from the given one, by which means we have $n+1$ equations, between which we can eliminate n constants, and the resulting equation will involve only $m-n$ constants; and, as we can eliminate any n of the m constants, it is plain that we can form as many equations as there are combinations of m things taken n and n together, that is, we can thus obtain

$$\frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots n} \text{ different equations.}$$

If, then, we deduce m derived equations from the given one, we have $m+1$ different equations, between which we can eliminate the m arbitrary constants, and thus obtain an equation involving only y and x and the successively derived functions, and independent of the constants. By a similar method also, if the original equation contain determinate functions of x, these may be eliminated, and an equation obtained involving only y and x and the derived functions. For examples, see Gregory, chapter iv.

32.] Third Application. — Suppose it is required to expand a given function of x, say $f(x)$, in rising powers of x of the following form,

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \text{\&c.},$$

subject to the following conditions:

1st. $A_0, A_1, A_2, A_3, \ldots$ are constants independent of x, and to be determined.

2dly. $A_0, A_1, A_2, A_3, \ldots$ are constants which do not become infinite for any value of x.

3dly. The series contains no terms involving either negative or fractional powers of x.

Then, forming the derived functions,

$$f'(x) = A_1 + 2.A_2 x + 3.A_3 x^2 + \text{\&c.}$$

$$f''(x) = 1.2.A_2 + 2.3.A_3 x + \text{\&c.}$$

$$f'''(x) = 1.2.3.A_3 + \text{\&c.}$$

In these expressions let x be equal to 0; then, since by conditions (2) and (3) none of the quantities assume the indeterminate value $\frac{0}{0}$, or become infinite, we have

$$f(0) = A_0,$$

$$f'(0) = A_1,$$

$$f''(0) = 1.2.A_2, \qquad \therefore \quad A_2 = \frac{f''(0)}{1.2},$$

$$f'''(0) = 1.2.3.A_3, \qquad \therefore \quad A_3 = \frac{f'''(0)}{1.2.3};$$

and so on.

Whence, substituting,

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{1.2} + f'''(0)\frac{x^3}{1.2.3}$$

$$+ f^{iv}(0)\frac{x^4}{1.2.3.4} \text{ \&c.}$$

a series which is called, after its inventor, Stirling's Series; and also Maclaurin's, having been introduced by him into his *Treatise of Fluxions.*

As this series is of great importance in the applications of the Differential Calculus, the proof of it must not rest upon any fallacious reasoning, or upon any vague assumption which may be either too wide or too narrow; and therefore a rigorous and exact proof of the truth of it, and one too that limits the extent of its applicability, will be given hereafter. The explanation of it given above is to be taken only as presumptive evidence that it is likely to be true. For examples, see Hind, chap. iv. section iii., and Gregory, chap. v. section ii.

33.] Thus far, then, we have considered the derivation of these functions one from the other to be simply an algebraical artifice; but a question arises, what is *meant* by $f'(x)$, $f''(x)$, &c.; and what connexion, besides the algebraical relation, is there between any one function and its derived functions? To this subject we shall devote the next chapter. Also, we have made no supposition as to the absolute values of the quantities dy and dx; it is true that they are infinitesimals, but, as we said before, are all infinitesimals equal? Not necessarily so; and, from what follows, it will be seen that our calculation will be greatly simplified by assuming certain conditions in accordance with which y and x shall vary.

Let $y = f(x)$ be an explicit function of x, and suppose it to be finite and continuous for all values of x, under which we consider it; whether any other values may make it infinite or discontinuous is a question with which we are not concerned in the present enquiry. Suppose that x receives an increment dx, in general $y = f(x)$ changes also, and becomes, according to what we have already explained, $y + dy$, so that we have the equation

$$y + dy = f(x + dx).$$

Conceive x to receive another increment, not necessarily equal to dx, then the function assumes the form

$$y+dy+d(y+dy)=f\{x+dx+d(x+dx)\},$$

$$y+2dy+d^2y=f(x+2dx+d^2x);$$

and let x receive another increment, and so on, and we have a series of equations,

$$y=f(x),$$

$$y+dy=f(x+dx),$$

$$y+2dy+d^2y=f(x+2dx+d^2x),$$

$$y+3dy+3d^2y+d^3y=f(x+3dx+3d^2x+d^3x),$$

. . . .

$$y+ndy+\frac{n(n-1)}{1.2}d^2y+\frac{n(n-1)(n-2)}{1.2.3}d^3y+\ldots\ldots$$

$$=f\{x+ndx+\frac{n(n-1)}{1.2}d^2x+\ldots\},$$

which equation we can prove to be true for all positive and integral values of n, in the same way as in Art. 30. we have shown the series for $f^n(x)$ to be true when $f(x)=uv$.

Hence we determine the several and successive changes and variations of an explicit function corresponding to the successive values of the variables. This will be much simplified by the following considerations.

34.] Suppose x to increase by constant increments, i. e. all the dx's to be equal; then d^2x, which is the increment of one dx over another, is equal to 0; and d^3x, which is the increment of one d^2x over another, $=0$; and similarly

$$d^4x=0, \qquad d^nx=0,$$

and the equation becomes

$$f(x+ndx)=y+ndy+\frac{n(n-1)}{1.2}d^2y$$

$$+\frac{n(n-1)(n-2)}{1.2.3}d^3y+\&c.$$

When x varies subject to these conditions it is called the variable of equal augments or of equal increments, and is what former writers have called the *independent* variable; which name, in deference to usage, we shall employ, but at the same time would have it understood that an independent variable is one which varies by equal augments, and of which each differential after the first vanishes. As this condition is of the greatest importance in the application of the Calculus to questions of Geometry and Physics, we will illustrate it as follows.

Suppose we are considering any function of x between the values x_1 and x_0, x_1 being the greater of the two, and the function remaining finite and continuous for all values of x between these limits, then we may resolve the difference $x_1 - x_0$ into small elements dx, the number of them being of course infinitely large when dx is infinitely small. It is at once plain that it is not necessary that all the dx's should be equal; they may vary in magnitude, and if they do, there will be an increment of one dx over another dx, that is, there will be a $d.dx$, which we will designate by d^2x. Neither, again, need all the d^2x's be equal, but, if they be not, there will again be an increment of one d^2x over another d^2x, that is, there will be a $d.d^2x$, which we will symbolise by d^3x, and so on, and these values we call the first, second, and third differentials of x; but if we once introduce the condition that a differential of any order shall be resolved into elements all equal to one another, then all the subsequent differentials vanish. Such have we done in the case above with dx; we resolve the difference $x_1 - x_0$ into elements dx all equal, and, therefore, there is no increment of one over another, whence we have

$$d^2x = 0, \quad d^3x = 0, \ldots d^nx = 0.$$

Or thus, again, conceive a small body, as a billiard-ball, to move over a finite distance in a straight line in a finite time; consider the straight line to be the axis of x; let the body at the beginning of the motion be at a distance x_0 from the origin, and at the end of the time to be at a distance x_1, and conceive the time of its passing over the distance $x_1 - x_0$ to be t; resolve this time into equal elements dt, and the space $x_1 - x_0$ into corresponding elements dx. If the body moves through the whole space at the same rate, viz. with the same velocity, then, during equal times dt, equal spaces dx will be described; but, if the velocity varies, equal spaces will not be passed over in equal times. On the first supposition, then, all the dx's will be equal, $d^2x=0$, and x is an independent variable; on the second the dx's vary, and d^2x, which is the increment of one dx passed over in a time dt, over another dx passed over in the preceding or succeeding time dt, as the case may be, is the measure of the increase of the rate of motion. If, then, all the d^2x's were equal, we should say that the velocity of motion is continually increasing, and at a constant rate; but if d^2x were not constant, then the rate of increase of the velocity of the ball is no longer constant, but varies according to some law on which the rate of increase depends. It will be observed, however, that if the whole time of motion be resolved into equal elements dt, the supposition of x being a variable of equal augments is incompatible with a varying velocity. Hence, too, it is manifest that, generally, we are not at liberty to consider more than one of the variables to increase or decrease by equal quantities: as, in the case above, if we resolve the time into equal elements, then, in general, unequal spaces will be passed over in equal times, and we cannot consider all the dx's to be equal, and therefore we cannot make $d^2x = 0$; and if we resolve the distance into equal parts, then, if the velocity varies, these equal spaces will be passed over in unequal times, and therefore all the dt's will not be equal, and we cannot put $d^2t=0$. In general, however, we are at liberty to choose, for an independent variable, whichever is most convenient.

35.] Let us now consider in what manner these considerations modify what has been said above. In the series of quantities in Art. 29.,

$$y = f(x),$$

$$dy = f'(x)\,dx,$$

$$\frac{d^2y}{dx^2} = f''(x), \text{ and so on.}$$

If we consider $f'(x)\,dx$ to be the product of two variable quantities, and differentiate it as such, we have, in accordance with our former notation making $f''(x)\,dx$ to be the symbol for $d.f'(x)$, and $f'''(x)\,dx$ that for $d.f''(x)$, and so on,

$$dy = f'(x)\,dx,$$
$$d^2y = f''(x)\,dx^2 + f'(x)\,d^2x,$$
$$d^3y = f'''(x)\,dx^3 + 3f''(x)\,dx\,d^2x + f'(x)\,d^3x,$$
$$d^4y = f^{iv}(x)\,dx^4 + 6f'''(x)\,dx^2\,d^2x + 3f''(x)\,(d^2x)^2$$
$$+ 4f''(x)\,dx\,d^3x + f'(x)\,d^4x,$$

and so on.

Now let dx be constant; whence $d^2x = 0$, $d^3x = 0$, &c.;

$$\therefore \quad dy = f'(x)\,dx,$$
$$d^2y = f''(x)\,dx^2,$$
$$d^3y = f'''(x)\,dx^3,$$
$$d^4y = f^{iv}(x)\,dx^4;$$

$\therefore$ $f''(x)$, or its equivalent $\frac{d^2y}{dx^2}$, is derived from $f'(x)$, on the supposition that x is the independent variable;

$f'''(x)$, or its equivalent $\frac{d^3y}{dx^3}$, is derived from $f''(x)$, on the same supposition;

and $f^n(x) = \frac{d^ny}{dx^n}$ is derived from $f^{n-1}(x)$, on the same supposition.

Whenever, therefore, we meet with these or similar symbols, it is to be borne in mind that they have been successively derived on the supposition that the variable x, that is the variable in the denominator, increases by constant increments.

Again; suppose (in Art. 33, 34.) that $f(x)$ is finite and continuous for all values of x between x and $x+h$; and suppose h to be equal to $n\,dx$,

whence $$n = \frac{h}{dx}$$

Then

$$f(x+h)=y+\frac{h}{dx}dy+\frac{\frac{h}{dx}\left(\frac{h}{dx}-1\right)}{1.2}d^2y$$

$$+\frac{\frac{h}{dx}\left(\frac{h}{dx}-1\right)\left(\frac{h}{dx}-2\right)}{1.2.3}d^3y+\&c.$$

$$=y+h\frac{dy}{dx}+\frac{h(h-dx)}{1.2}\frac{d^2y}{dx^2}$$

$$+\frac{h(h-dx)(h-2dx)}{1.2.3}\frac{d^3y}{dx^3}+\&c.$$

Let dx be very small, and therefore n very large; then, replacing y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. by their values $f(x)$, $f'(x)$, $f''(x)$, &c., and simplifying,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{1.2}f''(x)+\frac{h^3}{1.2.3}f'''(x)$$

$$+\frac{h^4}{1.2.3.4}f^{iv}(x)+\&c.;$$

which is the series known under the name of Taylor's Series; but, as it is of the utmost importance in the applications of the Calculus, another proof will be given in a subsequent chapter; and the present proof is to be considered of the same kind as that of Maclaurin's Series, given above in Art. 32.

36.] From this supposition, then, which we are allowed to make, of one variable increasing by equal increments, and, therefore, of the several differentials of it after the first vanishing, problems such as the following arise.

Having given an expression involving x, y $\{=f(x)\}$, and the several derived functions and differential coefficients, on the supposition that one of the variables is independent, to transform it into its equivalent, when neither of the variables is independent. Or,

To transform it into its equivalent, when the other variable is independent. Or, Having given an expression in which a variable and its differentials are involved, which is either an independent variable or not, and having given an equation connecting this variable with some other new variable, it is required to eliminate the old variable by means of these two equations, and to replace it in the original equation by its equivalents in terms of this new variable; this new variable being independent or not, as the case may be.

These several processes are called changes of the independent variable, and the best method is to replace the expression which has been simplified by the condition of a variable being independent, by its complete value when no such supposition has been made, and then to introduce whatever other conditions the problem requires.

As if, for instance, having given an equation involving x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c., the successive differential coefficients having been calculated on the supposition that x is an independent variable, it is required to replace these several differential coefficients by their equivalents when x is not independent.

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d.\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y\,dx - d^2x\,dy}{dx^3},$$

$$\frac{d^3y}{dx^2} = \frac{d.\left(\frac{d^2y}{dx^2}\right)}{dx} = \frac{(d^3x\,dy - d^3x\,dy)\,dx - 3\,(d^2y\,dx - d^2x\,dy)\,d^2x}{dx^5},$$

and so on, we must replace the several quantities $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c., by their equivalents on the right-hand side of the above equations, in which expressions neither x nor y is an independent variable: and if y is to be an independent variable in the new expression, $d^2y=0$, $d^3y=0$, &c., and the equivalents are as follow; viz.

$\frac{d^2y}{dx^2}$ must be replaced by $-\frac{d^2x\,dy}{dx^3}$,

$\frac{d^3y}{dx^3}$ . . . . $\frac{3\,(d^2x)^2\,dy - d^3x\,dy\,dx}{dx^5}$,

and so on.

And, again, suppose an equation is given involving x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c., in which x is the independent variable, and suppose an equation is given connecting x and a new variable θ, viz. $x = f(\theta)$, if it is required to eliminate x between these two equations, we must first replace the several differential coefficients by their complete values, and then calculate dx, d^2x, d^3x, &c., in terms of θ, $d\theta$, $d^2\theta$, &c.; whence we shall have an equation involving only y and θ and their differentials, in which we are at liberty to make whatever supposition is allowable as to either y or θ being independent.

Ex. 1. To transform $\dfrac{\left\{1+\dfrac{dy^2}{dx^2}\right\}^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}}$ into its equivalent, when (α) neither x nor y is independent; (β) when y is the independent variable.

(α) Replace $\frac{d^2y}{dx^2}$ by its complete expression as found above, and we have

$$\frac{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\dfrac{d^2x\,dy-d^2y\,dx}{dx^3}} = \frac{(dy^2+dx^2)^{\frac{3}{2}}}{d^2x\,dy-d^2y\,dx};$$

and then, (β) if y is independent, $d^2y=0$, and the expression becomes

$$\frac{(dy^2+dx^2)^{\frac{3}{2}}}{d^2x\,dy} = \frac{\left(1+\dfrac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}.$$

Ex. 2. Having given $u+\frac{1}{x}\,\frac{du}{dx}+\frac{d^2u}{dx^2}=0$, and $x^2=4\theta$, to eliminate x, and to find the equivalents of the first when (α) neither u nor θ is independent; (β) when θ is independent; (γ) when u is independent.

The complete expression of the given equation is

$$u+\frac{1}{x}\;\frac{du}{dx}+\frac{d^2u\,dx-d^2x\,du}{dx^3}=0.$$

$$x^2=4\theta, \quad \therefore \; dx=\frac{d\theta}{\sqrt{\theta}}, \quad \therefore \; d^2x=\frac{2\theta\,d^2\theta-d\theta^2}{2\theta^{\frac{3}{2}}}:$$

whence, by substitution,

$$(\alpha) \quad u\,d\theta^3+du\,d\theta^2+\theta\,d^2u\,d\theta-\theta\,du\,d^2\theta=0;$$

and, if θ is independent, $d^2\theta=0$, and we have

$$u\,d\theta^3+du\,d\theta^2+\theta\,d^2u\,d\theta=0;$$

$$(\beta) \quad \text{or} \qquad u+\frac{du}{d\theta}+\theta\frac{d^2u}{d\theta^2}=0,$$

and, if u is independent, $d^2u=0$, and we have

$$u\,d\theta^3+du\,d\theta^2-\theta\,du\,d^2\theta=0;$$

$$(\gamma) \qquad \theta\,\frac{d^2\theta}{du^2}-\frac{d\theta^2}{du^2}-u\,\frac{d\theta^3}{du^3}=0.$$

For other examples, see Gregory, chap. iii. sect. i.; and Hind, page 34.

B. *Successive Differentiation of Functions of Two or more Variables.*

37.] Let $u=\mathrm{F}(x, y, z \ldots)$ be the function of several variables of which it is required to find the successive differentials. To fix our thoughts, let us take a function of two variables, viz.

$$u=\mathrm{F}(x, y).$$

Now the first total differential of this, being a function of two variables as well as of the differentials of the variables, itself also admits of being differentiated again; and so does the second total differential, and so the nth. It is also to be remarked, that the partial derived functions of u are in general functions of both variables; and, therefore, the differentials of these are to be calculated for variations of both. Before proceeding further, however, we must prove the following proposition.

The successive partial derived functions of any function are independent of the order in which the several variables are supposed to change; that is, e. g.,

$$\left(\frac{d^2 u}{dx dy}\right) = \left(\frac{d^2 u}{dy dx}\right),$$

$$d_x \text{F}(x, y, z \dots) = \text{F}(x + dx, y, z \dots) - \text{F}(x, y, z \dots),$$

$$d_y \text{F}(x, y, z \dots) = \text{F}(x, y + dy, z \dots) - \text{F}(x, y, z \dots);$$

$$\therefore\ d_y d_x \text{F}(x, y, z \dots) = \text{F}(x + dx, y + dy, z \dots) - \text{F}(x + dx, y, z \dots)$$
$$- \text{F}(x, y + dy, z \dots) + \text{F}(x, y, z \dots).$$

Similarly,

$$d_x d_y \text{F}(x, y, z \dots) = \text{F}(x + dx, y + dy, z \dots) - \text{F}(x, y + dy, z \dots)$$
$$- \text{F}(x + dx, y, z \dots) + \text{F}(x, y, z \dots);$$

$$\therefore\ d_x d_y \text{F}(x, y, z \dots) = d_y d_x \text{F}(x, y, z \dots),$$

or, writing the result according to the notation in Art. 25.,

$$\left(\frac{d^2 u}{dx\, dy}\right) = \left(\frac{d^2 u}{dy\, dx}\right).$$

As the proof here given does not depend on the function being of two variables only, it is plain that an analogous theorem is true for a function of any number of variables; so that we may always interchange, in whatever manner it is convenient, the order in which the several differentiations are performed: as, for instance,

$$d_x d_y d_z u = d_y d_z d_x u = d_z d_y d_x u, \text{ \&c.}$$

$$\left(\frac{d^3 u}{dx\, dy\, dz}\right) = \left(\frac{d^3 u}{dy\, dz\, dx}\right) = \left(\frac{d^3 u}{dz\, dx\, dy}\right).$$

Hence, also, it follows, that if successive partial differentiations are performed on a function of several variables, by making x alone to vary r times, by making y to vary s times, by making z to vary t times, the order of these may be interchanged in whatever way we please, and we have the same results; so that

$$\left(\frac{d^{r+s+t}u}{dx^r\,dy^s\,dz^t}\right) = \left(\frac{d^{s+t+r}u}{dy^s\,dz^t\,dx^r}\right) = \left(\frac{d^{t+r+s}u}{dz^t\,dx^r\,dy^s}\right).$$

Ex. $u = \sin(ax + by + cz)$,

$$\left(\frac{du}{dx}\right) = a\cos(ax + by + cz), \left(\frac{du}{dy}\right) = b\cos(ax + by + cz),$$

$$\left(\frac{d^2u}{dy\,dx}\right) = -ab\sin(ax + by + cz) = \left(\frac{d^2u}{dx\,dy}\right).$$

For other examples, see Gregory, chapter ii. sect. ii.

38.] Let, then, the function of two variables be

$$u = \mathrm{F}(x, y);$$

then, by Art. 26. $\mathrm{D}u = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy$;

and taking the total differentials of each of these quantities, bearing in mind that $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ are in general functions of both variables, and that x and y need not be increased by the same increments as at first, and, therefore, dx and dy are to be differentiated, we have

$$\begin{aligned}\mathrm{D}^2u &= \mathrm{D}\left(\frac{du}{dx}\right)dx + \mathrm{D}\left(\frac{du}{dy}\right)dy + \left(\frac{du}{dx}\right)d^2x + \left(\frac{du}{dy}\right)d^2y,\\ &= \left(\frac{d^2u}{dx^2}\right)dx^2 + 2\left(\frac{d^2u}{dx\,dy}\right)dx\,dy + \left(\frac{d^2u}{dy^2}\right)dy^2\\ &\qquad + \left(\frac{du}{dx}\right)d^2x + \left(\frac{du}{dy}\right)d^2y;\end{aligned}$$

the brackets indicating, as in Art. 26., that the derived functions within them are only partial ones. Similarly,

$$\begin{aligned}\mathrm{D}^3u = &\left(\frac{d^3u}{dx^3}\right)dx^3 + 3\left(\frac{d^3u}{dx^2\,dy}\right)dx^2dy + 3\left(\frac{d^3u}{dx\,dy^2}\right)dx\,dy^2\\ &+ \left(\frac{d^3u}{dy^3}\right)dy^3 + 3\left[\left(\frac{d^2u}{dx^2}\right)dx\,d^2x + \left(\frac{d^2u}{dx\,dy}\right)\{d^2x\,dy + d^2y\,dx\}\right.\\ &\left.+ \left(\frac{d^2u}{dy^2}\right)dy\,d^2y\right] + \left(\frac{du}{dx}\right)d^3x + \left(\frac{du}{dy}\right)d^3y + \&c.\end{aligned}$$

In like manner may the other successive total differentials be formed.

But if x and y, the variables involved in the function u, are such that we may consider them always to increase by constant increments, then dx and dy, upon this hypothesis, admit of no variation, and

$$d^2x = 0,\ d^3x = 0,\ \&c. \quad d^2y = 0,\ d^3y = 0,\ \&c.;$$

and the expressions become

$$u = \mathrm{F}(x,y),$$

$$\mathrm{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy,$$

$$\mathrm{D}^2u = \left(\frac{d^2u}{dx^2}\right) dx^2 + 2\left(\frac{d^2u}{dx\,dy}\right) dx\,dy + \left(\frac{d^2u}{dy^2}\right) dy^2,$$

$$\mathrm{D}^3u = \left(\frac{d^3u}{dx^3}\right) dx^3 + 3\left(\frac{d^3u}{dx^2\,dy}\right) dx^2\,dy + 3\left(\frac{d^3u}{dx\,dy^2}\right) dx\,dy^2 + \left(\frac{d^3u}{dy^3}\right) dy^3,$$

and so on; the law of the coefficients being the same as that of $(1+x)^n$, which may be proved to be true for positive integral values of the exponent, by a train of reasoning similar to that in Art. 30.; whence, writing down the nth differential, we have

$$\mathrm{D}^n u = \left(\frac{d^n u}{dx^n}\right) dx^n + n\left(\frac{d^n u}{dx^{n-1}dy}\right) dx^{n-1}dy + \frac{n(n-1)}{1\,.\,2}\left(\frac{d^n u}{dx^{n-2}dy^2}\right) dx^{n-2}dy^2 + \&c.$$

Similarly, if $u = \mathrm{F}(x, y, z \ldots)$, a function of several variables, and x, y, z, &c., increase by constant increments,

$$\mathrm{D}u = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz + \&c.,$$

$$\mathrm{D}^2u = \left(\frac{d^2u}{dx^2}\right) dx^2 + \left(\frac{d^2u}{dy^2}\right) dy^2 + \left(\frac{d^2u}{dz^2}\right) dz^2 + 2\left(\frac{d^2u}{dy\,dz}\right) dy\,dz + 2\left(\frac{d^2u}{dz\,dx}\right) dz\,dx + 2\left(\frac{d^2u}{dx\,dy}\right) dx\,dy + \&c.$$

and so on.

If x, y, z do not increase by constant increments, then terms must be added to these expressions analogous to those which have disappeared in the above corresponding expressions for functions of two variables.

These expressions may also be written as under, in forms similar to those in Art. 25.

$$\mathrm{D}u = d_x u + d_y u + d_z u + \&c.$$

$$\mathrm{D}^2 u = d_x^2 u + d_y^2 u + d_z^2 u + 2\, d_y d_z u + 2\, d_z d_x u + 2\, d_x d_y u + \&c.$$

and then assume such simple symbolical forms, that we ought not to omit writing them down; viz.,

$$\mathrm{D}u = (d_x + d_y + d_z + \ldots)\, u,$$
$$\mathrm{D}^2 u = (d_x + d_y + d_z \ldots)^2\, u,$$
$$\mathrm{D}^3 u = (d_x + d_y + d_z + \ldots)^3\, u,$$
$$\cdots\cdots\cdots$$
$$\mathrm{D}^n u = (d_x + d_y + d_z + \ldots)^n\, u.$$

Ex. $$u = e^{ax+by+cz+\&c.}$$

$$\left(\frac{du}{dx}\right) = a e^{ax+by+cz+\&c.}$$

$$\left(\frac{du}{dy}\right) = b e^{ax+by+cz+\&c.}$$

$$\left(\frac{du}{dz}\right) = c e^{ax+by+cz+\&c.}$$

$$\left(\frac{d^2u}{dx^2}\right) = a^2 e^{ax+by+cz+\&c.}$$

&c. &c.

$$\mathrm{D}u = e^{ax+by+cz+\ldots}\{a dx + b dy + c dz + \ldots\},$$

$$\mathrm{D}^2 u = e^{ax+by+cz+\ldots}\{a^2 dx^2 + b^2 dy^2 + c^2 dz^2 + 2\, bc dy dz + 2 ca dz dx + 2 ab dx dy + \ldots\},$$

$$= e^{ax+by+cz+\ldots}\{a dx + b dy + c dz + \ldots\}^2.$$

Similarly,

$$\mathrm{D}^3 u = e^{ax+by+cz+\ldots}\{a dx + b dy + c dz + \ldots\}^3,$$
$$\mathrm{D}^n u = e^{ax+by+cz+\ldots}\{a dx + b dy + c dz + \ldots\}^n.$$

For examples, see Gregory, chap. ii. sect. ii.

39.] Suppose that the function is of the form

$$u = \mathrm{F}(x, y) = c,$$

c being a constant, which is the general form of an implicit function of two variables, then

$$\mathrm{D}u = 0 = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy,$$

$$\mathrm{D}^2 u = 0 = \left(\frac{d^2 u}{dx^2}\right) dx^2 + 2\left(\frac{d^2 u}{dx\,dy}\right) dx\,dy + \left(\frac{d^2 u}{dy^2}\right) dy^2 + \left(\frac{du}{dx}\right) d^2x + \left(\frac{du}{dy}\right) d^2y,$$

$$\mathrm{D}^3 u = 0 = \left(\frac{d^3 u}{dx^3}\right) dx^3 + \&\text{c}.$$

whence the following expressions, considering x to increase by constant increments, and $\therefore\ d^2x = 0$,

$$\left(\frac{du}{dy}\right)\frac{dy}{dx} + \left(\frac{du}{dx}\right) = 0,$$

$$\left(\frac{du}{dy}\right)\frac{d^2y}{dx^2} + 2\left(\frac{d^2u}{dx\,dy}\right)\frac{dy}{dx} + \left(\frac{d^2u}{dy^2}\right)\frac{dy^2}{dx^2} + \left(\frac{d^2u}{dx^2}\right) = 0,$$

which give us the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, without solving the original equation, and putting it in the explicit form.

As e.g.
$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\left(\frac{du}{dx}\right) = \frac{2x}{a^2}, \qquad \left(\frac{du}{dy}\right) = \frac{2y}{b^2},$$

$$\left(\frac{d^2u}{dx^2}\right) = \frac{2}{a^2}, \qquad \left(\frac{d^2u}{dx\,dy}\right) = 0, \qquad \left(\frac{d^2u}{dy^2}\right) = \frac{2}{b^2};$$

$\therefore$ from the formulæ
$$\frac{dy}{dx} = -\frac{b^2x}{a^2y},$$

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$$

40.] First Application of the preceding theory to the Elimination of arbitrary Functions of two or more Variables.

Let us first take a simple case; viz. that in which u is an arbitrary function of two variables, x and y, of the form $u = f(x, y)$, where the form of the function is undetermined. Calculating the partial differential coefficients of u with respect to the variations of x and y, in accordance with what has been said in Art. 26., we shall obtain two equations involving the same derived function, viz. $f'(x, y)$, which we can eliminate between them, and thus obtain an equation involving $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, but independent of the original primitive function. Similarly, if u involves several arbitrary functions of x and y, we can, by forming the successive partial differential coefficients, obtain expressions involving the same derived functions of u, which we can eliminate, and obtain equations not involving the arbitrary functions. The following examples will explain our meaning.

Ex. 1.
$$u = ax + by + cf(ex + gy),$$
$$\left(\frac{du}{dx}\right) = a + ecf'(ex + gy),$$
$$\left(\frac{du}{dy}\right) = b + gcf'(ex + gy);$$
$$\therefore \quad g\left(\frac{du}{dx}\right) - e\left(\frac{du}{dy}\right) = ag - eb,$$

which is the final equation, independent of the arbitrary function, but involving the partial differential coefficients, viz. $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$.

Ex. 2.
$$u = f(ax + by) + \phi(bx - ay),$$
$$\left(\frac{du}{dx}\right) = af'(ax + by) + b\phi'(bx - ay),$$
$$\left(\frac{du}{dy}\right) = bf'(ax + by) - a\phi'(bx - ay),$$
$$\therefore \quad a\left(\frac{du}{dx}\right) + b\left(\frac{du}{dy}\right) = (a^2 + b^2)f'(ax + by).$$

And differentiating again:

$$a\left(\frac{d^2u}{dx^2}\right) + b\left(\frac{d^2u}{dx\,dy}\right) = a\,(a^2+b^2)\,f''(ax+by),$$

$$a\left(\frac{d^2u}{dy\,dx}\right) + b\left(\frac{d^2u}{dy^2}\right) = b\,(a^2+b^2)\,f''(ax+by);$$

$$\therefore \quad ab\left\{\left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right)\right\} + (b^2-a^2)\left(\frac{d^2u}{dx\,dy}\right) = 0.$$

41.] If three variables are implicitly involved in the given expression and the arbitrary functions, the same method as above may be followed in forming the partial derived functions, if we consider one of the variables to be a function of the other two, and calculate its partial variations due to the variations of the others on this supposition.

Ex. $$\frac{y-b}{z-c} = \phi\left(\frac{x-a}{z-c}\right).$$

Let us consider z to be a function of x and y, and calculate the partial derived functions of z on this supposition; then

$$-\frac{y-b}{(z-c)^2}\left(\frac{dz}{dx}\right) = \phi'\left(\frac{x-a}{z-c}\right)\frac{(z-c)-(x-a)\left(\frac{dz}{dx}\right)}{(z-c)^2},$$

$$\frac{(z-c)-(y-b)\left(\frac{dz}{dy}\right)}{(z-c)^2} = -\phi'\left(\frac{x-a}{z-c}\right)\frac{x-a}{(z-c)^2}\left(\frac{dz}{dy}\right);$$

whence, by division and reduction,

$$(x-a)\left(\frac{dz}{dx}\right) + (y-b)\left(\frac{dz}{dy}\right) = z-c.$$

By a similar process, if we had considered x to be a function of y and z, and had taken the partial differential coefficients $\left(\frac{dx}{dy}\right)$ and $\left(\frac{dx}{dz}\right)$, we should have found,

$$x-a = \left(\frac{dx}{dz}\right)(z-c) + \left(\frac{dx}{dy}\right)(y-b).$$

Similarly, also, might y have been considered a function of x and z; and, if the successive derived functions had been formed on this supposition, we should have obtained a final equation of the same form as the two above.

42.] Again, if the given equation involve an arbitrary function of three variables, x, y, z; that is, if the equation be of the form

$$u = f(x, y, z);$$

we can form three partial differential coefficients of u, according as each of the three variables changes; and each of these expressions will involve the same derived function, between any two of which it may be eliminated, and we shall thus obtain three several equations independent of the original arbitrary function: or, if the original equation contain two arbitrary functions, we can eliminate their derived functions between the three several partial derived functions, and thus obtain an expression independent of them. As, for instance,

$$u = f(ax^2+by^3+cz^4)+\phi(\cos lx+\cos my+\cos nz),$$

$$\left(\frac{du}{dx}\right) = 2axf'(ax^2+by^3+cz^4)$$
$$-l\sin lx\,\phi'(\cos lx+\cos my+\cos nz),$$

$$\left(\frac{du}{dy}\right) = 3by^2f'(ax^2+by^3+cz^4)$$
$$-m\sin my\,\phi'(\cos ly+\cos my+\cos nz),$$

$$\left(\frac{du}{dz}\right) = 4cz^3f'(ax^2+by^3+cz^4)$$
$$-n\sin nz\,\phi'(\cos lx+\cos my+\cos nz);$$

$\therefore$ by Lagrange's method of cross-multiplication (see Preliminary Proposition IV.) we have

$$\left(\frac{du}{dx}\right)\{4cmz^3\sin my-3bny^2\sin nz\}+\left(\frac{du}{dy}\right)\{2anx\sin nz$$
$$-4clz^3\sin lx\}+\left(\frac{du}{dz}\right)\{3bly^2\sin lx-2amx\sin my\}=0,$$

an equation independent of the arbitrary functions, and therefore expressive of the properties of such functions, whatever be their specific forms. By similar methods we may eliminate arbitrary functions of any number of variables.

43.] In general, for determining to what order of differentiation we must proceed to eliminate any number of arbitrary functions from an expression containing two variables, let the following considerations suffice.

Suppose $u=0$ to comprise m arbitrary functions of x and y, then it is plain that each successive differentiation introduces m other arbitrary functions, which are the derived of the given functions; so that by proceeding to the nth order of differentiation, we have $(n+1)\,m$ different arbitrary functions: but, as the original equation $u=0$ gives one relation amongst these functions, so do

$$\left(\frac{du}{dx}\right)=0, \qquad \left(\frac{du}{dy}\right)=0,$$

$$\left(\frac{d^2u}{dx^2}\right)=0, \quad \left(\frac{d^2u}{dx\,dy}\right)=0, \quad \left(\frac{d^2u}{dy^2}\right)=0,$$

$$\cdots\cdots\cdots\cdots$$

$$\left(\frac{d^nu}{dx^n}\right)=0, \left(\frac{d^nu}{dx^{n-1}dy}\right)=0, \ldots \ldots \left(\frac{d^nu}{dy^n}\right)=0,$$

give us other relations; so that by means of n differentiations we have the number of relations equal to

$$1+2+3+\ldots\ldots+(n+1)=\frac{(n+1)(n+2)}{2}.$$

And in order that we may be able to eliminate all these, we must evidently have

$$\frac{(n+1)(n+2)}{1.2} > (n+1)\,m;$$

that is $\qquad n+2 > 2m,$

$$n > 2m-2;$$

that is n, which expresses the order of differentiation, must $=$ $2m-1$ at least, and we shall then have a sufficient number of equations to eliminate the arbitrary functions from. Thus, if the original equation involve but one arbitrary function, then $m=1$, and we need differentiate but once; if it involve two arbitrary functions, we must in the general case differentiate thrice, and so on. An example is subjoined in which three differentiations are required:

$$u = f(x+y) + xy\,\phi(x-y).$$

$$\left(\frac{du}{dx}\right) = f'(x+y) + y\,\phi(x-y) + xy\,\phi'(x-y),$$

$$\left(\frac{du}{dy}\right) = f'(x+y) + x\,\phi(x-y) - xy\,\phi'(x-y),$$

$$\left(\frac{du}{dx}\right) - \left(\frac{du}{dy}\right) = (y-x)\,\phi(x-y) + 2xy\,\phi'(x-y),$$

$$\therefore\ \left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dx\,dy}\right) = (y-x)\,\phi'(x-y) - \phi(x-y) + 2y\,\phi'(x-y) + 2xy\,\phi''(x-y);$$

$$\left(\frac{d^2u}{dx\,dy}\right) - \left(\frac{d^2u}{dy^2}\right) = -(y-x)\,\phi'(x-y) + \phi(x-y) + 2x\,\phi'(x-y) - 2xy\,\phi''(x-y),$$

$$\therefore\ \left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right) = 2(x+y)\,\phi'(x-y).$$

And differentiating again,

$$\left(\frac{d^3u}{dx^3}\right) - \left(\frac{d^3u}{dx\,dy^2}\right) = 2\phi'(x-y) + 2(x+y)\,\phi''(x-y),$$

$$\left(\frac{d^3u}{dy\,dx^2}\right) - \left(\frac{d^3u}{dy^3}\right) = 2\phi'(x-y) - 2(x+y)\,\phi''(x-y),$$

$$\therefore\ \left(\frac{d^3u}{dx^3}\right) + \left(\frac{d^3u}{dx^2\,dy}\right) - \left(\frac{d^3u}{dx\,dy^2}\right) - \left(\frac{d^3u}{dy^3}\right) = 4\phi'(x-y);$$

$$\text{but}\quad \phi'(x-y) = \frac{1}{2(x+y)}\left\{\left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right)\right\},$$

$$\therefore \quad \left(\frac{d^3u}{dx^3}\right) + \left(\frac{d^3u}{dx^2\,dy}\right) - \left(\frac{d^3u}{dx\,dy^2}\right) - \left(\frac{d^3u}{dy^3}\right)$$
$$= \frac{2}{x+y}\left\{\left(\frac{d^2u}{dx^2}\right) - \left(\frac{d^2u}{dy^2}\right)\right\}.$$

For examples, see Gregory's *Collection*, p. 46. ex. 15. ed. 1.

44.] Second Application. — To transform an expression involving partial Differentials and partial derived Functions into their Equivalents, when equations are given connecting the original variables with new variables.

Let us, as in the last Article, first consider the more simple case of a function of two variables. Suppose, for instance, an equation to be given involving $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$, dx, dy; $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ having been calculated on the supposition that x varied while y remained constant, and y varied while x was constant, and dx and dy being partial changes in x and y on similar suppositions. It is manifest that the very fact of there being such expressions as $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ imports that there is such an expression as $u = f(x,y)$; whether the expression can be found or not is immaterial.

Suppose that the variables x and y are connected with other new variables r and θ by means of equations of the form

$$x = \phi_1(r,\theta),$$
$$y = \phi_2(r,\theta).$$

If we substitute these values of x and y in the expressed or understood function $u = f(x,y)$, the equation becomes of the form

$$u = \text{F}(r,\theta);$$

the total differential of which is

$$\text{D}u = \left(\frac{d\text{F}}{dr}\right) dr + \left(\frac{d\text{F}}{d\theta}\right) d\theta;$$

by means of which we have, dividing through successively by dx and dy, and changing $\frac{\mathrm{D}u}{dx}$ into $\left(\frac{du}{dx}\right)$, and $\frac{\mathrm{D}u}{dy}$ into $\left(\frac{du}{dy}\right)$, since in these cases we have the ratio of the variations of u and of x, and of u and of y, in accordance with what has been said in Art. 25., and bracketing them to indicate that they are partial differential coefficients,

$$\left(\frac{du}{dx}\right) = \left(\frac{d\mathrm{F}}{dr}\right)\frac{dr}{dx} + \left(\frac{d\mathrm{F}}{d\theta}\right)\frac{d\theta}{dx},$$

$$\left(\frac{du}{dy}\right) = \left(\frac{d\mathrm{F}}{dr}\right)\frac{dr}{dy} + \left(\frac{d\mathrm{F}}{d\theta}\right)\frac{d\theta}{dy};$$

remembering that $\frac{dr}{dx}$, $\frac{d\theta}{dx}$ are to be calculated on the supposition that y does not vary, that is, that $dy = 0$; and $\frac{dr}{dy}$, $\frac{d\theta}{dy}$, on the supposition that $dx = 0$. Since then, from the two equations given above, we have

$$dx = \left(\frac{d\phi_1}{dr}\right)dr + \left(\frac{d\phi_1}{d\theta}\right)d\theta,$$

$$dy = \left(\frac{d\phi_2}{dr}\right)dr + \left(\frac{d\phi_2}{d\theta}\right)d\theta,$$

$\therefore$ to calculate dx, let $dy = 0$, and eliminating $d\theta$ and dr between these two equations on this supposition we have

$$\frac{dx}{dr} = \frac{\frac{d\phi_1}{dr}\frac{d\phi_2}{d\theta} - \frac{d\phi_2}{dr}\frac{d\phi_1}{d\theta}}{\frac{d\phi_2}{d\theta}},$$

$$\frac{dx}{d\theta} = \frac{\frac{d\phi_1}{d\theta}\frac{d\phi_2}{dr} - \frac{d\phi_2}{d\theta}\frac{d\phi_1}{dr}}{\frac{d\phi_2}{dr}}.$$

Similarly, $$\frac{dy}{dr} = \frac{\frac{d\phi_1}{d\theta}\frac{d\phi_2}{dr} - \frac{d\phi_2}{d\theta}\frac{d\phi_1}{dr}}{\frac{d\phi_1}{d\theta}},$$

$$\frac{dy}{d\theta} = \frac{\frac{d\phi_2}{d\theta}\frac{d\phi_1}{dr} - \frac{d\phi_1}{d\theta}\frac{d\phi_2}{dr}}{\frac{d\phi_1}{dr}}.$$

All these differential coefficients being partial ones, if we substitute these quantities in the expressions above for $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, the resulting expressions will be the equivalents of $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$, when x and y are replaced by their equivalents in terms of r and θ.

45.] If the expression which is to be transformed into its equivalent involve only the partial differentials dx and dy, then dx and dy must be calculated as in the last six equations, on the supposition that each remains constant while the other varies.

If the expression to be transformed involves three variables x, y, z, and these are given in terms of three other variables r, θ, ϕ, the operations are to be effected in a similar manner; but as a consideration of the particular forms of the functions connecting the variables will usually shorten the process, and as the principle is the same as that involved in the last Article, it is of little use to calculate the general expressions, and therefore we forbear to give them.

Ex. 1. To transform $\left(\frac{d\mathrm{R}}{dx}\right)$ and $\left(\frac{d\mathrm{R}}{dy}\right)$ into their equivalents, having given,

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

In this problem there is implied a function $\mathrm{R} = f(x, y)$, which becomes, when x and y are replaced by their equivalents,

$$\text{R} = \text{F}(r, \theta);$$

$$\therefore \quad \left(\frac{d\text{R}}{dx}\right) = \left(\frac{d\text{R}}{dr}\right)\frac{dr}{dx} + \left(\frac{d\text{R}}{d\theta}\right)\frac{d\theta}{dx},$$

$$\left(\frac{d\text{R}}{dy}\right) = \left(\frac{d\text{R}}{dr}\right)\frac{dr}{dy} + \left(\frac{d\text{R}}{d\theta}\right)\frac{d\theta}{dy}.$$

$$dx = dr\cos\theta - r\sin\theta\, d\theta,$$

$$dy = dr\sin\theta + r\cos\theta\, d\theta.$$

To calculate dx, let $dy = 0$, and eliminating dr and $d\theta$ between the equations in turn, we have

$$\frac{dx}{dr} = \frac{1}{\cos\theta}, \qquad \frac{dx}{d\theta} = -\frac{r}{\sin\theta};$$

similarly we have

$$\frac{dy}{dr} = \frac{1}{\sin\theta}, \qquad \frac{dy}{d\theta} = \frac{r}{\cos\theta};$$

$$\therefore \quad \left(\frac{d\text{R}}{dx}\right) = \left(\frac{d\text{R}}{dr}\right)\cos\theta - \left(\frac{d\text{R}}{d\theta}\right)\frac{\sin\theta}{r},$$

$$\left(\frac{d\text{R}}{dy}\right) = \left(\frac{d\text{R}}{dr}\right)\sin\theta + \left(\frac{d\text{R}}{d\theta}\right)\frac{\cos\theta}{r};$$

whence we have two transformations useful in the planetary theory, viz.

$$x\left(\frac{d\text{R}}{dy}\right) - y\left(\frac{d\text{R}}{dx}\right) = \left(\frac{d\text{R}}{d\theta}\right),$$

$$x\left(\frac{d\text{R}}{dx}\right) + y\left(\frac{d\text{R}}{dy}\right) = r\left(\frac{d\text{R}}{dr}\right).$$

Ex. 2. Having given $x = r\cos\theta$, $y = r\sin\theta$, to transform $dx\,dy$ into its equivalent, subject to the conditions that y does not vary when x varies, and x does not vary when y varies.

$$dx = dr\cos\theta - r\sin\theta\, d\theta,$$

$$dy = dr\sin\theta + r\cos\theta\, d\theta.$$

$$\therefore \quad \text{as before,} \quad dx = \frac{1}{\cos\theta}\, dr.$$

Whence it follows that when $dx = 0$, that is, when y varies, $dr = 0$.

$$\therefore \quad dy = r\cos\theta\, d\theta.$$

$$\therefore \quad dy\,dx = r\,dr\,d\theta.$$

For more examples, see Gregory's *Collection*, p. 37.

CHAP. IV.

ON CERTAIN RELATIONS BETWEEN FUNCTIONS AND DERIVED FUNCTIONS, ON WHICH THE APPLICATIONS OF THE DIFFERENTIAL CALCULUS DEPEND.

IN the last two chapters rules have been constructed whereby functions may be differentiated and derived functions formed; it remains now to prove certain relations amongst them, and their applications, in order to which the following theorems are necessary.

46.] THEOREM I. Given that $y = f(x)$ is a continuous function of x, for $x = x_0$, and for values a little greater and a little less than x_0: then, if $f'(x_0)$ is positive, x and $f(x)$ are, for that particular value, increasing or decreasing simultaneously; and if $f'(x_0)$ is negative, as x increases and passes through x_0, $f(x)$ is decreasing, or *vice versâ.*

Let Δy and Δx be, as before, the simultaneous and finite changes in the values of y and x; then it is plain that according as $\frac{\Delta y}{\Delta x}$ is positive or negative, so is $\frac{dy}{dx}$ or $f'(x)$ which is its limit. Since, then,

$$y = f(x),$$

$$y + \Delta y = f(x + \Delta x),$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x}.$$

On the supposition that $\frac{\Delta y}{\Delta x}$ is positive, the numerator and denominator of the fraction must have the same signs; and therefore if $x + \Delta x$ is $> x$, that is if x increases, $f(x + \Delta x)$ is $>$ than $f(x)$: but if $x + \Delta x$ is $< x$, that is if x decreases, then $f(x + \Delta x)$

is $<$ than $f(x)$; and the same is true of the limiting value when Δx and Δy become dx and dy. So, again, if $\frac{\Delta y}{\Delta x}$ be negative, the numerator and denominator must have different signs; and therefore, if x increases, $f(x)$ must decrease; and if x decreases, $f(x)$ increases, and the same will be true in the limit. Hence, we conclude that if $\frac{dy}{dx} = f'(x)$ be positive for $x = x_0$, at that particular value x and $f(x)$ are increasing or decreasing simultaneously; and if $f'(x)$ is negative when $x = x_0$, as x increases $f(x)$ decreases, or as x decreases $f(x)$ increases.

Hence, if there is a function of x, $y = f(x)$, which is continuous for all values of x from $x = x_0$ to $x = x_1$, x_1 being $>$ than x_0, as long as $f'(x)$ is positive, x and $f(x)$ are increasing or decreasing simultaneously; and if it be negative, then as x increases $f(x)$ decreases, and *vice versâ*.

Hence, too, it is plain that if $f(x) = 0$ when $x = 0$, if $f'(x)$ be positive and finite, $f(x)$ is > 0 when $x > 0$, and $f(x) < 0$ when x is < 0; but if, on the contrary, $f'(x)$ be finite and negative, $f(x)$ is < 0 when x is > 0, and $f(x)$ is > 0 when x is < 0.

As illustrations of this theorem, let us consider the following examples.

$$\text{Let} \quad y = f(x) = \sin x, \qquad \therefore \quad \frac{dy}{dx} = f'(x) = \cos x.$$

First, let x have any value between 0 and $\frac{\pi}{2}$; then $\cos x$ is positive, and x and $\sin x$ are increasing or decreasing simultaneously. Secondly, let x have a value between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, for which values $\cos x$ is negative, and as x increases $\sin x$ decreases; and so on for other values of the arc: and therefore in this particular example, as long as $f'(x)$ is positive, the function and the variable are increasing or decreasing simultaneously; and if $f'(x)$ is negative, there is an increase of the one corresponding to a decrease of the other. Similarly, too, if $f(x) = \cos x$, in which case $f'(x) = -\sin x$, the cosine decreases as the arc increases from 0 to π, and increases as the arc increases from π to 2π.

It is necessary to observe also that $f'(x)$ cannot change its sign without passing through 0 or $\frac{1}{0}$ and therefore, if x is continually increasing, there cannot be a change in the corresponding increase or decrease of $f(x)$, unless $f'(x)$ becomes 0 or $\frac{1}{0}$.

47.] THEOREM II. If $F(x)$ and $f(x)$ be two functions of x, continuous in value for all values of x between x_0 and $x_0 + h$, and if their first-derived functions $F'(x)$ and $f'(x)$ be the same, and if, in addition, $f(x)$ is always increasing or decreasing with x between these limits, that is $f'(x)$ does not change its sign, then

$$\frac{F(x_0 + h) - F(x_0)}{f(x_0 + h) - f(x_0)} = \frac{F'(x_0 + \theta h)}{f'(x_0 + \theta h)},$$

where θ is some numerical quantity greater than 0 and less than 1.

Let us divide the quantity h into n equal parts, each equal to dx (see Art. 34.), so that $n dx = h$. Consider the series of fractions:

$$\frac{F(x_0 + dx) - F(x_0)}{f(x_0 + dx) - f(x_0)}, \frac{F(x_0 + 2dx) - F(x_0 + dx)}{f(x_0 + 2dx) - f(x_0 + dx)}, \ldots$$

$$\ldots \frac{F(x_0 + ndx) - F\{x_0 + (n-1)dx\}}{f(x_0 + ndx) - f\{x_0 + (n-1)dx\}},$$

which is a series of unequal fractions, the denominators of which are always affected with the same signs, since between the limits $f(x)$ is always increasing or decreasing with x, and therefore by Preliminary Proposition I. the ratio of the sum of the numerators to the sum of the denominators is equal to some mean value of the fractions; but the ratio of the sum of the numerators to the sum of the denominators is, bearing in mind that $n dx = h$, $\frac{F(x_0 + h) - F(x_0)}{f(x_0 + h) - f(x_0)}$, and therefore this ratio is equal to some quantity greater than the least, and less than the greatest, of the series of fractions. These fractions, however, can be put under other forms. Divide every numerator and every denominator by dx, and we have

$$\frac{\frac{F(x_0+dx)-F(x_0)}{dx}}{\frac{f(x_0+dx)-f(x_0)}{dx}},\quad \frac{\frac{F(x_0+2dx)-F(x_0+dx)}{dx}}{\frac{f(x_0+2dx)-f(x_0+dx)}{dx}},\ \ldots$$

$$\ldots\ldots\ \frac{\frac{F(x_0+ndx)-F\{x_0+(n-1)dx\}}{dx}}{\frac{f(x_0+ndx)-f\{x_0+(n-1)dx\}}{dx}};$$

which become, when dx is very small, that is, when the quantity h is resolved into a very large number of elements,

$$\frac{F'(x_0)}{f'(x_0)},\ \frac{F'(x_0+dx)}{f'(x_0+dx)},\ \ldots\ldots\ \frac{F'\{x_0+(n-1)dx\}}{f'\{x_0+(n-1)dx\}}:$$

the last of which is $\frac{F'(x_0+h-dx)}{f'(x_0+h-dx)}$, and therefore a mean value is $\frac{F'(x_0+\theta h)}{f'(x_0+\theta h)}$, θ being some numerical quantity greater than 0 and less than 1: greater than 0, I say, because $\frac{F'(x_0)}{f'(x_0)}$ would be the value only if all were equal, which is not in general the case; and less than 1, because all the fractions are not equal to $\frac{F'(x_0+h)}{f'(x_0+h)}$; and therefore we conclude

$$\frac{F(x_0+h)-F(x_0)}{f(x_0+h)-f(x_0)} = \frac{F'(x_0+\theta h)}{f'(x_0+\theta h)}, \qquad (1)$$

subject to the following conditions: $F(x)$, $F'(x)$, $f(x)$, $f'(x)$ being continuous and finite for all values of x between x_0 and x_0+h, and $f'(x)$ not changing sign within these limits.

It will be observed that we have divided the quantity h into *equal* parts; but the theorem is equally true in whatever manner it be divided.

In order the better to understand this important proposition let us consider the following instance:

$F(x) = x^4$, $f(x) = x^2$, $F'(x) = 4x^3$, $f'(x) = 2x$; and the requisite conditions are fulfilled when x is considered to be some positive quantity.

$$\frac{\mathrm{F}(x_0+h)-\mathrm{F}(x_0)}{f(x_0+h)-f(x_0)} = \frac{(x_0+h)^4-x_0^4}{(x_0+h)^2-x_0^2} = (x_0+h)^2+x_0^2$$
$$= 2x_0^2+2x_0h+h^2;$$

and $$\frac{\mathrm{F}'(x_0+\theta h)}{f'(x_0+\theta h)} = \frac{4(x_0+\theta h)^3}{2(x_0+\theta h)} = 2(x_0+\theta h)^2 = 2x_0^2+4x_0\theta h$$
$$+ 2\theta^2 h^2;$$

which two quantities being equated to one another in accordance with the theorem above, we have

$$2x_0+h = 4x_0\theta + 2h\theta^2;$$

and θ is manifestly some quantity greater than 0 and less than 1.

48.] *Corollary.* If the two functions $\mathrm{F}(x)$, $f(x)$, be such that $\mathrm{F}(x_0)=0$, $f(x_0)=0$, we have

$$\frac{\mathrm{F}(x_0+h)}{f(x_0+h)} = \frac{\mathrm{F}'(x_0+\theta h)}{f'(x_0+\theta h)};$$

and putting $\theta h = h_1$, h_1 being less than h,

$$\frac{\mathrm{F}(x_0+h)}{f(x_0+h)} = \frac{\mathrm{F}'(x_0+h_1)}{f'(x_0+h_1)}.$$

Suppose that, besides all these conditions, $\mathrm{F}'(x_0)=0$, $f'(x_0)=0$: then, considering $\mathrm{F}'(x)$ and $f'(x)$ as new functions of x, having for derived functions $\mathrm{F}''(x)$ and $f''(x)$, and being continuous and finite between the limits x_0 and x_0+h_1; and if, in addition, $f'(x)$ is always increasing or decreasing between these limits, so that $f''(x)$ does not become 0 or $\frac{1}{0}$ by changing its sign; then

$$\frac{\mathrm{F}'(x_0+h_1)}{f'(x_0+h_1)} = \frac{\mathrm{F}''(x_0+\theta h_1)}{f''(x_0+\theta h_1)};$$

and putting $\theta h_1 = h_2$, h_2 being less than h_1,

$$\frac{\mathrm{F}(x_0+h)}{f(x_0+h)} = \frac{\mathrm{F}''(x_0+h_2)}{f''(x_0+h_2)}.$$

In the same way, if $\mathrm{F}''(x_0)=0$, $f''(x_0)=0$; and their derived functions, $\mathrm{F}'''(x)$, $f'''(x)$, are finite and continuous for all values of x between the limits x_0 and x_0+h_2; and if, in addition, $f''(x)$ always increases or decreases between these same limits; we have

$$\frac{\mathrm{F}''(x_0+h_2)}{f''(x_0+h_2)}=\frac{\mathrm{F}'''(x_0+\theta h_2)}{f'''(x_0+\theta h_2)}=\frac{\mathrm{F}'''(x_0+h_3)}{f'''(x_0+h_3)},$$

and so on. Now, remembering that h_3 is less than h_2, and that h_2 is less than h_1, which is itself less than h; and that, therefore, h_3 is of the form θh, where θ is some positive quantity less than 1; if these several conditions, and other similar to them, hold good up the nth derived functions exclusively, we have the following result. If there are two functions of x, $\mathrm{F}(x)$ and $f(x)$ finite and continuous, for all values of x between x_0 and x_0+h; and if all the derived functions up to the nth inclusively are finite and continuous also; and if, besides,

$$\mathrm{F}(x_0)=0,\quad \mathrm{F}'(x_0)=0,\quad \mathrm{F}''(x_0)=0,\quad \ldots\ldots \mathrm{F}^{n-1}(x_0)=0,$$

$$f(x_0)=0,\quad f'(x_0)=0,\quad f''(x_0)=0,\quad \ldots\ldots f^{n-1}(x_0)=0;$$

and $f(x)$, $f'(x)$, $f''(x)\ldots f^{n-1}(x)$ are such functions as always to increase or decrease between these same limits; then

$$\frac{\mathrm{F}(x_0+h)}{f(x_0+h)}=\frac{\mathrm{F}^n(x_0+\theta h)}{f^n(x_0+\theta h)}. \tag{2}$$

49.] If all the preceding conditions are fulfilled, except that $\mathrm{F}(x_0)$ and $f(x_0)$ did not vanish, a similar proposition is true, for $\frac{\mathrm{F}(x_0+h)-\mathrm{F}(x_0)}{f(x_0+h)-f(x_0)}$ would be equal to $\frac{\mathrm{F}'(x_0+\theta h)}{f'(x_0+\theta h)}$; and this latter would, if $\mathrm{F}'(x_0)=0$, &c., be equal to $\frac{\mathrm{F}''(x_0+\theta h)}{f''(x_0+\theta h)}$, and so on; and at last we should have

$$\frac{\mathrm{F}(x_0+h)-\mathrm{F}(x_0)}{f(x_0+h)-f(x_0)}=\frac{\mathrm{F}^n(x_0+\theta h)}{f^n(x_0+\theta h)}. \tag{3}$$

Suppose that in the results of this Article and the last the lower limit $x_0=0$, then the equations respectively become

$$\frac{\mathrm{F}(h)}{f(h)}=\frac{\mathrm{F}^n(\theta h)}{f^n(\theta h)},\text{ or writing } x \text{ for } h\ \frac{\mathrm{F}(x)}{f(x)}=\frac{\mathrm{F}^n(\theta x)}{f^n(\theta x)}; \tag{4}$$

$$\text{and} \qquad \frac{\mathrm{F}(h)-\mathrm{F}(0)}{f(h)-f(0)}=\frac{\mathrm{F}^n(\theta h)}{f^n(\theta h)},$$

and, writing x for h,
$$\frac{\mathrm{F}(x)-\mathrm{F}(0)}{f(x)-f(0)}=\frac{\mathrm{F}^n(\theta x)}{f^n(\theta x)}. \qquad (5)$$

50.] We proceed to give a specific form to $f(x)$, such as will satisfy all the conditions we have made, and to deduce from the last Articles some propositions which will be useful in the sequel of the work.

Take the general result of Art. 49.; all the conditions will be fulfilled if

$$f(x)=(x-x_0)^n, \qquad \therefore \ f(x_0)=0, \text{ and } f(x_0+h)=h^n.$$
$$f'(x)=n(x-x_0)^{n-1}, \qquad f'(x_0)=0.$$
$$f''(x)=n(n-1)(x-x_0)^{n-2}, \qquad f''(x_0)=0.$$
$$f^{n-1}(x)=n(n-1)(n-2)\ldots 3.2.(x-x_0), \qquad f^{n-1}(x_0)=0.$$
$$f^n(x)=n(n-1)(n-2)\ldots 3.2.1,$$
$$f^n(x_0)=n(n-1)(n-2)\ldots 3.2.1.$$

And as this last does not involve x, it is the same whatever x be.

$$\therefore \quad f^n(x_0+\theta h)=n(n-1)(n-2)\ldots 3.2.1.$$

And substituting these values in the general equation of Art. 49., we have

$$\mathrm{F}(x_0+h)-\mathrm{F}(x_0)=\frac{h^n}{1.2.3\ldots(n-1)n}\mathrm{F}^n(x_0+\theta h); \qquad (6)$$

the conditions to which this equation is subject being that $\mathrm{F}(x)$, $\mathrm{F}'(x)$, $\mathrm{F}''(x) \ldots \mathrm{F}^n(x)$ are finite and continuous for all values of x between x_0 and x_0+h, and that $\mathrm{F}'(x_0)=0$, $\mathrm{F}''(x_0)=0$, up to $\mathrm{F}^{n-1}(x_0)=0$; but $\mathrm{F}^n(x_0)$ does not vanish.

51.] The following are particular cases of this important result.

Suppose that none of the derived functions of $\mathrm{F}(x)$ vanish, but that $\mathrm{F}(x_0)$ and $\mathrm{F}'(x_0)$ are both finite, then $n=1$, and we have

$$\mathrm{F}(x_0+h)-\mathrm{F}(x_0)=h\,\mathrm{F}'(x_0+\theta h). \qquad (7)$$

It is worth while to compare this with what is said in Art. 9.; and, if the inferior limit $x_0 = 0$, we have

$$F(h) - F(0) = h F'(\theta h);$$

and, writing x for h, $F(x) - F(0) = x F'(\theta x)$. (8)

Suppose that only $F'(x_0) = 0$, and that $F''(x_0)$ did not vanish, then $n = 2$, and

$$F(x_0 + h) - F(x_0) = \frac{h^2}{1.2} F''(x_0 + \theta h), \tag{9}$$

and so on.

Again, suppose that in the general expression, $F(x_0) = 0$, then

$$F(x_0 + h) = \frac{h^n}{1.2 \ldots (n-1)n} F^n(x_0 + \theta h); \tag{10}$$

and if the inferior limit $x_0 = 0$, then

$$F(h) = \frac{h^n}{1.2.3..(n-1)n} F^n(\theta h); \tag{11}$$

and, writing x for h,

$$F(x) = \frac{x^n}{1.2.3 \ldots (n-1)n} F^n(\theta x), \tag{12}$$

remembering that this equation is subject to the conditions $F(0) = 0$, $F'(0) = 0$, $F''(0) = 0, \ldots F^{n-1}(0) = 0$.

Suppose that in this last expression $F(0) = 0$, but that $F'(0)$ does not vanish, then

$$F(x) = x F'(\theta x); \tag{13}$$

and therefore every function of a variable x, which vanishes when $x = 0$, has x for a factor, unless the first-derived function $= \frac{1}{0}$.

CHAP. V.

FIRST APPLICATION OF THE PRECEDING THEOREMS TO THE COMPARISON OF ORDERS OF INFINITESIMALS, AND TO THE EVALUATION OF QUANTITIES OF THE FORMS

$$\frac{0}{0},\ \pm\frac{\frac{1}{0}}{\frac{1}{0}},\ 0\times\frac{1}{0},\ 0^0,\ \left(\frac{1}{0}\right)^0,\ 1^{\frac{1}{0}},\ 0^{\frac{1}{0}}.$$

52.] In questions such as those proposed for discussion in this chapter, it is important to observe what is the exact meaning of the numerical unit; namely, that it is the ratio of equality, and independent of the particular magnitudes of the quantities which are compared. Hence it follows that it is immaterial whether the quantities are infinitesimal, finite, or infinite, provided that we can assure ourselves that they are equal. Whenever, therefore, the same factor is involved in the numerator and denominator of a fraction, be it of any magnitude and kind, it may be divided out, and the value will not be changed by the division: by this means expressions on which operations are to be performed can often be simplified; an instance will illustrate our meaning. Suppose it is required to determine the value of

$$\frac{x-a+\sqrt{(2x^2-2ax)}}{\sqrt{(x^2-a^2)}},$$

when $x=a$, in which case the fraction assumes the form $\frac{0}{0}$; but why? Because both numerator and denominator involve a factor $\sqrt{(x-a)}$, which is equal to 0 when $x=a$; but the factor in the numerator is exactly equal to the factor in the denominator, therefore the ratio of one to the other is unity, by which therefore $\frac{\sqrt{(x-a)}}{\sqrt{(x-a)}}$ may be replaced, and the fraction becomes

$$\frac{\sqrt{(x-a)}+\sqrt{(2x)}}{\sqrt{(x+a)}},$$

which is equal to 1, when $x=a$.

53.] Def. Suppose, now, we symbolise an infinitesimal by i, and take it as the standard to which all other infinitesimals are to be compared, and which we will therefore call the base of the system, and let a be any constant quantity; then, if

$$\frac{f(i)}{i^r}$$

be equal to 0 for all values of r less than a, but equal to $\frac{1}{0}$ for all values of r greater than a, then $f(i)$ is what we call an infinitesimal of the order a.

Hence it follows that every quantity which does not vanish nor become infinite when $i = 0$, is to be regarded as an infinitesimal of the order 0; and, therefore, the form which all finite quantities assume in this point of view is 0^0. As is evident from the example in the preceding Article, both numerator and denominator $= 0$, when $x = a$, and the fraction is of the form $\frac{0 \times k}{0 \times k'}$ (k and k' being constants); and as the infinitesimals, being exactly equal, are of the same order, the form of the fraction is $0^0 \frac{k}{k'}$, but 0^0 is equal to 1 in this case, and the true value of the fraction is $\frac{k}{k'}$.

54.] The Equation (12) in Art. 51. will enable us to determine the order of infinitesimals, viz.

$$\text{F}(x) = \frac{x^n}{1.2.3\ldots n}\text{F}^n(\theta x).$$

The conditions subject to which this expression has been determined are

$$\text{F}(0) = 0, \text{F}'(0) = 0, \ldots\ldots \text{F}^{n-1}(0) = 0,$$

and all these functions must be continuous and not infinite for values of x between the limits 0 and x, and $\text{F}^n(x)$ is the first derived function which does not $= 0$, and does not become infinite, when $x = 0$. It is also to be remembered that the limits are x and 0, but as we have to evaluate expressions when $x = 0$, it is plain that we may make the difference between the superior and inferior limits as small as we please.

In the above equation for x write i, and change the functional symbol from F to f, and we have

$$f(i) = \frac{i^n}{1.2.3 \ldots\ldots n} f^n(\theta i);$$

and putting $i = 0, f(i) = f(0) = 0$, that is, becomes an infinitesimal, and is an infinitesimal of the nth order, since

$$\frac{f(i)}{i^n} = \frac{f^n(\theta i)}{1.2.3 \ldots\ldots n};$$

and as the right-hand side of the equation is neither 0 nor $\frac{1}{0}$ when $i = 0$, it is plain that $\frac{f(i)}{i^r} = 0$ for all values of r less than n, and $= \frac{1}{0}$ for all values of r greater than n.

Hence we conclude that if $f(i)$ be a function of i, such that $f(0) = 0, f'(0) = 0, f''(0) = 0$, and all the derived functions vanish up to the $(n-1)$th inclusively, but that $f^n(i)$ does not vanish nor become infinite when $i = 0$, then

$$f(i) = \frac{i^n}{1.2.3 \ldots\ldots n} f^n(\theta i),$$

and if i be taken as the base of infinitesimals, $f(i)$ is an infinitesimal of the nth order.

If, when $i = 0, f(i)$ is not 0, but assumes an indeterminate form $\frac{0}{0}$, it must be evaluated first by the methods which are explained at the end of this chapter.

Ex. 1. To determine the order of infinitesimal of $\sin i$, compared with i, when $i = 0$.

$$f(i) = \sin i = 0, \text{ when } i = 0,$$

$$f'(i) = \cos i = 1, \ldots\ldots,$$

$$\therefore \; f(i) = \sin i = \frac{i}{1} \cos(\theta i).$$

$$\therefore \; \text{when } i = 0, \qquad \sin 0 = \frac{0^1}{1} \times 1;$$

therefore $\sin i$ is an infinitesimal of the first order, taking i as the base, when $i = 0$.

Ex. 2. To determine the order of infinitesimal of $\tan i - \sin i$, taking i as the base, when $i = 0$.

$$f(i) = \tan i - \sin i = 0, \text{ when } i = 0,$$

$$f'(i) = \sec^2 i - \cos i = 0, \quad . \quad . \quad . \quad .,$$

$$f''(i) = 2 \tan i \sec^2 i + \sin i = 0, \quad . \quad .,$$

$$f'''(i) = 6 \sec^4 i - 4 \sec^2 i + \cos i = 3, \text{ when } i = 0;$$

$$\therefore \quad f(i) = \tan i - \sin i = \frac{i^3}{1.2.3}\{6 \sec^4 \theta i - 4 \sec^2 \theta i + \cos \theta i\},$$

$$\therefore \quad \text{if } i = 0, \tan 0 - \sin 0 = \frac{0^3}{1.2.3} \times 3.$$

$\therefore$ if i be the base, $\tan i - \sin i$ is an infinitesimal of the third order when $i = 0$.

55.] If any other value of i, as e. g. $i = a$, renders $f(i)$ and its several derived functions $= 0$, and fulfils all the other conditions enumerated above, we may replace $i - a$ by i, and reduce the case to the form already discussed.

56.] Thus far we have arrived at these two results; first, the ratio of two quantities which are exactly equal to one another, be they infinitesimal, finite, or infinite, is unity, and may be replaced by 1 without affecting the truth of the expression in which the ratio is involved; secondly, we have discovered a method of determining the order of infinitesimals; we proceed to apply these to the evaluation of functions which, for particular values of the variable or variables on which they depend, assume the indeterminate forms which are written at the heading of this chapter.

57.] Evaluation of quantities of the form $\frac{0}{0}$.

If a quotient of two functions of a variable or variables for particular values of these assumes the form $\frac{0}{0}$, it is plain that such is the case only because certain factors in the numerator and denominator become 0 for these particular values, that is, become infinitesimals. It is plain, too, from what has been said, that, if these infinitesimals are of the same order, the fraction will be a finite quantity; and, if that in the numerator be of a higher order than that in the denominator, the value of the fraction is 0; and, if that in the denominator be of a higher order than that in the numerator, the fraction is $\frac{1}{0}$: an example will render this plain.

Suppose we have to evaluate $\frac{(x-a)^m \text{M}}{(x-a)^n \text{N}}$ when $x = 0$, M and N being functions of x or not, as the case may be, but not involving any factor of the form $(x-a)$, and not vanishing when $x = a$; the fraction assumes the form $\frac{0}{0}$, but the law of indices authorises us to put it under the form

$$(x-a)^{m-n}\frac{\text{M}}{\text{N}};$$

and if $x = a$, it $= 0$, if m be $> n$,

$$\ldots\ldots = \frac{\text{M}}{\text{N}}, \text{ if } m = n,$$

$$\ldots\ldots = \frac{1}{0}, \text{ if } m \text{ be } < n.$$

It will be seen from Article 59. that similar results are true, if the numerator and denominator be infinities.

Hence, then, the first step towards the evaluation of such quantities is to detect, if possible, the factors common to both the numerator and the denominator, and to divide them out, and then to evaluate the resulting fraction by giving to the variables the assigned values, as e. g.

$$\frac{(a^2-x^2)^{\frac{1}{2}} + (a-x)}{(a-x)^{\frac{1}{2}} + (a^3-x^3)^{\frac{1}{2}}} = \frac{0}{0},$$

if $x = a$; the common factor is $(a-x)^{\frac{1}{2}}$, divide numerator and denominator by it, and we have

$$\frac{\sqrt{(a+x)} + \sqrt{(a-x)}}{1 + \sqrt{(a^2+ax+x^2)}} = \frac{\sqrt{2a}}{1 + a\sqrt{3}}, \text{ if } x = a.$$

The following is an easy method of applying this process. If $x = a$ causes the fraction to assume the form $\frac{0}{0}$, for $x - a$ write h, that is, for x substitute $a + h$ in both numerator and denominator, develope both in a series of terms of ascending powers of h, divide through by the power of h which is common to them, let $h = 0$, and the result is the true value of the fraction.

58.] In cases, however, where it is difficult to detect the common factors, as well as in all cases where the necessary conditions are fulfilled, the theorems of the preceding chapter enable us to evaluate these quantities.

Suppose $f(x)$ and $\phi(x)$ to be two functions of x which $= 0$, when $x = x_0$. Suppose, also, that for the same value of x their several derived functions vanish up to the $(n-1)$th inclusively, but that the nth neither vanishes nor becomes infinite: then, by the theorem contained in Equation (10) Art. 51.,

$$f(x_0 + h) = \frac{h^n}{1.2.3\ldots n} f^n(x_0 + \theta h),$$

$$\phi(x_0 + h) = \frac{h^n}{1.2.3\ldots h} \phi^n(x_0 + \theta h);$$

$$\therefore \quad \frac{f(x_0 + h)}{\phi(x_0 + h)} = \frac{f^n(x_0 + \theta h)}{\phi^n(x_0 + \theta h)}.$$

Suppose, now, that h, the difference between the superior and inferior limits, diminishes without limit; then, putting $h = 0$,

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f^n(x_0)}{\phi^n(x_0)};$$

that is, the true value of the ratio $\frac{f(x_0)}{\phi(x_0)}$, which presents itself under the indeterminate form $\frac{0}{0}$, is the ratio of the values which, when $x = x_0$, the derived functions of the numerator and denominator have, which are the first not to vanish when $x = x_0$.

The following are particular examples. If the functions themselves vanish when $x = x_0$, but not the first derived, then

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f'(x_0)}{\phi'(x_0)};$$

and if the functions themselves and their first derived vanish, but not the second, then

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f''(x_0)}{\phi''(x_0)},$$

and so on.

If the same order of derived functions do not simultaneously vanish, then the indeterminate form has for its real value either 0 or $\frac{1}{0}$, according as the denominator or numerator has first ceased to vanish. This is plain from the proof we have given; because h will be of different powers in the equivalents of $f(x_0 + h)$ and $\phi(x_0 + h)$, and, therefore, will not divide out in the ratio.

Ex. 1.

$$\frac{f(x)}{\phi(x)} = \frac{e^x - e^{-x}}{\sin x} = \frac{0}{0}, \text{ when } x = 0;$$

$$\frac{f'(x)}{\phi'(x)} = \frac{e^x + e^{-x}}{\cos x} = 2, \text{ and therefore the real value is 2.}$$

Ex. 2.

$$\frac{f(x)}{\phi(x)} = \frac{x - \sin x}{x^3} = \frac{0}{0}, \text{ when } x = 0$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{1 - \cos x}{3x^2} = \frac{0}{0}$$

$$= \frac{f''(x)}{\phi''(x)} = \frac{\sin x}{6x} = \frac{0}{0}$$

$$= \frac{f'''(x)}{\phi'''(x)} = \frac{\cos x}{6} = \frac{1}{6}, \text{ when } x = 0;$$

$\therefore$ the true value of $\frac{x - \sin x}{x^3}$, when $x = 0$, is $\frac{1}{6}$.

Ex. 3.

$$\frac{f(x)}{\phi(x)} = \frac{(x-a)^n}{e^x - e^a} = \frac{0}{0}, \text{ when } x = a,$$

$$= \frac{f'(x)}{\phi'(x)} = \frac{n(x-a)^{n-1}}{e^x},$$

which, when $x = a$, is either 0, ne^{-a}, or $\frac{1}{0}$, according as n is greater than, equal to, or less than 1.

59.] Suppose, however, that the two functions $f(x)$ and $\phi(x)$ become $\frac{1}{0}$ when $x = x_0$, in which case their reciprocals become 0, then the fraction may be put under the form $\frac{0}{0}$ as follows, and evaluated as before:

$$\frac{f(x_0)}{\phi(x_0)} = \frac{\dfrac{1}{\phi(x_0)}}{\dfrac{1}{f(x_0)}} = \frac{0}{0} = \frac{-\dfrac{\phi'(x_0)}{\{\phi(x_0)\}^2}}{-\dfrac{f'(x_0)}{\{f(x_0)\}^2}} = \frac{\phi'(x_0)}{f'(x_0)} \frac{\{f(x_0)\}^2}{\{\phi(x_0)\}^2};$$

$$\therefore \quad \frac{f(x_0)}{\phi(x_0)} = \frac{f'(x_0)}{\phi'(x_0)};$$

If the first-derived functions $f'(x_0)$, $\phi'(x_0)$ also become infinite, they must be evaluated in the same way as $\frac{f(x_0)}{\phi(x_0)}$, and we shall have

$$\frac{f'(x_0)}{\phi'(x_0)} = \frac{f''(x_0)}{\phi''(x_0)};$$

and if the several derived functions become infinite up to the nth, when $x = x_0$, but the nth are finite,

$$\frac{f(x_0)}{\phi(x_0)} = \frac{f'(x_0)}{\phi'(x_0)} = \cdots\cdots = \frac{f^n(x_0)}{\phi^n(x_0)}.$$

The true value, therefore, of such indeterminate quantities is the ratio of their derived functions of the same order, both of which do not simultaneously become infinite when $x = x_0$.

Ex.

Evaluate $\frac{\log x}{\cot x}$, when $x = 0$.

$$\frac{f(x)}{\phi(x)} = \frac{\log x}{\cot x} = \frac{-\frac{1}{0}}{\frac{1}{0}}, \text{ when } x = 0,$$

$$\therefore \quad \frac{f'(x)}{\phi'(x)} = -\frac{\sin^2 x}{x} = \frac{0}{0}, \text{ when } x = 0;$$

$$\frac{f''(x)}{\phi''(x)} = -\frac{2 \sin x \cos x}{1} = 0, \text{ when } x = 0,$$

$\therefore$ 0 is the true value.

Evaluate $\frac{x^n}{e^x}$, when $x = \frac{1}{0}$.

$$\frac{f(x)}{\phi(x)} = \frac{x^n}{e^x} = \frac{\frac{1}{0}}{\frac{1}{0}}, \text{ when } x = \frac{1}{0},$$

$$\frac{f'(x)}{\phi'(x)} = \frac{nx^{n-1}}{e^x} = \frac{\frac{1}{0}}{\frac{1}{0}}, \text{ when } x = \frac{1}{0},$$

and so on, until

$$\frac{f^n(x)}{\phi^n(x)} = \frac{n(n-1)(n-2)\ldots 4.3.2.1}{e^x} = 0, \text{ when } x = \frac{1}{0};$$

therefore the true value is 0.

60.] If there be two functions of x, $f(x)$ and $\phi(x)$, which respectively become 0 and $\frac{1}{0}$, when $x = x_0$, then their product $f(x) \times \phi(x)$ may be put under the form

$$\frac{f(x)}{\frac{1}{\phi(x)}}$$

which takes the form $\frac{0}{0}$ when $x = x_0$, and is to be evaluated by the method explained in Art. 58., by taking the derived functions of the numerator and denominator.

Similarly, if $f(x)$ and $\phi(x)$ be two functions of x, which when $x = x_0$ become 0, $\frac{1}{f(x_0)} - \frac{1}{\phi(x_0)}$ becomes $\frac{1}{0} - \frac{1}{0}$, but may be put under the form

$$\frac{\phi(x_0) - f(x_0)}{f(x_0)\,\phi(x_0)}, \text{ which } = \frac{0}{0},$$

and may be evaluated as the like forms in Art. 58.

Ex. 1.

When $\quad x = \frac{1}{0}, e^{-x} \log x = 0 \times \frac{1}{0},$

but $\quad e^{-x} \log x = \frac{\log x}{e^x} = \frac{\frac{1}{0}}{\frac{1}{0}} = \frac{1}{x e^x} = 0$, when $x = \frac{1}{0}$.

Ex. 2.

Evaluate $x \log x$, when $x = 0$.

$$x \log x = \frac{\log x}{\frac{1}{x}} = \frac{-\frac{1}{0}}{\frac{1}{0}} = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = -0, \text{ when } x = 0;$$

$\therefore$ the true value of $x \log x$, when $x = 0$, is 0.

61.] Lastly, we may determine as follows the true values of functions of x, which offer themselves under the general form $f(x)^{\phi(x)}$, and which, for particular values of the variable, assume one or other of the forms 0^0, $\left(\frac{1}{0}\right)^0$, $1^{\pm\frac{1}{0}}$, $0^{\frac{1}{0}}$.

Let $\quad y = f(x)^{\phi(x)},$

$$\therefore \quad \log_e y = \phi(x) \log_e f(x);$$

and as the logarithm has singular values when $f(x)=0$, or $=1$, or $=\frac{1}{0}$, we may express the last equation in the form

$$\log_e y = \frac{\log_e f(x)}{\frac{1}{\phi(x)}},$$

which, for the particular value x_0, will be of the form $\frac{0}{0}$ or $\frac{\frac{1}{0}}{\frac{1}{0}}$, and may be evaluated according to the methods of Art. 58, or 59.

Ex. 1.

Evaluate x^x, when $x = 0$.

$$\text{Let} \qquad y = x^x, \qquad \therefore \quad \log_e y = x \log_e x;$$

and by last Article, Ex. 2.,

$$x \log_e x = -0, \qquad \therefore \quad \log_e y = -0;$$

$$\therefore \quad y = x^x = 1.$$

Ex. 2.

Evaluate $x^{\frac{1}{1-x}}$, when $x = 1$.

$$y = x^{\frac{1}{1-x}}, \quad \therefore \log_e y = \frac{1}{1-x} \log_e x = \frac{\log_e x}{1-x} = \frac{0}{0}, \text{ when } x = 1,$$

$$\therefore \log_e y = \frac{\log_e x}{1-x} = -\frac{1}{x} = -1, \quad \therefore y = x^{\frac{1}{1-x}} = \frac{1}{e}, \text{ when } x = 1.$$

For other examples illustrative of these processes see Mr. Gregory's *Collection*, chap. vi., and Mr. Hind's *Examples*, chap. vi.

62.] We proceed to determine the true value of certain expressions which assume indeterminate forms in functions of two variables.

$$\text{Let} \qquad u = \text{F}(x,y) = c.$$

Then, by the method explained in Art. 27., we have

$$\frac{dy}{dx} = -\frac{\left(\frac{d\text{F}}{dx}\right)}{\left(\frac{d\text{F}}{dy}\right)};$$

and as it often happens that for particular values of x and y, which simultaneously satisfy the equation $\text{F}(x,y) = c$, $\frac{dy}{dx}$ assumes an indeterminate form $\frac{0}{0}$ or $\frac{\frac{1}{0}}{\frac{1}{0}}$, this must be evaluated according to the methods of Art. 58. and 59, by taking the total differentials of the numerator and denominator respectively; the process will be best explained by an example.

Suppose it is required to find the value of $\frac{dy}{dx}$ when $x=0$ and $y=0$, having given

$$\text{F}(x,y) = ay^2 - x^3 - bx^2 = 0,$$

$$\therefore \quad \text{when } x = 0, y = 0.$$

$$\left(\frac{d\text{F}}{dx}\right) = -3x^2 - 2bx, \qquad \left(\frac{d\text{F}}{dy}\right) = 2ay;$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2 + 2bx}{2ay} = \frac{0}{0}, \text{ when } x = 0 \text{ and } y = 0.$$

Taking the total differentials of the numerator and denominator, and dividing by dx,

$$\frac{dy}{dx} = \frac{6x + 2b}{2a\frac{dy}{dx}} = \frac{b}{a\frac{dy}{dx}}, \text{ when } x = y = 0;$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{b}{a}, \text{ and } \frac{dy}{dx} = \pm\sqrt{\frac{b}{a}},$$

Ex. 2. To determine the values of $\frac{dy}{dx}$, corresponding to $x = 0$ and $y = 0$, having given

$$x^4 + ay^3 - 2axy^2 - 3ax^2y = 0.$$

Differentiating as before, we find

$$\frac{dy}{dx} = \frac{4x^3 - 2ay^2 - 6axy}{3ax^2 + 4axy - 3ay^2} = \frac{0}{0}, \text{ when } x = y = 0,$$

$$= \frac{12x^2 - 4ay\frac{dy}{dx} - 6ay - 6ax\frac{dy}{dx}}{6ax + 4ax\frac{dy}{dx} + 4ay - 6ay\frac{dy}{dx}} = \frac{0}{0}, \text{ when } x = y = 0.$$

And here it is to be remarked that in the next differentiation $\frac{dy}{dx}$ is to be considered constant; for although $\frac{dy}{dx}$ may have several values corresponding to the particular values of x and y, yet these values do not vary with small variations of x and y, and therefore are to be considered invariable when x and y vary.

$$\therefore \quad \frac{dy}{dx} = \frac{24x - 4a\frac{dy^2}{dx^2} - 6a\frac{dy}{dx} - 6a\frac{dy}{dx}}{6a + 4a\frac{dy}{dx} + 4a\frac{dy}{dx} - 6a\frac{dy^2}{dx^2}}$$

$$= \frac{-4a\frac{dy^2}{dx^2} - 12a\frac{dy}{dx}}{6a + 8a\frac{dy}{dx} - 6a\frac{dy^2}{dx^2}}, \text{ when } x = y = 0.$$

$\therefore$ multiplying and reducing,

$$\frac{dy^3}{dx^3} - 2\frac{dy^2}{dx^2} - 3\frac{dy}{dx} = 0;$$

$$\therefore \quad \frac{dy}{dx} = 0 \text{ and } = 3 \text{ and } = -1.$$

Hence, then, it appears that we may determine the true value of $\frac{dy}{dx}$ by differentiating successively the numerator and denominator of the fraction which gives the indeterminate result, until we arrive at an equation which gives certain definite values. We shall meet with more examples of this process in future chapters.

CHAP. VI.

SECOND APPLICATION OF THE PRECEDING THEOREMS TO THE EXPANSION OF FUNCTIONS. TAYLOR'S SERIES. MACLAURIN'S SERIES. LAGRANGE'S THEOREM.

A. *Of Functions of One Variable.*

63.] ON referring to Art. 9., and writing h for Δx, it will be seen that if $F(x)$ be a function of x, which is finite and continuous for all values of x between $x = x$ and $x = x + h$, and $Rh = R_1$,

$$F(x + h) - F(x) = hF'(x) + R_1, \qquad (1)$$

R_1 being a quantity which diminishes without limit, as h diminishes without limit: but if h be finite, R_1 does not vanish, and our object in the present chapter is to determine its value; or, in other words, given that $F(x)$ is continuous and finite for all values of x between x and $x + h$, it is required to expand $F(x + h)$ in a series of ascending powers of h.

We may also thus arrive at the Equation (1) above.

If $F'(x)$ does not vanish, and remains finite and continuous for all values of x between x and $x + h$, then by Equation (7) Art. 51.,

$$F(x + h) - F(x) = h\,F'(x + \theta h),$$

which may be written in the form

$$F(x + h) - F(x) = h\,F'(x) + R_1.$$

We proceed to determine R_1:

$$R_1 = F(x + h) - F(x) - h\,F'(x),$$

therefore R_1 is a function of h which $= 0$, when $h = 0$; also its first-derived

$$\frac{dR_1}{dh} = F'(x + h) - F'(x) = 0, \text{ when } h = 0;$$

and its second-derived $\frac{d^2 R_1}{dh^2} = F''(x+h) = F''(x)$, when $h = 0$, subject to the conditions that $F'(x)$ and $F''(x)$ are continuous and finite for all values of x between the limits $x = x$ and $x = x + h$. Therefore by the theorem expressed in Equation (11) Art. 51.,

$$R_1 = \frac{h^2}{1.2} F''(x + \theta h); \tag{2}$$

and substituting this value in Equation (1) above,

$$F(x+h) = F(x) + h F'(x) + \frac{h^2}{1.2} F''(x + \theta h), \tag{3}$$

which may be written in the form

$$F(x+h) = F(x) + h F'(x) + \frac{h^2}{1.2} F''(x) + R_2.$$

Whence $R_2 = F(x+h) - F(x) - h F'(x) - \frac{h^2}{1.2} F''(x);$ (4)

and therefore R_2 is a function of h, which vanishes when $h = 0$;

and its 1st derived $\frac{dR_2}{dh} = F'(x+h) - F'(x) - h F''(x) = 0$,

when $h = 0$,

and its 2d derived $\frac{d^2 R_2}{dh^2} = F''(x+h) - F''(x) = 0$, when $h = 0$;

and its 3d derived $\frac{d^3 R_2}{dh^3} = F'''(x+h) = F'''(x)$, when $h = 0$.

$$\therefore \text{ as before,} \qquad R_2 = \frac{h^3}{1.2.3} F'''(x + \theta h),$$

subject to the conditions that $F'(x)$, $F''(x)$, $F'''(x)$ are continuous and finite between the assigned limits. Substituting this value of R_2, we have

$$F(x+h) - F(x) - h F'(x) - \frac{h^2}{1.2} F''(x) = \frac{h^3}{1.2.3} F'''(x + \theta h); \tag{5}$$

and continuing in the same manner, if all the derived functions of $F(x)$ are finite and continuous between the assigned limits up to the nth inclusively, we have

$$F(x+h) - F(x) - F'(x)\frac{h}{1} - \dots - \frac{h^{n-1}}{1.2.3\dots\dots(n-1)}F^{n-1}(x)$$

$$= \frac{h^n}{1.2.3\dots n}F^n(x+\theta h);$$

and therefore,

$$F(x+h) = F(x) + \frac{h}{1}F'(x) + \frac{h^2}{1.2}F''(x) + \frac{h^3}{1.2.3}F'''(x) + \dots$$

$$\dots + \frac{h^{n-1}}{1.2\dots(n-1)}F^{n-1}(x) + \frac{h^n}{1.2\dots n}F^n(x+\theta h); \quad (6)$$

particularly bearing in mind the conditions subject to which this equation is true. This expression, then, gives the expansion of $F(x+h)$, and shows in what cases it is possible.

64.] If the last term of this series, viz. $\frac{h^n}{1.2\dots n}F^n(x+\theta h)$, diminishes without limit, as n increases without limit, then

$$F(x+h) = F(x) + \frac{h}{1}F'(x) + \frac{h^2}{1.2}F''(x) + \frac{h^3}{1.2.3}F'''(x) + \&c. \quad (7)$$

which is Taylor's Series for the expansion of $F(x+h)$. But we must remember that the Series (7) is equivalent to $F(x+h)$ only when $F(x)$ and *all* its derived functions are continuous and finite between x and $x+h$, and when $\frac{h^n}{1.2\dots(n-1)n}F^n(x+\theta h)$ diminishes without limit as n increases without limit.

65.] On comparing this proof with the imperfect one given in Art. 35. it is seen that in the Equation (6) of Art. 63. one side is exactly equal to the other; that is, the right-hand side consists of a finite number of terms, the sum of which is exactly equal to $F(x+h)$; whereas in Art. 35. we had no means of determining to how many terms it might be necessary to proceed, or what error would be committed by stopping at a particular term. The above formula shows what is the sum of all the terms after the nth, viz. $\frac{h^n}{1.2.3\dots(n-1)n} \mathrm{F}^n(x+\theta h)$, which is to be added to the first n terms, that the sum may be exactly equal to $\mathrm{F}(x+h)$. This term is known by the name of Lagrange's expression for the limits of Taylor's Series, having been first determined by that eminent mathematician; and although the expression is somewhat indeterminate, because θ is *some* positive numerical quantity less than 1, yet the error can usually be ascertained within very small limits.

66.] Failure of the Theorem.—The Equation (6) requires certain conditions to be fulfilled up to $\mathrm{F}^n(x)$, but none as to any subsequent derived functions, and therefore these may be discontinuous or infinite between the assigned limits without affecting the truth of the series. Thus the expansion may be correct up to a certain term, but would be incorrect if it were carried beyond that term.

Suppose, for example, it is required to expand

$$\mathrm{F}(x) = f(x) + (x-a)^{m+\frac{p}{q}} \phi(x); \qquad (8)$$

m being an integral positive number, and $\frac{p}{q}$ being a proper fraction, and neither $f(x)$ nor $\phi(x)$ involving factors of the form $(x-a)$. Or, again, suppose that $\mathrm{F}(x)$ involves factors of the form $(x-a)^{-n}$.

In the latter case it is plain that if the values of x between the limits include the particular value a, the conditions are not satisfied, for $\mathrm{F}(x)$ involving $(x-a)^{-n}$ ceases to be finite when $x=a$; and in this case the theorem contained in Equation (6) fails. So, again, in the former case, if the limits include the value a, all the derived functions will be finite for the particular value $x=a$ up to $\mathrm{F}^m(x)$, but the subsequent ones will be infinite, and therefore will not satisfy the conditions under which the Equation (6) has been formed. The expansion, therefore, must not be carried beyond the mth term; but the addition of $\frac{h^m}{1.2\ldots m}\,\mathrm{F}^m(x+\theta h)$ will make the equation exact. An example illustrative of what has been said is subjoined.

Let it be required to expand $\mathrm{F}\,(x+h)$ for the particular value $x=a$, having given

$$\mathrm{F}(x) = x^4 + (x-a)^{\frac{5}{2}}\sin x,$$

$$\mathrm{F}'(x) = 4x^3 + \frac{5}{2}(x-a)^{\frac{3}{2}}\sin x + (x-a)^{\frac{5}{2}}\cos x,$$

$$\mathrm{F}''(x) = 12x^2 + \frac{15}{4}(x-a)^{\frac{1}{2}}\sin x + 5(x-a)^{\frac{3}{2}}\cos x$$
$$- (x-a)^{\frac{5}{2}}\sin x.$$

But if we form $\mathrm{F}'''(x)$ it will involve $(x-a)^{-\frac{1}{2}}$, which will become infinite when $x=a$, and therefore fails to fulfil the conditions under which Equation (6) has been determined: therefore, in this case,

$$\mathrm{F}(x+h) = \mathrm{F}(x) + \frac{h}{1}\mathrm{F}'(x) + \frac{h^2}{1.2}\mathrm{F}''(x+\theta h);$$

$\therefore$ substituting the specific values above given, and putting $x=a$,

$$\mathrm{F}(a+h) = a^4 + 4a^3\frac{h}{1} + \frac{h^2}{1.2}\{12(a+\theta h)^2$$

$$+ \frac{15}{4}(\theta h)^{\frac{1}{2}}\sin(a+\theta h) + 5(\theta h)^{\frac{3}{2}}\cos(a+\theta h) - (\theta h)^{\frac{5}{2}}\sin(a+\theta h)\},$$

θ being a positive value less than unity; whence it appears that the failure is due to the fact that the real expansion of $\mathrm{F}\,(a+h)$ involves fractional powers of h.

67.] If in Formula (6) Art. 63. we put $x=0$, that is, make zero the inferior limit, and write x for h, and bear in mind that x is the superior limit; and therefore the conditions are that none of the functions or derived functions must be infinite or discontinuous between $x=0$ and $x=x$, then

$$\mathrm{F}(x) = \mathrm{F}(0) + x\mathrm{F}'(0) + \frac{x^2}{1.2}\mathrm{F}''(0) + \frac{x^3}{1.2.3}\mathrm{F}'''(0) + \ldots.$$

$$\ldots + \frac{x^{n-1}}{1.2.3\ldots(n-1)}\mathrm{F}^{n-1}(0) + \frac{x^n}{1.2.3\ldots.n}\mathrm{F}^n(\theta x). \quad (9)$$

The following are particular cases of this expansion:

$$\text{F}(x) = \text{F}(0) + x\,\text{F}'(\theta x),$$

$$\text{F}(x) = \text{F}(0) + x\,\text{F}'(0) + \frac{x^2}{1.2}\text{F}''(\theta x),$$

and so on. Thus for the functions e^x, $\cos x$, $\sin x$, $\log_e x$:

$$\frac{e^x - 1}{x} = e^{\theta x}, \quad \frac{1-\cos x}{x} = \sin\theta x, \quad \frac{\sin x}{x} = \cos\theta x,$$

$$\frac{\log_e(1+x)}{x} = \frac{1}{1+\theta x}.$$

68.] If, as n increases without limit, $\dfrac{x^n}{1.2.3.\ldots n}\text{F}^n(\theta x)$ decreases without limit, then $\text{F}(x)$ will be expressed by the equation

$$\text{F}(x) = \text{F}(0) + \frac{x}{1}\text{F}'(0) + \frac{x^2}{1.2}\text{F}''(0) + \frac{x^3}{1.2.3}\text{F}'''(0) + \&\text{c.} \quad (10)$$

that is, the limit of the sum of all the terms on the right-hand side of the equation is equal to $\text{F}(x)$, and is the same series as was before deduced, but by an imperfect method, in Art. 32.

69.] Equation (9) shows that the error committed by stopping at the nth term, or, in other words, that the sum of all the terms after the nth, is $\dfrac{x^n}{1.2.\ldots n}\text{F}^n(\theta x)$. The same indefiniteness is involved in this expression as was in that for the limits of Taylor's Series, because θ is *some* positive quantity less than unity, of which we cannot determine the exact value.

70.] The series given in Equation (6) of this chapter may be made to assume another form which is sometimes convenient, and by means of it the sum of the terms of the expansion after the nth may be otherwise expressed. This has been done as follows by Cauchy.

In Equation (6) let $h = a - x$

$$\therefore \quad \text{F}(a) = \text{F}(x) + (a-x)\text{F}'(x) + \frac{(a-x)^2}{1.2}\text{F}''(x) + \ldots\ldots$$

$$\ldots + \frac{(a-x)^{n-1}}{1.2.3.\ldots(n-1)}\text{F}^{n-1}(x) + \frac{(a-x)^n}{1.2.3\ldots n}\text{F}^n\{(x+\theta(a-x)\}.$$

For x write a, and for a write x:

$$\text{F}(x) = \text{F}(a) + (x-a)\,\text{F}'(a) + \frac{(x-a)^2}{1.2}\,\text{F}''(a) + \dots\dots$$

$$+ \frac{(x-a)^{n-1}}{1.2\dots(n-1)}\,\text{F}^{n-1}(a) + \frac{(x-a)^n}{1.2\dots(n-1)n}\,\text{F}^n\{a+\theta(x-a)\}, \quad (11)$$

the superior limit in this case being x and the inferior a; so that it is for all values of x between these that the conditions are to be satisfied.

As particular cases of the formula we have

$$\text{F}(x) = \text{F}(a) + (x-a)\,\text{F}'\{a+\theta\,(x-a)\}. \quad (12)$$

$$\text{F}(x) = \text{F}(a) + (x-a)\,\text{F}'(a) + \frac{(x-a)^2}{1.2}\,\text{F}''\{a+\theta\,(x-a)\}. \quad (13)$$

As an example of the Formula (11) take $\log x$, and we have

$$\log x = \log a + \frac{x-a}{a} - \frac{1}{2}\left(\frac{x-a}{a}\right)^2 + \frac{1}{3}\left(\frac{x-a}{a}\right)^3 - \&c.\dots.$$

$$\dots\dots \pm \frac{1}{n-1}\left(\frac{x-a}{a}\right)^{n-1} \mp \frac{1}{n}\left(\frac{x-a}{a+\theta\,(x-a)}\right)^n.$$

71.] Let us symbolise the last term of (11) by $\phi(a)$, then we have

$$\phi(a) = \frac{(x-a)^n}{1.2.3..n}\,\text{F}^n\,\{a+\theta(x-a)\} \quad (14)$$

$$\text{F}(x) = \text{F}(a) + \frac{x-a}{1}\,\text{F}'(a) + \frac{(x-a)^2}{1.2}\,\text{F}''(a) + \dots.$$

$$\dots + \frac{(x-a)^{n-1}}{1.2.3\dots.(n-1)}\,\text{F}^{n-1}(a) + \phi(a).$$

Differentiate this, making a the variable, and we have

$$\frac{(x-a)^{n-1}}{1.2.3\dots.(n-1)}\,\text{F}^n(a) + \phi'(a) = 0. \quad (15)$$

It is plain from (14) that $\phi(x) = 0$; therefore, writing in (12) ϕ for F, we have

$$\phi(a) + (x-a)\,\phi'\{a + \theta_1(x-a)\} = 0,$$

θ_1 being different from θ in (14);

$$\therefore\quad \phi(a) = -(x-a)\,\phi'\{a+\theta_1(x-a)\}. \quad (16)$$

In (15) for a write $a+\theta_1(x-a)$, and we have

$$\frac{(x-a)^{n-1}(1-\theta_1)^{n-1}}{1.2.3.\dots(n-1)}\mathrm{F}^n\{a+\theta_1(x-a)\}+\phi'\{a+\theta_1(x-a)\}=0; \quad (17)$$

whence, by elimination between (16) and (17),

$$\phi(a)=\frac{(1-\theta_1)^{n-1}(x-a)^n}{1.2.3.\dots(n-1)}\mathrm{F}^n\{a+\theta_1(x-a)\}; \quad (18)$$

and substituting in (11) this particular form of the remainder, we have

$$\mathrm{F}(x)=\mathrm{F}(a)+\frac{x-a}{1}\mathrm{F}'(a)+\frac{(x-a)^2}{1.2}\mathrm{F}''(a)+\dots\dots$$

$$+\frac{(x-a)^{n-1}}{1.2\dots(n-1)}\mathrm{F}^{n-1}(a)+\frac{(x-a)^n(1-\theta)^{n-1}}{1.2.\dots(n-1)}\mathrm{F}^n\{a+\theta(x-a)\}. \quad (19)$$

Substitute in this series $a+h$ for x, and subsequently write x for a, and we have

$$\mathrm{F}(x+h)=\mathrm{F}(x)+h\,\mathrm{F}'(x)+\frac{h^2}{1.2}\mathrm{F}''(x)+\dots\dots$$

$$\dots+\frac{h^{n-1}}{1.2.\dots(n-1)}\mathrm{F}^{n-1}(x)+\frac{h^n(1-\theta)^{n-1}}{1.2.\dots(n-1)}\mathrm{F}^n(x+\theta h); \quad (20)$$

and the corresponding expression for the remainder, in Maclaurin's Series, is

$$\frac{(1-\theta)^{n-1}x^n}{1.2.3.\dots(n-1)}\mathrm{F}^n(\theta x). \quad (21)$$

We add two examples; one of the series which is numbered (9), using the form of the remainder numbered (21), and the second of the series numbered (20); and refer for others to chap. v. of Mr. Gregory's *Collection.*

To expand a^x in a series of ascending powers of x.

Let $\mathrm{F}(x)=a^x$, $\therefore$ $\mathrm{F}'(x)=a^x\log_e a$, $\mathrm{F}''(x)=a^x(\log_e a)^2\dots$

$$\dots\mathrm{F}^n(x)=a^x(\log_e a)^n;$$

$\therefore$ $\mathrm{F}(0)=1$, $\mathrm{F}'(0)=\log_e a$, $\mathrm{F}''(0)=(\log_e a)^2$, $\mathrm{F}'''(0)=(\log_e a)^3$, and so on; substituting which, we have

$$a^x = 1 + x \log_e a + \frac{x^2}{1.2}(\log_e a)^2 + \frac{x^3}{1.2.3}(\log_e a)^3 + \dots\dots$$

$$+ \frac{x^{n-1}}{1.2.3\,..\,(n-1)}(\log_e a)^{n-1} + \frac{(1-\theta)^{n-1}\,x^n}{1.2.3\,..\,(n-1)}(\log_e a)^n\, a^{\theta x}.$$

And as the remainder diminishes without limit as n increases without limit, we may write

$$a^x = 1 + x \log_e a + \frac{x^2}{1.2}(\log_e a)^2 + \frac{x^3}{1.2.3}(\log_e a)^3 + \&c.$$

Ex. 2. To expand $\log(x+h)$.

$$F(x) = \log x, \qquad \therefore\ F'(x) = \frac{1}{x} = x^{-1},\ F''(x) = -x^{-2},$$

$$F'''(x) = (-)^2 1.2\,x^{-3}, \qquad F^{\text{IV}}(x) = (-)^3 1.2.3\,x^{-4}, \&c.$$

$$F^n(x) = (-)^{n-1}\, 1.2.3.4 \dots (n-1)x^{-n};$$

$$\therefore\quad \log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \dots\dots$$

$$\dots (-)^{n-2}\frac{h^{n-1}}{(n-1)x^{n-1}} (-)^{n-1}\frac{h^n(1-\theta)^{n-1}}{n}\;\frac{1}{(x+\theta h)^n}.$$

If x be greater than h the last term will diminish without limit, and we may write

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \&c.$$

B. *Of Functions of Two or more Variables.*

72.] We proceed now to extend our methods of expansion to functions of two or more variables; and, first, of two variables.

Having given $F(x,y)$, to find the value of $F(x+h, y+k)$, h and k being finite increments of x and y, $F(x,y)$ being finite and continuous for all values of the variables between the limits x,y and $x+h, y+k$. It is supposed, also, that the several partial derived functions of $F(x,y)$ are finite and continuous between the same limits.

Let us consider the finite increments of x and y to be ht and kt, so that, finally, they will be reduced to h and k by putting

$t = 1$; then our object is to expand $F(x + ht, y + kt)$, and let us assume

$$f(t) = F(x + ht, y + kt),$$

so that $$f(0) = F(x,y).$$

By the Series (9) Art. 67. we have

$$f(t) = f(0) + tf'(0) + \frac{t^2}{1.2}f''(0) + \dots + \frac{t^{n-1}}{1.2\dots(n-1)}f^{n-1}(0)$$
$$+ \frac{t^n}{1.2.3\dots n}f^n(t\theta).$$

By the formula for the differentiation of a function of two variables, proved in Art. 38., considering t to increase by constant increments, which is a condition requisite for the truth of the series above, and for sake of convenience omitting the brackets indicative of partial differentiation,

$$\frac{dF}{dt} = f'(t) = \frac{dF}{d(x+ht)}\frac{d(x+ht)}{dt} + \frac{dF}{d(y+kt)}\frac{d(y+kt)}{dt}$$
$$= \frac{dF}{d(x+ht)}h + \frac{dF}{d(y+kt)}k;$$
$$\therefore \quad f'(0) = \frac{dF}{dx}h + \frac{dF}{dy}k.$$

Similarly,

$$\frac{d^2F}{dt^2} = \frac{d^2F}{\{d(x+ht)\}^2}h^2 + 2\frac{d^2F}{d(x+ht)d(y+kt)}hk$$
$$+ \frac{d^2F}{\{d(y+kt)\}^2}k^2;$$

$$\therefore \quad f''(0) = \frac{d^2F}{dx^2}h^2 + 2\frac{d^2F}{dx\,dy}hk + \frac{d^2F}{dy^2}k^2,$$

$$f'''(0) = \frac{d^3F}{dx^3}h^3 + 3\frac{d^3F}{dx^2\,dy}h^2k + 3\frac{d^3F}{dx\,dy^2}hk^2 + \frac{d^3F}{dy^3}k^3$$

. . . .

$$f^n(0) = \frac{d^nF}{dx^n}h^n + n\frac{d^nF}{dx^{n-1}dy}h^{n-1}k + \frac{n(n-1)}{1.2}\frac{d^nF}{dx^{n-2}dy^2}h^{n-2}k^2$$
$$+ \&c.$$

And substituting these values in the equation above, we have

$$\text{F}(x+ht,\,y+kt) = \text{F}(x,y) + \frac{t}{1}\left\{\frac{d\text{F}}{dx}h + \frac{d\text{F}}{dy}k\right\}$$

$$+ \frac{t^2}{1.2}\left\{\frac{d^2\text{F}}{dx^2}h^2 + 2\frac{d^2\text{F}}{dx\,dy}hk + \frac{d^2\text{F}}{dy^2}k^2\right\}$$

$$+ \frac{t^3}{1.2.3}\left\{\frac{d^3\text{F}}{dx^3}h^3 + 3\frac{d^3\text{F}}{dx^2dy}h^2k + \ldots\ldots\right\}\ldots$$

$$\ldots + \frac{t^n}{1.2.3\ldots n}\left\{\frac{d^n\text{F}}{dx^n}h^n + n\frac{d^n\text{F}}{dx^{n-1}dy}h^{n-1}k + \ldots\right.$$

$$\left.\ldots + \frac{d^n\text{F}}{dy^n}k^n\right\}_{\left\{\begin{matrix}x+\theta ht,\\ y+\theta kt,\end{matrix}\right\}}$$

the meaning of the notation in the last line being, that x and y are replaced by $x+\theta ht$, $y+\theta kt$, in the coefficient of t^n; let $t=1$, whence

$$\text{F}(x+h,y+k) = \text{F}(x,y) + \frac{d\text{F}}{dx}h + \frac{d\text{F}}{dy}k$$

$$+ \frac{1}{1.2}\left\{\frac{d^2\text{F}}{dx^2}h^2 + 2\frac{d^2\text{F}}{dx\,dy}hk + \frac{d^2\text{F}}{dy^2}k^2\right\}$$

$$+ \frac{1}{1.2.3}\left\{\frac{d^3\text{F}}{dx^3}h^3 + 3\frac{d^3\text{F}}{dx^2dy}h^2k + \ldots\ldots\right\}$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\ldots + \frac{1}{1.2.3\ldots n}\left\{\frac{d^n\text{F}}{dx^n}h^n + n\frac{d^n\text{F}}{dx^{n-1}dy}h^{n-1}k + \ldots\ldots\right.$$

$$\left.\ldots + \frac{d^n\text{F}}{dy^n}k^n\right\}_{\left\{\begin{matrix}x+\theta h.\\ y+\theta k.\end{matrix}\right\}} \qquad (22)$$

73.] If it be required to expand $\text{F}(x+h,\,y+k,\,z+l,\,\ldots\ldots)$, then, by a similar process, we have

$$\text{F}(x+h,\,y+k,\,z+l\ldots\ldots) = \text{F}(x,y,z\ldots\ldots)$$

$$+\left\{\frac{d\text{F}}{dx}h + \frac{d\text{F}}{dy}k + \frac{d\text{F}}{dz}l + \ldots\ldots\right\}$$

$$+\frac{1}{1.2}\left\{\frac{d^2\text{F}}{dx^2}h^2+\frac{d^2\text{F}}{dy^2}k^2+\ldots\ldots+2\frac{d^2\text{F}}{dx\,dy}hk\right.$$

$$\left.+2\frac{d^2\text{F}}{dx\,dz}hl+\ldots\right\}$$

$$+\frac{1}{1.2.3}\left\{\frac{d^3\text{F}}{dx^3}h^3+\ldots\ldots+3\frac{d^3\text{F}}{dx^2dy}h^2k+\ldots\ldots\right.$$

$$\left.\ldots+3\frac{d^3\text{F}}{dx\,dy^2}hk^2+\ldots\right\}$$

$$+\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.$$

$$\ldots+\frac{1}{1.2.3\ldots n}\left\{\frac{d^n\text{F}}{dx^n}h^n+\frac{d^n\text{F}}{dy^n}k^n+\ldots+n\frac{d^n\text{F}}{dx^{n-1}dy}h^{n-1}k\right.$$

$$\left.+\ldots\ldots\right\}\left\{\begin{matrix}x+\theta h,\\ y+\theta k,\\ z+\theta l,\\ .\quad.\quad.\\ .\quad.\quad.\end{matrix}\right\} \qquad (23)$$

replacing $x, y, z \ldots$ in the last term by $x+\theta h$, $y+\theta k$, $z+\theta l, \ldots$

As the Equations (22) and (23) stand at present, each side is exactly equal to the other; but if we can assure ourselves that as n increases without limit, each term of the remainder decreases without limit, then the remainders may be neglected, and the equations will be modified accordingly.

74.] If in the preceding Formula (22) we make $x=0$, $y=0$, and then change h and k into x and y, we have

$$\text{F}(x,y)=\text{F}(0,0)+\left(\frac{d\text{F}}{dx}\right)_0 x+\left(\frac{d\text{F}}{dy}\right)_0 y,$$

$$+\frac{1}{1.2}\left\{\left(\frac{d^2\text{F}}{dx^2}\right)_0 x^2+2\left(\frac{d^2\text{F}}{dx\,dy}\right)_0 xy+\left(\frac{d^2\text{F}}{dy^2}\right)_0 y^2\right\}$$

$$+\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.$$

$$\ldots+\frac{1}{1.2\ldots n}\left\{\frac{d^n\text{F}}{dx^n}x^n+n\frac{d^n\text{F}}{dx^{n-1}dy}x^{n-1}y+\ldots+\frac{d^n\text{F}}{dy^n}y^n\right\}\left\{\begin{matrix}\theta x,\\ \theta y,\end{matrix}\right\} \qquad (24)$$

where we have to replace x and y by the value 0 in all the partial derived functions, except in the last term, where they are to be replaced by θx and θy; and if this last term decreases without limit, as n increases without limit, then the remainder may be neglected, and the series may be written without the last term. Similarly, if the functions were of more than two variables.

75.] As an application of the Series (23), we will prove the following important properties of homogeneous functions, which are due to Euler, and are known by the name of Euler's Theorems of Homogeneous Functions.

Suppose $F(x, y, z \dots)$ to be a homogeneous function of n dimensions and r variables: for x write $(1 + k)x$, for y $(1 + k)y$, and similarly for the others; then the function becomes $F\{(1+k)x, (1 + k)y, (1 + k)z, \dots\}$, and may be put under the forms $(1 + k)^n F(x, y, z \dots)$, since it is homogeneous and of n dimensions, and $F(x + kx, y + ky, z + kz, \dots)$, which are of course equal. The first of these may be expanded by the binomial theorem, and the second by the Equation (23), kx, ky, kz being the respective increments of x, y, z; and therefore we have

$$\left\{1+nk+\frac{n(n-1)}{1.2}k^2+\frac{n(n-1)(n-2)}{1.2.3}k^3+\dots\right\}F(x,y,z\dots)$$

$$=F(x,y,z\dots)+\frac{dF}{dx}kx+\frac{dF}{dy}ky+\frac{dF}{dz}kz+\dots$$

$$\dots+\left\{\frac{d^2F}{dx^2}k^2x^2+\frac{d^2F}{dy^2}k^2y^2+\dots+2\frac{d^2F}{dx\,dy}k^2xy+\dots\right\}\frac{1}{1.2}$$

$$+\left\{\frac{d^3F}{dx^3}k^3x^3+\frac{d^3F}{dy^3}k^3y^3+\dots+3\frac{d^3F}{dx^2dy}k^3x^2y+\dots\right\}\frac{1}{1.2.3}$$

$$+\ \dots\dots\dots\dots ,$$

whence, equating coefficients of the same powers of k, we have

$$nF(x, y, z\dots)=x\frac{dF}{dx}+y\frac{dF}{dy}+z\frac{dF}{dz}+\&c.,$$

$$n(n-1)\,\mathrm{F}(x, y, z\ldots) = x^2\frac{d^2\mathrm{F}}{dx^2} + y^2\frac{d^2\mathrm{F}}{dy^2} + \ldots + 2\,xy\frac{d^2\mathrm{F}}{dx\,dy} + \&\text{c}.$$

$$n(n-1)(n-2)\,\mathrm{F}(x, y, z\ldots) = x^3\frac{d^3\mathrm{F}}{dx^3} + \ldots + 3\,x^2y\frac{d^3\mathrm{F}}{dx^2dy} + \ldots$$

$$\ldots + 3\,xy^2\frac{d^3\mathrm{F}}{dx\,dy^2} + \&\text{c}.$$

and so on for the other terms.

As an example illustrative of this, let us take

$$\mathrm{F}(x, y, z) = \mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + \mathrm{D}yz + \mathrm{E}zx + \mathrm{G}xy;$$

$$\therefore\quad \left(\frac{d\mathrm{F}}{dx}\right) = 2\,\mathrm{A}x + \mathrm{E}z + \mathrm{G}y,$$

$$\left(\frac{d\mathrm{F}}{dy}\right) = 2\,\mathrm{B}y + \mathrm{D}z + \mathrm{G}x,$$

$$\left(\frac{d\mathrm{F}}{dz}\right) = 2\,\mathrm{C}z + \mathrm{D}y + \mathrm{E}x;$$

$$\therefore\quad x\left(\frac{d\mathrm{F}}{dx}\right) + y\left(\frac{d\mathrm{F}}{dy}\right) + z\left(\frac{d\mathrm{F}}{dz}\right) = 2\,\{\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + \mathrm{D}yz$$

$$+ \mathrm{E}zx + \mathrm{G}xy\} = 2\,\mathrm{F}(x, y, z).$$

For other examples, see Mr. Gregory's *Collection*, chap. ii. ex. 19.

76.] Lagrange's Theorem for the Expansion of Implicit Functions.

Suppose that $y = z + x\phi(y)$, in which equation y is an implicit function of two variables z and x, which are supposed to have no other relation to each other besides that given by this equation, so that they may vary independently of each other; it is required to determine in ascending powers of x another function of y, viz. $f(y)$.

Let $u = f(y)$, and therefore u is a function of x; whence by Stirling's Series

$$u = [u] + \left[\frac{du}{dx}\right]\frac{x}{1} + \left[\frac{d^2u}{dx^2}\right]\frac{x^2}{1.2} + \left[\frac{d^3u}{dx^3}\right]\frac{x^3}{1.2.3} + \&\text{c.} \quad (25)$$

bracketing the quantities, to indicate that particular values of them are to be taken; viz. when $x = 0$, that is, if

$u = \mathrm{F}(x)$, $[u] = \mathrm{F}(0)$, $\left[\frac{du}{dx}\right] = \mathrm{F}'(0)$, and so on. Hence we

have the following data:

$$u = f(y), \qquad y = z + x\phi(y); \tag{26}$$

$$\therefore \quad \text{when} \quad x = 0,\ y = z,$$

$$\therefore \quad [u] = f(z),$$

forming the derived functions of (26), first by making x to vary, and then z.

$$\frac{dy}{dx} = \phi(y) + x\phi'(y)\frac{dy}{dx}, \qquad \therefore \frac{dy}{dx} = \frac{\phi(y)}{1 - x\phi'(y)};$$

$$\frac{dy}{dz} = 1 + x\phi'(y)\frac{dy}{dz}, \qquad \therefore \frac{dy}{dz} = \frac{1}{1 - x\phi'(y)};$$

$$\therefore \quad \frac{dy}{dx} = \phi(y)\frac{dy}{dz}. \tag{27}$$

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx},$$

$\therefore$ substituting from (27), $\qquad \dfrac{du}{dx} = \dfrac{du}{dy}\phi(y)\dfrac{dy}{dz}.$

Let $x = 0,\ y = z, \qquad \therefore \quad dy = dz$ and $u = [u] = f(z)$;

$$\therefore \quad \left[\frac{du}{dx}\right] = \frac{d.f(z)}{dz}\phi(z).$$

Again, since $\dfrac{du}{dx} = \dfrac{du}{dy}\phi(y)\dfrac{dy}{dz}$,

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left\{\frac{du}{dy}\phi(y)\frac{dy}{dz}\right\}$$

$$= \frac{d\left\{\frac{du}{dy}\phi(y)\right\}}{dy}\frac{dy}{dx}\frac{dy}{dz} + \frac{du}{dy}\phi(y)\frac{d^2y}{dxdz}$$

$$= \frac{d\left\{\frac{du}{dy}\phi(y)\right\}}{dy}\frac{dy}{dz}\frac{dy}{dx} + \frac{du}{dy}\phi(y)\frac{d^2y}{dz\,dx}$$

$$= \frac{d}{dz}\left\{\frac{du}{dy}\phi(y)\frac{dy}{dx}\right\},$$

and substituting for $\frac{dy}{dx}$ from (27)

$$\frac{d^2u}{dx^2} = \frac{d}{dz}\left\{\frac{du}{dy}\{\phi(y)\}^2\frac{dy}{dz}\right\}.$$

Let $x = 0$, in which case $y = z$, $dy = dz$, and $u = f(z)$;

$$\therefore \left[\frac{d^2u}{dx^2}\right] = \frac{d}{dz}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^2\right\}.$$

In a similar way differentiating again $\frac{d^2u}{dx^2}$, and substituting from (27), we shall find

$$\frac{d^3u}{dx^3} = \frac{d^2}{dz^2}\left\{\frac{du}{dy}\{\phi(y)\}^3\frac{dy}{dz}\right\};$$

and therefore $\left[\frac{d^3u}{dx^3}\right] = \frac{d^2}{dz^2}\left\{\frac{d.f(z)}{dz}\{\phi(z)\}^3\right\}$.

Let us, then, assume that the formula is true for $\frac{d^{n-1}u}{dx^{n-1}}$:

$$\frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-2}}{dz^{n-2}}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dz}\right\},$$

$$\frac{d^nu}{dx^n} = \frac{d}{dx}\frac{d^{n-2}}{dz^{n-2}}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dz}\right\},$$

and since the order of differentiation is indifferent,

$$= \frac{d^{n-2}}{dz^{n-2}}\frac{d}{dx}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dz}\right\}$$

$$= \frac{d^{n-2}}{dz^{n-2}}\left\{\frac{d\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\right\}}{dy}\frac{dy}{dx}\frac{dy}{dz} + \frac{du}{dy}\{\phi(y)\}^{n-1}\frac{d^2y}{dx\,dz}\right\}$$

$$= \frac{d^{n-2}}{dz^{n-2}}\left\{\frac{d\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\right\}}{dy}\frac{dy}{dz}\frac{dy}{dx} + \frac{du}{dy}\phi\{(y)\,^{n-1}\frac{d^2y}{dzdx}\right\}$$

$$= \frac{d^{n-2}}{dz^{n-2}}\frac{d}{dz}\left\{\frac{du}{dy}\{\phi(y)\}^{n-1}\frac{dy}{dx}\right\}$$

$$= \frac{d^{n-1}}{dz^{n-1}}\left\{\frac{du}{dy}\{\phi(y)\}^{n}\frac{dy}{dz}\right\}, \text{ by virtue of Equation (27);}$$

$$\therefore \quad \left[\frac{d^n u}{dx^n}\right] = \frac{d^{n-1}}{dz^{n-1}}\left\{\frac{d\,.f(z)}{dz}\{\phi(z)\}^n\right\}.$$

If, therefore, the formulæ are true for $n-1$, they are true for n; they are true when $n = 3$, therefore they are true when $n = 4$, and therefore are true for all positive integral values of n, the only cases in which it is necessary for us to find them. Substituting, then, in Equation (25) the values above determined, we have

$$f(y) = f(z) + \frac{d\,.f(z)}{dz}\phi(z)\frac{x}{1} + \frac{d}{dz}\left\{\frac{d\,.f(z)}{dz}\{\phi(z)\}^2\right\}\frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}\left\{\frac{d\,.f(z)}{dz}\{\phi(z)\}^3\right\}\frac{x^3}{1.2.3} + \dots$$

$$\dots + \frac{d^{n-1}}{dz^{n-1}}\left\{\frac{d\,.f(z)}{dz}\{\phi(z)\}^n\right\}\frac{x^n}{1.2.3\dots n} + \&c. \qquad (28)$$

Ex. Given $y = a + e \sin y$, to find $\sin y$.

On comparing these with Equations (26),

$$\left.\begin{matrix} f(y) = \sin y \\ \phi(y) = \sin y \end{matrix}\right\}, \qquad \left.\begin{matrix} z = a \\ x = e \end{matrix}\right\};$$

$$\therefore \quad \sin y = \sin z + \cos z \, \sin z\frac{x}{1} + \frac{d}{dz}(\cos z \sin^2 z)\frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}(\cos z \, \sin^3 z)\frac{x^3}{1.2.3} + \&c.$$

$$= \sin z + \cos z \sin z \frac{x}{1} - (3 \sin^3 z - 2 \sin z)\frac{x^2}{1.2}$$

$$- \cos z\,(16 \sin^3 z - 6 \sin z)\,\frac{x^3}{1.2.3} + \&c.$$

$$\therefore \quad \sin y = \sin a + \cos a \sin a \frac{e}{1} - \sin a\,(3 \sin^2 a - 2)\,\frac{e^2}{1.2}$$

$$- \cos a\,(16 \sin^3 a - 6 \sin a)\,\frac{e^3}{1.2.3} + \&c.$$

CHAP. VII.

THIRD APPLICATION OF PRECEDING THEOREMS TO THE DETERMINATION OF MAXIMA AND MINIMA VALUES OF FUNCTIONS.

77.] Def. When a particular value of a function is greater than all its values in the immediate neighbourhood, that is to say, than all its values when the variables are made to increase or decrease by small quantities, this value is said to be a *maximum*.

And if the particular value be less than all the values which correspond to the variables, when they are made to increase or decrease by small quantities, then such a value is called a *minimum*.

In reference to these definitions, then, it is to be borne in mind that maxima and minima are not terms used absolutely, but in reference to the values of the functions which are immediately adjacent to those to which the terms are applied.

As a simple illustration of our meaning, let us consider $\sin x$. Let the radius of the circle be unity, then, when the arc $= 0$, the sine $= 0$, but as the arc increases up to $\frac{\pi}{2}$, the sine increases and at last becomes 1, which is its maximum, for as the arc becomes larger the sine becomes smaller, and continually decreases, passing through 0 when the arc $= \pi$, until, when the arc $= \frac{3\pi}{2}$, the sine $= -1$; after which it begins to increase, and so on through 0, when the arc $= 2\pi$, until, when the arc $= \frac{5\pi}{2}$, the sine $= +1$, and so on. Thus as the arc increases, the sine periodically attains to maxima and minima values.

A. *Of Functions of One Variable.*

78.] Let $y = \text{F}(x)$ be the function of which the maxima and minima values are to be determined.

From our definition it is plain that if, as x increases up to a certain value x_0, $\text{F}(x)$ increases, and afterwards as x increases, $\text{F}(x)$ decreases, then $\text{F}(x)$ has attained a maximum value at $x = x_0$; so again, if, as x increases up to a certain value x_0, $\text{F}(x)$ decreases and afterwards increases, then $\text{F}(x_0)$ is a minimum value.

Now Theorem I. Chapter IV. is immediately applicable to determine these conditions: if x and $\text{F}(x)$ are simultaneously increasing, $\text{F}'(x)$ is positive; but if as x increases $\text{F}(x)$ decreases, $\text{F}'(x)$ is negative.

If, therefore, at any point $x = x_0$, $\text{F}'(x)$ changes its sign from $+$ to $-$, we have a maximum value; and if $\text{F}'(x)$ changes its sign from $-$ to $+$, we have a minimum value: and, as changes of sign can take place only when the quantity passes through 0 or $\frac{1}{0}$, we have the following rule to determine maxima and minima;*

Find every value of x which renders $\text{F}'(x) = 0$ or $= \frac{1}{0}$; if such a value makes $\text{F}'(x)$ change its sign, we have a maximum or minimum, but if there is no change of sign, there are no such corresponding values; and if, as x increases, $\text{F}'(x)$ changes its sign from $+$ to $-$, there is a corresponding maximum value, but if $\text{F}'(x)$ changes sign from $-$ to $+$, there is a minimum value.

As an instance, consider the case given above, viz. $y = \sin x$.

$$\therefore \quad \frac{dy}{dx} = f'(x) = \cos x = 0, \text{ if}$$

$x = \frac{\pi}{2}$, and $f'(x)$ changes sign from $+$ to $-$, $\therefore$ a maximum;

$x = \frac{3\pi}{2}$ . . . . $-$ to $+$, $\therefore$ a minimum;

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* More will be said in a subsequent chapter on $\text{F}(x)$ not being a maximum nor a minimum when $\text{F}(x) = \frac{1}{0}$.

$x = \frac{5\pi}{2}$, and $f'(x)$ changes sign from + to −, ∴ a maximum;

and so on. Because, if x is rather less than $\frac{\pi}{2}$ or $\frac{5\pi}{2}$, &c., $\cos x$ is +ve; and if x is a little greater than $\frac{\pi}{2}$ or $\frac{5\pi}{2}$, &c., $\cos x$ is −ve; whereas if x is a little less than $\frac{3\pi}{2}$ or $\frac{7\pi}{2}$, $\cos x$ is −ve, and when x is a little greater than these values, $\cos x$ is +ve.

The change of sign can often be conveniently determined as follows. From Art. 29. it appears that $F''(x) = \frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}$; therefore, by Theorem I. Chap. IV., supposing $F''(x)$ not to vanish when $F'(x) = 0$, if as x increases $\frac{dy}{dx}$ passes from − to +, i. e. increases, $\frac{d^2y}{dx^2}$ is positive, but if it passes from + to −, it is decreasing, and $\frac{d^2y}{dx^2}$ is negative; therefore if $F'(x_0) = 0$, and $F''(x_0)$ is positive, $F(x_0)$ is a minimum, but if $F''(x_0)$ is negative, $F(x_0)$ is a maximum.

79.] When $F'(x)$ is an algebraical function, and admits of being resolved into its factors, it is easy to determine what values of the variable give maxima and minima. Corresponding to every factor of the form $(x - x_0)^{2m+1}$, i. e. to every factor of uneven dimensions, there is a change of sign, and therefore a maximum or minimum value; but to factors of even dimensions of the form $(x - x_0)^{2m}$ there is no change of sign, and therefore no maximum nor minimum; as, for instance, suppose

$$F'(x) = x^3 (x-1)^2 (x-2)^3 (x-4)^4,$$

which is equal to 0, if

$x = 0$, and gives a change of sign from + to −, ∴ a maximum;

$x = 1$, and gives no change of sign, ∴ no max. nor minimum;

I 2

$x = 2$, and gives a change of sign from $-$ to $+$, $\therefore$ a minimum;

$x = 4$, and gives no change of sign, $\therefore$ no max. nor minimum.

80.] The meaning of the several conditions of maxima and minima are illustrated geometrically in figs. 6, 7, 8, 9.

Suppose $y = \text{F}(x)$ to represent a curve such as those drawn in the figs.

Let $\text{OM}_0 = x_0$, $\text{M}_0\text{P}_0 = y_0$, the corresponding ordinate.
Then in fig. 6. as x increases up to x_0, $y = \text{F}(x)$ increases, and therefore $\text{F}'(x)$ is $+$ve; but as soon as x has passed the value x_0, y begins to decrease, and $\text{F}'(x)$ is negative, and the ordinate y or $\text{F}(x)$ has manifestly attained a maximum value at x_0.

In fig. 7. the reverse is the case; as x has increased up to x_0 $y = \text{F}(x)$ has been decreasing, but as soon as x is greater than x_0, $\text{F}(x)$ increases, and thus the sign of $\text{F}'(x)$ has changed from $-$ to $+$ at x_0, and the corresponding value of $\text{F}(x)$ is a minimum.

Fig. 8. illustrates the case of $\text{F}'(x)$ being positive up to x_0, and although $\text{F}'(x) = 0$, yet it does not change its sign, but continues positive afterwards, and therefore we have no maximum value.

In the curve drawn in fig. 9., $\text{F}'(x)$ is negative throughout; at P_0 it is $=$ to 0, but, as it does not change its sign, there is no minimum value.

Two examples are subjoined.

Ex. 1. $$y = x^2 - 2x - 3,$$

$$\frac{dy}{dx} = 2x - 2 = 2(x-1) = 0, \text{ if}$$

$x = 1$, and changes sign from $-$ to $+$, which indicates a minimum, in which case $y = -4$.

Ex. 2. To divide the number a into two such parts, x and $a - x$, that $x^n(a-x)^m$ may be a maximum.

$$y = x^n(a-x)^m,$$

$$\therefore \quad \frac{dy}{dx} = x^{n-1}(a-x)^{m-1}(na - nx - mx)$$

$$= (n+m)\,x^{n-1}(a-x)^{m-1}\left(\frac{na}{m+n} - x\right),$$

which $= 0$ if $x = 0$, and changes sign from $-$ to $+$ if n be an even number, which indicates a minimum, but undergoes no change of sign if n be an odd number. Also $\frac{dy}{dx} = 0$, if $x = a$, and changes sign from $-$ to $+$ if m be even, which indicates a minimum, but undergoes no change of sign if m be odd. Also $\frac{dy}{dx} = 0$, if $x = \frac{na}{m+n}$, and changes sign from $+$ to $-$, which indicates a maximum.

It is recommended to the reader to illustrate these principles and criteria by drawing the lines whose equations are respectively $y = ax$, $y = ax^2$, $y = ax^3$, $y = ax^4$, and showing in what cases $x = 0$ gives maxima and minima values of y.

81.] We have given this method of determining maxima and minima, because it is plainer to the perception, and depends on the use of algebraical symbols, which may be worked with, though not understood, less than the common method, of which the following is a modification.

Let $F(x)$ be the function of which the maximum and minimum values are to be determined; then by Equation (7) Art. 51.,

$$F(x+h) - F(x) = hF'(x+\theta h).$$

As h diminishes without limit $F'(x+\theta h)$ approaches to its limiting value, $F'(x)$.

Suppose now that x_0 is such a value of x that $F(x_0)$ is a maximum or minimum; then if h diminishes without limit, and $F'(x_0)$ does not vanish, we have

$$F(x_0+h) - F(x_0) = hF'(x_0);$$

and if these conditions are satisfied, according as h is positive or negative will $F(x_0)$ be less than $F(x_0+h)$ or greater than $F(x_0-h)$, and therefore $F(x_0)$ will be neither a maximum nor a minimum. Suppose, however, $F'(x_0) = 0$, then, by Equation (9) Art. (51.), if $F''(x_0)$ does not vanish,

$$F(x_0+h) - F(x_0) = \frac{h^2}{1.2} F''(x_0+\theta h);$$

which becomes, when h diminishes without limit,

$$F(x_0+h)-F(x_0)=\frac{h^2}{1.2}F''(x_0).$$

If, therefore, $F''(x_0)$ is positive, $F(x_0)$ is less than both $F(x_0+h)$ and $F(x_0-h)$; but if $F''(x_0)$ is negative, $F(x_0)$ is greater than both $F(x_0+h)$ and $F(x_0-h)$; whence we conclude,

If $F'(x_0)=0$, and $F''(x_0)$ does not vanish, if $F''(x_0)$ is negative, $F(x_0)$ is a maximum, and if $F''(x_0)$ is positive, $F(x_0)$ is a minimum.

But if, again, $F''(x_0)=0$, and $F'''(x_0)$ does not vanish,

$$F(x_0+h)-F(x_0)=\frac{h^3}{1.2.3}F'''(x_0+\theta h);$$

in which case, as h^3 changes its sign with h, it is plain that there is no maximum nor minimum value: but if $F'''(x_0)=0$, and $F^{iv}(x_0)$ does not vanish, then

$$F(x_0+h)-F(x_0)=\frac{h^4}{1.2.3.4}F^{iv}(x_0+\theta h);$$

in which case, as before, there will be a maximum or minimum value of $F(x_0)$, according as $F^{iv}(x_0)$ is negative or positive.

And thus, generally, if the value x_0, which makes $F'(x)=0$, so affects $F''(x)$, $F'''(x)$, up to $F^{n-1}(x)$ that all vanish, but that $F^n(x_0)$ does not vanish, then we have

$$F(x_0+h)-F(x_0)=\frac{h^n}{1.2.3\ldots n}F^n(x_0+\theta h),$$

and if n be an odd number, there is no maximum nor minimum value; but if n be an even number, $F(x_0)$ is a maximum if $F^n(x_0)$ is negative, and a minimum if $F^n(x_0)$ is positive.

In the application of this theory to questions of geometrical maxima and minima, it will subsequently appear that figs. 6. and 7. correspond to the analytical conditions of every derived function vanishing, when $x=x_0$, up to one of an odd order inclusively, and the next derived function of an even order remaining finite; and figs. 8. and 9. correspond to the condition that the first-derived function that does not vanish is of an odd order.

Ex.

$$\text{F}(x) = y = \frac{x}{1+x^2},$$

$$\text{F}'(x) = \frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2} = 0, \text{ if } x = +1, \text{ and if } x = -1,$$

$$\text{F}''(x) = \frac{d^2y}{dx^2} = -\frac{6x+2x^3}{(1+x^2)^3}.$$

If $x = +1, \quad \frac{d^2y}{dx^2} = -1, \quad \therefore \quad y = \frac{1}{2}$, a maximum;

$x = -1, \quad \frac{d^2y}{dx^2} = +1, \qquad y = -\frac{1}{2}$, a minimum.

For examples, see Mr. Gregory's *Collection*, chap. vii.

B. *Implicit Functions of a Single Variable.*

82.] Suppose that the equation connecting y and x is an implicit one, of the form

$$u = \text{F}(x, y) = c,$$

then by the expression in Art. 27. we have

$$\frac{dy}{dx} = -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)};$$

and as a necessary condition of a maximum and minimum is that $\frac{dy}{dx}$ must change sign, and as it can only change sign by passing through 0 or $\frac{1}{0}$, we must have either $\left(\frac{du}{dx}\right) = 0$ or $\frac{1}{0}$, or $\left(\frac{du}{dy}\right) = 0$ or $\frac{1}{0}$, but $\frac{dy}{dx}$ must not be of the form $\frac{0}{0}$ or $\frac{\frac{1}{0}}{\frac{1}{0}}$;

and, again, on referring to Art. 39., if $\left(\frac{du}{dx}\right) = 0$,

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{d^2u}{dx^2}\right)}{\left(\frac{du}{dy}\right)}, \qquad \because \quad \frac{dy}{dx} = 0;$$

and if this be positive we have a minimum, but if negative, a maximum value.

This method, however, of determining maxima and minima is very incomplete, as it does not discuss the cases where $\left(\frac{du}{dy}\right)$ or any other of the partial derived functions become infinite; and it is, therefore, to be taken as a suggestion of the manner in which such problems are to be solved: the best plan is to determine the special maxima and minima values for each problem separately, as follows.

Ex. Required to find the maxima and minima values of y, having given

$$y^3 + x^3 - 3\,axy = 0.$$

$$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax},$$

which $= 0$ if $x^2 = ay$, i. e. if $y = \frac{x^2}{a}$, whence we have from the equation

$$x^6 = 2\,a^3 x^3$$

$$\therefore \quad \left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\}, \text{ or } \left.\begin{matrix} x = 2^{\frac{1}{3}}a \\ y = 2^{\frac{2}{3}}a \end{matrix}\right\}.$$

The latter values render the denominator of $\frac{dy}{dx}$ +ve, and therefore there is a change of sign of $\frac{dy}{dx}$ from + to −, and, therefore, these values correspond to a maximum.

If $x = 0$, $y = 0$, $\frac{dy}{dx} = \frac{0}{0}$, which must be evaluated as in Art. 62.

$$\frac{dy}{dx} = -\frac{2\,x - a\,\frac{dy}{dx}}{2\,y\,\frac{dy}{dx} - a} = -\frac{dy}{dx}, \text{ if } x = 0,\ y = 0,$$

$$\therefore \quad \frac{dy}{dx} = 0, \text{ or } = \frac{1}{0}.$$

If $\frac{dy}{dx} = 0$, it changes sign from $-$ to $+$, which indicates a minimum.

For other examples, see Mr. Gregory's *Collection*, chap. vii. sect. ii.

C. *Maxima and Minima of Functions of Two or more Variables.*

83.] First, of two variables. It is advisable that the reader should be familiar with geometry of three dimensions, or with some subject in which a variable is a function of two or more variables, as the problems we have to solve are such as the following.

Let it be required to find the maximum value of z, having given $z = f(x, y)$, which equation represents a surface, and, therefore, we have to find what are the values of x and y, that the corresponding value of z shall be greater than its values when x and y either increase or decrease by a small quantity. As, for instance, suppose the plane of xy to pass through the centre of a sphere and the origin to be on the circumference, z will have a maximum value when the corresponding values of x and y refer to the centre of the sphere.

Let $u = \text{F}(x, y)$ be the function of two variables of which the maxima and minima values are to be determined. Then it is plain, from what has been said in Art. 77., that if $x_0\, y_0$ are the particular values of x and y, to which corresponds a maximum or minimum value of u, then $\text{F}(x_0, y_0)$ is greater or less than the values corresponding to the variables whether x varies when y is constant, or y varies when x is constant, or whether both increase or decrease together, or whether one increases and the other decreases; that is, $\text{F}(x_0 + h, y_0 + k)$ is to be less or greater than $\text{F}(x_0, y_0)$, whatever be the signs of h and k, and in whatever manner these signs are combined. Therefore, if $\text{F}(x_0, y_0)$ be a maximum, $\text{F}(x_0, y_0)$ is greater than $\text{F}(x_0 \pm h, y_0 \pm k)$; and if $\text{F}(x_0, y_0)$ be a minimum, $\text{F}(x_0, y_0)$ is less than $\text{F}(x_0 \pm h, y_0 \pm k)$.

By the Expansion (22) in Art. 72. we have

$$\text{F}(x_0 + h, y_0 + k) - \text{F}(x_0, y_0) =$$

$$\left(\frac{d\text{F}}{dx}\right) h + \left(\frac{d\text{F}}{dy}\right) k$$

$$+ \left\{\left(\frac{d^2\text{F}}{dx^2}\right) h^2 + 2\left(\frac{d^2\text{F}}{dx\,dy}\right) hk + \left(\frac{d^2\text{F}}{dy^2}\right) k^2\right\} \frac{1}{1.2}$$

$$+ \left\{\left(\frac{d^3\text{F}}{dx^3}\right) h^3 + 3\left(\frac{d^3\text{F}}{dx^2\,dy}\right) h^2k + 3\left(\frac{d^3\text{F}}{dx\,dy^2}\right) hk^2 + \left(\frac{d^3\text{F}}{dy^3}\right) k^3\right\} \frac{1}{1.2.3}$$

$$+ \ldots\ldots\ldots\ldots\ldots,$$

the several partial differential coefficients having values corresponding to x_0, y_0.

But, as before observed, $\text{F}(x_0 \pm h, y_0 \pm k) - \text{F}(x_0, y_0)$ must have the same sign, whatever be the signs of h and k, and in whatever manner combined; and as h and k may manifestly be taken so small, that the sign of the whole expression may be made to depend on that of the term involving the lowest powers of h and k, it is plain that this requisite condition cannot be fulfilled, unless

$$\left(\frac{d\text{F}}{dx}\right) h + \left(\frac{d\text{F}}{dy}\right) k = 0;$$

and, as no relation is given between h and k, we must have

$$\left(\frac{d\text{F}}{dx}\right) = 0, \qquad \left(\frac{d\text{F}}{dy}\right) = 0.$$

And suppose that the values of x and y, which satisfy these conditions, do not make $\left(\frac{d^2\text{F}}{dx^2}\right) = \left(\frac{d^2\text{F}}{dx\,dy}\right) = \left(\frac{d^2\text{F}}{dy^2}\right) = 0$, then the next term, which gives the sign to the whole expression, is

$$\left\{\left(\frac{d^2\text{F}}{dx^2}\right) h^2 + 2\left(\frac{d^2\text{F}}{dx\,dy}\right) hk + \frac{d^2\text{F}}{dy^2} k^2\right\} \frac{1}{1.2};$$

and this must not change sign when the signs of h and k change, for which it is requisite that

1st, $\left(\frac{d^2\text{F}}{dx^2}\right)$ and $\left(\frac{d^2\text{F}}{dy^2}\right)$ should be of the same sign;

2d, $\left(\frac{d^2 F}{dx\,dy}\right)$ should not be of such relative magnitude to the other terms as to affect the sign of the whole, which condition will be satisfied if the whole expression can be put under the form of two squares; i. e. if four times the product of the first and last terms is greater than the square of the middle term, that is, if

$$\left(\frac{d^2 F}{dx^2}\right)\left(\frac{d^2 F}{dy^2}\right) \text{ is greater than } \left(\frac{d^2 F}{dx\,dy}\right)^2.$$

This relation, having been determined by Lagrange, is known by the name of Lagrange's Condition.

If these several conditions are fulfilled, and $\left(\frac{d^2 F}{dx^2}\right)$ and $\left(\frac{d^2 F}{dy^2}\right)$ are positive, $F(x_0, y_0)$ is less than $F(x_0 \pm h, y_0 \pm k)$, and is a minimum; but, if $\left(\frac{d^2 F}{dx^2}\right)$ and $\left(\frac{d^2 F}{dy^2}\right)$ are negative, $F(x_0, y_0)$ is a maximum. Hence arises the following rule for the determination of maxima and minima of a function of two unconnected variables. Let $u = F(x, y)$ be the function; determine the values of x and y, which render $\left(\frac{dF}{dx}\right) = 0$ and $\left(\frac{dF}{dy}\right) = 0$; if these do not make to vanish $\left(\frac{d^2 F}{dx^2}\right)$, $\left(\frac{d^2 F}{dx\,dy}\right)$, and $\left(\frac{d^2 F}{dy^2}\right)$, and if $\left(\frac{d^2 F}{dx^2}\right)\left(\frac{d^2 F}{dy^2}\right)$ be greater than $\left(\frac{d^2 F}{dx\,dy}\right)^2$, then, according as $\left(\frac{d^2 F}{dx^2}\right)$ and $\left(\frac{d^2 F}{dy^2}\right)$ are negative or positive, will $F(x, y)$ be a maximum or a minimum.

Ex. $u = F(x, y) = x^3 + y^3 - 3axy$,

$$\left.\begin{aligned} \therefore\quad \left(\frac{dF}{dx}\right) &= 3x^2 - 3ay = 0, \text{ if } y = \frac{x^2}{a} \\ \left(\frac{dF}{dy}\right) &= 3y^2 - 3ax = 0, \text{ if } y^2 = ax \end{aligned}\right\}, \quad \therefore\quad x^4 = a^3 x.$$

$$\therefore\quad \left.\begin{aligned} x &= 0 \\ y &= 0 \end{aligned}\right\} \text{ and } \left.\begin{aligned} x &= a. \\ y &= a. \end{aligned}\right\}$$

$$\left.\begin{array}{l}\left(\frac{d^2 F}{dx^2}\right) = 6x \\ \left(\frac{d^2 F}{dx\,dy}\right) = -3a \\ \left(\frac{d^2 F}{dy^2}\right) = 6y\end{array}\right\}, \therefore \left(\frac{d^2 F}{dx^2}\right)\left(\frac{d^2 F}{dy^2}\right) - \left(\frac{d^2 F}{dx\,dy}\right)^2 = 36xy - 9a^2.$$

If $x = 0$, $y = 0$, Lagrange's condition is not satisfied; therefore there is no corresponding maximum nor minimum. If $x = a$, $y = a$, $\left(\frac{d^2 F}{dx^2}\right) = \left(\frac{d^2 F}{dy^2}\right) = 6a$, and are both positive, the condition becomes $27a^2$, and is therefore satisfied, and we have a minimum value of u, viz. $u = -a^3$.

If the values of x and y, which make $\left(\frac{dF}{dx}\right) = 0$ and $\left(\frac{dF}{dy}\right) = 0$, also make $\left(\frac{d^2 F}{dx^2}\right)$, $\left(\frac{d^2 F}{dy^2}\right)$, and $\left(\frac{d^2 F}{dx\,dy}\right)$ equal to 0, then the first terms in the expansion that do not vanish are those involving h^3, $h^2 k$, $h k^2$, and k^3, which manifestly change sign with h and k; and, therefore, in the same way as was argued in Art. 81., there cannot be a maximum or minimum value of the function unless $\left(\frac{d^3 F}{dx^3}\right)$, $\left(\frac{d^3 F}{dx^2\,dy}\right)$, &c. also vanish, and the terms involving h^4, &c. do not vanish, and a relation must exist between them analogous to Lagrange's condition; and so, in general, the function can only have a maximum or minimum value when, for those values of x and y which make $\left(\frac{dF}{dx}\right)$ and $\left(\frac{dF}{dy}\right)$ equal to 0, the first derived functions that do not vanish are of an even order.

84.] The geometrical meaning of Lagrange's condition is as follows.

Conceive a point on a surface, at which $\left(\frac{dF}{dx}\right) = 0$, and $\left(\frac{dF}{dy}\right) = 0$, and a normal to be drawn. If sections be made of the surface by planes passing through the normal, it is manifest that, in general, the curvature of these will be different, and different not only in amount, but also in direction: the radius of curvature of one section may be in one direction, and of that of another section in the contrary direction. Now, if the curvature of every normal section is turned the same way, then $\left(\frac{d^2\text{F}}{dx^2}\right)$ $\left(\frac{d^2\text{F}}{dy^2}\right)$ is greater than $\left(\frac{d^2\text{F}}{dx\,dy}\right)^2$, and it is plain that in such a case there is a maximum or minimum value of the function; but if $\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right)$ be less than $\left(\frac{d^2\text{F}}{dx\,dy}\right)^2$, the curvature of some sections is turned one way, and of other sections the opposite way, in which case some sections would give maxima and others minima values of the function. The ellipsoid and hyperboloid of one sheet are good instances of these two cases respectively. But if $\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right)=\left(\frac{d^2\text{F}}{dx\,dy}\right)^2$, a series of maxima or minima values of the function is indicated; such as, if $z=\text{F}(x, y)$ represent a surface formed by the revolution of an ellipse about an axis parallel to its minor axis, the extremities of the minor axis generate circles which are loci of maxima and minima values of z. The proof of these statements belongs to the province of solid geometry.

85.] If the function, of which the maxima and minima values are to be determined, is of three variables, viz. $u = \text{F}(x, y, z)$, the conditions for a maximum or minimum are determined as before in Art. 83., and become

$$\left(\frac{d\text{F}}{dx}\right) = 0, \quad \left(\frac{d\text{F}}{dy}\right) = 0, \quad \left(\frac{d\text{F}}{dz}\right) = 0;$$

and there must be no change of sign, whatever be the signs of h, k, l, in

$$\left(\frac{d^2\text{F}}{dx^2}\right)h^2 + \left(\frac{d^2\text{F}}{dy^2}\right)k^2 + \left(\frac{d^2\text{F}}{dz^2}\right)l^2 + 2\left(\frac{d^2\text{F}}{dy\,dz}\right)kl$$
$$+ 2\left(\frac{d^2\text{F}}{dz\,dx}\right)lh + 2\left(\frac{d^2\text{F}}{dx\,dy}\right)hk.$$

Which condition will be fulfilled if

$$4h^2\left(\frac{d^2\text{F}}{dx^2}\right)\left\{\left(\frac{d^2\text{F}}{dy^2}\right)k^2+\left(\frac{d^2\text{F}}{dz^2}\right)l^2+2\left(\frac{d^2\text{F}}{dy\,dz}\right)kl\right\} \text{ be}$$

$$> 4h^2\left\{\left(\frac{d^2\text{F}}{dz\,dx}\right)l+\left(\frac{d^2\text{F}}{dy\,dx}\right)k\right\}^2,$$

$$\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right)k^2+\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dz^2}\right)l^2+2\left(\frac{d^2\text{F}}{dy\,dz}\right)\left(\frac{d^2\text{F}}{dx^2}\right)kl \text{ be}$$

$$> \left(\frac{d^2\text{F}}{dz\,dx}\right)^2 l^2+2\left(\frac{d^2\text{F}}{dy\,dx}\right)\left(\frac{d^2\text{F}}{dz\,dx}\right)kl+\left(\frac{d^2\text{F}}{dy\,dx}\right)^2 k^2,$$

$$\left\{\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy^2}\right)-\left(\frac{d^2\text{F}}{dy\,dx}\right)^2\right\}\left\{\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dz^2}\right)-\left(\frac{d^2\text{F}}{dx\,dz}\right)^2\right\} \text{ be}$$

$$> \left\{\left(\frac{d^2\text{F}}{dx^2}\right)\left(\frac{d^2\text{F}}{dy\,dz}\right)-\left(\frac{d^2\text{F}}{dz\,dx}\right)\left(\frac{d^2\text{F}}{dy\,dx}\right)\right\}^2.$$

In like manner, if the function be of more variables, may the conditions of a maximum and minimum be determined.

86.] When, however, the problem is to determine the maxima and minima of a function of several variables, it frequently happens that certain equations are given between the variables, so that the number of independent variables is less than the number in the given function. Thus, suppose we have to determine the maxima and minima values of

$$u = \text{F}(x, y, z \,.\,.\,.\,.\,.),$$

a function of n variables $x, y, z, \ldots\ldots$; and suppose, besides, we have m equations given, connecting these variables, viz.

$$\begin{aligned} \text{F}_1\,(x, y, z, \ldots\ldots) &= 0, \\ \text{F}_2\,(x, y, z, \ldots\ldots) &= 0, \\ \ldots\ldots\ldots & \\ \text{F}_m\,(x, y, z, \ldots\ldots) &= 0. \end{aligned}$$

In order to apply the method which has been explained in the last Article, it would be necessary to eliminate m variables between $m+1$ equations, by which means u would become a function of $n-m$ variables; and then forming the partial derived functions $\left(\frac{d\text{F}}{dx}\right)$, $\left(\frac{d\text{F}}{dy}\right)$, $\left(\frac{d\text{F}}{dz}\right)$, &c. the number of which is

$n-m$, and equating each to 0, we should have $n-m$ equations, from which we could (theoretically at least) determine the $n-m$ variables. This method, however, though theoretically possible, is frequently attended with great difficulty on account of elimination; and, if the original expressions be symmetrical, the symmetry is destroyed by it: in which case we may proceed as follows.

It is plain from what has been said, that, as there are $n-m$ variables entirely independent in their variations, we have $n-m$ conditions to make; which will be equivalent to equating to 0 the $(n-m)$ partial derived functions, with respect to these variables, of $\mathrm{F}(x, y, z, \ldots)$. Differentiating the functions in order, and remembering that $\mathrm{D}u=0$, because u is to be a maximum or minimum, we have

$$\mathrm{D}u = 0 = \left(\frac{d\mathrm{F}}{dx}\right) dx + \left(\frac{d\mathrm{F}}{dy}\right) dy + \left(\frac{d\mathrm{F}}{dz}\right) dz + \&c.$$

$$0 = \left(\frac{d\mathrm{F}_1}{dx}\right) dx + \left(\frac{d\mathrm{F}_1}{dy}\right) dy + \left(\frac{d\mathrm{F}_1}{dz}\right) dz + \&c.$$

$$0 = \left(\frac{d\mathrm{F}_2}{dx}\right) dx + \left(\frac{d\mathrm{F}_2}{dy}\right) dy + \left(\frac{d\mathrm{F}_2}{dz}\right) dz + \&c.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 = \left(\frac{d\mathrm{F}_m}{dx}\right) dx + \left(\frac{d\mathrm{F}_m}{dy}\right) dy + \left(\frac{d\mathrm{F}_m}{dz}\right) dz + \&c.$$

The meaning of which is, that x, y, z, &c. do not vary independently of each other, but consistently with the conditions involved in the last m equations. Hence, to eliminate dx, dy, dz, &c. multiply these equations by indeterminate quantities, $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_m$, and add to the first; and, collecting the coefficients of dx, dy, dz, &c. we have

$$\left\{ \left(\frac{d\mathrm{F}}{dx}\right) + \lambda_1 \left(\frac{d\mathrm{F}_1}{dx}\right) + \lambda_2 \left(\frac{d\mathrm{F}_2}{dx}\right) + \ldots\ldots + \lambda_m \left(\frac{d\mathrm{F}_m}{dx}\right) \right\} dx$$

$$+ \left\{ \left(\frac{d\mathrm{F}}{dy}\right) + \lambda_1 \left(\frac{d\mathrm{F}_1}{dy}\right) + \lambda_2 \left(\frac{d\mathrm{F}_2}{dy}\right) + \ldots\ldots + \lambda_m \left(\frac{d\mathrm{F}_m}{dy}\right) \right\} dy$$

$$+\left\{\left(\frac{d\text{F}}{dz}\right)+\lambda_1\left(\frac{d\text{F}_1}{dz}\right)+\lambda_2\left(\frac{d\text{F}_2}{dz}\right)+\ldots\ldots+\lambda^m\left(\frac{d\text{F}_m}{dz}\right)\right\}dz$$
$$+\left\{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \right\}\ldots$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \ldots=0.$$

Which equation is subject to n conditions, viz. $n-m$, on account of $n-m$ independent variables being involved, and m on account of our having introduced m indeterminate multipliers $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_m$. Let these conditions be that the coefficient of each differential is equal to 0;

$$\therefore\quad \left(\frac{d\text{F}}{dx}\right)+\lambda_1\left(\frac{d\text{F}_1}{dx}\right)+\lambda_2\left(\frac{d\text{F}_2}{dx}\right)+\ldots\ldots+\lambda_m\left(\frac{d\text{F}_m}{dx}\right)=0,$$

$$\left(\frac{d\text{F}}{dy}\right)+\lambda_1\left(\frac{d\text{F}_1}{dy}\right)+\lambda_2\left(\frac{d\text{F}_2}{dy}\right)+\ldots\ldots+\lambda_m\left(\frac{d\text{F}_m}{dy}\right)=0;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

between which equations $\lambda_1, \lambda_2 \ldots \lambda_m$ are to be eliminated, and x, y, z, &c. determined, which will be the values corresponding to a maximum or minimum value of $\text{F}(x, y, z \ldots)$. The sign of the second differential coefficient will determine whether the particular value be a maximum or a minimum: but, in most cases where this method is applicable, the form of the function at once decides whether it admits of a maximum or of a minimum.

Ex. 1. Suppose a, b, c, p to be constants, and it is required to determine the minimum value of $u^2=x^2+y^2+z^2$, having given

$$p = ax + by + cz.$$

$$\text{D}(u^2) = 0 = 2x\,dx + 2y\,dy + 2z\,dz,$$
$$0 = a\,dx + b\,dy + c\,dz;$$

$$\therefore\quad (2x+\lambda a)\,dx + (2y+\lambda b)\,dy + (2z+\lambda c)\,dz = 0,$$

$$\therefore\quad 2x+\lambda a=0,\qquad 2y+\lambda b=0,\qquad 2z+\lambda c=0,$$

$$\therefore\quad \frac{2x}{a}=\frac{2y}{b}=\frac{2z}{c}=\frac{2\,(ax+by+cz)}{a^2+b^2+c^2}=\frac{2p}{a^2+b^2+c^2}$$

$$=\frac{2\sqrt{(x^2+y^2+z^2)}}{\sqrt{(a^2+b^2+c^2)}}=\frac{2u}{\sqrt{(a^2+b^2+c^2)}},$$ by Preliminary Proposition III.

$$\therefore \quad x = \frac{ap}{a^2+b^2+c^2}, \; y = \frac{bp}{a^2+b^2+c^2}, \; z = \frac{cp}{a^2+b^2+c^2},$$

$$u = \frac{p}{\sqrt{(a^2+b^2+c^2)}}.$$

Ex. 2. To determine the lengths of the Semi-axes of an Ellipse referred to its Centre and Conjugate Diameters.

Let ω be the angle between the conjugate diameters; then, if r be the distance of any point in the curve from the centre, the maximum and minimum values of r will be respectively the semi-major and semi-minor axes. Let a_1 and b_1 be the semi-conjugate axes; then we have

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1,$$

$$r^2 = x^2 + 2xy\cos\omega + y^2;$$

$$\therefore \quad \frac{x}{a_1^2}\,dx + \frac{y}{b_1^2}\,dy = 0;$$

$$(x + y\cos\omega)\,dx + (y + x\cos\omega)\,dy = r\,dr = 0;$$

$$\therefore \left\{\frac{x}{a_1^2} + \lambda(x + y\cos\omega)\right\}dx + \left\{\frac{y}{b_1^2} + \lambda(y + x\cos\omega)\right\}dy = 0.$$

$$\text{Let } \left.\begin{aligned} \frac{x}{a_1^2} + \lambda(x + y\cos\omega) &= 0 \\ \frac{y}{b_1^2} + \lambda(y + x\cos\omega) &= 0 \end{aligned}\right\}$$

Multiply the first by x, the second by y, and add:

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \lambda(x^2 + 2xy\cos\omega + y^2) = 0.$$

$$1 + \lambda r^2 = 0; \qquad \therefore \quad \lambda = -\frac{1}{r^2}.$$

Whence substituting,

$$\left(\frac{1}{a_1^2} - \frac{1}{r^2}\right)x - \frac{1}{r^2}\cos\omega\, y = 0,$$

$$-\frac{1}{r^2}\cos\omega\, x + \left(\frac{1}{b_1^2} - \frac{1}{r^2}\right)y = 0:$$

and, by cross-multiplication,

$$\left(\frac{1}{a_1^2} - \frac{1}{r^2}\right)\left(\frac{1}{b_1^2} - \frac{1}{r^2}\right) - \frac{1}{r^4}\cos^2\omega = 0;$$

$$\therefore \quad r^4 - (a_1^2 + b_1^2)\, r^2 + a_1^2 b_1^2 \sin^2\omega = 0.$$

And, since the roots of this equation are a^2 and b^2, we have by the theory of equations

$$a^2 + b^2 = a_1^2 + b_1^2,$$

$$ab = a_1 b_1 \sin\omega,$$

two well known properties of the ellipse.

For other examples illustrative of the processes explained in this chapter, see Mr. Gregory's *Collection*, chap. vii.

PART II.

Geometrical Applications.

CHAP. VIII.

ON THE GEOMETRICAL INTERPRETATION OF SYMBOLS.

On Geometrical Principles.

87.] In the first chapter of this work, when we were discussing the subject matter of the Differential Calculus, we laid down that the magnitudes which we symbolise are of two kinds, constants and variables; and we said that the variables of which we were going to treat change value in accordance with the law of continuity. It appears, also, from what was said in a subsequent article of the same chapter, that the variations of these variable quantities are confined within certain limits, which cannot be more accurately defined than that they are infinity and zero, that is, $\frac{1}{0}$ and 0. Our object in the present chapter is so to enlarge our conceptions of geometry and geometrical magnitudes as to adjust them to what we have thus made *principles* of the Calculus; but as we are not writing a treatise on the difficulties of elementary geometry, we shall rather state results than discuss the methods by which we have arrived at them, adding, however, a few illustrations to render our conceptions more sensible. Much confusion seems to have arisen in this branch of mathematics from writers not accurately distinguishing between *the mode of generating* geometrical quantities and geometrical quantities *when generated:* in the following remarks the ideas of motion and of limits are introduced; motion as having to do with the generation of quantities, and limits as a property of them when generated. This, however, does not encroach at all on the province of mechanics, wherein we treat of motion as the effect of certain causes, and discuss its circumstances; as e. g. the law of the force which produces it, the velocity with which the moving material changes position, which necessarily involves time, and so on: but in what follows we consider motion as a simple act, a primary conception as a quality of matter; and if it tends, as it does, to give clearness to our first geometrical conceptions, it is nothing but a servile adherence to an inferior, though customary method, which would hinder us from introducing it.

It is conceived that all geometrical quantity, whether linear, superficial, or spatial, is, from its very nature, capable of increase or decrease to an indefinite extent. A line may be very long, nay, of an infinite length, or very short; space may be very small, such as, so to speak, it would require a microscope of almost infinite power to render visible, or it may be very large: whenever such quantities vary, they do so in accordance with the law of continuity (see Art. 4.); they cannot pass from one magnitude to another without passing through all intermediate magnitudes; they "grow" larger and larger, or less and less. This capability of increase or decrease is involved in our idea of geometrical quantity; it is a property of it; and it is contrary to our conceptions to say that such change can be only by alternations. Space may become larger without first becoming smaller, and a line may continue to *grow* longer and longer, without ever becoming shorter, and *vice versâ*; that is, the increase may be progressive, and so may the decrease.

There are, however, limits within which this variation is included: the superior limit of geometrical magnitude of the concrete kind, called *space*, is infinite space; so of superficial and linear magnitudes, the superior limits are respectively infinite superficies, and a line of infinite length.

The inferior limit of all these is the same, the geometrical zero, a *point*. We wish all that has been said in the first chapter on infinity and infinitesimals to be transferred to this place, with the single exception that what has there been laid down, having had reference to abstract numerical quantities and zeros, is now to be referred to concrete geometrical magnitudes.

88.] Of the Definitions of Geometrical Quantities founded on these conceptions, the following are useful to our present purpose.

I. A *point* is the inferior limit of geometrical space.

II. A *sphere* is the locus of a point in space, which is always at the same distance from a given point.

III. A *plane* is the surface of a sphere, the radius of which is infinitely great.

IV. A *circle* is the locus of a point, which is always at the same distance from a given point, all the points being in one plane.

V. A *straight line* is the arc of a circle, the radius of which is infinitely great.

VI. A *triangle* is a plane figure contained by three straight lines meeting one another, two and two.

VII. And, if the triangle be isosceles, the sides of that triangle, having a finite base and the vertex at an infinite distance, are *parallel straight lines*.

As this is not intended to be an accurate treatise on the principles of geometry, many words are used which have not been defined, as line, locus, &c.; these, however, are to be taken in their ordinary significations, and it is to be observed, with respect to these definitions and conceptions, that the surfaces, lines, &c. they refer to, are only approximations to the accurate ones. But they are such approximations as may differ from the real ones by quantities as small as we please; and as these small quantities may be infinitesimals, such that it would require an infinity of them to make a finite quantity, and we do not take an infinite number of them, these differences may, in conformity with what has been said in the first chapter, be neglected, and our definitions are, *practically*, rigorously exact.

Having defined a plane, as we have done, to be the limiting spherical surface when the radius becomes infinitely great, it follows that the extreme positive side of the plane, when continued, runs into the extreme negative side; that is, having traced the plane as far as we can on the positive side, we meet it again on the negative; and, although the surface appears to be discontinuous, it is not in reality so, the positive side being continued into the negative, and the apparent discontinuity arises from the defects in our powers of apprehending and symbolising such quantities. Thus, then, if we have any continuous curve traced on the plane, and the curve runs off to the extreme positive side of the plane, we ought not to consider it to stop or to have points of discontinuity, but we must consider the branches of it to be continued, and must look for them on the negative side of the plane. We may borrow from the figure of the earth, and our mode of determining position on its surface, an illustration of what is here intended. We measure, say from the meridian of Greenwich, degrees along the equator to 180° east longitude; and then, instead of proceeding further on, and measuring in the same direction, we measure backwards, and reckon degrees of longitude west; and what would be 181° east longitude becomes 179° west. If, then, east corresponds to the positive direction, west does to the negative.

It is worth remarking how exactly our ideas of a plane coincide with the definition I have given. We speak of the surface of water as a plane, and consider it to be *level*, whereas it is a portion of the surface of a sphere, whose radius is very large compared with the area we take, say 4000 miles compared with a few inches.

So, again, as to our conception of a straight line. A straight line being a particular instance of a circle is a continuous line; it does not terminate at positive infinity nor at negative infinity, but the two branches of the line are connected with one another, running, if we may so speak, round the circle of which the radius is infinity, and joining together. If, then, we take any given point on the circle as the origin, the distance to the opposite extremity of the circle is positive infinity, and we do not measure or follow the line farther in this direction, but considering the line to be continued beyond that point, we meet it on the opposite side, and measure it backwards. There is no point of discontinuity in the line: the line proceeds in the same direction; it has been positive infinity; the pole or extremity of the diameter of the circle has been passed, the line becomes negative infinity. The illustration above given from the figure of the earth aptly illustrates our meaning in this case. Considering any meridian to be the very large circle, and taking any place on it to be the origin, the "antipodes" to it becomes either positive infinity or negative infinity according as we measure in the positive or negative direction; the sign of the quantity changes immediately after the pole has been passed, and what was positive infinity becomes negative infinity. Therefore, in this point of view, infinity is not a quantity incapable of increase, for the line may be continued round and round the meridianal circle as often as we please; there is no limit to the quantity, the limit is to our powers of symbolising such quantities.

It is worth observing, too, that the definition given of parallel straight lines enables us to avoid the difficulty connected with our first introduction to the theory and properties of such lines. Having shown, as Playfair has done, that the exterior angles of a triangle are together equal to four right angles, it follows that the interior angles are equal to two right angles: but if the base of a triangle remains finite, and the vertex is removed further and further, the vertical angle becomes less and less, and diminishes without limit, in which case the sum of the base angles is equal to two right angles, and the sides become parallel straight lines, and thus their properties, which are enuntiated in the XXIXth proposition of the first book of Euclid, immediately follow.

A good illustration of this theory occurs in the phenomena of parallax. If the angles subtended at the centre of the earth by the sun and any fixed star, whose parallax has not been discovered, be observed when the earth is in perihelion and at aphelion, it is found that, notwithstanding the extreme delicacy of our instruments, the sum of these two angles is exactly equal to two right angles. Taking, then, the two positions of the earth to be the extremities of the base of a triangle, and the line passing through the sun's centre and terminated by them to be the base, and the fixed star to be the vertex, it appears that, although the base of the triangle be 190,000,000 of miles, the angle subtended by it at the vertex is nothing, and the two lines drawn to the star from the earth, at the two positions of it, are parallel straight lines.

89.] In corroboration, also, of what has here been stated, the following are a few out of a great many striking instances.

In differentiating $\tan\theta$, we have

$$\frac{d.\tan\theta}{d\theta} = \sec^2\theta,$$

which is necessarily a positive quantity; and therefore, by Theorem I. Chap. IV., θ and $\tan\theta$ are always increasing and decreasing simultaneously, and therefore as θ increases $\tan\theta$ increases. Now, as θ approaches to 90°, $\tan\theta$ becomes $+\frac{1}{0}$, and immediately after θ has passed 90°, $\tan\theta$ becomes $-\frac{1}{0}$, indicating that negative infinity is positive infinity increased, that is, as θ has increased, and passed through 90°, $\tan\theta$ has increased from $+\frac{1}{0}$ to $-\frac{1}{0}$; and, so again, as θ increases from 90° to 180° $\tan\theta$ is continually increasing from $-\frac{1}{0}$ to 0, and passes through 0, and increases to $+\frac{1}{0}$, which is the value of $\tan\theta$, when $\theta =$ 270°; and so on, as θ increases $\tan\theta$ is continually increasing, travelling, if we may so say, round the circle of which it is conceived the straight line along which $\tan\theta$ lies is the limit when the radius of the circle is infinitely great. It is impossible not to remark how exactly this illustration agrees with what has been said in Chapter I. on the order of infinitesimals. Corresponding to every 180° through which θ turns, $\tan\theta$ passes from 0 to $\frac{1}{0}$, and on through $\frac{1}{0}$ to 0 again; that is, the path through which $\tan\theta$ has travelled is infinite, although θ has passed over only a finite quantity. When, then, θ has revolved through 360° and 540° and 720°, and so on, $\tan\theta$ has travelled over a length of line equal to twice, three times, four times, &c., the infinite length corresponding to a revolution of θ through 180°, and thus we have infinities bearing a finite ratio to each other. Conceive, moreover, θ to have revolved an infinite number of times through 180°, then the distance over which $\tan\theta$ will have travelled will be an infinity of infinities, that is, will be $(\text{infinity})^2$, and thus we obtain different orders of infinity.

Again, suppose we had given the following problem: to find the maximum and minimum values of y, when

$$y = \frac{(x+2)^2}{(x-3)^3},$$

$$\frac{dy}{dx} = -\frac{(x+2)(x+12)}{(x-3)^4},$$

$y = \frac{1}{0}$, when $x = 3$, but as $\frac{dy}{dx}$ does not change its sign, this value of y is neither a maximum nor a minimum. How then is the result to be interpreted? As follows: since $\frac{dy}{dx}$ is negative, y decreases as x increases, and when x is a little less than 3, $y = -\frac{1}{0}$, but when x is a little greater than 3, $y = +\frac{1}{0}$; therefore, as x has passed through 3, the value of y has changed from $-\frac{1}{0}$ to $+\frac{1}{0}$, but y has decreased during this progressive increase of x, therefore $+\frac{1}{0}$ is $-\frac{1}{0}$ decreased; therefore y has not reached a minimum or maximum value when $x = 3$, because it has not become $-\frac{1}{0}$, and then returned, but it has gone on decreasing. And if we draw a graphical representation of the curve corresponding to the equation, such as in fig. 10., the phenomena explain themselves. The curve on the negative side of the axis of y is of the form CB, where OB $= 2$; and if OA $= 3$, the curve is continually approaching the line drawn through A parallel to the axis of y, and when x is nearly 3, y is $-\frac{1}{0}$, but when x is greater than 3, y is $+\frac{1}{0}$; that is, the curve has crossed the asymptote at the pole of the circle of infinite radius opposite to A, and has returned in the direction EF, the branch in the direction of E being a continuation of that in the direction of D. Similarly the branch in the direction F would, if produced, unite itself to that in the direction C, having crossed the axis of x at the pole opposite to O.

In corroboration of this theory, it will appear that if the criteria, which will be discussed in the next chapter, be applied, whenever a curve is of the form fig. 10., at such points as where the branch E meets the branch D, and crosses the asymptote, we have all the characteristics of a point of inflexion; and if the curve be such as in fig. 11., we have the characteristics of a point of *embrassement;* and whenever such as is represented in fig. 12., all the conditions of a maximum and minimum ordinate.

And so, again, whenever a branch of a curve continues to infinity, it always returns in some way or another; and, in whatever manner a rectilineal asymptote be drawn, no branch of the curve ever goes off asymptotic to it without returning in one of the ways indicated in the figures 10, 11, and 12.; and it seems impossible to account for such phenomena except on the theory explained above, viz. that the plane and the straight line are respectively the superior limits of the sphere and the circle, when the radii become very large.

On the Interpretation of Symbols of Direction.

90.] In algebraical geometry, and, therefore, in the applications of the Differential Calculus to the theory of plane curves, we meet with symbols of two distinct characters: symbols of quantity, such as $a, b, c, \dots x, y, z, \theta, \phi, \psi, \dots$, when symbolical respectively of lines and angles; and symbols of direction, $+, -, +\sqrt{(-)}, -\sqrt{(-)}$, &c. Our object is so to enlarge our method of interpreting symbols of this second kind, as to comprehend those which are usually called impossible, of which, however, we shall discuss only two, viz. $+\sqrt{-}, -\sqrt{-}$, or as they may be written, in accordance with the index law, $+(-)^{\frac{1}{2}}, -(-)^{\frac{1}{2}}$. For a fuller explanation of the principles of explaining these symbols, we would refer to *Etudes Philosophiques sur la Science du Calcul*, par M. F. Vallès; and for the general theory on the meaning of $(+)^{\frac{p}{q}}$ to Dr. Peacock's *Algebra*, Mr. Warren's treatise on the subject, and to several papers in the *Cambridge Mathematical Journal.*

As to symbols of quantity, it is to be observed, that, when we symbolise a line by a, we do not mean that a is the absolute length of the line; for all lengths can only be relative, and there must be some modulus or standard to compare them with: but we intend a line which is in length a times some arbitrary, though for the time fixed, standard unit. So, a line symbolised by b is a line b times in length some unit. Thus, then, a, b are numerical quantities, not concrete magnitudes, but abstract quantuplicities, the subject matter of arithmetical algebra, and, therefore, subject to its laws; they do not designate the absolute lengths of lines, but the number of times a certain concrete unit is to be taken. So, again, if an area be symbolised by $a\,b$, a and b are abstract numbers, which must be multiplied together by the laws of arithmetical algebra, and their product is the number of times the superficial unit is to be taken. Let it, then, be carefully borne in mind that this is the meaning of the several symbols of quantity, whether constant or variable, which we shall use in the next and following chapters. Suppose, then, we have a line symbolised by a, and we fix upon a certain point as the origin from which lines are to be measured, any line drawn from it, equal in length to a times the linear unit, will fulfil the requirements of the single symbol a. Inasmuch, then, as an indefinite number of equal lines may be drawn from any one point, thus far we have no means of determining which of all such lines is intended; hence arises the necessity of some other symbols to indicate direction, or, as they are called, *symbols of affection*. One or two of the most simple cases of these we proceed to explain; feeling assured that the principle of explanation is so entirely in harmony with the usual meaning of $+$ and $-$, that it ought not to be omitted in an elementary treatise; and also because it enables us to show that an algebraical curve, though apparently discontinuous and confined within certain fixed limits, is not in reality so, but extends to infinity in all directions. Other parts of the theory (some of which are as yet not sufficiently established) we omit, as unsuited to our present object.

91.] Suppose O, fig. 13., to be the point from which lines are to be measured, and $\text{OA} = a$ times the linear unit to be drawn from O towards the right hand. Now, since, as we said above, any line drawn from O, a times the linear unit in length, will be symbolised by a, it is necessary to fix on some originating direction; suppose this to be OA, and any line measured from O towards A to be affected with the symbol of direction $+$; if, then, after a line has undergone any operation or a series of operations, it comes into the position OA, it is still to be symbolised by $+$; and, if the line be a, by $+a$. Such an operation we may conceive to be a reciprocating one, the line at one time being in the position OA, and at another in the position O′A′, having moved sideways, and assumed all intermediate positions. Or, we can conceive that the line OA has revolved round the point O, and, having turned on the plane of the paper through 360°, has again come into its original position, and so on continually; and it is manifest that as often as it has revolved through any multiple of 360°, it has assumed its original position O A, and is therefore to be symbolised by $+ a$. So, also, there are many conceivable ways in which the line may have moved, and that periodically, and at the end of a complete period be in the position O A. But have we any other customary mode of indicating direction, to serve as a guide which of these conceivable operations to take? We have. Whenever a line equal in length to a is measured from O towards the left, we symbolise it by $-a$; if, therefore, either $(-)$ were a symbol for the operation of one oscillation having been performed on the line, i. e. the line having passed into the position OA_1, or $(-)$ symbolised the line O A having been turned through 180°, either would account for the negative sign of affection, and $(-)$ would be the symbol of the operation; but, under the first hypothesis, the line at one stage of the process will be half on the positive side of the origin, and half on the negative. If, therefore, the operation be continuous, which it is, in passing from + to −, there should be some symbol to indicate that particular stage; it does not, however, appear that we have any symbol of the kind; and such a motion, and a line in such a state, are what in our ordinary geometrical conceptions we do not use nor contemplate. Let us, therefore, consider whether we have not symbols to indicate a line in any intermediate position between O A and OA_1, conceiving the line to pass from the one position to the other by means of revolving through 180°.

As we said before, whenever the line is measured from O in the direction O A, it is to be affected with a + sign. Taking, therefore, O as the origin of line, and O A as the *direction line* from which symbols and operations of affection are to be originated, whenever a line, as e. g. O A, has turned an integral number of times through 360°, it is to be affected with the sign with which it started. If, therefore, it was affected with the + sign at first, indicating that it started from O A, and if + be the symbol of turning through 360°, after one revolution the symbol of affection is + on the back of +, i. e. according to the index law $+^2$; similarly after two revolutions, $+^3$; and after $(n-1)$ revolutions, $+^n$. Supposing, therefore, that the line which is of the length a, when along the originating direction O A, is unaffected with any sign, $+a$ means that the line has turned through 360°, and has come again into the position whence it started; and so $+^n a$ means that a line of length a has revolved n times from the direction of origination, and is in the position OA; whence it appears (in accordance with the arithmetical meaning and law of $+$) that $+$ is, for symbolical purposes of direction, equivalent to $+^n$, n being a whole number.

In conformity, then, with the algebraical law of indices $+^{\frac{1}{2}}$ is the symbol of that operation, which, being performed twice, one on the back of the other, brings the symbol into the value $+$; that is, if $+$ signifies turning the line through 360°, $(+)^{\frac{1}{2}}$ indicates turning it through 180°, but $-$ symbolises this operation,

$$\therefore \quad +^{\frac{1}{2}} = -, \text{ and } (-)^2 = +;$$

or the operation symbolised by $(-)$ performed twice, one upon another, is equivalent to the operation signified by $+$, and means turning a line through 360°. Similarly, again, $(+)^{\frac{2n+1}{2}}$ is equivalent to $-$, for it is equivalent to $+^{n+\frac{1}{2}} = +^n (+)^{\frac{1}{2}} = +^n -$; and this coincides with the ambiguity we have always in the sign of $+^{\frac{1}{2}}$, for it may, as far as the form $+^{\frac{1}{2}}$ teaches, be either $+$ or $-$; if therefore the $+$, whose root has to be extracted, be raised to an even power, its root is to be affected with a positive sign; but if the $+$ be $+^{2n+1}$, then the square root is $+^n +^{\frac{1}{2}}$, which is equivalent to $-$, and the root must be affected with the negative sign. Hence, also, it is plain that $\sqrt{(-a)}\sqrt{(-a)}$, which equals $\sqrt{a^2}$, can only be $-a$, because the $+$, with which a^2 is affected under the radical, is of only the first power.

Therefore we have shown that in symbolical geometry

$$\left.\begin{array}{ll} \text{1st,} & +^n = + \\ \text{2d,} & +^{\frac{2n+1}{2}} = - \end{array}\right\}.$$

92.] So, again, $+^{\frac{1}{4}}$ symbolises that operation which, being performed twice, one on the back of the other, is equivalent to $+^{\frac{1}{2}}$, i. e. to $(-)$, and, being performed four times successively one on the back of the other, is equivalent to $+$;

$$\therefore \quad +^{\frac{1}{4}} = (-)^{\frac{1}{2}};$$

and, therefore, as $-$ indicates that a line is to be turned through 180°, so $(-)^{\frac{1}{2}}$ means that a line is to be turned through 90°. Whenever, then, a line is affected with $(-)^{\frac{1}{2}}$, which is equivalent to $+^{\frac{1}{4}}$, as its symbol of direction, that line is to be drawn at right angles to the original direction of origination, viz. in the direction OA_2 (see fig. 14.); and whenever the symbol of direction is $+^{\frac{3}{4}}$, which $= +^{\frac{1}{2}} +^{\frac{1}{4}} = -(-)^{\frac{1}{2}}$, the line which is affected with it is to be drawn as the direction OA_3. Similarly, $+^{\frac{4n+1}{4}}$ indicates a line drawn in the direction OA_2; and $+^{\frac{4n+3}{4}}$ a line drawn in the direction OA_3. So, also, $+^{\frac{1}{n}}$ means that that line with which it is affected is to be drawn at an angle of $\frac{360°}{n}$ to the originating direction OA.

93.] To apply these principles to the delineation of plane curves from their equations, suppose $y=f(x)$ to be the equation to the curve; since x and y have already preoccupied the two directions at right angles to each other in the plane of the paper which is called the *plane of reference*, we must seek for some other means by which a line which has been measured in the positive direction may be made to turn through 180° into the negative. Such will be the case if it is made to revolve in a plane to which the other axis is perpendicular; as, for instance, if x revolves in a plane at right angles to the axis of y, by which means, whenever x is affected with $\pm(-)^{\frac{1}{2}}$, it is measured in a plane passing through the axis of y, and perpendicular to the axis of x. Similarly, if y be affected with $\pm(-)^{\frac{1}{2}}$, it is to be drawn in the plane passing through the axis of x, and perpendicular to the axis of y. Thus it appears that an equation between x and y not only represents a curve in the plane of the paper, but also curves in the planes at right angles to it, passing through the axes of x and y. An example is subjoined.

To trace the curve whose equation is

$$y^2 + x^2 = a^2.$$

$$\therefore \quad y = \pm \sqrt{(a^2 - x^2)};$$

and this represents a circle in the plane of the paper for all values of x less than a: but, when x is $> a$, we have (see fig. 15.)

$$y = \pm (-)^{\frac{1}{2}} \sqrt{(x^2 - a^2)};$$

which represents an equilateral hyperbola (indicated by the dotted line in the figure) in a plane passing through O A, and at right angles to the plane of the paper. Similarly, we have

$$x = \pm \sqrt{(a^2 - y^2)};$$

which represents the same circle as before in the plane of the paper for all values of y less than a: but, when y is $>$ than a,

$$x = \pm (-)^{\frac{1}{2}} \sqrt{(y^2 - a^2)};$$

which represents an equilateral hyperbola, and which is to be drawn in the plane passing through O B, at right angles to the plane of the paper, and is represented by the dotted line in the figure. The straight lines are drawn to represent the asymptotes, and are to be considered to be in the planes in which the hyperbolæ are drawn. Their equations are

$$y = \pm(-)^{\frac{1}{2}}x, \qquad x = \pm(-)^{\frac{1}{2}}y.$$

CHAP. IX.

ON PROPERTIES OF PLANE CURVES AS DEFINED BY THE EQUATIONS REFERRED TO RECTANGULAR CO-ORDINATES.

In the following discussion we shall find it convenient to use sometimes the explicit or resolved form of the equation to a curve, viz. $y=f(x)$; and sometimes the implicit or unresolved form, viz. $u = \text{F}(x, y) = c$, c being a constant, and admitting of the particular value 0. As, e. g., the equation to the ellipse may be put under either of the forms,

$$\left.\begin{array}{l} y = \pm \frac{b}{a} \sqrt{(a^2 - x^2)} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{array}\right\}$$

94.] To find the Equation to the Tangent to a given Curve.

Def. The tangent to a curve is that line which passes through two points of the curve which are infinitely near to each other.

Let ξ, η be the current co-ordinates to the tangent; and, first, let us consider the two points through which the line is to pass to be at a finite distance apart.

Let x, y be the co-ordinates to one point,

$x + \Delta x, y + \Delta y$ . . . . the other.

Then the equation to the secant is $\eta - y = \frac{\Delta y}{\Delta x}(\xi - x)$. (1)

If the two points become infinitely near to each other, according to the principles of the Calculus, $\frac{\Delta y}{\Delta x}$ becomes $\frac{dy}{dx}$, the secant becomes a tangent, and the equation becomes

$$\eta - y = \frac{dy}{dx}(\xi - x); \tag{2}$$

or, as it may be written, $\frac{\eta - y}{dy} = \frac{\xi - x}{dx}$. (3)

If, therefore, the equation to the curve be given under the explicit form $y = f(x)$, by differentiation we have

$$\frac{dy}{dx} = f'(x),$$

and the equation to the tangent may be found immediately from (2); but if the equation to the curve be given under the implicit form $u = \text{F}(x, y) = c$,

then $$\text{D}u = \left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy = 0;$$

and substituting for dx and dy their proportional values given in Equation (3), the equation to the tangent assumes the form

$$(\xi - x)\left(\frac{d\text{F}}{dx}\right) + (\eta - y)\left(\frac{d\text{F}}{dy}\right) = 0. \tag{4}$$

If the equation to the curve be a homogeneous function of n dimensions, then, since by the property of such functions proved in Art. 75.,

$$x\left(\frac{d\text{F}}{dx}\right) + y\left(\frac{d\text{F}}{dy}\right) = nu,$$

the equation to the tangent (4) assumes the simple form

$$\xi\left(\frac{d\text{F}}{dx}\right) + \eta\left(\frac{d\text{F}}{dy}\right) = nu. \tag{5}$$

Ex. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 = \text{F}(x, y) = u.$$

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) = \frac{2x}{a^2}, \qquad \left(\frac{d\text{F}}{dy}\right) = \frac{2y}{b^2}, \qquad n = 2;$$

$$\therefore \quad \frac{x\xi}{a^2} + \frac{y\eta}{b^2} = 1$$

is the equation to the tangent of an ellipse.

95.] From the definition of a tangent which has been given, there immediately follows a remarkable theorem as to the limit of the number of tangents which can be drawn to a curve from any one point.

Suppose the equation to the curve to be of the nth order: then a line drawn through any point cannot cut the curve in more than n points; and, therefore, there cannot be drawn more lines cutting the curve in two points, than there are permutations of these n points of section taken two and two together; that is, there cannot be more than $n(n-1)$: but as each of these secants, if the two points of section can be made to coalesce, may become a tangent, and as there can be no other tangents, it follows that there cannot be more than $n(n-1)$ tangents drawn to a curve of the nth degree through a given point. The theorem does not prove that the curve is such as to admit of *all* these tangents, but limits the number that can be drawn under the most favourable circumstances.

96.] From Equation (2) it appears that $\frac{dy}{dx}$ is the trigonometrical tangent of the angle made with the axis of x by the tangent to the curve. If, therefore, at any point $(x=a, y=b)$, $\frac{dy}{dx}$ has a finite value, the curve at that point is inclined to the axis of x at a finite angle, as in fig. 17.; and if $\frac{dy}{dx}$ be positive, which indicates that as x increases y increases also, the curve must be such as that delineated in fig. 8.; and if $\frac{dy}{dx}$ be negative, that is, as x increases, y decreases, or *vice versâ*, the curve is such as that in fig. 9.

If $\frac{dy}{dx} = 0$, the tangent, and therefore the curve at the point of contact, is parallel to the axis of x; and if $\frac{dy}{dx}$ changes its sign, there is a maximum or minimum ordinate, such as we have drawn in figs. 6. and 7.; but if $\frac{dy}{dx}$ does not change sign, then the form of the curve will be such as in figs. 8. and 9., according as $\frac{dy}{dx}$ is positive or negative. If $\frac{dy}{dx}=\frac{1}{0}$, the tangent, and therefore the curve at the point of contact, is perpendicular to the axis of x, and is such as one or other of the forms drawn in fig. 16.; viz. if $\frac{dy}{dx}=\frac{1}{0}$, and changes sign from + to −, we have the form marked α; and if the change of sign be from − to +, that marked β: but if $\frac{dy}{dx}=\frac{1}{0}$, and does not change sign, but is + throughout, then the form of the curve is that marked γ; and, if negative throughout, the form is that marked δ.

The case in which $\frac{dy}{dx}=\frac{0}{0}$, for any particular finite values of the ordinate and abscissa, and in which, therefore, $\left(\frac{dF}{dx}\right)=0$, $\left(\frac{dF}{dy}\right)=0$, and which indicates an indefiniteness as to the direction of the tangent, we shall discuss hereafter.

97.] Let Δs be the distance between the two points in the curve through which the secant of Art. 94. passes, that is, the length of the chord joining them; then

$$\Delta s^2 = \Delta x^2 + \Delta y^2.$$

Let the two points approach infinitely near to each other; in which case, according to the notation we have adopted, Δx, Δy, and Δs will become respectively dx, dy, and ds, and we shall have

$$ds^2 = dx^2 + dy^2; \qquad (6)$$

and ds becomes the distance between these two points, which are infinitely near to each other; that is, it becomes an element of the curve, an infinitesimal arc: it is, in fact, the small portion of the tangent line which is common to the tangent and the curve. Or under another mode of considering the curve, that is, of conceiving it to be generated by a point moving according to a given law, ds is the distance between two successive positions of the point; and if these two positions are taken so near to each other that only an infinitesimal instant of time has elapsed during the passage from one to the other, it is impossible to conceive but that the moving point has passed in a straight line from one to the other; the length of which straight line is ds.

If, then, we use the character τ, to symbolise the angle made by the tangent with the axis of x, we have, from Equation (2),

$$\tan \tau = \frac{dy}{dx};$$

and therefore, by means of (6), we have

$$\cos \tau = \pm \frac{dx}{ds} = \frac{\pm \left(\frac{dF}{dy}\right)}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}};$$

$$\sin \tau = \pm \frac{dy}{ds} = \frac{\mp \left(\frac{dF}{dx}\right)}{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2\right\}^{\frac{1}{2}}}.$$

98.] To find the Equation to the Normal to a Plane Curve.

DEF. The *normal* to a plane curve is the straight line perpendicular to the tangent, which passes through the point of contact.

Let ξ, η be the current co-ordinates to the normal, and x, y the co-ordinates to the point of contact; then, if the equation to the curve be explicit, viz. $y = f(x)$, the equation to the normal is

$$\eta - y = -\frac{dx}{dy}(\xi - x); \qquad (7)$$

and if the equation to the curve be in the implicit form, then

$$\frac{\eta - y}{\left(\frac{dF}{dy}\right)} = \frac{\xi - x}{\left(\frac{dF}{dx}\right)}. \qquad (8)$$

With respect to these equations, it is to be observed that there are critical states of them, as of the equations to the tangent, corresponding to singular values of $\frac{dy}{dx}$, and of $\left(\frac{dF}{dx}\right)$ and $\left(\frac{dF}{dy}\right)$.

As we have fully discussed these latter in Art. 96., and as the results will be similar in this case, it is unnecessary to repeat them.

From the equations to the tangent and the normal it appears that whenever x and y become affected with the symbols $\pm\sqrt{-}$, η or ξ will be also; and, therefore, whenever the curve is not in the plane of reference, the tangent and the normal are not.

From the above expressions it is plain that the equation to a straight line passing through the origin and perpendicular to the tangent may be put under either of the forms,

$$\left.\begin{aligned} \eta\, dy + \xi\, dx = 0, \\ \frac{\eta}{\left(\frac{d\text{F}}{dy}\right)} = \frac{\xi}{\left(\frac{d\text{F}}{dx}\right)}. \end{aligned}\right\} \qquad (9)$$

99.] To discuss the Equations to the Tangent and the Normal.

The several properties which the expressions marked (2) (4) (7) (8) involve might be deduced from the equations themselves; but as the following method addresses itself more directly to the senses and to geometrical construction, it is thought to be preferable.

Let AP be the curve which is represented in fig. 17., PT the tangent line, PG the normal line; MT is called the subtangent, MG the subnormal, PT the tangent, PG the normal.

Let $\left.\begin{aligned}\text{OM} = x \\ \text{MP} = y\end{aligned}\right\}$ and OY = perpendicular from origin on tangent $= p$.

Let $\text{OT} = \xi_0$ = intercept of axis of x by tangent.

$\text{OT}' = \eta_0 =$ . . . . y . . . .

Then, since $\tan \text{PTM} = \tan \text{GPM} = \frac{dy}{dx}$, and $\tan \text{TPM} = \tan \text{PGM} = \frac{dx}{dy}$, we have the following values from the geometry of the figure.

$$\left.\begin{aligned}
&\text{Subtangent} = \text{MT} = \text{MP} \tan \text{MPT} = y\frac{dx}{dy},\\
&\text{Subnormal} = \text{MG} = \text{MP} \tan \text{MPG} = y\frac{dy}{dx},\\
&\text{Tangent} = y \sec \tan^{-1}\frac{dx}{dy} = y\sqrt{1+\frac{dx^2}{dy^2}} = y\frac{ds}{dy},\\
&\text{Normal} = y \sec \tan^{-1}\frac{dy}{dx} = y\sqrt{1+\frac{dy^2}{dx^2}} = y\frac{ds}{dx},\\
&\xi_0 = \text{OT} = \text{MT} - \text{OM} = y\frac{dx}{dy} - x.
\end{aligned}\right\}$$

But since it is measured in a negative direction from the origin, its sign must be changed, and we have (10)

$$\xi_0 = x - y\frac{dx}{dy},$$

$$\eta_0 = \text{OT}' = \text{MP} - \text{PR} = y - x\frac{dy}{dx},$$

$$p = \text{OY} = \text{OT} \sin \text{PTO} = \pm\left(x - y\frac{dx}{dy}\right)\sin\tan^{-1}\frac{dy}{dx}$$

$$= \pm\frac{x\dfrac{dy}{dx} - y}{\sqrt{\left(1+\dfrac{dy^2}{dx^2}\right)}} = \pm\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right).$$

If the equation to the curve be an implicit function, substitutions must be made for these several values in terms of their equivalents. It is unnecessary to write them down, except the last, which assumes a very elegant form; for since

$$\frac{dy}{dx} = -\frac{\left(\dfrac{d\text{F}}{dx}\right)}{\left(\dfrac{d\text{F}}{dy}\right)},$$

$$p = \frac{x\left(\dfrac{d\text{F}}{dx}\right) + y\left(\dfrac{d\text{F}}{dy}\right)}{\left\{\left(\dfrac{d\text{F}}{dx}\right)^2 + \left(\dfrac{d\text{F}}{dy}\right)^2\right\}^{\frac{1}{2}}} = \frac{nu}{\left\{\left(\dfrac{d\text{F}}{dx}\right)^2 + \left(\dfrac{d\text{F}}{dy}\right)^2\right\}^{\frac{1}{2}}},$$

if $u = F(x, y)$ be a homogeneous function of n dimensions. Similarly might the values of other lines be found, were they required. For examples, see chap. ix. of Mr. Gregory's *Collection*, and chap. viii. of Mr. Hind's *Series*.

100.] On Asymptotes to Curves.

DEF. A line is said to be an *asymptote to a curve*, when the curve approaches continually nearer and nearer to it, but does not meet it within a finite distance. It is manifest from the definition, that there are two classes of asymptotes, rectilinear and curvilinear, which it will be convenient to discuss separately.

A. *Rectilinear Asymptotes.*

If the curve has asymptotes which are either the axes themselves or lines parallel to them at finite distances, they are easily determined. If $y = 0$ renders x infinite, the axis of x is asymptotic to the curve; similarly, if y is infinite when $x = 0$, the axis of y is an asymptote; such we have in the curve whose equation is $xy = k^2$. And, again, if $y = \frac{1}{0}$ when $x = a$ (a being some finite quantity), then the line parallel to the axis of y, at a distance a, is an asymptote. Similarly, if $x = \frac{1}{0}$ when $y = b$, the line parallel to the axis of x, at a distance b from it, is an asymptote. Such lines are drawn in fig. 18., the equation to which curve is

$$xy - ay - bx = 0,$$

which may be put under either of the forms,

$$y = \frac{bx}{x-a}, \qquad x = \frac{ay}{y-b};$$

whence, $y = \frac{1}{0}$ when $x = a = \text{OA}$; and $x = \frac{1}{0}$ when $y = b = \text{OB}$.

If, however, a curve has rectilinear asymptotes not parallel to the axes of co-ordinates, they are to be determined by one or the other of the following methods.

Since the difference between the ordinate to the asymptote and the ordinate to the curve diminishes without limit, as the abscissa is infinitely increased, if by any artifice, as the binomial theorem or Maclaurin's Series, we can expand the equation to the curve in a series of terms of descending powers of x, of the form

$$y = a_1 x + a_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \&c.; \qquad (11)$$

then all and every term after the first two diminish without limit, as the abscissa is infinitely increased, and the line whose equation is

$$y = a_1 x + a_0 \qquad (12)$$

is an asymptote to the curve.

And according as the first term after a_0, be it $\frac{b_1}{x}$ or $\frac{b_2}{x^2}$ &c., is positive or negative, so will the ordinate to the curve be greater or less than the ordinate to the asymptote, and the curve will be above or below the asymptote.

The Equation (12) is to be constructed in the ordinary way.

Ex.

$$y^3 = ax^2 - x^3$$

$$= -x^3\left(1 - \frac{a}{x}\right),$$

$$\therefore \; y = -x\left(1 - \frac{a}{x}\right)^{\frac{1}{3}}$$

$$= -x\left(1 - \frac{a}{3x} - \frac{a^2}{9x^2} - \ldots\right)$$

$$= -x + \frac{a}{3} + \frac{a^2}{9x} - \&c.$$

$\therefore$ the equation to the asymptote is

$$y = -x + \frac{a}{3},$$

which represents a line inclined at 135° to the axis of x, and cutting the axis of y at a distance $= \frac{a}{3}$ from the origin. The next term in the series, being affected with a positive sign, shows that the ordinate to the curve is greater than the ordinate to the asymptote, and therefore that the curve lies above the asymptote.

If, however, the equation to the curve is such as not without difficulty to admit of expansion in the form (11), then, considering the asymptote to be the tangent line to the curve, at an infinite distance from the origin, the problem resolves itself into the construction of this particular tangent; and since the tangent makes with the axis of x an angle $= \tan^{-1}\frac{dy}{dx}$, and its intercepts on the axes are η_0 and ξ_0, of which the values are given in Equations (10), the determination of any two of these three quantities corresponding to $x = \frac{1}{0}$ and $y = \frac{1}{0}$ will enable us to construct the line. For the infinite values of the co-ordinates these quantities will assume indeterminate forms, which must be evaluated according to the methods explained in Chap. V., and especially Art. 62. Two examples are subjoined; for others, see the *Collections* of Mr. Gregory and of Mr. Hind.

Ex. 1. To find the Asymptote to the Curve

$$y^3 + x^3 - 3axy = 0.$$

$$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax} = \frac{\frac{1}{0}}{\frac{1}{0}}, \text{ when } x = \frac{1}{0}, y = \frac{1}{0}.$$

Therefore, differentiating numerator and denominator successively, as in Art. 62., and bearing in mind that $\frac{dy}{dx}$ is a definite quantity not admitting of differentiation, and therefore to be considered constant,

$$\frac{dy}{dx} = -\frac{2x - a\dfrac{dy}{dx}}{2y\dfrac{dy}{dx} - a} = \frac{\frac{1}{0}}{\frac{1}{0}}$$

$$= -\frac{2}{2\dfrac{dy^2}{dx^2}};$$

$$\therefore \frac{dy^3}{dx^3} = -1, \text{ and } \frac{dy}{dx} = -1,$$

omitting the other two values of $\frac{dy}{dx}$, because they refer to branches of the curve out of the plane of reference; therefore the asymptote makes an angle of 135° with the axis of x.

To determine η_0

$$\eta_0 = y - x\frac{dy}{dx} = \frac{axy}{y^2 - ax} = \frac{\frac{1}{0}}{\frac{1}{0}}, \text{ when } x = \frac{1}{0}, \text{ and } y = \frac{1}{0}.$$

$\therefore$ evaluating as before, $\eta_0 = -a$,

and the equation to the asymptote is

$$y = -x - a.$$

Ex. 2.

$$ay^3 - bx^3 + c^2xy = 0.$$

$$\frac{dy}{dx} = -\frac{c^2y - 3bx^2}{c^2x + 3ay^2} = \frac{\frac{1}{0}}{\frac{1}{0}}, \text{ when } x = \frac{1}{0}, \text{ and } y = \frac{1}{0},$$

$$= -\frac{c^2\dfrac{dy}{dx} - 6bx}{c^2 + 6ay\dfrac{dy}{dx}} = \frac{\frac{1}{0}}{\frac{1}{0}}$$

$$= -\frac{-6b}{6a\dfrac{dy^2}{dx^2}}$$

$$\therefore \quad \frac{dy^3}{dx^3} = \frac{b}{a}, \text{ and } \frac{dy}{dx} = \left(\frac{b}{a}\right)^{\frac{1}{3}},$$

$$\eta_0 = y - x\frac{dy}{dx}$$

$$= -\frac{c^2xy}{c^2x+3ay^2} = \frac{\frac{1}{0}}{\frac{1}{0}};$$

and evaluating as before,

$$\eta_0 = -\frac{c^2}{3a^{\frac{2}{3}}b^{\frac{1}{3}}};$$

and therefore the equation to the asymptote is

$$y = \left(\frac{b}{a}\right)^{\frac{1}{3}}\left\{x - \frac{c^2}{3a^{\frac{1}{3}}b^{\frac{2}{3}}}\right\}.$$

101.] If, having applied one or the other of these methods to the determination of asymptotes, we arrive at results affected with $\pm \sqrt{-}$, the lines must be drawn in their own planes, as was indicated at the close of Art. 93.; sometimes, however, curves have branches out of the plane of reference, which are asymptotic to straight lines in the plane, as in the following example. These, however, must be determined in one or other of the methods which have been just explained.

Ex. $$x^4(y-x)^2 = a^2-x^2,$$

$$\therefore \quad y - x = \pm\frac{\sqrt{(a^2-x^2)}}{x^2}.$$

When x is infinitely great, the right-hand side of the equation vanishes, and we have $y = x$, which is the equation to a line passing through the origin at 45° to the axis of x, and which is asymptotic to two branches of the curve; but, for all values of x not within the limits $\pm a$, the curve lies out of the plane of reference, and, therefore, the asymptote is that to which these branches are continually approaching. The form of the curve is given in fig. 19., the dotted lines representing the branches in a plane perpendicular to the plane of the paper.

B. *Of Curvilinear Asymptotes.*

102.] Two curves may also be asymptotic to each other; as e. g. suppose that the form of the equation to the given curve, when expanded in descending powers of x, is

$$y = a_2 x^2 + a_1 x + a_0 + \frac{b_1}{x} + \&c.$$

Then if we neglect on the right-hand side of the equation all terms after the first three, which is equivalent to making $x = \frac{1}{0}$, the curve whose equation is

$$y = a_2 x^2 + a_1 x + a_0,$$

which is a parabola, is asymptotic to the given curve.

Also, if in Equation (11) we take account of the first three terms, and multiply through by x, and neglect all the subsequent terms, then we have the equation to a hyperbola, viz.

$$xy = a_1 x^2 + a_0 x + b_1 ;$$

and this curve is asymptotic to the given curve, because the difference between the lengths of their ordinates is a quantity which diminishes without limit as x increases without limit. And so again, if we take account of the first four terms of (11), and neglect all subsequent ones, we shall obtain the equation to a curve which is nearer to the given curve than either the rectilinear asymptote or the hyperbola; and thus, by a similar process, we obtain a series of curves more and more asymptotic to the given curve. Thus, also, we often find curves which have cubical and semicubical parabolas asymptotic to them: we subjoin an example.

$$y^2 = x^2 \sqrt{(x^2 - a^2)},$$

$$\therefore \quad y = \pm x^{\frac{3}{2}} \left\{1 - \frac{a^2}{x^2}\right\}^{\frac{1}{4}}$$

$$= \pm x^{\frac{3}{2}} \left\{1 - \frac{a^2}{4x^2} - \dots\right\}$$

$$= \pm \left\{x^{\frac{3}{2}} - \frac{a^2}{4\sqrt{x}} - \dots\right\};$$

$\therefore$ the equation to the asymptotic curve is

$$y = \pm x^{\frac{3}{2}}.$$

103.] On Direction of Curvature and Points of Inflexion.

Having in the preceding articles determined what is the absolute inclination to the co-ordinate axis of any element of a plane curve, we proceed in the present to discuss criteria for the convexity or concavity of a curve towards fixed axes or in fixed directions; and to facilitate our calculations we shall consider x to increase by constant increments, i. e. dx to be constant or x to be the *independent* variable.

Since then, under this supposition, $\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} =$ the ratio of the increment of $\tan\tau$ to the increment of x, and since τ and $\tan\tau$ always increase and decrease simultaneously (see Art. 89.), it follows that, if $\frac{d^2y}{dx^2}$ is positive, τ and x are increasing or decreasing simultaneously, and if $\frac{d^2y}{dx^2}$ is negative, as x increases τ decreases, and *vice versâ:* but from the geometry it is plain that, if x and τ are increasing together, the form of the curve must be such as that in fig. 20., which is convex downwards; and if as x increases τ decreases, the form is that in fig. 21., which is concave downwards; and therefore, if $\frac{d^2y}{dx^2}$ is positive, the curve is convex downwards, and if $\frac{d^2y}{dx^2}$ is negative, it is concave downwards. If, therefore, at any point, say $x = a, y = b$, $\frac{d^2y}{dx^2}$ changes its sign, which it can do only by passing through 0 or $\frac{1}{0}$, the direction of the curvature changes; and there is what is called *a point of inflexion,* that is, the curve having been convex downwards becomes concave downwards, or *vice versâ.* Such points are delineated in fig. 22. Hence we have the following rules to determine a curve's direction.

If $\frac{d^2y}{dx^2}$ be positive, the curve is convex downwards.

If $\frac{d^2y}{dx^2}$ be negative, . . . concave . . . .

And, to determine points of inflexion, equate $\frac{d^2y}{dx^2}$ to 0 and to $\frac{1}{0}$: if the values of x, which satisfy either of these conditions, also make $\frac{d^2y}{dx^2}$ to change its sign, then there is a point of inflexion.

This is also evident from the following considerations.

104.] Let $y = f(x)$ be the equation to a curve, and suppose that we not only consider the curve at the point (x, y), but also at another point $(x+h, y+k)$; then, by Taylor's Series, writing $\frac{dy}{dx}$ for $f'(x)$, $\frac{d^2y}{dx^2}$ for $f''(x)$, &c., all these differential coefficients being finite, and since $y + k = f(x + h)$, we have

$$y + k = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c. \quad (13)$$

And if a tangent be drawn to the curve at the point x, y, its equation is

$$\eta = y + \frac{dy}{dx}(\xi - x);$$

and, therefore, its ordinate, when ξ becomes $x+h$, is given by the equation

$$\eta = y + \frac{dy}{dx}h. \quad (14)$$

Subtracting, then, (14) from (13), by which means we shall get the difference between the ordinates to the curve and to the tangent corresponding to the abscissa $x+h$, we have

$$y + k - \eta = \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.; \quad (15)$$

and taking h very small, which is equivalent to considering only the point in the curve which is next to $(x+dx, y+dy)$, we have, neglecting all terms after the first,

$$y + k - \eta = \frac{d^2y}{dx^2}\frac{h^2}{1.2}.$$

And, therefore, if $\frac{d^2y}{dx^2}$ be positive, the ordinate to the curve is greater than the ordinate to the tangent, and, therefore, the curve lies above the tangent, whatever be the sign of h, that is, the curve is convex downwards, as in fig. 20.; but if $\frac{d^2y}{dx^2}$ be negative, contrary results follow, and the curve is concave downwards, as in fig. 21.

If, therefore, at any point the curve passes through the tangent so as to be above it on one side of the point of contact and section, but below it on the other, then $y + k - \eta$ must change sign as h changes sign; which can only be the case when $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3}$ is a finite quantity, or, in general, when the term in the expansion (15), which gives sign to the whole, is one involving an odd power of h, in which case the curve will pass from below the tangent to above it, or *vice versâ*, in one or other of the manners indicated in fig. 22. It is plain that, at a point of inflexion, $\frac{dy}{dx}$, and therefore the angle τ, attains to a maximum or minimum value.

In the diagrams we have drawn to illustrate the general theory explained above, we have considered $\tan \tau$ to be some finite quantity, but if τ be 0 or 90°, i.e. if $\frac{dy}{dx} = 0$ or $\frac{1}{0}$, and $\frac{d^2y}{dx^2} = 0$ or $\frac{1}{0}$ and change its sign, then the curves are such as those in figs. 8. and 9., and in γ and δ of fig. 16.

The curvature of a curve towards the right or the left may be determined in a similar manner by discussing the values and signs of $\frac{d^2x}{dy^2}$.

Ex. To determine the Direction of Curvature, and the Points of Inflexion of the Curve

$$y = x + \cos x.$$

$$\frac{dy}{dx} = 1 - \sin x,$$

$$\frac{d^2y}{dx^2} = -\cos x.$$

$\therefore$ $\frac{d^2y}{dx^2}$ is negative, and the curve is concave downwards for all values of x between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$, $\frac{7\pi}{2}$ and $\frac{9\pi}{2}$; that is, in general, for all values of x between $\frac{(4n-1)\pi}{2}$ and $\frac{(4n+1)\pi}{2}$: and $\frac{d^2y}{dx^2}$ is positive, and the curve is convex downwards, for all values of x between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, $\frac{5\pi}{2}$ and $\frac{7\pi}{2}$, in general, between $\frac{(4n+1)\pi}{2}$ and $\frac{(4n+3)\pi}{2}$; and whenever $x = \frac{\pi}{2}$, or $\frac{3\pi}{2}$, or $\frac{5\pi}{2}$, or, in general, $\frac{(2n+1)\pi}{2}$, $\frac{d^2y}{dx^2} = 0$, and changes sign, and there are points of inflexion.

If the equation to the curve be given in the implicit form, $u = \mathrm{F}(x,y) = c$, considering, as we have in this article, x to be a variable of equal augments dx, so that $d^2x = 0$, the second differential of $\mathrm{F}(x,y) = c$ is, as was shown in Art. 39.

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right) dx^2 + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) dx\,dy + \left(\frac{d^2\mathrm{F}}{dy^2}\right) dy^2 + \left(\frac{d\mathrm{F}}{dy}\right) d^2y = 0.$$

Whence, dividing by $(dx)^2$, we have the following expression to determine $\frac{d^2y}{dx^2}$,

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right) + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)\frac{dy}{dx} + \left(\frac{d^2\mathrm{F}}{dy^2}\right)\frac{dy^2}{dx^2} + \left(\frac{d\mathrm{F}}{dy}\right)\frac{d^2y}{dx^2} = 0;$$

and substituting for $\frac{dy}{dx}$ its equivalent, viz.

$$-\frac{\left(\frac{dF}{dx}\right)}{\left(\frac{dF}{dy}\right)},$$

we have

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{d^2F}{dx^2}\right)\left(\frac{dF}{dy}\right)^2 - 2\left(\frac{d^2F}{dx\,dy}\right)\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right) + \left(\frac{d^2F}{dy^2}\right)\left(\frac{dF}{dx}\right)^2}{\left(\frac{dF}{dy}\right)^3}.$$

As long, then, as this quantity is positive, the curve is convex downwards; and, if it be negative, the curve is concave downwards; and if, therefore, at any point on the curve,

$$\left(\frac{d^2F}{dx^2}\right)\left(\frac{dF}{dy}\right)^2 - 2\left(\frac{d^2F}{dx\,dy}\right)\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right) + \left(\frac{d^2F}{dy^2}\right)\left(\frac{dF}{dx}\right)^2 = 0,$$

and changes sign, and at the same time $\left(\frac{dF}{dy}\right)$ does not $= 0$, then there is a point of inflexion.

Similarly might we find $\frac{d^2x}{dy^2}$, and determine whether a curve is convex or concave towards the right or the left.

105.] Before we proceed to the discussion of points through which two or more branches of a curve pass, it will be worth while to examine the diagram marked 23., which is intended to be an infinitely magnified drawing of a curve of the kind we have been considering as being generated by a point moving subject to some continuous law; we have placed the curve as in the figure, in order that $\frac{d^2y}{dx^2}$ may be affected with a positive sign.

Let $y = f(x)$ be the equation to the curve, $\left.\begin{matrix}\text{OK} = x \\ \text{KP} = y\end{matrix}\right\}$, and, for simplicity's sake, let us consider the successive augments of x to be constant, viz. KL = LM = MN = ... = dx, so that, in accordance with what has been said in Art. 34., $d^2x = d^3x = \ldots \ldots = 0$; and let P, Q, R, S be points on the curve corresponding to the successively increased values of x: and, conceiving the successive elements to be infinitely magnified, PQ, QR, RS are such quantities as we have in Art. 97. symbolised by ds. In the same manner, then, as in Art. 34.

$$f(x) = y = \text{KP},$$
$$f(x+dx) = y+dy = \text{LQ},$$
$$f(x+2dx) = y+2dy+d^2y = \text{MR},$$
$$f(x+3dx) = y+3dy+3d^2y+d^3y = \text{NS},$$

and so on; whence by subtraction we have

$$dy = \text{LQ} - \text{KP} = \text{QU} = \text{XV} = \text{TW},$$

and $\text{MR} = \text{MZ} + \text{RZ}$: but $\text{MZ} = \text{MV} + 2\,\text{VX} = y + 2dy$;

$\therefore$ substituting for MR from above, we have

$$y + 2dy + d^2y = y + 2dy + \text{RZ},$$
$$\therefore \quad d^2y = \text{RZ}.$$

As we have not deduced any geometrical properties from d^3y, it is unnecessary to do more than to show what line is represented by it;

From above we have $\text{NS} = y + 3dy + 3d^2y + d^3y$:

but $\quad \text{NW} = y, \quad \text{WG} = 3dy, \quad \text{GE} = 2d^2y,$

$$\therefore \quad \text{SE} = d^2y + d^3y$$
$$= \text{RZ} + d^3y;$$
$$\therefore \quad d^3y = \text{SE} - \text{RZ}.$$

Hence it is manifest that $\frac{dy}{dx} = \frac{\text{QU}}{\text{PU}} = \tan \text{QPU} = \tan \text{PJO}$ = trigonometrical tangent of the angle made with axis of x by tangent to curve; and, therefore, if at any point on a curve $\frac{dy}{dx} = 0$, as we pass from one point to the next consecutive point, y does not increase; and, therefore, the element of the curve

PQ is along the line PU, and is parallel to the axis of x. Similarly, whenever $\frac{dy}{dx} = \frac{1}{0}$, the element PQ is perpendicular to PU.

And since $d^2y = \text{RZ}$, it is plain that d^2y represents the deflexion of the curve from the tangent line: and, therefore, if $d^2y = 0$, three consecutive points are in the same straight line, and the curve has for those three points become straight; and if d^2y be positive, the line RZ is to be measured up from the tangent, and the curve lies above the tangent; but, if d^2y be negative, it must be measured downwards, and the curve lies below the tangent; if, therefore, d^2y is positive on both sides of the point P, the curve is convex downwards, and is such as is drawn in fig. 20., and if d^2y is negative on both sides of the point (x, y), then the curve is concave downwards as in fig. 21.; and if d^2y changes its sign at the point, by passing through 0 or $\frac{1}{0}$, the curve is above the tangent on one side of the point, and below it on the other, and, therefore, there is a point of inflexion.

Hence, then, we learn the geometrical meaning of the process of differentiation; it implies a passage from one point of a curve to the next consecutive point, and, as often as we differentiate, we pass to successive points, and we obtain expressions which represent deflexions from straight lines, and so on.

Thus, by means of one differentiation, we consider the curve with respect to two points on it, by two differentiations we consider the curve at three points, and so on. More will be said hereafter on the properties of curves under this mode of considering them. Let the reader who is acquainted with the analytical expressions of velocity and accelerating force interpret them by means of the diagram 23.

106.] Thus far, then, we have considered what geometrical properties belong to the first and second differential coefficients, when they have determinate values; but suppose at any point in a curve, say $(x = a, y = b)$, which we shall call a *critical* point, $\frac{dy}{dx}$ assumes the indeterminate form $\frac{0}{0}$, a question arises, what geometrical meaning belongs to it. This we proceed to explain; and we shall find it more convenient to use the implicit form of the equation to the curve, viz.

$$u = \text{F}(x, y) = c, \tag{16}$$

whence, by differentiation,

$$\text{D}u = 0 = \left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy;$$

$$\therefore \frac{dy}{dx} = -\frac{\left(\frac{d\text{F}}{dx}\right)}{\left(\frac{d\text{F}}{dy}\right)}.$$

If, then, at the point under consideration, $\frac{dy}{dx} = \frac{0}{0}$, it is necessary that $\left(\frac{d\text{F}}{dx}\right) = 0$ and $\left(\frac{d\text{F}}{dy}\right) = 0$; then, evaluating the expression above in accordance with the principles laid down in Art. 62., by differentiating the numerator and denominator, we have

$$\frac{dy}{dx} = -\frac{\left(\frac{d^2\text{F}}{dx^2}\right) dx + \left(\frac{d^2\text{F}}{dx\,dy}\right) dy}{\left(\frac{d^2\text{F}}{dy^2}\right) dy + \left(\frac{d^2\text{F}}{dx\,dy}\right) dx}$$

$$= -\frac{\left(\frac{d^2\text{F}}{dx^2}\right) + \left(\frac{d^2\text{F}}{dx\,dy}\right)\frac{dy}{dx}}{\left(\frac{d^2\text{F}}{dy^2}\right)\frac{dy}{dx} + \left(\frac{d^2\text{F}}{dx\,dy}\right)}. \tag{17}$$

And suppose, first, that this quantity does not become $\frac{0}{0}$ at the critical point in question, that is, that all the second partial differential coefficients do not vanish, then, multiplying and reducing, we have

$$\left(\frac{d^2\text{F}}{dy^2}\right)\frac{dy^2}{dx^2} + 2\left(\frac{d^2\text{F}}{dx\,dy}\right)\frac{dy}{dx} + \left(\frac{d^2\text{F}}{dx^2}\right) = 0, \tag{18}$$

a quadratic in $\frac{dy}{dx}$, giving, therefore, two values for $\frac{dy}{dx}$; thereby showing that two branches of the curve pass through the point, which is called a *double point*, admitting of several varieties, according as the roots of (18) are real and unequal, real and equal, or impossible, and according as the curve extends in the plane of reference or not on both sides of the point in question. Now the roots of (18) are

$$\left.\begin{array}{l}\text{real and unequal}\\ \text{real and equal}\\ \text{impossible}\end{array}\right\}\text{according as }\left(\frac{d^2\mathrm{F}}{dy^2}\right)\left(\frac{d^2\mathrm{F}}{dx^2}\right)\text{ is }\left\{\begin{array}{c}<\\=\\>\end{array}\right\}\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)^2.$$

Let us first consider the case of the two roots being real and unequal, and we have

$$\frac{dy}{dx}=\frac{-\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)\pm\sqrt{\left\{\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)^2-\left(\frac{d^2\mathrm{F}}{dx^2}\right)\left(\frac{d^2\mathrm{F}}{dy^2}\right)\right\}}}{\frac{d^2\mathrm{F}}{dy^2}}$$

If the curve extends in the plane of reference on both sides of the point in question, as P_0 in fig. 24., then the point is called a *real double point;* but if the curve is in the plane of reference on one side of the point, but is in another plane on the other side, as is indicated in fig. 25., where the dotted lines show the course of the curve out of the plane of reference, then such a point is called a *point saillant;* and if the curve is out of the plane of reference on both sides of the point in question, but pierces the plane at the point, then we have what is called a *conjugate point,* which, however, corresponds to the case of two roots of Equation (18), being impossible.

Secondly, if two roots be real and equal, we have

$$\frac{dy}{dx}=-\sqrt{\frac{\left(\frac{d^2\mathrm{F}}{dx^2}\right)}{\left(\frac{d^2\mathrm{F}}{dy^2}\right)}},$$

and there are two branches passing through the point and having the same tangent.

If these branches are in the plane of reference on both sides of the point, the curve is such as one or the other of those delineated in fig. 26., and such are called points of *osculation* and of *embrassement;* and if they are in the plane of reference on one side of the point, and on the other side pass out of it, then the curve at the point is such as one or other of those drawn in fig. 27., where the dotted lines indicate the course of the curve out of the plane of the paper, and the points are called *cusps*, or *points de rebroussement*, fig. α of the first species, fig. β of the second species. Varieties of cusps are delineated in fig. 28.

Thirdly, if two roots of (17) are impossible, both branches of the curve are out of the plane of reference on both sides of the point in question, but pierce it at the point; then there is a conjugate point, of which there are varieties according as both branches have the same or different tangent lines.

These several subordinate varieties of double points must be distinguished by examining the form and nature of the equation to the curve, and of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, when $x = a \pm h$, $y = b \pm k$, h and k being taken very small: as e.g. in fig. 27., if when $x = a + h$, $\frac{d^2y}{dx^2}$ is positive for one and negative for the other branch of the curve, and when $x = a - h$, $\frac{dy}{dx}$ is affected with $\sqrt{-}$, we have fig. α; but fig. β, if $\frac{d^2y}{dx^2}$ is positive for both branches of the curve, and the curve is out of the plane of the paper when x is less than a.

Subjoined are a few examples illustrative of the processes of which we have here given a sketch, and in which $\frac{dy}{dx}$ is evaluated according to the method in Art. 62.

Ex. 1. To determine the Nature of the Point at the Origin of the Curve whose Equation is $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

$$\therefore \quad \frac{dy}{dx} = \frac{a^2x - 2x(x^2+y^2)}{a^2y + 2y(x^2+y^2)} = \frac{0}{0}, \text{ when } x = 0 \text{ and } y = 0.$$

$$= \frac{a^2 - \&c.}{a^2\frac{dy}{dx} + \&c.} = \frac{1}{\frac{dy}{dx}}, \text{ when } x = 0 \text{ and } y = 0.$$

$$\therefore \quad \frac{dy^2}{dx^2} = 1 \text{ and } \frac{dy}{dx} = \pm 1.$$

$\therefore$ two branches of the curve pass through the origin, cutting the axis of x at angles respectively of 45° and 135° (see fig. 29.); and there is a real *double* point.

Ex. 2. Discuss the Nature of the Point, $y = b$, $x = a$, of the Curve whose Equation is $(y-b)^2 = (x-a)^3$.

$$\therefore \quad \frac{dy}{dx} = \frac{3(x-a)^2}{2(y-b)} = \frac{0}{0} \text{ at the critical point,}$$

$$= \frac{6(x-a)}{2\frac{dy}{dx}} = 0 \text{ at the critical point.}$$

$$\therefore \quad \frac{dy^2}{dx^2} = 3(x-a), \text{ and } \frac{dy}{dx} = \pm\sqrt{(x-a)},$$

which is affected with $\sqrt{-}$ when x is $< a$, but $+$ or $-$ for all values of $x > a$, and $= 0$ when $x = a$; $\therefore$ we have a cusp, the tangent to the branches of which is parallel to the axis of x.

Also $\frac{d^2y}{dx^2} = \pm \frac{3}{4(x-a)^{\frac{1}{2}}}$, which is therefore positive for one branch and negative for the other; $\therefore$ the cusp is of the first species, such as is delineated in fig. 30.

Ex. 3. $(y - x^2)^2 = \pm (x-1)^{\frac{5}{2}}$, required to determine the Nature of the Point, $x = 1$, $y = 1$.

$$\frac{dy}{dx} = \frac{\pm 5(x-1)^{\frac{3}{2}} + 8x(y-x^2)}{4(y-x^2)} = \frac{0}{0} \text{ at the critical point,}$$

$$= \frac{\pm\frac{15}{2}(x-1)^{\frac{1}{2}} + 8(y-x^2) + 8x\left(\frac{dy}{dx} - 2x\right)}{4\left(\frac{dy}{dx} - 2x\right)}.$$

$\therefore$ at the critical point $\frac{dy}{dx} = 2$; and when x is less than 1 $\frac{dy}{dx}$ is affected with $\sqrt{-}$, but is *real* when x is > 1; and the form of the last equation shows that the critical point is a double point, which is $\therefore$ a cusp.

Also, since $\frac{d^2y}{dx^2} = 2 \pm \frac{15}{16}(x-1)^{-\frac{1}{4}}$, at the critical point $\frac{d^2y}{dx^2} = 2$; and if x be a little $>$ than 1, $\frac{d^2y}{dx^2}$ is still positive; $\therefore$ the cusp is of the second species, with both branches convex downwards, and touches a line which is inclined to the axis of x at $\tan^{-1} 2$. (See fig. 31.)

For other examples on this and the following articles, see chap. x. of Mr. Gregory's *Collection*, and chapters x. and xi. of Mr. Hind's *Series*.

107.] Suppose, however, that at the critical point under discussion

$$\left(\frac{d^2\text{F}}{dx^2}\right) = 0, \quad \left(\frac{d^2\text{F}}{dx\,dy}\right) = 0, \quad \left(\frac{d^2\text{F}}{dy^2}\right) = 0,$$

then the value of $\frac{dy}{dx}$ given by the expression (17) again assumes the form $\frac{0}{0}$, and the numerator and denominator of it must be differentiated again; in which operation, however, it is to be borne in mind that $\frac{dy}{dx}$ does not vary with x and y near to the critical point, and is therefore to be considered constant; the true meaning and effect of their successive differentiations being as follows. Several branches of the curve have certain consecutive points in common, and certain elements in common; whilst, therefore, we are considering the curve, as to its duration at one or more of these common points, it is indeterminate for which branch of the curve we are considering it; and therefore we must pass on from these common points to those contiguous ones which are on different branches of the curve, and then the tangent lines drawn through these become separate for each branch, and the direction of each thereby becomes determined. Let the reader try to draw for himself an infinitely magnified diagram of such points and curves in the same manner as we have drawn fig. 23.

Differentiating, therefore, the numerator and denominator of the right-hand member of Equation (17), and dividing through by dx,

$$\frac{dy}{dx} = -\frac{\left(\frac{d^3 F}{dx^3}\right) + 2\left(\frac{d^3 F}{dx^2\, dy}\right)\frac{dy}{dx} + \left(\frac{d^3 F}{dx\, dy^2}\right)\frac{dy^2}{dx^2}}{\left(\frac{d^3 F}{dy^3}\right)\frac{dy^2}{dx^2} + 2\left(\frac{d^3 F}{dx\, dy^2}\right)\frac{dy}{dx} + \left(\frac{d^3 F}{dx^2\, dy}\right)}; \qquad (19)$$

whence, multiplying and reducing,

$$\left(\frac{d^3 F}{dy^3}\right)\frac{dy^3}{dx^3} + 3\left(\frac{d^3 F}{dx\, dy^2}\right)\frac{dy^2}{dx^2} + 3\left(\frac{d^3 F}{dx^2 dy}\right)\frac{dy}{dx} + \left(\frac{d^3 F}{dx^3}\right) = 0, \qquad (20)$$

a cubic in $\frac{dy}{dx}$, and therefore with three roots, showing that three branches of the curve pass through the critical point, which is called a *triple* point; the three branches being all in the plane of reference, or one in and two out of the plane, according as the roots of (20) are all *real*, or one *real* and two impossible.

As the criteria of this division lead to a long and complicated expression, it is needless to investigate it here; and, moreover, as the determination of the several values of $\frac{dy}{dx}$ corresponding to the several branches of the curve is not difficult, we shall only add an example.

Similarly, again, if the several third partial differential coefficients vanish at the point under discussion, we must differentiate again the numerator and denominator of the right-hand member of (19); by which means we shall obtain a biquadratic in $\frac{dy}{dx}$, indicating that four branches of the curve pass through the point, which is therefore called a *quadruple* point.

Ex. 1. To determine the Nature of the Point at the Origin of the Curve whose Equation is

$$x^4 - ayx^2 + by^3 = 0.$$

$$\frac{dy}{dx} = \frac{4x^3 - 2axy}{ax^2 - 3by^2} = \frac{0}{0} \text{ at origin,}$$

$$= \frac{12x^2 - 2ax\dfrac{dy}{dx} - 2ay}{2ax - 6by\dfrac{dy}{dx}} = \frac{0}{0}$$

$$= \frac{24x - 4a\dfrac{dy}{dx}}{2a - 6b\dfrac{dy^2}{dx^2}} = \frac{-4a\dfrac{dy}{dx}}{2a - 6b\dfrac{dy^2}{dx^2}}, \text{ when } x = y = 0;$$

$$\therefore \quad 6a\frac{dy}{dx} - 6b\frac{dy^3}{dx^3} = 0:$$

whence $\quad \dfrac{dy}{dx} = 0$, and $\dfrac{dy}{dx} = \pm\sqrt{\left(\dfrac{a}{b}\right)}$.

The curve is such as we have drawn in fig. 32.

108.] Such, then, is the general theory of multiple points; of which the analytical notes are, the vanishing of successive partial differential coefficients of the implicit equation to the curve. That such must vanish, owing to the circumstances of several branches passing through the same point, may thus be shown *a priori*.

If a curve be such that, when $x = a \pm h$, y has many values, or, to borrow language from the theory of equations, the equation formed in powers of y has several unequal roots, but when $x = a$ several of these values of y become equal, say $y = b$, then in this case as many roots which before were unequal must become equal, as there are branches passing through the point; and thus there will be several equal factors multiplied together, which will produce a factor of the form $(y-b)^n$. By a similar train of reasoning we may prove that at such a point several factors which at other points are unequal become equal, and we shall have a factor of the form $(x-a)^m$, m and n being some numerical quantities at least greater than 1; and, since the differentiation diminishes the exponent of such a quantity only by unity, it is plain that $\left(\dfrac{dF}{dx}\right)$

will, at the point in question, have a factor of the form $(x-a)^{m-1}$, and therefore will $=0$. Similarly $\left(\frac{d\text{F}}{dy}\right)$ will have a factor of the form $(y-b)^{n-1}$, and will $=0$ also: and according to the numerical magnitudes of m and n will be the number of branches passing through the point, and the number of successive partial differential coefficients which $=0$, for the values $x=a$, $y=b$.

109.] We proceed now to analyse the equation to a curve, and to trace the figure which the analytical expression represents, as far as the results of common algebraical geometry, and what has been said in the preceding articles, enable us to do. *All* curves we cannot trace; the problem is as general as the resolution of equations of all degrees; and, therefore, what will be said is to be considered only an explanation of the slight means we possess of discussing some few simple curves.

1) If the equation admits of being simplified by a change of origin, or by a transformation from rectangular to polar coordinates, let this be effected before we begin to analyse it; as e. g. the equation $x^2-2ax+y^2+2bx=0$ admits of being discussed with greater facility when for x we write $x+a$, and for y, $y-b$; and the result is $x^2+y^2=a^2+b^2$. So, again, as to the Spiral of Archimedes, the curve is more easily traced when the equation is under the form of $r=a\theta$ than when it is in the form $\sqrt{(x^2+y^2)}=a\tan^{-1}\frac{y}{x}$.

So, again, if the equation to the curve be of the form $y=f(x)\pm\phi(x)$, in which case the curve $y=f(x)$ is diametral to the curve to be traced, the most convenient method is to trace separately $y=f(x)$ and $y=\phi(x)$, and then to add and subtract the ordinate of the latter curve from that of the former.

2) Let the equation to the curve, if possible, be put under the explicit form $y=f(x)$; determine all the points where the curve meets the co-ordinate axis, by finding the several values of x which render $y=0$, and the several values of y which render $x=0$; and by examining the change or continuation of sign determine whether the curve passes from above to below,

or *vice versâ*, the axis of x, or from the right to the left, or *vice versâ*, of the axis of y, or whether it touches the axes; and, if it cuts the axis, determine by the value of $\frac{dy}{dx}$ at the point at what angle it cuts. Again, if for all values of x from $-\frac{1}{0}$ to $+\frac{1}{0}$, y is unaffected with $\pm\sqrt{-}$, the curve extends infinitely in both directions in the plane of the paper; but if $x=a$, $y=b$ be a point such that for values of $x=a+h$, y is affected with $\pm\sqrt{-}$, but for $x=a-h$, y is not so affected, then at that point the curve leaves the plane of the paper, and the point is called a *point d'arrêt*, or *point de rupture*, such as is at the origin of the curve whose equation is $y=x^2 \log_e x$, which is drawn in fig. 33., and such as in the line which is represented by the two equations

$$\left.\begin{aligned} y &= 0, \\ x &= a\cos\theta, \end{aligned}\right\}$$

when x is not within the limits $+a$ and $-a$. Such points, however, seem to arise from our deficient knowledge of the properties of logarithms of negative quantities. And if two branches pass into another plane, then the point at which they do so is a cusp, or *point saillant*, such as are figured in diagrams 25, 27, 28. The distinctive characters of these points, of course, depend on the value or values of $\frac{dy}{dx}$ at the points. And lastly, if the equation to the curve is satisfied by $x=a, y=b$, but when x is increased or decreased by a quantity however small y is affected with $\sqrt{-}$, then at such a point the curve which lies in some other plane pierces the plane of reference; such a point is called a conjugate or isolated point; and, of course, one or two or more branches of a curve may pass through such a point: as, for instance, if the equation to a line be

$$y - b = (-)^{\frac{1}{2}} (x - a),$$

the equation is satisfied by $x=a$, $y=b$, which indicates a point in the plane of reference; but every other point of the line is in the plane passing through the line BD (see fig. 34., OA $=a$, AB$=b$), and perpendicular to the plane of the paper.

When two branches of the curve simultaneously pierce the plane of the paper, the two roots of (18) Art. 106. are impossible, as is the case in Example 1., which is traced below in Art. 110. And a curve may have an infinite number of such conjugate points, the curve continually passing from above to below, and *vice versâ*, through the plane of the paper, such as in the subjoined example:

$$y = ax^2 \pm \sqrt{(bx)} \sin x.$$

The curve is traced in fig. 35., the dotted line indicating the branches in a plane perpendicular to the plane of the paper. $y=ax^2$, which represents the diametral curve, is a parabola, drawn as in the fig.; and the ordinate to the curve is periodically reduced to its ordinate when $x = 0$, or $= \pi$, or $= 2\pi$, ... or $=$ any multiple of π; but when x is negative, the part of the ordinate to be added or subtracted to the ordinate of the parabola is affected with $(-)^{\frac{1}{2}}$, except at the points where $x =$ some multiple of π, at which the branches of the curve pierce the plane of reference: and thus it continues *ad infinitum*, the curve itself being continuous, but there being a series of discontinuous points, if we consider only those points which the plane of the paper contains.

3) On the method of determining the relative increase and decrease of x and y nothing more need be said; but we must be careful to investigate the points at which $\frac{dy}{dx} = 0$ and $= \frac{1}{0}$, and to observe whether or not there is a change of sign, as it is the criterion of maxima and minima. With this object we shall equate $\frac{dy}{dx}$ to 0 and $\frac{1}{0}$, and examine the course the curve takes at these critical points.

4) In regard to asymptotes, and the course of the curve with respect to them, we must examine for what finite values of x y is infinite, and for what values of y x is infinite, as such will be asymptotes parallel to the axes of y and x respectively; and by investigating whether $\frac{dy}{dx}$ changes sign or not for these asymptotic values we shall determine whether the infinite ordi-

nate is a maximum or minimum; that is, whether it returns, or whether it continues round the circle of infinite radius, such as we described in the last chapter, and which of the forms delineated in figs. 10, 11, 12, 18, the curve takes. We must also be careful to determine whether there are rectilineal asymptotes inclined at oblique angles to the axes of co-ordinates, and whether the curve be above them or below them. It may happen that two distinct branches of a curve will approach the same asymptote; sometimes, also, a curve will cut its asymptote; as e. g. if $y = a\,\frac{\sin x}{x}$, the axis of x is an asymptote, and the curve cuts it whenever $x =$ a multiple of π.

5) The general character of a curve, with regard to the curvature of it in a particular direction, and its points of inflexion, has been sufficiently discussed above in Arts. 103. and 104. Should, however, $\frac{d^2 y}{dx^2} = 0$, and not change sign at any point in the curve, we may conclude that more than two elements of the curve are in one and the same straight line. (See Art. 105.)

6) There is nothing more to be added on the theory of multiple points and their varieties.

7) As a general rule, however, it is of little use to examine the values of x and y, except in such critical states as we have above described.

110.] In the discussion of any equation representing a plane curve, the method indicated by the following rules will be found the most convenient one to adopt.

I. Reduce the equation, if possible, to the explicit form, and simplify it, as far as may be, by means of a change of origin, or by a transformation into polar co-ordinates.

II. Discover, arrange, and tabulate with their proper signs, all the critical values of y and x, both in and out of the plane of reference.

III. Discuss and tabulate the critical values of $\frac{dy}{dx}$, as e. g. determine at what angles the curve cuts the axis, the maximum and minimum ordinates, &c.

IV. Find the equations to the asymptotes, and determine whether the curve is above or below them.

V. Find (if it be possible in a convenient form) $\frac{d^2y}{dx^2}$; thence determine the direction of curvature, and points of inflexion.

VI. If at any point $\frac{dy}{dx} = \frac{0}{0}$, evaluate the quantity, and determine the several double, triple points, &c.

Ex. 1. Discuss the Curve whose Equation is

$$y^2(x^2 - a^2) = x^4.$$

$$\therefore \quad y = \pm \frac{x^2}{\sqrt{(x^2 - a^2)}},$$

Since the given equation is not changed when we write $-x$ and $-y$ for $+x$ and $+y$, it appears that the curve is situated symmetrically in the four quadrants. Differentiating, we have

$$\frac{dy}{dx} = \pm \frac{x(x^2 - 2a^2)}{(x^2 - a^2)^{\frac{3}{2}}}, \text{ also } \frac{dy}{dx} = \frac{2x^3 - xy^2}{y(x^2 - a^2)},$$

$$\frac{d^2y}{dx^2} = \pm \frac{a^2(x^2 + 2a^2)}{(x^2 - a^2)^{\frac{5}{2}}},$$

To find the equations to the asymptote

$$y = \pm \frac{x^2}{x\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}}} = \pm x\left(1 - \frac{a^2}{x^2}\right)^{-\frac{1}{2}}$$

$$= \pm x\left(1 + \frac{a^2}{2x^2} + \ldots\right)$$

$$= \pm \left(x + \frac{a^2}{2x} + \ldots\right),$$

$\therefore$ the equations to the asymptotes are $y = \pm x$; and as the sign of next term, viz. $\frac{a^2}{2x}$, is positive, the curve lies above the asymptote.

When $x = 0$ and $y = 0$, $\frac{dy}{dx} = \frac{0}{0}$; $\therefore$ evaluating the second value of $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{6x^2 - y^2 - 2xy\frac{dy}{dx}}{\frac{dy}{dx}(x^2 - a^2) + 2xy} = \frac{0}{-a^2\frac{dy}{dx}}, \text{ when } x = 0, \text{ and } y = 0;$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = -\frac{0}{a^2} \text{ and } \frac{dy}{dx} = \pm\sqrt{-\left(\frac{0}{a^2}\right)};$$

indicating that two branches of the curve touch the axis of x at the origin, since $\frac{dy}{dx} = 0$, but that both are out of the plane of the paper.

It appears, also, that the curve is in a plane perpendicular to the plane of the paper, for all values of x between $+a$ and $-a$. Hence we tabulate as follows:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1.	0	$\pm\sqrt{-}, 0, \pm\sqrt{-}$	$\pm\sqrt{-}, 0, \pm\sqrt{-}$	$\pm\sqrt{-}$
2.	$-a$	$\pm, \frac{1}{0}, \pm\sqrt{-}$	$\pm, \frac{1}{0}, \pm\sqrt{-}$	$\pm$
3.	$+a$	$\pm\sqrt{-}, \frac{1}{0}, \pm$	$\pm\sqrt{-}, \frac{1}{0}, \pm$	$\pm$
4.	$+\sqrt{2}.a$	$\pm 2a$	$\mp, 0, \pm$	$\pm$
5.	$-\sqrt{2}.a$	$\pm 2a$	$\mp, 0, \pm$	$\pm$
6.	$\pm\frac{1}{0}$	$\pm\frac{1}{0}$	± 1	$\pm, 0, \mp$

From 1. it appears that the curve passes through the origin in two branches, which are out of the plane of reference, and touch the axis of x; whence, as 2. and 3. show, the curve is receding further from the axis of x, and when $x = +a$ and $x = -a$, $y = \pm\frac{1}{0}$, and we have two asymptotes parallel to

the axis of y. For values of x outside of these lines, the curve is in the plane of reference, and returns towards the axis of x, and reaches minimum and maximum values when $x = a\sqrt{2}$, as is shown by (4) and (5), whence it recedes again towards the asymptotes whose equations are $y = \pm x$, and intersects them at $\frac{1}{0}$ in a point of inflexion, as shown by (6), the curve lying above the asymptote in the first quadrant, and being symmetrically situated in the others. Its course is traced in fig. 36., where OA $= a$, OB $= \sqrt{2}.a$, OC $= 2a$, and where the dotted line represents the curve out of the plane of reference.

If the equation to be discussed had been

$$y^2(a^2 - x^2) = x^4,$$

the branches of the curve which are in the plane of reference would have been out of it, and *vice versâ*. The continuity of curve is remarkable in both cases.

Ex. 2. Discuss the Curve whose Equation is

$$y^3 = 2ax^2 - x^3.$$

$$\therefore \quad y = x^{\frac{2}{3}}(2a - x)^{\frac{1}{3}},$$

$$\frac{dy}{dx} = \frac{4a - 3x}{3x^{\frac{1}{3}}(2a - x)^{\frac{2}{3}}}, \text{ also } \frac{dy}{dx} = \frac{4ax - 3x^2}{3y^2},$$

$$\frac{d^2y}{dx^2} = -\frac{8a^2}{9x^{\frac{4}{3}}(2a - x)^{\frac{5}{3}}}.$$

To find the equation to the asymptote:

$$y^3 = -x^3\left(1 - \frac{2a}{x}\right)$$

$$\therefore \quad y = -x\left(1 - \frac{2a}{x}\right)^{\frac{1}{3}},$$

$$= -x\left(1 - \frac{2a}{3x} - \frac{4a^2}{9x^2} - \ldots\right);$$

$\therefore$ the equation to the asymptote is

$$y = -x + \frac{2a}{3};$$

and, as the sign of the next term is positive, the curve lies above the asymptote.

$$\frac{dy}{dx} = \frac{4ax - 3x^2}{3y^2} = \frac{0}{0}, \text{ when } x = 0 \text{ and } y = 0,$$

$$= \frac{4a - 6x}{6y\frac{dy}{dx}} = \frac{4a}{6y\frac{dy}{dx}}, \text{ when } x = y = 0;$$

$$\therefore \quad \frac{dy}{dx} = \pm\sqrt{\frac{2a}{3y}} = \pm\frac{1}{0}, \text{ when } x = y = 0.$$

Therefore there are at the origin two branches touching the axis of y, and the form of $\frac{dy}{dx}$ shows that if y is negative $\frac{dy}{dx}$ is affected with $\pm\sqrt{-}$, therefore the origin is a cusp of the first species.

Hence, to tabulate the critical values:

	x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1.	0	$+, 0, +$	$\frac{0}{0} = \pm\frac{1}{0}$	$-$
2.	$2a$	$+, 0, -$	$-, \frac{1}{0}, -$	$-, \frac{1}{0}, +$
3.	$\frac{4a}{3}$	$+$	$+, 0, -$	$-$
4.	$+\frac{1}{0}$	$-\frac{1}{0}$	-1	$+, 0, -$
5.	$-\frac{1}{0}$	$+\frac{1}{0}$	-1	$-, 0, +$

Hence it appears from (1) that the curve passes through the origin, which is a cusp of the first species, the two branches of which touch the axis of y, and are above the axis of x, both branches being concave downwards; and the curve, having been above the axis of x, from $x = 0$ to $x = 2a$, at this last point cuts the axis of x at right angles, and changes its curvature, from having been concave downwards, to being convex downwards. (3) Shows that the curve has attained to a maximum ordinate when $x = \frac{4a}{3}$; the curve approaches to the asymptote whose equation is $y = -x + \frac{2a}{3}$, which, as is shown by (4) and (5), it cuts and touches at $\frac{1}{0}$, where is a point of inflexion, and thus the two asymptotic branches unite. We have traced the curve only in the plane of reference, as we have not discussed the geometrical meaning of the cube roots of $+$. For the course of the curve, see fig. 37., $\text{OA} = 2\text{A}$, $\text{OB} = \frac{4a}{3}$, $\text{OC} = \frac{2a}{3}$.

CHAP. X.

ON PROPERTIES OF PLANE CURVES AS DEFINED BY EQUATIONS REFERRED TO POLAR CO-ORDINATES.

111.] In this chapter our object is to investigate, for curves referred to polar co-ordinates, formulæ somewhat analogous to those which have been discussed in the last for curves referred to rectangular co-ordinates; but, before doing so, it is necessary to say a few words on the method of interpreting polar equations.

Let $r=f(\theta)$ be the equation to the curve. Then, taking a fixed point S as origin, which is called the *pole*, and a fixed line SX passing through it as the line of origination, which is called the *prime radius* (see fig. 38.), it is manifest that the moveable radius, which is symbolised by r, may revolve about S in two directions: and thus, if the only datum be that r makes an angle θ with the prime radius, it is undetermined whether r is above or below SX; that is, whether r revolves *up* from SX from right to left, or *down* from left to right. Hence arises the necessity of some symbol of the direction in which r turns, so that angles formed in one direction may be differently symbolised to those formed in another. This indefiniteness will be avoided if we call angles positive when measured *up* from SX, as in fig. 38., that is, when the radius vector revolves round S in the direction indicated by the curved arrow; and negative when they are measured *down* from SX, and the radius vector revolves in the direction indicated by the curved arrow in fig. 39. In this case, then, + and −, as affecting angles, indicate the two different directions in which r can revolve in the plane of the paper. But suppose, for any given value of θ, r is affected with a negative sign, in what direction is r to be measured? If r be affected with a positive sign, the length of it, determined by the equation to the curve, is to be measured along the revolving radius, which is inclined at the given angle to the prime radius; as e. g. if a polar equation between r and θ is such that, when $\theta = \frac{\pi}{4}$, $r = a$, then a length $= a$ is to be measured from the pole along the revolving radius, which is inclined at 45° to the prime radius. From analogy, therefore, to what has been said in Art. 91. on the signs + and −, $-r$ must be measured along the radius vector produced backwards; i. e. if, when $\theta = \frac{\pi}{4}$, $r = -a$, a line equal to a must be measured from the pole along the revolving radius produced backwards, that is, in a direction making an angle of 135° with the prime radius. In order the better to avoid confusion on this subject, conceive the revolving radius to be an arrow of variable length, such as we have drawn in figs. 38. and 39., the pole being a fixed point in it; then, if θ be the angle between the prime radius and the part of the arrow towards the barbed end, lines measured from S in the direction SP will be positive, and in the direction SQ negative. If, therefore, r is affected with a positive sign, it is to be measured towards the barbed end, but if with a negative sign towards the feathered end, of the arrow. In figs. 38. and 39. different positions of the arrow are drawn, to indicate different positive and negative directions of r.

In the following discussion we shall omit those particular values of r which are affected with $\pm\sqrt{-}$, and consider only those affected with $\pm$; being careful, however, to make r revolve in both the positive and negative directions, otherwise the curve will appear to be discontinuous at certain points. As an example, let us take the Spiral of Archimedes, the equation to which is $r = a\theta$. It is plain that as θ increases in the positive direction r is positive, and increasing also in the same proportion; also when $\theta = 0$, $r = 0$; also, if θ be negative, r is negative, and is to be measured backwards. The curve is drawn in fig. 40., the dotted line being the branch of the curve corresponding to θ measured in the negative direction.

112.] Determination of certain Geometrical Lines in Curves referred to Polar Co-ordinates.

Let $r = f(\theta)$ be the equation, and suppose that the curve it represents is such as that drawn in fig. 41., which the reader is recommended to examine well; for the values of the several lines which we require will be deduced from the geometry, as were those in Art. 99.

Let S be the pole, SX prime radius, APQ the curve, $\text{PSX} = \theta$, $\text{SP} = r$. Let XSP be increased by a small angle $\text{QSP} = d\theta$, then $\text{SQ} = f(\theta + d\theta) = r + dr$. From centre S and with radius $\text{SP} = r$ describe the small circular arc PR, subtending $d\theta$.

$$\therefore \quad \text{PR} = r\,d\theta, \tag{1}$$

$$\text{RQ} = dr; \tag{2}$$

and for the value of PQ, which is equal to the element of the arc of the curve, we have

$$\text{PQ} = ds = \sqrt{(dr^2 + r^2 d\theta^2)}. \tag{3}$$

Through the two points P Q on the curve, which are infinitely near to each other, draw the line QPT, which is the tangent to the curve at the point P; and through P draw the normal PG, and through S draw TSG perpendicular to the radius vector SP, and Sy perpendicular to the tangent PT. The lengths PT and PG are respectively called the polar tangent and polar normal; SG is called the polar subnormal; ST the polar subtangent; and Sy, the perpendicular from the pole on the tangent, is symbolised by p. These lines we proceed to determine.

Since $\tan \text{PQR} = \dfrac{\text{PR}}{\text{RQ}}$, we have, from (1) and (2),

$$\tan \text{PQR} = \frac{r\,d\theta}{dr}.$$

And since $\text{SPT} = \text{SQT} + \text{PSQ} = \text{PQR} + d\theta$; therefore SPT and PQR being in general finite angles, and $d\theta$ being an infinitesimal angle and diminishing without limit, we may, in accordance with the principle and laws of Art. 8., neglect $d\theta$ in the above equation, and write

$$\text{SPT} = \text{PQR}, \text{ in the limit};$$

and SPT is the angle contained between the curve and the radius vector;

$$\left.\begin{aligned} \therefore \quad \tan \text{SPT} &= \frac{r\,d\theta}{dr}, \\ \sin \text{SPT} = \frac{r\,d\theta}{ds},\ \cos \text{SPT} &= \frac{dr}{ds}, \end{aligned}\right\} \qquad (4)$$

$$\left.\begin{aligned} \text{ST} &= \text{Polar subtangent} = \text{SP} \tan \text{SPT} = \frac{r^2\,d\theta}{dr}, \\ \text{SG} &= \text{Polar subnormal} = \text{SP} \tan \text{SPG} = \text{SP} \cot \text{SPT} = \frac{dr}{d\theta}, \\ \text{PT} &= \text{Polar tangent} = \text{SP} \sec \text{SPT} = \frac{r\,ds}{dr}, \\ \text{PG} &= \text{Polar normal} = \text{SP} \operatorname{cosec} \text{SPT} = \frac{ds}{d\theta}, \\ \text{S}y &= p = \text{SP} \sin \text{SP}y = \frac{r^2\,d\theta}{ds} = \frac{r^2\,d\theta}{\sqrt{(dr^2 + r^2\,d\theta^2)}}, \\ \text{P}y &= \text{SP} \cos \text{SP}y = \frac{r\,dr}{ds} = \sqrt{(r^2 - p^2)}. \end{aligned}\right\} \qquad (5)$$

Similarly may the values of other lines be determined, if they are required.

The value of p may be put under another form which is often very convenient. Let $u = \frac{1}{r}$, the reciprocal of the radius vector.

$$\therefore \quad du = -\frac{dr}{r^2};$$

and from above, $\frac{1}{p^2} = \frac{dr^2 + r^2\,d\theta^2}{r^4\,d\theta^2} = \frac{1}{r^2} + \frac{1}{r^4}\frac{dr^2}{d\theta^2}$;

$\therefore$ substituting, in terms of u,

$$\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2}. \qquad (6)$$

The value of p in (5) might also have been deduced, as follows, from the expression for p in Equation (10) Art. 99. viz.

$$p = x\frac{dy}{ds} - y\frac{dx}{ds}:$$

for since $x = r\cos\theta$, $y = r\sin\theta$,

$$dx = dr\cos\theta - r\sin\theta\, d\theta, \qquad dy = dr\sin\theta + r\cos\theta\, d\theta;$$

$$\therefore \quad x\,dy - y\,dx = r^2\,d\theta;$$

$$\therefore \quad p = \frac{r^2\,d\theta}{ds}.$$

For examples in which the foregoing expressions are applied, see Mr. Gregory's *Collection*, chapter ix. section 2., and Mr. Hind's *Series*, chapter ix.

113.] Asymptotes to Polar Curves.

Curves referred to polar co-ordinates of course admit of rectilinear and curvilinear asymptotes in the same manner as curves referred to rectangular co-ordinates. Curvilinear asymptotes, however, are of little service in determining the course or the peculiarities of a curve, and therefore we shall say nothing of them in general: but there is one remarkable species, viz. the asymptotic circle, which we shall explain at greater length.

A. *Rectilinear Asymptotes.*

As the rectilinear asymptote is a tangent to a curve at an infinite distance, the Formulæ (5) above will enable us to determine whether there are such asymptotes to a polar curve.

If for any *finite* value of θ, say $\theta = \alpha$, r is infinite, then either the radius vector itself, or a line parallel to it, is an asymptote to the curve: and since the polar subtangent, which is equal to $r^2\dfrac{d\theta}{dr}$, is the perpendicular distance from the pole on a tangent, if the value of $\dfrac{r^2\,d\theta}{dr}$, corresponding to $\theta = \alpha$ and $r = \dfrac{1}{0}$, be finite, the line can be drawn, and if $\dfrac{r^2\,d\theta}{dr} = 0$ the radius vector itself is the asymptote; but if it be equal to $\dfrac{1}{0}$, the asymptote, being at an infinite distance from the pole, cannot be constructed. An inspection of fig. 42. will render this plain; in which SP is the infinite radius vector, TL the asymptote, ST the value of $\frac{r^2 d\theta}{dr}$, when $\theta = \alpha$ and $r = \frac{1}{0}$. If there are several values of θ for which r is infinite, there may be several rectilinear asymptotes. Hence, to determine them,

Find what finite values of θ render $r = \frac{1}{0}$. If the polar subtangent, corresponding to such infinite values of r and finite values of θ, be finite, then there are rectilinear asymptotes which may be drawn.

It is to be borne in mind that when $\frac{r^2 d\theta}{dr}$ is positive, the asymptote lies below the radius vector, as in fig. 42.; and, if it be negative, the asymptote lies above it.

Ex. $$r = \frac{a}{\sqrt{\theta}}, \quad \therefore \quad \frac{r^2 dr}{d\theta} = -2 a \theta^{\frac{1}{2}};$$

$$\therefore \quad \text{when } \theta = 0,\ r = \frac{1}{0}, \text{ and polar subtangent} = 0:$$

showing that the prime radius itself is a rectilinear asymptote to the curve, which is delineated in fig. 43. It is to be observed that there are in the curve two discontinuous points; one at the origin, and the other where the infinite branch ends: these apparent anomalies are owing to our not having discussed the means of interpreting r when affected with $\pm\sqrt{-}$, which it will be when θ is negative.

B. *Asymptotic Circles.*

114.] Suppose that as θ increases without limit r approaches without limit to a finite value a, then the spiral is more and more nearly approaching to a circle whose radius is a; if the curve approaches to it from the outside, the circle is called an interior asymptotic circle, and if from the inside, an exterior asymptotic circle.

Ex. $$r = a\left(\frac{1+\theta}{\theta}\right),$$

which may be written under the form

$$r = a + \frac{a}{\theta}.$$

First, let θ be positive, then r is always $>$ than a, and when $\theta = 0$, r is $\frac{1}{0}$, and $r^2 \frac{d\theta}{dr} = -a$, showing that the line parallel to the prime radius at a distance a above it is an asymptote to the curve; and when $\theta = \frac{1}{0}$, $r = a$; whence we have an interior asymptotic circle, such as is drawn in fig. 44.

Secondly, let θ be negative, then

$$r = a - \frac{a}{\theta},$$

and, therefore, when $\theta = 0$, $r = -\frac{1}{0}$, and r is negative as θ increases until $\theta = 1$, in which case $r = 0$, and thence r is always less than a until $\theta = \frac{1}{0}$ when $r = a$. Thus we have such a curve as that dotted in the figure, and with an exterior asymptotic circle (the continuity of the two branches of the curve is worth remarking) of radius SA $= a$.

115.] On Concavity and Convexity, and Points of Inflexion.

On an inspection of the figures marked 45. and 46. it will be manifest that if a curve referred to polar co-ordinates is concave towards the pole, as r increases, p increases also, and therefore $\frac{dr}{dp}$ is positive; and if the curve be convex towards the pole, as r increases, p decreases, and *vice versâ*, and therefore $\frac{dr}{dp}$ is negative. If, therefore, the equation to the curve be given in the form $r = f(\theta)$, in order to determine whether the curve is concave or convex towards the pole, we must transform the equation into one between r and p, by means of the relations given in (5) or (6) of Art. 112., and thence find $\frac{dr}{dp}$, and for all values for which

$\frac{dr}{dp}$ is positive, the curve is concave towards the pole,

$\frac{dr}{dp}$ is negative, ... convex . . . . . ;

and, therefore, if at any point $\frac{dr}{dp}$ changes sign by passing through 0 or $\frac{1}{0}$, at such a point the direction of curvature changes, and there is a point of inflexion: hence, to determine such points, equate $\frac{dr}{dp}$ to 0 and to $\frac{1}{0}$, and examine whether $\frac{dr}{dp}$ changes sign; if it does, there is a point of inflexion.

We subjoin an example, and refer for others to chapter x. of Mr. Gregory's *Collection*, and to chapters ix. and x. of Mr. Hind's *Series*.

Ex. $$r = \frac{a}{\sqrt{\theta}}. \quad \text{Let } \frac{1}{r} = u,$$

$$\therefore \quad a^2 u^2 = \theta,$$

$$\therefore \quad 2a^2 u\,du = d\theta;$$

$\therefore$ substituting in (6) of Art. 112., and reducing,

$$p = \frac{2a^4 r}{\sqrt{(4a^4 + r^4)}};$$

whence $$\frac{dp}{dr} = \frac{2a^4(4a^4 - r^4)}{(4a^4 + r^4)^{\frac{3}{2}}};$$

$\therefore \quad \frac{dp}{dr} = 0$, and changes sign when $r = a\,(2)^{\frac{1}{2}}$.

The curve is represented in fig. 43. We have, therefore, a point of inflexion at Q, where SQ $= a\sqrt{2}$, and $\frac{dr}{dp}$ is positive, that is, the curve is concave towards the pole, when r is less than $a\sqrt{2}$, and $\frac{dr}{dp}$ is negative, or the curve is convex towards the pole, when r is greater than $a\sqrt{2}$.

116] We have now discussed all the peculiarities that curves referred to polar co-ordinates generally admit of, and we proceed to give general rules for the analysis of such equations,

and for tracing the curves which are represented by them; with this object in view we have to make the following suggestions.

1) That if the equation be of the form $r = f(\theta) \pm \psi(\theta)$, so that $r = f(\theta)$ represents a curve which is diametral to the curve which is to be traced, we had better trace the curves separately, or at least separately arrange the lengths of the radii vectores corresponding to the several values of θ, and then add and subtract the radii vectores of the second from those of the first.

2) Investigate the several values of θ which make $r = 0$ or $= \frac{1}{0}$; and in the latter case, if the value of θ be finite, determine whether the polar subtangent is finite or not, as this is the criterion whether the rectilinear asymptote can be constructed or not. Give such particular values to θ as the equation suggests, as e. g. if the equation involves a function of 3θ, put $\theta = 15°$, $30°$, $45°$, &c.; or if the equation involves a function of $\frac{\theta}{4}$, put $\theta =$ $60°$, $90°$, $120°$, $180°$, and so on. In general give to θ such values that r may be constructed; and, by giving to θ the values 0 and $n\pi$, we find the values of r when the curve cuts the prime radius, or the prime radius produced backwards; and remember to make r revolve in both directions.

3) It is convenient to find $\frac{dr}{d\theta}$, as it is the ratio of the corresponding increments of r and θ; and, therefore, if it is positive, as θ increases r increases; and if it is negative, r decreases as θ increases; and if $\frac{dr}{d\theta} = 0$, we have no increase of r corresponding to an increase of θ, that is, the curve is at right angles to the radius vector, which is also manifest from Equation (4) Art. 112., because, at such a point, $\tan \text{SPT} = \frac{1}{0}$. And if $\frac{dr}{d\theta} = 0$ and changes its sign, we have a maximum or minimum value of r, which point is called an *apse:* and we have instances of such in the ellipse, if the focus be the pole, at the extremities of the

major axis; and in the circle, if the centre be the pole, every point is an apse.

4) Nothing more need be said on the subject of rectilinear asymptotes and asymptotic circles; nor

5) On the direction of curvature and points of inflexion.

117.] Hence, then, to trace a curve referred to polar co-ordinates,

I. Investigate, arrange, and tabulate, with their proper signs, all the particular values of θ which render $r = 0$, or $= \frac{1}{0}$, or equal to a value that may be constructed without difficulty.

II. Find $\frac{dr}{d\theta}$; examine its sign, and at what values of θ it is equal to 0, and to $\frac{1}{0}$, and whether it changes its sign; if it does, at such points there are maximum and minimum radii vectores.

III. Determine whether any finite values of θ render $r = \frac{1}{0}$; if so, find the value of $r^2 \frac{d\theta}{dr}$ corresponding to this value of θ, and construct the asymptote: examine whether there is an asymptotic circle.

IV. Transform the equation into its equivalent between r and p: find $\frac{dr}{dp}$, and examine its sign for the purpose of determining whether the curve is convex or concave towards the pole, also examine whether $\frac{dr}{dp}$ changes its sign by passing through 0 or $\frac{1}{0}$, as such will be a point of inflexion.

V. Trace the curve in a similar manner, by making r to revolve in a negative direction.

Ex. 1. $$r = a \sin \frac{\theta}{2},$$

$$\frac{dr}{d\theta} = \frac{a}{2} \cos \frac{\theta}{2}.$$

r is never greater than a, and $r = 0$ when $\theta = 0, = 2\pi, = 4\pi, = \ldots\ldots 2n\pi$; and $\frac{dr}{dp} = 0$ when $\theta = \pi, = 3\pi, = \ldots = (2n+1)\pi$. Therefore we have the following table.

	θ	r	$\frac{dr}{d\theta}$
1.	0	$-, 0, +$	$+$
2.	$\frac{\pi}{2}$	$+\frac{a}{\sqrt{2}}$	$+$
3.	π	$+a$	$+, 0, -$
4.	$\frac{3\pi}{2}$	$+\frac{a}{\sqrt{2}}$	$-$
5.	2π	$+, 0, -$	$-$
6.	$\frac{5\pi}{2}$	$-\frac{a}{\sqrt{2}}$	$-$
7.	3π	$-a$	$-, 0, +$
8.	$\frac{7\pi}{2}$	$-\frac{a}{\sqrt{2}}$	$+$
9.	4π	$-, 0, +$	$+$

Hence it appears that r attains a maximum value a when $\theta = \pi, = 5\pi$, &c., and a minimum value when $\theta = 3\pi, = 7\pi$, &c. The form of the curve is that in fig. 47.

Ex. 2. Trace the Curve whose Equation is

$$r = a\frac{\theta + \sin\theta}{\theta - \sin\theta}; \qquad \therefore \quad \frac{dr}{d\theta} = \frac{2a(\theta\cos\theta - \sin\theta)}{(\theta - \sin\theta)^2}.$$

r is always positive, since the arc is always greater than the sine; and since for all values of θ in the first and second quadrants $\sin\theta$ is positive, and for values in the third and fourth quadrants $\sin\theta$ is negative; therefore in the first and second quadrants r is always greater than a, and in the third and fourth r is less than a.

And since, when $\theta = 0$, $\frac{\sin\theta}{\theta} = 1$, (Lemma II. Art. 12.)

$\therefore$ when $\theta = 0$, $r = \frac{1}{0}$: hence, to determine the corresponding polar subtangent,

$$r^2\frac{d\theta}{dr} = \frac{a}{2}\frac{(\theta+\sin\theta)^2}{\theta\cos\theta - \sin\theta} = \frac{0}{0}, \text{ when } \theta = 0,$$

$$= \frac{a}{2}\frac{2(\theta+\sin\theta)(1+\cos\theta)}{-\theta\sin\theta} = \frac{0}{0}, \text{ when } \theta = 0,$$

$$= a\frac{(1+\cos\theta)^2 - \sin\theta(\theta+\sin\theta)}{-\sin\theta - \theta\cos\theta} = \frac{1}{0}, \text{ when } \theta = 0;$$

$\therefore$ the rectilinear asymptote cannot be drawn.

When $\theta = \frac{1}{0}$, $r = a$, therefore there is an asymptotic circle whose radius is a. Hence we tabulate as follows:

	θ	r
1.	0	$\frac{1}{0}$
2.	$\frac{\pi}{2}$	$a\frac{\pi+2}{\pi-2}$
3.	π	a
4.	$\frac{3\pi}{2}$	$a\frac{3\pi-2}{3\pi+2}$
5.	2π	a
6.	$\frac{5\pi}{2}$	$a\frac{5\pi+2}{5\pi-2}$
7,	3π	a
8.	$\frac{1}{0}$	a

It appears, then, that the curve starts from infinity, as delineated in fig. 48.; and periodically when $\theta = \pi, = 2\pi, = 3\pi, =$ &c., passes through the two points A and B, which are the extremities of the diameter of the circle whose radius is a, to which circle it continually approaches; being outside in the first and second quadrants, and inside the circle in the third and fourth. We have then this peculiarity, that the curve on the outside is becoming gradually nearer and nearer to the circle; and on the inside of the circle, as θ increases it recedes further and further from the diameter towards the circumference.

CHAP. XI.

ON CURVATURE OF PLANE CURVES.

A. *Curves referred to Rectangular Co-ordinates.*

118.] THE considerations on which we entered in Art. 103. Chapter IX. enable us to determine in what direction, with reference to fixed co-ordinate axes, the curvature of a curve is turned, and also to determine points of inflexion; we now proceed to discuss the *amount* of curvature: but, as a new affection of a plane curve is hereby introduced, the amount being either great or small, and these being relative terms, it is necessary to fix on some standard with which to compare it; the circle naturally suggests itself as a convenient measure, inasmuch as the curvature of it is the same at every point of the same circle, and, therefore, depends on some *essential* property of the circle. With a view to the determination of this property, let us conceive a circle of radius r to be drawn, and a straight line touching it a given point. If the radius increases, the circle approaches nearer and nearer to the tangent line, and the curvature becomes less and less, and when the radius becomes infinitely great the circle degenerates into a straight line, and the curvature vanishes (see Art. 88.): also, as the radius decreases the curvature increases, and ultimately, when $r = 0$, the curvature is infinitely great; hence the curvature is some function of $\frac{1}{r}$, and it is usual and natural to take the most simple function, viz. $\frac{1}{r}$, as the measure of it.

A question then arises, in what manner most convenient to our present purpose is this Measure of a Circle's Curvature to be determined? The relation between a circular arc, the radius of a circle, and the angle subtended at the centre, is

$$\text{arc} = \text{rad.} \times \text{angle}.$$

If, therefore, ds = the small arc between two points (x, y) $(x+dx, y+dy)$, $d\psi$ the small angle subtended at the centre, and ρ the radius,

$$ds = \pm \rho\, d\psi;$$

whence
$$\frac{1}{\rho} = \pm \frac{d\psi}{ds},$$

using the positive or negative sign as is requisite, since ρ is to be an absolute length, i. e. independent of direction.

Let us now consider how we are hereby enabled to determine the curvature at any point on a given curve. Of course, the above ratio $\frac{d\psi}{ds}$ is no longer constant, but varies at every point of the curve. Conceive, then, two normals to be drawn at two consecutive points of a curve. These will in general meet at a finite distance, and $d\psi$ will be the small angle they include; and, as the two points become infinitely near to one another, the ratio $\frac{d\psi}{ds}$ will converge towards some determinate limit, and will become the measure of the curvature at the point, or rather will measure the *mean* curvature, and no appreciable error will be committed by our taking it as the measure of the actual curvature; whence we have

$$\rho = \pm \frac{ds}{d\psi}; \tag{1}$$

and calling the line ρ, as we may do from the analogy of the circle, the *radius of curvature*, and the point at which the consecutive normals intersect the *centre of curvature*, we may define as follows:

Def. The distance from the curve at which two consecutive normals of a plane curve intersect, is the *radius of curvature* at that point.

119.] We proceed to determine its Value. (See fig. 49.)

Let the equation to the curve be $y = f(x)$, and P, Q be the two points on it at which the normals PO, QO are drawn, O their point of intersection, which is therefore the centre of curvature; through G draw a line KG parallel to QO; then $PQ = ds$, $POQ = d\psi = PGK = d \tan^{-1} \frac{dx}{dy}$, $PO = QO = \rho =$ radius of curvature;

$$\therefore \quad ds = \rho d \tan^{-1} \frac{dx}{dy} \tag{2}$$

$$= \rho \frac{d^2x dy - d^2y dx}{dy^2} \frac{dy^2}{dx^2 + dy^2},$$

$$\therefore \quad ds^3 = \rho (d^2x dy - d^2y dx),$$

$$\therefore \quad \rho = \frac{ds^3}{d^2x dy - d^2y dx}, \tag{3}$$

in which expression neither x, y, nor s is an independent variable, and therefore the above is the most general value of ρ.

If x be taken as the independent variable $d^2x = 0$, and Equation (3) becomes

$$\rho = \frac{ds^3}{-d^2y dx} = -\frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}, \tag{4}$$

which is the value of the radius of curvature most generally used; and, if y be the independent variable, $d^2y = 0$, and

$$\rho = \frac{ds^3}{d^2x dy} = \frac{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}. \tag{5}$$

We have omitted the $\pm$ signs, because they indicate only the direction in which the radius of curvature is to be drawn, whether *down* from the curve, as in fig. 49., or *up*, as would be the case in fig. 54.; an inspection, however, of the curve in fig. 49., and of the Equation (4), shows that if ρ is to be measured downwards, and then reckoned positive, as therein delineated, Equation (4) must be affected with the negative sign.

On comparing Equations (1) and (2), it appears that the angle which we have denoted by $d\psi$, and which is the angle between two consecutive normals, $= d \tan^{-1} \frac{dx}{dy}$, whence (see fig. 49.)

$$\tan^{-1} \frac{dx}{dy} = \psi = \text{PGM}:$$

but, according to the notation in Art. 97. Chap. IX.,

$$\tan^{-1} \frac{dy}{dx} = \tau = \text{PTM};$$

$$\therefore \quad \tau + \psi = \frac{\pi}{2},$$

$$\therefore \quad d\tau + d\psi = 0,$$

whence
$$\rho = - \frac{ds}{d\tau}; \tag{6}$$

and $d\tau$, which $= \text{T}'\text{PT}$, and is the angle of inclination of two consecutive tangents, is called *the angle of contingence.*

The value of ρ given in (3) may be put under another very elegant form, which is sometimes convenient:

$$\therefore \quad ds^2 = dx^2 + dy^2,$$

$$ds\, d^2 s = dx\, d^2 x + dy\, d^2 y, \tag{7}$$

$$\therefore \quad ds^2 (d^2 s)^2 = dx^2 (d^2 x)^2 + 2\, dx dy\, d^2 x\, d^2 y + dy^2 (d^2 y)^2,$$

squaring (3), $\frac{ds^6}{\rho^2} = dy^2 (d^2 x)^2 - 2\, dx dy\, d^2 x d^2 y + dx^2 (d^2 y)^2,$

$\therefore$ by addition,

$$ds^2 \left\{ (d^2 s)^2 + \frac{ds^4}{\rho^2} \right\} = (dx^2 + dy^2) \{(d^2 x)^2 + (d^2 y)^2\},$$

whence
$$\frac{1}{\rho^2} = \frac{1}{ds^4} \{(d^2 x)^2 + (d^2 y)^2 - (d^2 s)^2\},$$

$$\frac{1}{\rho} = \frac{1}{ds^2} \{(d^2 x)^2 + (d^2 y)^2 - (d^2 s)^2\}^{\frac{1}{2}}; \tag{8}$$

therefore, if s be the independent variable, $d^2 s = 0$, and

$$\frac{1}{\rho} = \frac{1}{ds^2} \{(d^2x)^2 + (d^2y)^2\}^{\frac{1}{2}},$$

$$\frac{1}{\rho} = \left\{ \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 \right\}^{\frac{1}{2}}. \qquad (9)$$

As an example, let us apply Equation (4) to the circle,

$$y = \sqrt{(a^2 - x^2)},$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{(a^2 - x^2)}}, \qquad \frac{d^2y}{dx^2} = \frac{-a^2}{(a^2 - x^2)^{\frac{3}{2}}};$$

$$\therefore \quad \rho = a.$$

For other examples, see Gregory's *Collection*, chap. xii., and Hind's *Series*, chap. viii.

If the equation to the curve be given under the implicit form

$$u = \mathrm{F}(x, y) = c,$$

we must substitute as follows in the general value of ρ given in Equation (3); since, as shown in Art. 38.,

$$\left(\frac{d\mathrm{F}}{dx}\right) dx + \left(\frac{d\mathrm{F}}{dy}\right) dy = 0,$$

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right) dx^2 + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) dx\,dy + \left(\frac{d^2\mathrm{F}}{dy^2}\right) dy^2 + \left(\frac{d\mathrm{F}}{dx}\right) d^2x$$
$$+ \left(\frac{d\mathrm{F}}{dy}\right) d^2y = 0;$$

$$\therefore \quad \left(\frac{d\mathrm{F}}{dx}\right) d^2x + \left(\frac{d\mathrm{F}}{dy}\right) d^2y = -\left\{\left(\frac{d^2\mathrm{F}}{dx^2}\right) dx^2\right.$$
$$\left. + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) dx\,dy + \left(\frac{d^2\mathrm{F}}{dy^2}\right) dy^2\right\}; \qquad (10)$$

from which equations,

$$-\frac{dx}{\left(\frac{d\mathrm{F}}{dy}\right)} = \frac{dy}{\left(\frac{d\mathrm{F}}{dx}\right)} = \frac{d^2x\,dy - d^2y\,dx}{d^2x\left(\frac{d\mathrm{F}}{dx}\right) + d^2y\left(\frac{d\mathrm{F}}{dy}\right)}$$

$$= \frac{ds}{\left\{\left(\frac{d\mathrm{F}}{dx}\right)^2 + \left(\frac{d\mathrm{F}}{dy}\right)^2\right\}^{\frac{1}{2}}},$$

∴ substituting from (3) and (10), the third of these equalities becomes

$$-\frac{1}{\rho}\,\frac{ds^3}{\left(\frac{d^2\text{F}}{dx^2}\right)dx^2+2\left(\frac{d^2\text{F}}{dx\,dy}\right)dx\,dy+\left(\frac{d^2\text{F}}{dy^2}\right)dy^2}\,.$$

Whence, substituting and reducing, we have

$$\frac{1}{\rho}=\frac{\left(\frac{d\text{F}}{dy}\right)^2\left(\frac{d^2\text{F}}{dx^2}\right)-2\left(\frac{d\text{F}}{dx}\right)\left(\frac{d\text{F}}{dy}\right)\left(\frac{d^2\text{F}}{dx\,dy}\right)+\left(\frac{d^2\text{F}}{dy^2}\right)\left(\frac{d\text{F}}{dx}\right)^2}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2\right\}^{\frac{3}{2}}}. \qquad (11)$$

As an example, let us apply this expression to the ellipse,

$$\text{F}(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

$$\left(\frac{d\text{F}}{dx}\right)=\frac{2x}{a^2},\qquad \left(\frac{d\text{F}}{dy}\right)=\frac{2y}{b^2},$$

$$\left(\frac{d^2\text{F}}{dx^2}\right)=\frac{2}{a^2},\quad \left(\frac{d^2\text{F}}{dx\,dy}\right)=0,\quad \left(\frac{d^2\text{F}}{dy^2}\right)=\frac{2}{b^2};$$

whence $$\frac{1}{\rho}=\frac{(ab)^4}{(a^4y^2+b^4x^2)^{\frac{3}{2}}}.$$

120.] It is manifest that, in general, as the point at which the radius of curvature is drawn moves along the curve, the centre of curvature moves also, and traces out a continuous curve if the point moves continuously along the first curve. We proceed now to determine the Position of the Centre of Curvature, and its locus as the point at which the radius of curvature is drawn moves.

Let x and y be the co-ordinates to the point on the curve, ξ and η the co-ordinates to the centre of curvature, then, on referring to fig. 50., it will be seen that

$$\text{OM}=x,\qquad \text{ON}=\xi,\qquad \text{PGM}=\psi,$$

$$\text{MP}=y,\qquad \text{N}\Pi=\eta,\qquad \text{PTM}=\tau,$$

$$\text{P}\Pi=\rho.$$

Then, since $\frac{dx}{ds}$, $\frac{dy}{ds}$ are respectively the sine and cosine of the angle PGM, we have, from the geometry of the figure,

$$\left.\begin{aligned} \xi &= x + \rho \frac{dy}{ds}, \\ \eta &= y - \rho \frac{dx}{ds}, \end{aligned}\right\} \qquad (12)$$

which formulæ determine the position of the centre of curvature corresponding to any point of the curve; and, by eliminating x and y between these equations and the equation to the curve, viz. $y = f(x)$, there will result an equation involving ξ and η, which will represent the locus of the centre of curvature. This new curve is called *the evolute* to the original curve, for a reason which will be shortly explained.

The Equations (12) assume various forms, according to the value we give to ρ; i. e. whether we express ρ by one or other of its values (3), (4), (5), (9), (11). Thus, applying the value of the radius of curvature given in (4) when x is the independent variable, we have

$$\left.\begin{aligned} \xi &= x - \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}} \frac{dy}{dx}, \\ \eta &= y + \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}, \end{aligned}\right\} \qquad (13)$$

and, if the equation to the curve be given in the implicit form, the equations must be modified according to the Equation (11), and those by means of which (11) has been determined.

Let us take the two following examples of applying the formulæ (13) and determining the equations to the evolutes of given curves.

Ex. 1. To find the Evolute of the Ellipse whose Equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y},$$

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3};$$

whence $$\xi = \frac{a^2-b^2}{a^4}x^3,$$

$$\eta = -\frac{a^2-b^2}{b^4}y^3;$$

whereby we have determined the position of the centre of curvature of any point in the ellipse; and since

$$\frac{x^2}{a^2} = \left(\frac{a\xi}{a^2-b^2}\right)^{\frac{2}{3}},$$

$$\frac{y^2}{b^2} = \left(\frac{b\eta}{a^2-b^2}\right)^{\frac{2}{3}},$$

Adding, and bearing in mind the equation to the ellipse, we have

$$(a\xi)^{\frac{2}{3}} + (b\eta)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}},$$

which is the equation to the evolute of the ellipse, and is represented in fig. 51.

Ex. 2. To find the Equation to the Evolute of the Cycloid,

$$y = \sqrt{(2ax-x^2)} + a\,\text{versin}^{-1}\frac{x}{a}.$$

The origin being placed at O, the vertex of the cycloid, and the curve being placed as in fig. 52., $\text{OM} = x$, $\text{MP} = y$, $\text{ON} = \xi$, $\text{NQ} = \eta$, $\text{OA} = 2a$, $\text{AO}' = \pi a$; by differentiation

$$\frac{dy}{dx} = \left(\frac{2a-x}{x}\right)^{\frac{1}{2}},$$

$$\frac{d^2y}{dx^2} = -\frac{a}{x(2ax-x^2)^{\frac{1}{2}}};$$

whence, substituting in Equation (13),

$$\xi = 4a - x,$$

$$\eta = y - 2(2ax - x^2)^{\frac{1}{2}};$$

whence $x = 4a - \xi,$

$$y = \eta + 2\{(\xi - 2a)(4a - \xi)\}^{\frac{1}{2}};$$

and substituting in the equation to the cycloid,

$$\eta = -\{(\xi - 2a)(4a - \xi)\}^{\frac{1}{2}} + a \operatorname{versin}^{-1} \frac{4a - \xi}{a},$$

an equation representing a cycloid exactly equal and similarly placed to the original one, but having its vertex at O′, as in the figure, which may be shown as follows. Transfer the origin to O′, by writing

$$\text{for } \xi, \quad 2a + \xi',$$

$$\text{. . } \eta, \quad \pi a - \eta';$$

whence we have

$$\pi a - \eta' = -\{\xi'(2a - \xi')\}^{\frac{1}{2}} + a \operatorname{versin}^{-1} \frac{2a - \xi'}{a}$$

$$= -(2a\xi' - \xi'^2)^{\frac{1}{2}} + \pi a - a \operatorname{versin}^{-1} \frac{\xi'}{a};$$

$$\therefore \quad \eta' = (2a\xi' - \xi'^2)^{\frac{1}{2}} + a \operatorname{versin}^{-1} \frac{\xi'}{a};$$

which, being an equation of exactly the same form as the equation to the given cycloid, represents an equal and similarly placed curve, viz. O′Q.

Theoretically, the equations to the evolutes of all curves may be found by means of the Equations (12), but in most cases the difficulty of elimination is so great as to be beyond the present powers of analysis. For other examples, see Gregory, chapter xii. section 2., and Hind, chapter viii.

121.] We proceed, now, to discuss the general Properties of the Curve whose current co-ordinates are ξ and η.

From Equation (12) we have

$$\left.\begin{aligned} \xi - x &= \rho\frac{dy}{ds}, \\ \eta - y &= -\rho\frac{dx}{ds}; \end{aligned}\right\} \qquad (14)$$

$\therefore$ squaring and adding,

$$(\xi - x)^2 + (\eta - y)^2 = \rho^2, \qquad (15)$$

also $$(\xi - x)dx + (\eta - y)dy = 0, \qquad (16)$$

$$(\xi - x)d^2x + (\eta - y)d^2y = \frac{\rho}{ds}(d^2x\,dy - d^2y\,dx),$$

$$= ds^2 \text{ by condition (3).} \quad (17)$$

Hence it appears, that as (17) is the differential of (16), and (16) is the differential of (15), the differentiations being performed on the supposition that ξ, η, and ρ are constant, and x and y vary, (15) represents a circle whose radius is ρ, and the co-ordinates to whose centre are ξ and η, and which passes through the three points on the original curve (x,y), $(x + dx, y + dy)$, $(x + 2dx + d^2x, y + 2dy + d^2y)$, which circle is called *the circle of curvature.* This result is in accordance with what was laid down in Art. 118. and 119.; for since ξ and η refer to the point of meeting of two consecutive normals, and each normal implies a tangent passing through two points, there must be three consecutive points in the curve for which ξ, η, and ρ do not vary. It is also to be observed that (16) is the equation to the normal, if ξ and η are its current co-ordinates, and, therefore, the centre of curvature is on the normal.

122.] Again, suppose λ and μ to be the angles made by the line PΠ, which is the radius of curvature in fig. 50., with the co-ordinate axes, and let us consider s to be the independent variable; then, since

$$ds^2 = dx^2 + dy^2,$$

$$\therefore \quad 0 = d^2x\,dx + d^2y\,dy. \qquad (18)$$

Comparing this with (16), as both equations are simultaneously true, we have

$$\frac{d^2x}{\xi - x} = \frac{d^2y}{\eta - y} = \frac{\{(d^2x)^2 + (d^2y)^2\}^{\frac{1}{2}}}{\rho} = \frac{ds^2}{\rho^2} \text{ by Equation (9)};$$

$$\left.\begin{aligned} \therefore \quad \frac{\xi - x}{\rho} &= \rho \frac{d^2x}{ds^2}, \\ \frac{\eta - y}{\rho} &= \rho \frac{d^2y}{ds^2}; \end{aligned}\right\} \qquad (19)$$

$$\text{but} \qquad \frac{\xi - x}{\rho} = \cos\lambda, \text{ and } \frac{\eta - y}{\rho} = \cos\mu;$$

$$\therefore \quad \cos\lambda = \rho\frac{d^2x}{ds^2}, \ \cos\mu = \rho\frac{d^2y}{ds^2}. \qquad (20)$$

Again, since the new curve is the locus of the point of intersection of any two consecutive normals of the original curve, if the new curve be continuous, each normal must pass through two points in the new curve which are infinitely near to one another. Hence, in the expressions (14), ξ, η, ρ, x, y may all vary simultaneously, and we have

$$d\xi = dx + d\rho\frac{dy}{ds} + \rho\frac{d^2y\,ds - d^2s\,dy}{ds^2};$$

and, substituting for ρ and d^2s from (3) and (7), we have

$$\left.\begin{aligned} d\xi &= \frac{dy}{ds}\,d\rho, \\ \text{and, similarly,} \qquad d\eta &= -\frac{dx}{ds}\,d\rho. \end{aligned}\right\} \qquad (21)$$

Whence it appears that we may differentiate (14) on the supposition that ρ, ξ, and η vary independently of x and y; that is, the normal passing through (x, y) passes through (ξ, η) and $(\xi + d\xi, \eta + d\eta)$; though, of course, as is plain from the figure 50., the length of ρ varies.

123.] Again, from (21),

$$d\xi\, dx + d\eta\, dy = 0, \qquad (22)$$

$$d\xi^2 + d\eta^2 = d\rho^2. \qquad (23)$$

Hence, from (23), if σ be the length of an arc of the new curve, and $d\sigma$ an element of the arc, then, since

$$d\sigma^2 = d\eta^2 + d\xi^2,$$

we have
$$d\sigma^2 = d\rho^2;$$

$$\therefore \quad d\sigma = \pm\, d\rho.$$

And taking the positive sign, in order to accommodate the analytical expression to the curve in fig. 50., where $\text{A}\Pi = \sigma$, and the element of the arc at $\Pi = d\sigma$, and therefore σ and ρ are increasing simultaneously, we have

$$d\sigma - d\rho = 0;$$

$$\therefore \quad \sigma - \rho = \text{a constant} = c:$$

and, therefore, the difference in length between the radius of curvature of the original curve and the length of the new curve is constant. If, therefore, a perfectly flexible and inextensible string of the length ρ were fixed at the point (ξ, η) on the new curve, and wrapped round the arc of the new curve, just so much will be taken off from the string by the wrapping that the remainder will equal the radius of the old curve, corresponding to the point in the new curve at which the wrapping ends; and, therefore, if an inextensible string be unwrapped from the new curve, as e. g. (see fig. 50.) from A Π, the length of which is exactly equal to the length of the new curve + A O, which is constant, and is the radius of curvature of O P at O, the extremity of it will generate the old curve, viz. O P. It is for this reason that the new curve is called the *evolute*, and the original curve is called the *involute* with respect to it.

It is manifest that the lengths of all evolutes can be determined, that is, the lengths can be compared with other lines, whence they are said to be *rectifiable;* and that the length of the evolute is equal to the difference of the radii of curvature of the involute corresponding to its two extremities. Thus, consider the evolute to the ellipse which is delineated in fig. 51.; the length of the branch DE = the radius of curvature of ellipse at B less the radius of curvature at A. And from the example in Art. 119.,

$$\text{Rad. of curv. at B} = \frac{a^2}{b} = \text{DB},$$

$$\text{. . . . . at A} = \frac{b^2}{a} = \text{AE};$$

$$\therefore \text{ length of branch of evolute DE} = \frac{a^3-b^3}{ab}.$$

Hence, whole length of evolute $= 4\,\dfrac{a^3-b^3}{ab}$.

124.] Again, from (22),

$$d\xi\, dx + d\eta\, dy = 0;$$

$$\therefore \frac{d\eta}{d\xi} = -\frac{dx}{dy}.$$

And since $\frac{d\eta}{d\xi}$ is the tangent of the angle made with the axis of x by the tangent to the evolute, and $\frac{dy}{dx}$ is the tangent of the angle between the axis of x and the tangent to the involute, it follows that the tangent to the evolute is perpendicular to the tangent to the involute, or that the normal to the involute is a tangent to the evolute. This result might have been anticipated from what is said in Art. 121.

125.] The following geometrical considerations will enable us better to understand some of the results we have arrived at in the preceding articles on curvature.

The Equations (14), (15), (16), (17), connecting x, y, ρ, ξ, and η, show that the centre of curvature is the centre of a circle passing through three consecutive points in the curve. These *three* points are necessary to render the circle definite. If it passes through only two points, its centre may be *any where* on the normal which is perpendicular to the tangent passing through the two points, and thus there may be an infinite number of circles satisfying the condition: but, if the circle is to pass through *three* points, its centre must be on the normal perpendicular to the tangent passing through the second and third points, as well as on the normal corresponding to the first and second points; and as these two normals will intersect in *one* point, this point must be the centre of the circle, and the circle becomes definite; in other words, the two consecutive elements of the curve which are delineated in fig. 53., viz. PQ and QR, will form two sides of a triangle, and by joining PR the triangle will be completed, and the circle described about this triangle will be a definite circle, and pass through the points P, Q, R, which are three consecutive points on the curve. We may by this means determine the radius of curvature as follows.

Using the same notation as in Equation (6) of this chapter, $d\tau$ is the small angle between PQ produced and QR; and as this angle is very small we are authorised by Lemma II. Chap. I. to use indifferently the angle or its sine. Whence, and by Equation (6),

$$\sin d\tau = \pm \frac{ds}{\rho}.$$

And, since PQ $= ds$, and QR $= ds + d^2s$,

the area of the triangle PQR $= \dfrac{ds}{2}(ds + d^2s)\sin d\tau$

$$= \pm \frac{ds^2}{2\rho}(ds + d^2s).$$

Also, since PS $= dx$, SQ $= dy$, SU $= dx + d^2x$, TR $= dy + d^2y$, the area of PQR $=$ PUR $-$ PSQ $-$ ST $-$ QTR

$$= \tfrac{1}{2}(2\,dx + d^2x)(2\,dy + d^2y) - \tfrac{1}{2}\,dy\,dx - dy\,(dx + d^2x)$$
$$- \tfrac{1}{2}(dy + d^2y)(dx + d^2x)$$
$$= \tfrac{1}{2}(dx\,d^2y - dy\,d^2x);$$

and equating these two values of the area of PQR,

$$\pm \frac{ds^2}{\rho}(ds + d^2s) = dx\,d^2y - dy\,d^2x,$$

and neglecting d^2s, because it is added to ds, we have

$$\frac{1}{\rho} = \pm \frac{d^2x\,dy - dx\,d^2y}{ds^3},$$

the same result as in Art. 119.

If we had applied the common trigonometrical formula for the radius of a circle circumscribed about a given triangle whose sides are a, b, c, and $a+b+c=2\text{s}$, viz.

$$\rho = \frac{abc}{4\sqrt{\{\text{s}(\text{s}-a)(\text{s}-b)(\text{s}-c)\}}},$$

to the triangle PQR, whose sides are

$$\text{PQ}=ds,\ \text{QR}=ds+d^2s,\ \text{PR}=\{(2\,dx+d^2x)^2+(2\,dy+d^2y)^2\}^{\frac{1}{2}},$$

we should have found, as in Equation (8),

$$\frac{1}{\rho} = \frac{1}{ds^2}\{(d^2x)^2+(d^2y)^2-(d^2s)^2\}^{\frac{1}{2}}.$$

126.] An inspection of the value of the radius of curvature (4) in Art. 119. shows that, if $\frac{dy}{dx}$ be not infinite, the radius of curvature becomes infinite, or the curvature vanishes whenever $\frac{d^2y}{dx^2}=0$; that is, the curve degenerates into a straight line. Such is the case at the point of inflexion, at which, also, as the sign of $\frac{d^2y}{dx^2}$ changes, the direction in which the radius of curvature is to be drawn changes also, and the evolute passes off into an infinite branch asymptotic to the normal at the point of inflexion, and passes round the sphere of which the plane in which the curve lies is the superior limit, whereby the evolute has no point of discontinuity. If $\frac{d^2y}{dx^2}=0$, and does not change its sign, then the branches of the evolute are related to the normal asymptote in the manner indicated in fig. 12.; and if at any point on the curve $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are both infinite, then the value of the radius of curvature must be determined by the methods explained in Chap. V.

B. *Curves referred to Polar Co-ordinates.*

127.] To determine the Length of the Radius of Curvature.

Let $r=f(\theta)$ be the equation to the curve (see fig. 54.); then, from Equation (6) Art. 119.,

$$\rho = \pm \frac{ds}{d\tau};$$

but τ = the angle PKX = PSX + SPK

$$= \theta + \tan^{-1}\frac{r\,d\theta}{dr};$$

$$\therefore \quad d\tau = d\theta + \frac{d\left(\frac{r\,d\theta}{dr}\right)}{1+\frac{r^2 d\theta^2}{dr^2}}$$

$$= \frac{r^2 d\theta^3 + r\,dr\,d^2\theta + 2\,dr^2\,d\theta - r\,d\theta\,d^2 r}{dr^2 + r^2 d\theta^2}.$$

And since, from Equation (3) Art. 112.,

$$ds = (dr^2 + r^2 d\theta^2)^{\frac{1}{2}},$$

$$\rho = \pm \frac{(dr^2 + r^2 d\theta^2)^{\frac{3}{2}}}{r^2 d\theta^3 + r\,dr\,d^2\theta + 2\,dr^2\,d\theta - r\,d\theta\,d^2 r}; \qquad (24)$$

and if θ be an independent variable, $d^2\theta = 0$, and we have

$$\rho = \pm \frac{(dr^2 + r^2 d\theta^2)^{\frac{3}{2}}}{r^2 d\theta^3 + 2\,dr^2\,d\theta - r\,d\theta\,d^2 r}. \qquad (25)$$

Similarly may the expression be modified if r be the independent variable.

The value (24) might also have been deduced from the Equation (3) Art. 119., by substituting

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$

and calculating the successive differentials of x and y.

128.] The expression for the radius of curvature may also be put under a very elegant and simple form, as follows:

Let the equation to the curve be changed into its equivalent between r and p, by means of either (5) or (6) in Art. 112., so that

$$r = f(p).$$

Then, since $\tau = \theta + \text{SP}y$

$$= \theta + \sin^{-1}\frac{p}{r};$$

$$\therefore \quad d\tau = d\theta + \frac{r\,dp - p\,dr}{r\sqrt{(r^2-p^2)}}:$$

and since, from the geometry of the figure, by equating the two values of tan SPy,

$$d\theta = \frac{p\,dr}{r\sqrt{(r^2-p^2)}},$$

$$\text{and} \qquad ds = \frac{r\,dr}{\sqrt{(r^2-p^2)}};$$

$$\therefore \quad d\tau = \frac{dp}{\sqrt{(r^2-p^2)}},$$

$$\therefore \quad \rho = \pm\frac{ds}{d\tau} = \pm\frac{r\,dr}{dp}. \qquad (26)$$

Which expression may also thus be obtained (see fig. 55.):

$$\left.\begin{array}{l}\text{SP} = r,\\ \text{S}y = p,\end{array}\right\} \qquad \text{PO} = \rho, \text{ the radius of curvature,}$$

O being the point at which two consecutive normals intersect. Draw Sz from S perpendicular to OP, then Pz = Sy = p; therefore, from the geometry,

$$\text{SO}^2 = \text{SP}^2 + \text{PO}^2 - 2\,\text{SP}\,\text{PO}\sin\text{SP}y$$

$$= r^2 + \rho^2 - 2r\rho\frac{p}{r}$$

$$= r^2 + \rho^2 - 2p\rho. \qquad (27)$$

Then, since O and ρ remain the same for the next consecutive point in the curve, but r and p vary, we may differentiate on these conditions, and we have

$$0 = 2r\,dr - 2\rho\,dp;$$

$$\therefore \quad \rho = \frac{r\,dr}{dp}. \qquad (28)$$

The geometrical meaning of which expression is evident from fig. 56.

Let Py, Py' be two tangents drawn at consecutive points on the curve, Sy, Sy' the corresponding perpendiculars from the pole, then yPy' is the angle of contingence, and therefore

$$ds = \rho \times \text{angle } y\text{P}y';$$

but the angle yPy' is subtended by the small arc $uy' = dp$, at the distance Py', which is equal to $\sqrt{(r^2-p^2)}$; therefore, yPy' may be replaced by

$$\frac{dp}{\sqrt{(r^2-p^2)}};$$

and replacing ds by its value found above, there results

$$\rho = \frac{r\,dr}{dp}.$$

129.] A comparison of the results we have arrived at with what has been said in Article 121. shows that O is the centre of a circle which passes through three consecutive points on the curve. Let this circle be drawn as in fig. 55.; then PV, the part of the radius vector SP which is intercepted by the circle, is called the *chord of the circle of curvature*. Its value is thus determined:

$$\text{PV} = 2\text{PU} = 2\rho\cos\text{OPS}$$

$$= 2\rho\sin\text{SP}y = 2\rho\frac{p}{r}$$

$$= 2p\frac{dr}{dp}. \qquad (29)$$

130.] To find the Equation to the Evolute of a Curve referred to Polar Co-ordinates.

The following is the most convenient method of determining the equation.

Let the original equation be transformed into its equivalent between r and p, so that

$$r = f(p); \tag{30}$$

then, bearing in mind what has been said in Articles 121—124. on the properties of evolutes which are general and independent of the particular system of determining position, the line PO in fig. 55., which is a normal to the involute, is a tangent to the evolute; therefore, referring the evolute to polar co-ordinates, r and p, if we take S to be the pole to the evolute, we have

$$\left.\begin{array}{l} \text{SO} = r' \\ \text{SZ} = p' \end{array}\right\} \text{co-ordinates to the evolute,}$$

and the following equations:

$$\rho = r\frac{dr}{dp}, \tag{31}$$

and from the geometry,

$$p' = (r^2 - p^2)^{\frac{1}{2}}, \tag{32}$$

and from (27)

$$r'^2 = r^2 + \rho^2 - 2\rho p; \tag{33}$$

from which four equations we can (theoretically at least) eliminate r, p, and ρ, and obtain a final equation involving only r' and p', which will be the polar equation to the evolute.

The several properties of evolutes, and the relation they bear to their respective involutes, might be deduced from the discussion of them referred to polar co-ordinates; but, as they have been fully explained in the former part of this chapter, it is not worth while to repeat them.

Two examples are subjoined of the equations which have been discussed in the last articles; for others, see Gregory, chap. xii., and Hind, chap. ix.

Ex. 1. To determine the Radius and Chord of Curvature, and the Evolute of the Logarithmic Spiral, whose Equation is

$$r = a^{\theta}.$$

For the sake of convenience, let $\log_e a = \text{A}$;

$$\therefore \quad \frac{dr}{d\theta} = \text{A}a = \text{A}r,$$

$\therefore$ by means of Equation (5) Art. 112.,

$$p = \frac{r}{(1 + \text{A}^2)^{\frac{1}{2}}},$$

$$\therefore \quad \frac{dr}{dp} = (1 + \text{A}^2)^{\frac{1}{2}},$$

$$\therefore \quad \rho = r(1 + \text{A}^2)^{\frac{1}{2}};$$

$$\text{chord of curvature} = 2r:$$

and from (32) and (33)

$$p' = \frac{\text{A}r}{(1 + \text{A}^2)^{\frac{1}{2}}}, \qquad \therefore \quad r = \frac{(1 + \text{A}^2)^{\frac{1}{2}}}{\text{A}} p';$$

$$r' = \text{A}r, \qquad \therefore \quad r = \frac{r'}{\text{A}}:$$

whence, $r' = (1 + \text{A}^2)p'$,

which, as is plain from the above equation between r and p, represents an equal logarithmic spiral.

Ex. 2. To find the Evolute of the Curve whose Equation is

$$r^2 - p^2 = a^2:$$

whence, from (32)

$$p' = a;$$

and therefore the evolute is a circle whose radius is a, the given equation having been that to the involute of the circle.

CHAP. XII.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO THE DISCUSSION OF PROPERTIES OF CURVED SURFACES.

131.] In order the better to grasp the full meaning of the equations, and the results which we shall arrive at in this and the following chapter, it will be well to write down the general forms of the equations we shall employ, and to indicate the geometrical meaning of the constants and variables.

α) To discuss the Equation to a Straight Line in Space.

Let ξ, η, ζ be the current co-ordinates to the straight line; x, y, z, the co-ordinates to a point through which the line passes; λ, μ, ν, the direction angles of the line, that is, the angles between a parallel line through the origin and the co-ordinate axes; and let r be the distance between (x, y, z) and (ξ, η, ζ); then the equations to the line are

$$\frac{x-\xi}{\cos\lambda} = \frac{y-\eta}{\cos\mu} = \frac{z-\zeta}{\cos\nu} = r, \qquad (1)$$

the last of the equalities following by Preliminary Proposition III.

If, therefore, the equations to a line be given under the form

$$\frac{x-\xi}{\text{L}} = \frac{y-\eta}{\text{M}} = \frac{z-\zeta}{\text{N}}, \qquad (2)$$

each of these equalities is equal to

$$\frac{r}{\sqrt{(\text{L}^2+\text{M}^2+\text{N}^2)}};$$

and therefore

$$\cos\lambda = \frac{\text{L}}{(\text{L}^2+\text{M}^2+\text{N}^2)^{\frac{1}{2}}}, \qquad \cos\mu = \frac{\text{M}}{(\text{L}^2+\text{M}^2+\text{N}^2)^{\frac{1}{2}}},$$

$$\cos\nu = \frac{\text{N}}{(\text{L}^2+\text{M}^2+\text{N}^2)^{\frac{1}{2}}}, \tag{3}$$

and therefore L, M, N are proportional to the direction cosines of the line.

β) To discuss the Equation to a Plane.

DEF. A *plane* is the surface generated by a straight line moving round another straight line, at right angles to it.

Let the origin be at a point O on the straight line OQ (see fig. 57.), round which the generating line QP turns, and let λ, μ, ν be the direction angles of OQ; let x, y, z be the co-ordinates to any point P, on the line PQ, in any position, and let $\text{OP} = r$, $\text{OQ} = \delta$; then the direction cosines of OP are

$$\frac{x}{r}, \frac{y}{r}, \frac{z}{r};$$

and since OQP is a right angle,

$$\text{OQ} = \text{OP} \cos \text{POQ}.$$

$$\delta = r\left(\frac{x}{r}\cos\lambda + \frac{y}{r}\cos\mu + \frac{z}{r}\cos\nu\right);$$

$$\therefore \quad \cos\lambda . x + \cos\mu . y + \cos\nu . z = \delta: \tag{4}$$

and as this relation holds good for every point in QP, and in every position of QP, it is, according to our definition, the equation to a plane; λ, μ, ν being the direction angles of the normal to the plane, and δ the length of the perpendicular from the origin on the plane.

If, therefore, the equation to the plane be given in the form

$$\text{A}x + \text{B}y + \text{C}z = \text{D}, \tag{5}$$

on comparing this with (4), we have

$$\frac{\text{A}}{\cos\lambda} = \frac{\text{B}}{\cos\mu} = \frac{\text{C}}{\cos\nu} = \frac{\text{D}}{\delta} = (\text{A}^2 + \text{B}^2 + \text{C}^2)^{\frac{1}{2}},$$

Whence it appears that A, B, C, D are proportional respectively to the direction cosines of the normal to the plane, and to the length of the perpendicular on the plane from the origin; and we have

$$\left.\begin{aligned} \cos\lambda &= \frac{\text{A}}{(\text{A}^2 + \text{B}^2 + \text{C}^2)^{\frac{1}{2}}}, \\ \cos\mu &= \frac{\text{B}}{(\text{A}^2 + \text{B}^2 + \text{C}^2)^{\frac{1}{2}}}, \\ \cos\nu &= \frac{\text{C}}{(\text{A}^2 + \text{B}^2 + \text{C}^2)^{\frac{1}{2}}}, \\ \delta &= \frac{\text{D}}{(\text{A}^2 + \text{B}^2 + \text{C}^2)^{\frac{1}{2}}}. \end{aligned}\right\} \qquad (6)$$

132.] To find the Equation to a Tangent Plane to a Surface at a given Point.

Let the equation to the surface be

$$\text{F}(x, y, z) = 0. \qquad (7)$$

Our present object is to show that if a straight line be drawn through a point on the surface (x, y, z), and through a second point $(x + dx, y + dy, z + dz)$ infinitely near to it, the locus of such *tangent* lines is in general a plane, and is what is called the tangent plane. Of course, it is manifest that there are, in general, an infinity of points $(x + dx, y + dy, z + dz)$ contiguous to the first point, and therefore there are an infinity of tangent lines.

Let ξ, η, ζ be the current co-ordinates to one of the tangent lines, and x, y, z the co-ordinates to the point of contact on the surface, then the equations to the line are

$$\frac{x-\xi}{\mathrm{L}} = \frac{y-\eta}{\mathrm{M}} = \frac{z-\zeta}{\mathrm{N}}, \tag{8}$$

and, on account of the line passing through the point $(x+dx, y+dy, z+dz)$, we have

$$\frac{dx}{\mathrm{L}} = \frac{dy}{\mathrm{M}} = \frac{dz}{\mathrm{N}}; \tag{9}$$

$\therefore$ by division,

$$\frac{x-\xi}{dx} = \frac{y-\eta}{dy} = \frac{z-\zeta}{dz} = \frac{r}{ds}, \tag{10}$$

r being the distance between (x, y, z) and (ξ, η, ζ), and ds being equal to $(dx^2+dy^2+dz^2)^{\frac{1}{2}}$.

But the variations of x, y, z must be consistent with the equation to the surface, viz. with Equation (7), which gives us, if $\left(\frac{d\mathrm{F}}{dx}\right)$, $\left(\frac{d\mathrm{F}}{dy}\right)$, $\left(\frac{d\mathrm{F}}{dz}\right)$ do not all vanish at the point of contact,

$$\left(\frac{d\mathrm{F}}{dx}\right) dx + \left(\frac{d\mathrm{F}}{dy}\right) dy + \left(\frac{d\mathrm{F}}{dz}\right) dz = 0. \tag{11}$$

Multiplying the several terms of which equation by the terms of equality (10), we have

$$(x-\xi)\left(\frac{d\mathrm{F}}{dx}\right) + (y-\eta)\left(\frac{d\mathrm{F}}{dy}\right) + (z-\zeta)\left(\frac{d\mathrm{F}}{dz}\right) = 0; \tag{12}$$

which, being of the first degree, represents a plane, and, being the locus of the tangent lines to the surface at the point in question, represents the tangent plane.

133.] On comparing this equation with (5), and with the results of Equations (6), if α, β, γ be the direction angles of the normal to the plane, and p the perpendicular from the origin on the plane, we have

$$\left.\begin{aligned}
\cos\alpha &= \frac{\left(\frac{d\text{F}}{dx}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2+\left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}},\\
\cos\beta &= \frac{\left(\frac{d\text{F}}{dy}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2+\left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}},\\
\cos\gamma &= \frac{\left(\frac{d\text{F}}{dz}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2+\left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}},
\end{aligned}\right\} \tag{13}$$

$$p = \frac{x\left(\frac{d\text{F}}{dx}\right)+y\left(\frac{d\text{F}}{dy}\right)+z\left(\frac{d\text{F}}{dz}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2+\left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}}; \tag{14}$$

and if the equation to the surface $\text{F}(x, y, z) = c$ be a homogeneous function of n dimensions, then, since by the theorem proved in Art. 75.,

$$nc = x\left(\frac{d\text{F}}{dx}\right)+y\left(\frac{d\text{F}}{dy}\right)+z\left(\frac{d\text{F}}{dz}\right),$$

we have in this case

$$p = \frac{nc}{\left\{\left(\frac{d\text{F}}{dx}\right)^2+\left(\frac{d\text{F}}{dy}\right)^2+\left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}}, \tag{15}$$

and the equation to the tangent plane becomes

$$\xi\left(\frac{d\text{F}}{dx}\right)+\eta\left(\frac{d\text{F}}{dy}\right)+\zeta\left(\frac{d\text{F}}{dz}\right) = nc. \tag{16}$$

If the equation to the surface be given in the explicit form

$$z = f(x, y),$$

then $$\text{F}(x, y, z) = z - f(x, y) = 0,$$

and $$\left(\frac{d\text{F}}{dx}\right) = -\frac{d.f(x, y)}{dx} = -\left(\frac{dz}{dx}\right),$$

$$\left(\frac{d\text{F}}{dy}\right) = -\frac{d.f(x, y)}{dy} = -\left(\frac{dz}{dy}\right),$$

$$\left(\frac{d\text{F}}{dz}\right) = -\frac{dz}{dz} = 1;$$

whence the equation to the tangent plane becomes

$$z - \zeta = (x - \xi)\left(\frac{dz}{dx}\right) + (y - \eta)\left(\frac{dz}{dy}\right), \qquad (17)$$

and the values (13) and (14) must be modified accordingly.

134.] To find the Equation to the Normal of a Curved Surface.

Def. The *normal* is the straight line passing through the point of contact, and at right angles to the tangent plane.

Let ξ, η, ζ be the current co-ordinates to the normal, and x, y, z the co-ordinates to the point of contact; then, since the direction cosines of the normal are, by Equations (13), proportional to $\left(\frac{d\text{F}}{dx}\right)$, $\left(\frac{d\text{F}}{dy}\right)$, $\left(\frac{d\text{F}}{dz}\right)$, its equations are

$$\frac{x - \xi}{\left(\frac{d\text{F}}{dx}\right)} = \frac{y - \eta}{\left(\frac{d\text{F}}{dy}\right)} = \frac{z - \zeta}{\left(\frac{d\text{F}}{dz}\right)}; \qquad (18)$$

and if the equation to the surface be given under the explicit form, these expressions must be modified accordingly, and we shall have

$$\left.\begin{aligned} x - \xi &= -\left(\frac{dz}{dx}\right)(z - \zeta), \\ y - \eta &= -\left(\frac{dz}{dy}\right)(z - \zeta). \end{aligned}\right\} \qquad (19)$$

Hence, also, from (18), the equations to a line passing through the origin, and at right angles to the tangent plane, are

$$\frac{\xi}{\left(\frac{d\mathrm{F}}{dx}\right)} = \frac{\eta}{\left(\frac{d\mathrm{F}}{dy}\right)} = \frac{\zeta}{\left(\frac{d\mathrm{F}}{dz}\right)}. \tag{20}$$

Hence, if the problem be to find the locus of intersection with tangent plane of perpendiculars on it from the origin, we must eliminate x, y, z from Equations (7), (12), and (20), and the resulting relation between ξ, η, and ζ will represent the locus sought.

For examples illustrative of the preceding equations, see Gregory's *Collection*, chap. xiii.; and Moigno, *Leçons du Calcul Differentiel*, art. 184. leçon 32.

135.] If at the point on the surface at which the tangent lines are drawn,

$$\left(\frac{d\mathrm{F}}{dx}\right) = 0, \quad \left(\frac{d\mathrm{F}}{dy}\right) = 0, \quad \left(\frac{d\mathrm{F}}{dz}\right) = 0,$$

Equation (11) does not give a definite result, and there is no relation between dx, dy, and dz, besides (10), which being equivalent to only two equations, and with three unknown quantities involved, gives no determinate result. Hence we must seek for some other relation between dx, dy, dz arising out of the equation to the surface. Such we have, if all the second differential co-efficients do not vanish at the point in question, in the differential of (11), which is also the third term in the expansion of $\mathrm{F}(x+h, y+k, z+l)$; viz.

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right) dx^2 + \left(\frac{d^2\mathrm{F}}{dy^2}\right) dy^2 + \left(\frac{d^2\mathrm{F}}{dz^2}\right) dz^2 + 2\left(\frac{d^2\mathrm{F}}{dy\,dz}\right) dy\,dz$$
$$+ 2\left(\frac{d^2\mathrm{F}}{dz\,dx}\right) dz\,dx + 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right) dx\,dy = 0;$$

and, multiplying through by corresponding terms of equality (10), we have

$$\left(\frac{d^2\mathrm{F}}{dx^2}\right)(x-\xi)^2 + \left(\frac{d^2\mathrm{F}}{dy^2}\right)(y-\eta)^2 + \left(\frac{d^2\mathrm{F}}{dz^2}\right)(z-\zeta)^2$$
$$+ 2\left(\frac{d^2\mathrm{F}}{dy\,dz}\right)(y-\eta)(z-\zeta) + 2\left(\frac{d^2\mathrm{F}}{dz\,dx}\right)(z-\zeta)(x-\xi)$$
$$+ 2\left(\frac{d^2\mathrm{F}}{dx\,dy}\right)(x-\xi)(y-\eta) = 0, \tag{21}$$

an equation of the second degree, showing therefore that the locus of the tangent lines is not a plane, but a surface of the second order.

Changing the origin to the point under consideration, the equation assumes the form

$$\mathrm{A}\xi^2 + \mathrm{B}\eta^2 + \mathrm{C}\zeta^2 + 2\mathrm{D}\eta\zeta + 2\mathrm{E}\zeta\xi + 2\mathrm{F}\xi\eta = 0, \qquad (22)$$

which represents a cone of the second degree, the vertex being at the point of contact; and it may happen that the coefficients have such relations that the equation is decomposable into two factors of the first degree, in which case it will represent two planes.

If all the second differential co-efficients vanish, we must proceed to a third differentiation, and then, in a similar manner, we shall arrive at a cone of the third order.

For examples on these singular points in curved surfaces, see Gregory's *Examples*, chap. xiii. section 3.

CHAP. XIII.

ON THE APPLICATION OF THE DIFFERENTIAL CALCULUS TO THE DETERMINATION OF PROPERTIES OF CURVES IN SPACE.

136.] A *curve in space*, which is also called a curve of *double curvature*, for a reason which will hereafter appear, may be defined in two ways: either by the intersection of two surfaces whose equations involving x, y, z are given, and therefore by a combination of these equations; or, what amounts to the same, one of the variables, as e. g. z, may have been eliminated between these two equations involving x,y,z, and an equation obtained involving only x and y, which will be the equation to the projection of the curve on the plane of (xy), and so with the other variables, whereby three equations may be formed, each containing two variables, which will severally represent the projections of the curve on the co-ordinate planes, and any two of these equations will be sufficient to define the curve; and according as one or the other method is adopted, our formulæ will assume different shapes.

137.] To find the Equation to a *Tangent Line* to a Curve in Space.

DEF. A *tangent* is the straight line passing through two points on the curve which are infinitely near to each other.

Let ξ, η, ζ be the current co-ordinates to the tangent line, and first let the two points through which the line is to pass be at a finite distance (Δs) apart; and let their co-ordinates be

$$x, \; y, \; z, \qquad x+\Delta x, \; y+\Delta y, \; z+\Delta z;$$

then the equations to the line are

$$\frac{x-\xi}{\Delta x} = \frac{y-\eta}{\Delta y} = \frac{z-\zeta}{\Delta z} = \frac{r}{\Delta s}, \tag{1}$$

where r is the distance between (x,y,z) and (ξ,η,ζ).

And, when these two points become infinitely near to one another, the secant becomes a tangent, and its equations become

$$\frac{x-\xi}{dx} = \frac{y-\eta}{dy} = \frac{z-\zeta}{dz} = \frac{r}{ds}; \tag{2}$$

where $$ds = \sqrt{(dx^2+dy^2+dz^2)},$$
and is the differential of the arc of the curve.

On comparing these equations with those of α) in Art. 131. Equations (3), if λ, μ, ν be the direction angles of the tangent,

$$\cos\lambda = \frac{dx}{ds}, \ \cos\mu = \frac{dy}{ds}, \ \cos\nu = \frac{dz}{ds}. \tag{3}$$

If, then, the equations to the curve are two equations, say, of the forms

$$f(x,z) = 0, \qquad \phi(y,z) = 0,$$

$\frac{dx}{dz}$ and $\frac{dy}{dz}$ can be found by differentiation, and Equations (2) and (3) can be determined for the particular curve. But if the equations to two surfaces are given of the forms

$$F_1(x,y,z) = 0, \qquad F_2(x,y,z) = 0, \tag{4}$$

then, since
$$\left.\begin{aligned}\left(\frac{dF_1}{dx}\right)dx+\left(\frac{dF_1}{dy}\right)dy+\left(\frac{dF_1}{dz}\right)dz = 0,\\ \left(\frac{dF_2}{dx}\right)dx+\left(\frac{dF_2}{dy}\right)dy+\left(\frac{dF_2}{dz}\right)dz = 0,\end{aligned}\right\} \tag{5}$$

we have by elimination the following system of equations,

$$\frac{dx}{\left(\frac{dF_1}{dy}\right)\left(\frac{dF_2}{dz}\right)-\left(\frac{dF_1}{dz}\right)\left(\frac{dF_2}{dy}\right)} = \frac{dy}{\left(\frac{dF_1}{dz}\right)\left(\frac{dF_2}{dx}\right)-\left(\frac{dF_1}{dx}\right)\left(\frac{dF_2}{dz}\right)}$$
$$= \frac{dz}{\left(\frac{dF_1}{dx}\right)\left(\frac{dF_2}{dy}\right)-\left(\frac{dF_1}{dy}\right)\left(\frac{dF_2}{dx}\right)}; \tag{6}$$

whence, multiplying the several terms of equality (2) by the several terms of this equality, dx, dy, dz will divide out, and we shall have the equations to the tangent in terms of the partial differential coefficients of the intersecting surfaces. Similarly may the direction cosines in (3) be modified.

138.] To find the Equation to the Normal Plane to a Curve in Space.

Def. The plane perpendicular to the tangent line, and passing through the point of contact, is called the *normal plane.*

Let ξ, η, ζ be its current co-ordinates, and x, y, z be the point of contact through which it passes; then, since it is to be perpendicular to the line whose direction cosines are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, its equation is

$$(x-\xi)\,dx+(y-\eta)\,dy+(z-\zeta)\,dz = 0. \tag{7}$$

139.] To find the Equation to the Osculating Plane to a Curve in Space.

In curves, such as we have discussed in previous chapters, all the points lie in one plane, and therefore they are called *plane curves.* This property, however, does not hold good for all curves in space; although every three consecutive points must be in one plane, yet the fourth may be out of it; or, in other words, every two consecutive tangents are in the same plane, but the next consecutive tangent is, in general, in a different one; our object is to determine the equation to the plane which contains two consecutive tangents, and which is called the osculating plane, and is defined as follows.

Def. The *osculating plane* is the plane containing three consecutive points on a curve.

Let the equation to the plane be

$$\text{A}\xi + \text{B}\eta + \text{C}\zeta = \text{D}, \tag{8}$$

and let it pass through the points on the curve

$$x, y, z,$$

$$x + dx, y + dy, z + dz,$$

$$x + 2dx + d^2x, y + 2dy + d^2y, z + 2dz + d^2z.$$

Whence we have $$\text{A}x + \text{B}y + \text{C}z = \text{D}, \tag{9}$$

$$\text{A}dx + \text{B}dy + \text{C}dz = 0, \tag{10}$$

$$\text{A}d^2x + \text{B}d^2y + \text{C}d^2z = 0. \tag{11}$$

Whence, subtracting (8) from (9),

$$\text{A}(x-\xi) + \text{B}(y-\eta) + \text{C}(z-\zeta) = 0; \tag{12}$$

and eliminating successively between (10) and (11) we have the system

$$\frac{\text{A}}{dy\,d^2z - dz\,d^2y} = \frac{\text{B}}{dz\,d^2x - dx\,d^2z} = \frac{\text{C}}{dx\,d^2y - dy\,d^2x}; \tag{13}$$

whence, dividing (12) by the several terms of equality (13), we have

$$(dy\,d^2z - dz\,d^2y)(x-\xi) + (dz\,d^2x - dx\,d^2z)(y-\eta)$$
$$+ (dx\,d^2y - dy\,d^2x)(z-\zeta) = 0, \tag{14}$$

which is the equation to the osculating plane.

The method by which we have deduced this equation is the same as if we had defined the osculating plane to be that in which two consecutive tangents lie, as will be apparent from what follows.

Let the equation to the plane passing through (x, y, z) be

$$\text{A}(x-\xi) + \text{B}(y-\eta) + \text{C}(z-\zeta) = 0;$$

and since it is to be that in which two consecutive tangents lie, whose direction cosines are respectively

$$\frac{dx}{ds}, \qquad \frac{dy}{ds}, \qquad \frac{dz}{ds},$$

$$\frac{dx + d^2x}{ds + d^2s}, \qquad \frac{dy + d^2y}{ds + d^2s}, \qquad \frac{dz + d^2z}{ds + d^2s},$$

we must have the conditions

$$\text{A}dx + \text{B}dy + \text{C}dz = 0,$$

$$\text{A}(dx + d^2x) + \text{B}(dy + d^2y) + \text{C}(dz + d^2z) = 0;$$

whence, by subtraction,

$$\mathrm{A}\,d^2x + \mathrm{B}\,d^2y + \mathrm{C}\,d^2z = 0;$$

which two relations between A, B, C are the same as those above marked (10) and (11), whence equality (13) follows, and therefore the equation to the osculating plane is the same.

Hence, if l, m, n are the direction angles of the normal to this plane, we have

$$\frac{\cos l}{dy\,d^2z - dz\,d^2y} = \frac{\cos m}{dz\,d^2x - dx\,d^2z} = \frac{\cos n}{dx\,d^2y - dy\,d^2x}$$

$$= \frac{1}{\{(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2\}^{\frac{1}{2}}}, \tag{15}$$

The denominator of which last expression may be modified as follows:

$$(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2$$

$$= (dx^2 + dy^2 + dz^2)\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2\}$$

$$- (dx\,d^2x + dy\,d^2y + dz\,d^2z)^2: \tag{16}$$

but since

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{17}$$

$$ds\,d^2s = dx\,d^2x + dy\,d^2y + dz\,d^2z, \tag{18}$$

and therefore the right-hand member of (16) becomes

$$ds^2\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}, \tag{19}$$

and, if s be taken to be an independent variable,

$$ds^2\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2\}; \tag{20}$$

whence

$$\frac{\cos l}{dy\,d^2z - dz\,d^2y} = \frac{\cos m}{dz\,d^2x - dx\,d^2z} = \frac{\cos n}{dx\,d^2y - dy\,d^2x}$$

$$= \frac{1}{ds\,\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}^{\frac{1}{2}}}. \tag{21}$$

140.] If z be considered an independent variable so that $d^2 z = 0$, then the equation to the osculating plane becomes, dividing through by dz^3,

$$-\frac{d^2 y}{dz^2}(x-\xi)+\frac{d^2 x}{dz^2}(y-\eta)+\left(\frac{dx}{dz}\frac{d^2 y}{dz^2}-\frac{dy}{dz}\frac{d^2 x}{dz^2}\right)(z-\zeta)=0. \tag{22}$$

And similarly will the Equation (14) become modified, if any other variable be considered independent.

141.] In connexion with the subject of the osculating plane we proceed to determine the analytical conditions that the curvature of a curve may lie entirely in one plane; or, in other words, that every four consecutive points on the curve may be in one plane.

Let the equation to the plane be

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}z=\mathrm{D}.$$

Then

$$\left.\begin{aligned}\mathrm{A}dx+\mathrm{B}dy+\mathrm{C}dz&=0.\\ \mathrm{A}d^2x+\mathrm{B}d^2y+\mathrm{C}d^2z&=0.\\ \mathrm{A}d^3x+\mathrm{B}d^3y+\mathrm{C}d^3z&=0.\end{aligned}\right\} \tag{23}$$

Whence, from the last three equations, by cross-multiplication,

$$dx(d^2y\,d^3z-d^2z\,d^3y)+dy(d^2z\,d^3x-d^2x\,d^3z)$$
$$+\,dz(d^2x\,d^3y-d^2y\,d^3x)=0; \tag{24}$$

which condition becomes, if z be taken to be an independent variable,

$$\frac{d^2x}{dz^2}\frac{d^3y}{dz^3}-\frac{d^2y}{dz^2}\frac{d^3x}{dz^3}=0. \tag{25}$$

For examples of the equations deduced in this chapter, see Gregory's *Collection*, chap. xiii.

CHAP. XIV.

ON CONTACT OF CURVES, AND ENVELOPES.

We proceed in this chapter to discuss generally two somewhat kindred subjects: those of contact of curves, and of envelopes; specific forms of which have arisen in previous chapters, and which now it is our object to generalise.

A. *Of the Theory of Contact of Plane Curves.*

142.] Having in Article 94. defined a tangent line to be that which passes through two consecutive points on a curve, it appears that there are two points common to the tangent and the curve; and, from what was said in Article 121., it appears that the circle of curvature, whose radius and the co-ordinates to whose centre are determined respectively by the Equations (3) and (12) of Chap. XI., has three points common to itself and the curve. This property of curves having consecutive points in common, or, as it is called, having *contact*, it is our object to generalise, and with reference to it we define as follows.

Def. Curves which have *two* consecutive points in common have contact of the *first* order; those which have *three* consecutive points in common have contact of the *second* order; those which have *four* in common, of the *third* order; and, similarly, those which have $(n+1)$ consecutive points in common have contact of the nth order.

Curves which possess these relative properties are also called *osculating curves*, and curves are said to *osculate* to each other.

Nothing is said as to curves having only *one* point in common, because such a condition implies no more than that they intersect, but does not enable us to determine the relative direction of the curves.

Hence, then, it appears that if for two curves whose equations are

$$y = f(x), \qquad \eta = \phi(\xi),$$

we have the series of common points indicated in the following table, viz.

$$\left.\begin{matrix}(x, y)\\(\xi, \eta)\end{matrix}\right\},$$ the two curves intersect;

$$\left.\begin{matrix}(x, y), (x+dx, y+dy)\\(\xi, \eta), (\xi+d\xi, \eta+d\eta)\end{matrix}\right\},$$ the two curves have contact of the first order;

$$\left.\begin{matrix}(x, y), (x+dx, y+dy), (x+2dx+d^2x, y+2dy+d^2y)\\(\xi, \eta), (\xi+d\xi, \eta+d\eta), (\xi+2d\xi+d^2\xi, \eta+2d\eta+d^2\eta)\end{matrix}\right\},$$ there is contact of the second order;

and similarly for contact of the third order, and of the nth order, for which it is necessary that the successive differentials up to the nth should be equal in both curves.

These conditions become greatly simplified if we consider x and ξ to increase by equal augments, in which case the several differentials of x and ξ, after the first, vanish, and we have, if

$$\left.\begin{matrix}\xi = x\\\eta = y\end{matrix}\right\},$$ intersection of the curves;

$$\left.\begin{matrix}\xi = x\\\eta = y\end{matrix}\right\} \text{ and } \frac{d\eta}{d\xi} = \frac{dy}{dx},$$ contact of the first order;

$$\left.\begin{matrix}\xi = x\\\eta = y\end{matrix}\right\}, \frac{d\eta}{d\xi} = \frac{dy}{dx}, \text{ and } \frac{d^2\eta}{d\xi^2} = \frac{d^2y}{dx^2},$$ contact of the second order;

and if, besides, all the several successive differential coefficients, up to the nth inclusively, are equal in both curves, there is contact of the nth order.

Hence, if two curves have contact of the first order, they have a point in common, and the same tangent at the point; and, therefore, the tangent has contact of the first order. And if two curves have contact of the second order, they have not only a common tangent at the common point, but the curvature is the same, and is turned in the same direction; that is, they have the same circle of curvature.

143.] Considering, then, contact to depend on the number of the successively derived functions which are equal in the equations to the curves, we are led to the following mode of viewing the subject; from which several important properties may be deduced.

Let $$y = f(x), \qquad \eta = \phi(\xi), \tag{1}$$

be the equations to the curves, at the common point $x=\xi$, $y=\eta$, and let y' and η' be the ordinates corresponding to the abscissa $x+h$; then

$$y' = f(x) + \frac{h}{1}f'(x) + \frac{h^2}{1.2}f''(x) + \ldots + \frac{h^n}{1.2\ldots.n}f^n(x+\theta h), \tag{2}$$

$$\eta' = \phi(x) + \frac{h}{1}\phi'(x) + \frac{h^2}{1.2}\phi''(x) + \ldots + \frac{h^n}{1.2\ldots.n}\phi^n(x+\theta h). \tag{3}$$

Then, if the contact be of the first order,

$$f(x) = \phi(x), \quad f'(x) = \phi'(x), \text{ and}$$

$$y' - \eta' = \frac{h^2}{1.2}\{f''(x+\theta h) - \phi''(x+\theta h)\}; \tag{4}$$

that is, the distance between the ordinates corresponding to $x+h$ is an infinitesimal of the second order; and no other line can pass between these two curves, unless it has with each of them a contact of at least the first order. For, suppose it were possible that the ordinate y_1 corresponding to the abscissa $x+h$ of the curve $y = \mathrm{F}(x)$ is such that $\mathrm{F}'(x)$ is not equal to $f'(x)$, then

$$y' - y_1 = \frac{h}{1}\{f'(x+\theta h) - \mathrm{F}'(x+\theta h)\}: \tag{5}$$

which difference is obviously greater than that given in (4), if h is very small; and, therefore, the curve $y = \mathrm{F}(x)$ does not come between the curves $y = f(x)$ and $y = \phi(x)$.

If the contact be of the second order, then, besides the former conditions,

$$f''(x) = \phi''(x),$$

and, subtracting (3) from (2), we have

$$y' - \eta' = \frac{h^3}{1.2.3}\{f'''(x+\theta h) - \phi'''(x+\theta h)\}; \tag{6}$$

that is, the difference between the ordinates corresponding to the abscissa $x+h$ is an infinitesimal of the third order: but if there is another curve, $y = \text{F}(x)$, such that $\text{F}''(x)$ is not equal to $f''(x)$, although $\text{F}'(x) = f'(x)$; then, if y_1 be the ordinate of this third curve corresponding to the abscissa $x+h$,

$$y' - y_1 = \frac{h^2}{1.2}\{f''(x+\theta h) - \text{F}''(x+\theta h)\}; \qquad (7)$$

which difference, being an infinitesimal of the second order, is obviously greater than that given by Equation (6); and, therefore, this third curve does not come between the first two curves. Similarly, if the contact between two curves be of the nth order, the difference of the abscissæ corresponding to $x+h$, when h is very small, is an infinitesimal of the $(n+1)$th order. Hence we have the following theorems.

"Two curves which have contact of the nth order are infinitely nearer to one another than two curves which have contact of an order lower than the nth."

"A curve which has contact of the nth order cannot come between two curves which have contact of an order higher than the nth."

144.] An inspection of the Equations (4) and (6) above, and of other equations formed in a similar manner, and giving the difference between the ordinates y' and η', corresponding to the several orders of contact, leads to the following theorem.

"If two curves have contact of an *odd* order, they touch, and do not intersect; but, if the contact be of an *even* order, the curves touch and intersect."

For, suppose the contact to be of the nth order,

$$y' - \eta' = \frac{h^{n+1}}{1.2.3\ldots(n+1)},\{f^{n+1}(x+\theta h) - \phi^{n+1}(x+\theta h)\}. \quad (8)$$

Then, if n be *even*, $y' - \eta'$ changes its sign as h changes sign; and, therefore, $f(x-h) - \phi(x-h)$ and $f(x+h) - \phi(x+h)$ have different signs, and therefore the curves intersect at the point of contact. But, if the contact be of an *odd* order, n is odd, and $n+1$ is even, and $y' - \eta'$ does not change sign with h; that is, the curve which was nearer to the axis of x before contact is nearer to it afterwards, and the curves do not intersect.

145.] Suppose it is required to determine the order of contact which a curve may have with a given curve. The order manifestly depends on the number of arbitrary constants involved in the equation to the curve. One constant will be determined by the condition that there be a point common to both; a second, that there be a second point in common to both, that is, that the first derived function be the same to both curves; a third, that there be three points in common, that is, in addition to the former conditions, that the second derived function be the same to both curves: and so, if the nth derived function be the same to both, there must be $(n+1)$ points in common, which will, in general, require $(n+1)$ arbitrary constants to be involved in the equation to the curve.

Hence, if an equation involve $(n+1)$ arbitrary constants, it may, in general, be made to pass through $(n+1)$ points of a given curve, which points may be brought infinitely near to each other; and thus there may be contact of the nth order. The following are examples in illustration of what has been said in this Article.

Ex. 1. To determine the Order of Contact which a Straight Line may have with a Curve,

Let the equation to the line be $\eta = \alpha \xi + b$,

and the equation to the curve $y = f(x)$;

then, as *two* arbitrary constants, α and b, are involved in the equation to the line, there can in general be contact of only the first order. By differentiation,

$$\frac{d\eta}{d\xi} = \alpha;$$

and, therefore, α is to be replaced by $\frac{dy}{dx}$, and the constant b is determined by one point being common to both; viz.

$$\eta = \alpha \xi + b,$$

$$y = \alpha x + b;$$

$$\therefore \quad \eta - y = \alpha(\xi - x)$$

$$= \frac{dy}{dx}(\xi - x);$$

which is the equation to the tangent to the curve, which line, therefore, has contact of the first order.

If at the point where the two curves meet, $\frac{d^2y}{dx^2} = 0$, since $\frac{d^2\eta}{d\xi^2} = 0$, there will be contact of the second order: hence, at a point of inflexion the tangent has contact of the second order with the curve. Similarly, if all the derived functions of $f(x)$, up to the nth inclusively, vanish at the point where the line meets the curve, then the contact is of the nth order.

Ex. 2. To determine the Order of Contact which a Circle may have with a Curve,

Let the equation to the curve be $y = f(x)$,
and the equation to the circle

$$(\xi - \alpha)^2 + (\eta - \beta)^2 = \rho^2 ; \tag{9}$$

taking x, and ξ which is equal to it, to increase by equal increments, we have

$$(\xi-\alpha)+(\eta-\beta)\frac{d\eta}{d\xi} = 0; \tag{10}$$

$$1+(\eta-\beta)\frac{d^2\eta}{d\xi^2} + \frac{d\eta^2}{d\xi^2} = 0. \tag{11}$$

Whence $$\eta - \beta = -\frac{1+\frac{d\eta^2}{d\xi^2}}{\frac{d^2\eta}{d\xi^2}}, \quad \xi - \alpha = \frac{1+\frac{d\eta^2}{d\xi^2}}{\frac{d^2\eta}{d\xi^2}}\frac{d\eta}{d\xi}. \tag{12}$$

And as three arbitrary constants, α, β, ρ, are involved in the equation to the circle, there may be contact of the second order; and therefore we may write, for

$$\xi, \ \eta, \ \frac{d\eta}{d\xi}, \ \frac{d^2\eta}{d\xi^2},$$

$$x, \ y, \ \frac{dy}{dx}, \ \frac{d^2y}{dx^2}.$$

Whence we have

$$\beta = y + \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}, \quad \alpha = x - \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}} \frac{dy}{dx}. \tag{13}$$

$$\rho = \pm \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}; \tag{14}$$

which are severally the co-ordinates to the centre and the radius of the circle which has contact of the second order with the given curve. In comparing these expressions with those in Chap. XI. marked (4) (13), it will be seen that they are the same as those there determined for the co-ordinates of the centre and radius of curvature, when x is considered an independent variable; and, therefore, the osculating circle, or that which has contact of the second order, is identical with the circle of which we spoke in Art. 121., and called the circle of curvature; and, of course, all the properties which we there proved to belong to the circle of curvature belong to the osculating circle. And if we had not taken x to be an independent variable, but had calculated the complete differentials of (9) and (10), we should have arrived at expressions the same as those marked (15), (16), and (17) in Art. 121.

The contact between a circle and a curve is of the third order whenever the radius of curvature is a maximum or a mininum; that is, whenever $d\rho = 0$. Since we have the following expression for the radius of curvature,

$$\rho^2 = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^3}{\left(\frac{d^2y}{dx^2}\right)^2};$$

$$\therefore \rho d\rho = 0$$

$$= \frac{\left(1 + \frac{dy^2}{dx^2}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^3} \left\{ 3 \frac{dy}{dx} \left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \frac{dy^2}{dx^2}\right) \frac{d^3y}{dx^3} \right\},$$

$$\therefore \quad 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \frac{dy^2}{dx^2}\right)\frac{d^3y}{dx^3} = 0; \qquad (15)$$

and, on differentiating the last equation of (12), we have

$$3\frac{d\eta}{d\xi}\left(\frac{d^2\eta}{d\xi^2}\right)^2 - \left(1 + \frac{d\eta^2}{d\xi^2}\right)\frac{d^3\eta}{d\xi^3} = 0; \qquad (16)$$

that is, (15) and (16) are identical when the several differential coefficients of the two curves are the same up to the third; that is, the contact is of the third order, if the radius of curvature be a maximum or minimum.

Ex. 3. Hence, also, we conclude, that, as the complete equation of the second degree is

$$A\eta^2 + B\xi\eta + C\xi^2 + D\eta + E\xi + F = 0,$$

which involves apparently six constants, but of which only *five* are arbitrary, as one involves the standard of comparison, a central curve of the second degree, with all its terms complete, may have contact of the fourth order with a given curve; but as, in the case of the parabola, a relation is given amongst the constants, and there are only four arbitrary ones, a parabola can have contact of only the third order. Similarly, with regard to the ellipse, if certain conditions are given, the number of disposable constants becomes diminished, and the order of contact is lowered; as for instance, an ellipse with its major axis parallel to the axis of x has for its equation

$$\frac{(\xi-\alpha)^2}{a^2} + \frac{(\eta-\beta)^2}{b^2} = 1,$$

with four arbitrary constants, α, β, a, b; and therefore there can be contact of only the third order. Similarly, if in addition the ellipse is to be of a given eccentricity, then a relation is given between a and b, and we have only three arbitrary constants, and there can be contact of only the second order.

The following problem is proposed for solution.

To determine the co-ordinates to the vertex and the latus rectum of a parabola, whose principal axis is parallel to the axis of x, which has contact of the second order with the curve

$$y = f(x);$$

and the locus of the vertex as the osculating parabola moves round a given ellipse.

The theory of contact of curves in space is precisely similar to that which we have here developed, and therefore it is not worth while to enter into it at length, as it would require more space than we can afford to give to it.

B. *Of the Theory of Envelopes.*

146.] In the discussion of the relative properties of evolutes and involutes, in Chap. XI. Art. 123. and 124., it was proved that the normal to the involute is a tangent to the evolute; or, in other words, that as each normal to the involute passes through two consecutive points of the evolute, the latter curve may be conceived to be made up of an infinite number of infinitely short straight lines, each of which is a part of a normal to the involute; thus we say that the evolute is formed by the intersection of consecutive normals, and is called their *envelope*. This property of a curve being generated by the ultimate intersections of a series of lines or curves drawn after some known law, we proceed to generalise and discuss.

Let the equation to the family of curves, of which it is our object to determine the envelope, be

$$\mathrm{F}(x, y, \alpha) = 0, \tag{17}$$

in which α is some variable parameter, so that for every value of α we have some particular curve: but, if we make α to vary continuously, we shall have a series of curves, the position of each one differing but little from that of the next. Suppose, then, that α receives a variation $d\alpha$, then the two curves whose equations are (17) and

$$\mathrm{F}(x, y, \alpha + d\alpha) = 0 \tag{18}$$

are in position infinitely near to another; but owing to the variation of α they will, in general, intersect in some point, which will be determined by x and y being the same in both (17) and (18), and which will be a point on the envelope. If, therefore, we eliminate α between (17) and (18), the resulting equation will involve only x and y, and will be the equation to the envelope. Before we proceed to apply the method, we may put (18) under a more convenient form. Since, by Equation (7) Art. 51.,

$$\text{F}(x, y, \alpha + d\alpha) = \text{F}(x, y, \alpha) + d\alpha\, \text{F}'(x, y, \alpha + \theta d\alpha) = 0,$$

subtracting (17), we have

$$\text{F}'(x, y, \alpha + \theta d\alpha) = 0;$$

and in the limit, when $d\alpha$ becomes infinitely small,

$$\text{F}'(x, y, \alpha) = 0. \qquad (19)$$

Hence we have the following rule to determine the envelope of a family of curves, such as that represented by (17), where α is the variable parameter. "Eliminate the variable parameter between the equation to the family of curves, and the derived of that equation with reference to the parameter."

Ex. 1. To determine the Equation to the Curve formed by the Intersection of the Straight Lines whose Equation is

$$y = \alpha x + \frac{m}{\alpha},$$

where α varies.

Differentiate with respect to α, x and y being constant.

$$0 = x - \frac{m}{\alpha^2};$$

$$\therefore \quad \alpha = \pm \sqrt{\frac{m}{x}},$$

$$\therefore \quad y = \pm \sqrt{(mx)} \pm \sqrt{(mx)}$$

$$= \pm 2 \sqrt{(mx)},$$

$$\therefore \quad y^2 = 4mx,$$

which is the equation to a parabola.

It will easily be seen from this example and from the next, that the problem of determining envelopes produced by the intersection of straight lines is the inverse one to that of finding the equation to a tangent to a curve. In this case, the general equation to the tangent is given, and our object is to determine the curve of which it is the tangent; in the other, the curve is given, and we have to determine the equation to the line passing through any two consecutive points on it. Thus the given equation to the line above expresses the following geometrical property. From a point in the axis of x, at a distance m from the origin, lines are drawn, cutting the axis of y; and at the point of intersection other lines are drawn at right angles to them, to find the curve to which these latter lines are tangents.

Ex. 2. Straight Lines are drawn, such that the perpendicular Distance upon them, from a given point, is constant; to find their Envelope.

Take the given point as the origin, and let k be the given perpendicular distance. Then, if α be the tangent of the angle between any one line (which we take as the type of all) and the axis of x, the equation to the line is

$$y = \alpha x + k\sqrt{(1+\alpha^2)};$$

$$\therefore \quad 0 = x + \frac{k\alpha}{\sqrt{(1+\alpha^2)}},$$

$$\alpha = \frac{\pm x}{\sqrt{(k^2-x^2)}},$$

$$\therefore \quad y = \frac{\pm x^2 + k^2}{\sqrt{(k^2-x^2)}} = \sqrt{(k^2-x^2)};$$

$$\therefore \quad x^2 + y^2 = k^2,$$

the equation to a circle, which the envelope manifestly ought to be.

147.] If the equation representing the family of curves involve several, say n, variable parameters, and these parameters are related by $(n-1)$ other and independent equations, which conditions are equivalent to there being only *one* variable parameter, instead of eliminating $(n-1)$ parameters, and then differentiating with respect to the remaining one, and proceeding as in the last article, the following method is more elegant.

Let the equation to the family of curves be

$$\mathrm{F}(x, y, \alpha_1, \alpha_2, \alpha_3 \; . \; . \; . \; \alpha_n) = 0, \tag{20}$$

and let the $(n-1)$ equations of condition be

$$\left.\begin{aligned} f_1(\alpha_1, \alpha_2 \;.\;.\;.\;.\; \alpha_n) &= 0, \\ f_2(\alpha_1, \alpha_2 \;.\;.\;.\;.\; \alpha_n) &= 0, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\; & \\ f_{(n-1)}(\alpha_1, \alpha_2 \;.\;.\;.\; \alpha_n) &= 0. \end{aligned}\right\} \qquad (21)$$

Conceive $\alpha_1, \alpha_2, \alpha_3 \;.\;.\;.\; \alpha_n$ to vary by very small variations, by which means the above equations become

$$\left(\frac{d\mathrm{F}}{d\alpha_1}\right) d\alpha_1 + \left(\frac{d\mathrm{F}}{d\alpha_2}\right) d\alpha_2 + \;.\;.\;.\; + \left(\frac{d\mathrm{F}}{d\alpha_n}\right) d\alpha_n = 0, \qquad (22)$$

$$\left.\begin{aligned} \left(\frac{df_1}{d\alpha_1}\right) d\alpha_1 + \left(\frac{df_1}{d\alpha_2}\right) d\alpha_2 + \;.\;.\;.\; + \left(\frac{df_1}{d\alpha_n}\right) d\alpha_n &= 0, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\; & \\ \left(\frac{df_{(n-1)}}{d\alpha_1}\right) d\alpha_1 + \left(\frac{df_{(n-1)}}{d\alpha_2}\right) d\alpha_2 + \,.\, + \left(\frac{df_{(n-1)}}{d\alpha_n}\right) d\alpha_n &= 0. \end{aligned}\right\} (23)$$

Then, if x and y be the same in (20) and (22), they refer to the point in the envelope where the two particular curves of the family intersect; and, therefore, if the several variable parameters can be eliminated, the resulting equation between x and y will represent the locus of the points of intersection, which will be the envelope required.

From Equations (20), (21), (22), (23) we have $2n$ different relations, from which $(2n-1)$ quantities, viz. $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$, $\frac{d\alpha_1}{d\alpha_n} \;.\;.\;.\; \frac{d\alpha_{n-1}}{d\alpha_n}$, are to be eliminated; which is, of course, a possible problem. To solve it, multiply the $(n-1)$ equations in (23) by $(n-1)$ indeterminate multipliers $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_{n-1}$, and add all and (22) together; by which means there will be n conditions involved in one equation; and, therefore, we are at liberty to make n new conditions. Let these be that the coefficients of the differentials, viz. $d\alpha_1, d\alpha_2 \ldots d\alpha_n$, be equal to zero; whence we have

$$\left.\begin{array}{l}
\left(\frac{d\mathrm{F}}{d\alpha_1}\right) + \lambda_1\left(\frac{df_1}{d\alpha_1}\right) + \lambda_2\left(\frac{df_2}{d\alpha_1}\right) + . + \lambda_{n-1}\left(\frac{df_{n-1}}{d\alpha_1}\right) = 0, \\
\left(\frac{d\mathrm{F}}{d\alpha_2}\right) + \lambda_1\left(\frac{df_1}{d\alpha_2}\right) + \lambda_2\left(\frac{df_2}{d\alpha_2}\right) + . + \lambda_{n-1}\left(\frac{df_{n-1}}{d\alpha_2}\right) = 0, \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\left(\frac{d\mathrm{F}}{d\alpha_n}\right) + \lambda_1\left(\frac{df_1}{d\alpha_n}\right) + \lambda_2\left(\frac{df_2}{d\alpha_n}\right) + . + \lambda_{n-1}\left(\frac{df_{n-1}}{d\alpha_n}\right) = 0;
\end{array}\right\} \quad (24)$$

and between the Equations (20), (21), and (24), which are $2n$ in number, we may eliminate the $(2n-1)$ quantities, $\alpha_1, \alpha_2 \ldots \alpha_n, \lambda_1, \lambda_2 \ldots\ldots \lambda_{n-1}$, and ultimately arrive at an equation between x and y only, which will be the required envelope to the family of curves.

Ex. 1. A Straight Line of given length slides down between two Rectangular Axes; to find the Envelope of the Line in all Positions.

Let k = the length of the line; a and b the intercepts of the line on the axes of x and y respectively, then the equation to the line is

$$\frac{x}{a} + \frac{y}{b} = 1. \qquad (25)$$

Also we have $$a^2 + b^2 = k^2. \qquad (26)$$

Differentiating (25) and (26), making a and b to vary, we have

$$\frac{x}{a^2}\,da + \frac{y}{b^2}\,db = 0,$$

$$a\,da + b\,db = 0;$$

and multiplying the second of these by an indeterminate multiplier λ, and adding,

$$\left(\frac{x}{a^2} + \lambda a\right) da + \left(\frac{y}{b^2} + \lambda b\right) db = 0.$$

Let $\left.\begin{aligned} \frac{x}{a^2} + \lambda a = 0, \\ \frac{y}{b^2} + \lambda b = 0, \end{aligned}\right\} \qquad \therefore \quad \left\{\begin{aligned} \frac{x}{a} + \lambda a^2 = 0, \\ \frac{y}{b} + \lambda b^2 = 0; \end{aligned}\right.$

$$\therefore \quad \text{by addition,} \quad 1 + \lambda k^2 = 0,$$

$$\therefore \quad \lambda = -\frac{1}{k^2},$$

$$\therefore \quad \frac{x}{a^2} = \frac{a}{k^2}, \qquad \therefore \quad a = x^{\frac{1}{3}} k^{\frac{2}{3}},$$

$$\frac{y}{b^2} = \frac{b}{k^2}, \qquad \therefore \quad b = y^{\frac{1}{3}} k^{\frac{2}{3}};$$

$$\therefore \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}}.$$

Ex. 2. The Equation to the Evolute of a Curve may be found by the method of determining envelopes, if we consider it to be the curve formed by the intersection of consecutive normals.

Let $y = f(x)$ be the equation to the curve; then the equation to the normal is, Art. 98. Equation (7),

$$(\eta - y)\frac{dy}{dx} + (\xi - x) = 0. \tag{27}$$

Differentiating, considering η and ξ the current co-ordinates to the normal,

$$-(\eta - y)\frac{d^2y}{dx^2} + 1 + \frac{dy^2}{dx^2} = 0. \tag{28}$$

Whence, eliminating x and y between (27), (28), and the equation to the curve, the resulting equation will contain only η and ξ, and will represent the envelope of the normals. It is easily seen that this method is identical with that explained in Art. 120.

Let us apply the method to determine the Evolute of the Ellipse.

$$\text{F}(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{29}$$

Then, the equation to the normal being

$$\frac{\xi - x}{\left(\frac{d\mathrm{F}}{dx}\right)} = \frac{\eta - y}{\left(\frac{d\mathrm{F}}{dy}\right)},$$

we have, in this case, $$\frac{a^2\xi}{x} - \frac{b^2\eta}{y} = a^2 - b^2. \qquad (30)$$

$\therefore$ differentiating (29) and (30), with respect to x and y,

$$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy = 0,$$

$$\frac{a^2\xi}{x^2}\,dx - \frac{b^2\eta}{y^2}\,dy = 0.$$

Using an indeterminate multiplier, and equating the coefficients of the differentials to 0,

$$\left.\begin{aligned} \frac{x}{a^2} + \lambda\frac{a^2\xi}{x^2} &= 0, \\ \frac{y}{b^2} - \lambda\frac{b^2\eta}{y^2} &= 0. \end{aligned}\right\}$$

Multiplying by x and y, and adding, by conditions (29) and (30) we have

$$1 + \lambda(a^2 - b^2) = 0;$$

$$\therefore \quad \lambda = -\frac{1}{a^2 - b^2};$$

$$\therefore \quad \frac{x^3}{a^3} = \frac{a\xi}{a^2 - b^2},$$

$$\frac{y^3}{b^3} = -\frac{b\eta}{a^2 - b^2};$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 = \frac{(a\xi)^{\frac{2}{3}} + (b\eta)^{\frac{2}{3}}}{(a^2 - b^2)^{\frac{2}{3}}};$$

$$\therefore \quad (a\xi)^{\frac{2}{3}} + b\eta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

which is the equation to the evolute. The manner in which the curve is generated is indicated in fig. 51.

Ex. 3. To determine the Equation to the Surface formed by the continued Intersection of Planes whose Equations are represented by

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

subject to the condition $abc = k^3$.

Differentiating the first of these, and taking the logarithmic differential of the second, we have

$$\frac{x}{a^2}\,da + \frac{y}{b^2}\,db + \frac{z}{c^2}\,dc = 0,$$

$$\frac{da}{a} + \frac{db}{b} + \frac{dc}{c} = 0.$$

Hence, using an indeterminate multiplier, adding and equating to zero the coefficients of the differentials,

$$\left.\begin{aligned} \frac{x}{a^2} + \frac{\lambda}{a} &= 0, \\ \frac{y}{b^2} + \frac{\lambda}{b} &= 0, \\ \frac{z}{c^2} + \frac{\lambda}{c} &= 0; \end{aligned}\right\} \quad \therefore \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = -\lambda = \frac{1}{3},$$

by the first of the equations.

$$\therefore \left.\begin{aligned} a &= 3x, \\ b &= 3y, \\ c &= 3z, \end{aligned}\right\} \qquad 27xyz = k^3,$$

which is the equation to the surface.

For other examples see Gregory's *Collection*, chap. xiv., and Hind's *Series*, chap. xv. Ex. 169—210.

CHAP. XV.

ON CURVATURE OF CURVES IN SPACE, AND OF CURVED SURFACES.

It is proposed in the following Articles to apply to curves in space, and to curved surfaces, the principles on which was founded the discussion of the kindred affection of plane curves in Chap. XI., and to deduce some of the simplest properties. It is not, however, our intention to enter deeply into the subject, as it would require more space than we are able to give to it; and, therefore, what follows is not to be considered a scientific treatise, but an attempt at proving some propositions which most frequently occur, and lie on the surface of the subject. For further information the reader is desired to consult Moigno's *Calcul Differentiel*, and a very elegant paper on curves in space, by M. de Saint Venant, xxxme cahier de *Journal de l'Ecole Polytechnique,* in which he will find most of the abstruse formulæ connected with the subject interpreted geometrically.

A. *Curves in Space, or Curves of double Curvature.*

148.] On referring to Art. 137. and 139. it will be seen that there are two affections of a curve in space which it is necessary for us to discuss; one arising out of the inclination to each other of two consecutive tangents, the other out of the inclination of two consecutive osculating planes. As to the first, two consecutive elements of the curve may be inclined to each other at a finite and determinable angle, on the relation of which to the element of the curve at the point in question will depend (see Art. 118.) the radius of curvature and the amount of curvature; for what was said in Art. 118. is immediately applicable to this case. The two consecutive elements lie in one plane, viz. the osculating plane; and, therefore, if normals to the curve are drawn in this plane they will meet, and the distance from the curve at which they intersect will be the radius of curvature of the curve at that point. On referring, also, to what has been said in Art. 121. on the circle of curvature, and in Art. 145. on the osculating circle, it will be seen that a circle may be drawn in the osculating plane passing through three consecutive points on the curve, and will be a definite circle, and the radius of it will be the radius of curvature at the point. We shall in the following Articles determine its length, which is called *the radius of absolute curvature*, in both these ways. As to the second affection, the position of the osculating plane in general changes as the point under consideration moves along the curve; for, if it did not change, every *four* consecutive points on the curve would be in the same plane, and the curve would be a plane curve. On the *rate*, then, at which, as we move along the curve, the position of the osculating plane changes, depends the deviation of the curve from a plane curve. Hence arises a new affection, which is called by various names: "torsion," "cambure," "flexure," "second curvature." We shall call it *torsion*. Hence, too, the reason why curves in space are called curves of *double curvature*. This species of curvature, of course, depends on the distance between two consecutive points, and on the inclination to each other of the consecutive osculating planes drawn at those two points. If, therefore, $d\omega$ is the angle contained between two consecutive osculating planes, and ds is the element of the curve, then the torsion varies directly as $d\omega$, and inversely as ds, and therefore may be represented by a function of $\frac{d\omega}{ds}$. Borrowing, then, from an analogous property of a circle, it is convenient to represent this function by $\frac{1}{\mathrm{R}}$; so that

$$\frac{1}{\mathrm{R}} = \frac{d\omega}{ds},$$

and R is called the radius of torsion, and $\frac{1}{\mathrm{R}}$ measures the amount of torsion. Hence, then, we define,

DEF. The radius of *absolute curvature* is the distance from the curve at which two consecutive normals, drawn in the osculating plane, intersect.

DEF. The radius of *torsion* is the ratio of an element of the curve to the angle at which two consecutive osculating planes, drawn at the extremities of the element, are inclined.

149.] To determine the Radius of absolute Curvature.

Let radius of absolute curvature $= \rho$, and the angle between two consecutive tangents or angle of contingence $= d\tau$; then, if ds be an element of the curve, according to our definition,

$$ds = \pm \rho\, d\tau. \qquad (1)$$

Now, the direction cosines of one tangent are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}; \qquad (2)$$

and of the consecutive tangent,

$$\frac{dx}{ds} + d\frac{dx}{ds}, \quad \frac{dy}{ds} + d\frac{dy}{ds}, \quad \frac{dz}{ds} + d\frac{dz}{ds}. \qquad (3)$$

Whence, by the ordinary property of such cosines,

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1, \qquad (4)$$

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} + 2\left(\frac{dx}{ds}d\frac{dx}{ds} + \frac{dy}{ds}d\frac{dy}{ds} + \frac{dz}{ds}d\frac{dz}{ds}\right)$$
$$+ \left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2 = 1. \qquad (5)$$

But, from (2) and (3),

$$\cos d\tau = 1 - 2\sin^2\frac{d\tau}{2}$$
$$= \frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} + \frac{dx}{ds}d\frac{dx}{ds} + \frac{dy}{ds}d\frac{dy}{ds} + \frac{dz}{ds}d\frac{dz}{ds} \qquad (6)$$
$$= 1 - \tfrac{1}{2}\left\{\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2\right\};$$

$$\therefore \quad 4\sin^2\frac{d\tau}{2} = \left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2. \qquad (7)$$

And as, when the angle is very small, we may use, indifferently, the angle and its sine (see Lemma II. Art. 12.), we have

$$d\tau = \pm \left\{ \left(d\frac{dx}{ds} \right)^2 + \left(d\frac{dy}{ds} \right)^2 + \left(d\frac{dz}{ds} \right)^2 \right\}^{\frac{1}{2}}; \qquad (8)$$

$$\therefore \quad \frac{1}{\rho} = \pm \frac{1}{ds} \left\{ \left(d\frac{dx}{ds} \right)^2 + \left(d\frac{dy}{ds} \right)^2 + \left(d\frac{dz}{ds} \right)^2 \right\}^{\frac{1}{2}}. \qquad (9)$$

Whence, omitting the signs, since we are calculating an absolute length, differentiating and reducing by means of Equations (16)—(19) Art. 139., we have

$$\frac{1}{\rho} = \frac{\{(dy d^2z - dz d^2y)^2 + (dz d^2x - dx\, d^2z)^2 + (dx d^2y - dy d^2x)^2\}^{\frac{1}{2}}}{ds^3} \qquad (10)$$

$$= \frac{\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}^{\frac{1}{2}}}{ds^2}; \qquad (11)$$

and, if s is taken to be the independent variable,

$$\frac{1}{\rho} = \frac{\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2\}^{\frac{1}{2}}}{ds^2}, \qquad (12)$$

which may be written under the form,

$$\frac{1}{\rho} = \left\{ \left(\frac{d^2x}{ds^2} \right)^2 + \left(\frac{d^2y}{ds^2} \right)^2 + \left(\frac{d^2z}{ds^2} \right)^2 \right\}^{\frac{1}{2}} \qquad (13)$$

As we shall frequently meet with the quantities which are symmetrically arranged in the numerator of (10), it will be convenient to symbolise them as follows;

$$\text{Let} \left. \begin{aligned} dy\,d^2z - dz\,d^2y &= \mathrm{X}, \\ dz\,d^2x - dx\,d^2z &= \mathrm{Y}, \\ dx\,d^2y - dy\,d^2x &= \mathrm{Z}, \end{aligned} \right\}, \text{ and } \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = \mathrm{P}^2; \qquad (14)$$

whence $$\mathrm{P}^2 = \frac{ds^6}{\rho^2}. \qquad (15)$$

150.] Thus, then, we have determined several expressions for the length of the radius of absolute curvature, having defined it to be the distance from the curve at which two consecutive normals drawn in the osculating plane meet. We proceed to deduce certain properties of it, considering it to be the radius of a circle which passes through three consecutive points on the curve, and which, therefore, lies in the osculating plane. Hence we have the following system of equations, considering x, y, z to be the point on the curve at which the circle is drawn, ξ, η, ζ to be the co-ordinates to the centre of the circle, and ρ its radius.

$$(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = \rho^2, \tag{16}$$

$$(\xi - x)\,dx + (\eta - y)\,dy + (\zeta - z)\,dz = 0, \tag{17}$$

$$(\xi - x)\,d^2x + (\eta - y)\,d^2y + (\zeta - z)\,d^2z = ds^2, \tag{18}$$

$$(\xi - x)\,\mathrm{X} + (\eta - y)\,\mathrm{Y} + (\zeta - z)\,\mathrm{Z} = 0; \tag{19}$$

(17) and (18) following from (16) by successive differentiation, and (19) being the equation to the osculating plane; whence, by cross-multiplication between the last three, we have

$$(\xi - x)\{dx(d^2y\,\mathrm{Z} - d^2z\,\mathrm{Y}) + d^2x\,(\mathrm{Y}\,dz - \mathrm{Z}\,dy) + \mathrm{X}(dy\,d^2z - dz\,d^2y)\}$$
$$= ds^2\,(\mathrm{Y}\,dz - \mathrm{Z}\,dy); \tag{20}$$

whence, arranging, we have

$$(\xi - x)\{\mathrm{X}(dy\,d^2z - dz\,d^2y) + \mathrm{Y}(dz\,d^2x - dx\,d^2z) + \mathrm{Z}(dx\,d^2y - dy\,d^2x)\}$$
$$= ds^2\,(\mathrm{Y}\,dz - \mathrm{Z}\,dy),$$

Similarly,

$$\left.\begin{aligned} (\xi - x)(\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2) &= ds^2\,(\mathrm{Y}\,dz - \mathrm{Z}\,dy), \\ (\eta - y)(\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2) &= ds^2\,(\mathrm{Z}\,dx - \mathrm{X}\,dz), \\ (\zeta - z)(\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2) &= ds^2\,(\mathrm{X}\,dy - \mathrm{Y}\,dx). \end{aligned}\right\} \tag{21}$$

But
$$\begin{aligned} \mathrm{Y}\,dz - \mathrm{Z}\,dy &= dz\,(dz\,d^2x - dx\,d^2z) - dy\,(dx\,d^2y - dy\,d^2x) \\ &= (dz^2 + dy^2)\,d^2x - dx\,(dy\,d^2y + dz\,d^2z) \\ &= ds^2\,d^2x - dx\,ds\,d^2s \\ &= ds^3\,d\frac{dx}{ds}; \end{aligned}$$

and similarly for the other quantities.

$$\left.\begin{aligned} \therefore \quad (\xi - x)\, \mathrm{P}^2 &= ds^5\, d\,\frac{dx}{ds}, \\ (\eta - y)\, \mathrm{P}^2 &= ds^5\, d\,\frac{dy}{ds}, \\ (\zeta - z)\, \mathrm{P}^2 &= ds^5\, d\,\frac{dz}{ds}; \end{aligned}\right\} \qquad (22)$$

whence, squaring and adding, by means of (15) and (16),

$$\frac{1}{\rho} = \frac{1}{ds^2}\left\{\left(d\,\frac{dx}{ds}\right)^2 + \left(d\,\frac{dy}{ds}\right)^2 + \left(d\,\frac{dz}{ds}\right)^2\right\}^{\frac{1}{2}}, \quad (23)$$

the same value as (9), and which may be reduced in the same way.

By the equations above we are also enabled to determine the direction of the radius of absolute curvature. Let λ, μ, ν be its direction angles; then, substituting above in (22) for P^2 from (15), we have

$$\left.\begin{aligned} \cos\lambda &= \frac{\xi - x}{\rho} = \rho\,\frac{d\,\frac{dx}{ds}}{ds}, \\ \cos\mu &= \frac{\eta - y}{\rho} = \rho\,\frac{d\,\frac{dy}{ds}}{ds}, \\ \cos\nu &= \frac{\zeta - z}{\rho} = \rho\,\frac{d\,\frac{dz}{ds}}{ds}; \end{aligned}\right\} \qquad (24)$$

and, if s be the independent variable,

$$\left.\begin{aligned} \cos\lambda &= \rho\,\frac{d^2x}{ds^2}, \\ \cos\mu &= \rho\,\frac{d^2y}{ds^2}, \\ \cos\nu &= \rho\,\frac{d^2z}{ds^2}, \end{aligned}\right\} \qquad (25)$$

similar results to those obtained in the discussion of properties of plane curves, see Art. 122. It is worth observing, that by squaring and adding the several terms of (24), and also of (25), the results are in exact agreement with the equations above, marked (9) and (13).

151.] To determine the Radius of *Torsion*.

Applying the symbolical notation which we have adopted in Equations (14), the equation to the osculating plane is

$$\mathrm{X}(\xi-x)+\mathrm{Y}(\eta-y)+\mathrm{Z}(\zeta-z)=0; \qquad (26)$$

and therefore the direction cosines of its normal are

$$\frac{\mathrm{X}}{\mathrm{P}}, \quad \frac{\mathrm{Y}}{\mathrm{P}}, \quad \frac{\mathrm{Z}}{\mathrm{P}}, \qquad (27)$$

and the direction cosines of the normal of the consecutive osculating plane are

$$\frac{\mathrm{X}}{\mathrm{P}}+d\frac{\mathrm{X}}{\mathrm{P}}, \quad \frac{\mathrm{Y}}{\mathrm{P}}+d\frac{\mathrm{Y}}{\mathrm{P}}, \quad \frac{\mathrm{Z}}{\mathrm{P}}+d\frac{\mathrm{Z}}{\mathrm{P}}. \qquad (28)$$

Therefore, if $d\omega$ is the angle contained between two consecutive osculating planes, following precisely the same method as that used to determine $d\tau$ in Art. 149., we have

$$d\omega = \pm\left\{\left(d\frac{\mathrm{X}}{\mathrm{P}}\right)^2+\left(d\frac{\mathrm{Y}}{\mathrm{P}}\right)^2+\left(d\frac{\mathrm{Z}}{\mathrm{P}}\right)^2\right\}^{\frac{1}{2}}; \qquad (29)$$

and, therefore, if R be the radius of torsion, according to what was laid down in Art. 148.,

$$\frac{1}{\mathrm{R}}=\frac{1}{ds}\left\{\left(d\frac{\mathrm{X}}{\mathrm{P}}\right)^2+\left(d\frac{\mathrm{Y}}{\mathrm{P}}\right)^2+\left(d\frac{\mathrm{Z}}{\mathrm{P}}\right)^2\right\}^{\frac{1}{2}}; \qquad (30)$$

$$\therefore \frac{ds^2}{\mathrm{R}^2}=\left(\frac{\mathrm{P}\,d\mathrm{X}-\mathrm{X}\,d\mathrm{P}}{\mathrm{P}^2}\right)^2+\left(\frac{\mathrm{P}\,d\mathrm{Y}-\mathrm{Y}\,d\mathrm{P}}{\mathrm{P}^2}\right)^2+\left(\frac{\mathrm{P}\,d\mathrm{Z}-\mathrm{Z}\,d\mathrm{P}}{\mathrm{P}^2}\right)^2 \qquad (31)$$

$$=\frac{\mathrm{P}^2(d\mathrm{X}^2+d\mathrm{Y}^2+d\mathrm{Z}^2)-\mathrm{P}^2\,d\mathrm{P}^2}{\mathrm{P}^4} \qquad (32)$$

$$= \frac{(X^2+Y^2+Z^2)(dX^2+dY^2+dZ^2)-(X\,dX+Y\,dY+Z\,dZ)^2}{(X^2+Y^2+Z^2)^2} \tag{33}$$

$$= \frac{(Y\,dZ-Z\,dY)^2+(Z\,dX-X\,dZ)^2+(X\,dY-Y\,dX)^2}{(X^2+Y^2+Z^2)^2}. \tag{34}$$

But, differentiating the several terms of (14), we have

$$\left.\begin{aligned} dX &= dy\,d^3z - dz\,d^3y, \\ dY &= dz\,d^3x - dx\,d^3z, \\ dZ &= dx\,d^3y - dy\,d^3x; \end{aligned}\right\} \tag{35}$$

$$\begin{aligned} \therefore\quad Y\,dZ-Z\,dY &= Y(dx\,d^3y-dy\,d^3x)-Z(dz\,d^3x-dx\,d^3z) \\ &= dx(X\,d^3x+Y\,d^3y+Z\,d^3z)-d^3x(X\,dx+Y\,dy \\ &\qquad +Z\,dz): \end{aligned} \tag{36}$$

but $\qquad X\,dx+Y\,dy+Z\,dz = 0;$

$$\therefore\quad Y\,dZ-Z\,dY = dx\,(X\,d^3x+Y\,d^3y+Z\,d^3z), \tag{37}$$

and similar values for the other terms of the numerator of (34),

$$\therefore\quad \frac{ds^2}{R^2} = \frac{ds^2\,(X\,d^3x+Y\,d^3y+Z\,d^3z)^2}{(X^2+Y^2+Z^2)^2};$$

$$\therefore\quad \frac{1}{R} = \frac{X\,d^3x+Y\,d^3y+Z\,d^3z}{X^2+Y^2+Z^2}. \tag{38}$$

Also, by means of (30) and (15),

$$\frac{ds}{R} = \left\{\left(d\,\frac{\rho X}{ds^3}\right)^2+\left(d\,\frac{\rho Y}{ds^3}\right)^2+\left(d\,\frac{\rho Z}{ds^3}\right)^2\right\}^{\frac{1}{2}}. \tag{39}$$

152.] In reference to the values of $\frac{1}{\rho}$ and $\frac{1}{R}$, which we have determined, it is to be observed that, if $\frac{1}{\rho} = 0$, the curve becomes a straight line; of which the analytical conditions are, as is manifest from (10),

$$dy\,d^2z - dz\,d^2y = 0,$$

$$dz\,d^2x - dx\,d^2z = 0,$$

$$dx\,d^2y - dy\,d^2x = 0;$$

or, which are equivalent to them,

$$d\,\frac{dy}{dz} = 0, \qquad d\,\frac{dz}{dx} = 0, \qquad d\,\frac{dx}{dy} = 0; \qquad (40)$$

conditions which are plainly verified in the case of a straight line: and if the curvature lies entirely in one plane, that is, if the curve is a plane curve,

$$\mathrm{X}\,d^3x + \mathrm{Y}\,d^3y + \mathrm{Z}\,d^3z = 0; \qquad (41)$$

which is the same condition as that determined before in Art. 141.

B. *Of Curvature of Curved Surfaces.*

153.] In applying to curved surfaces the principles on which we have founded the discussion on curvature of curves plane and of double curvature, a difficulty meets us: we have made the radius of curvature to depend on the distance from the curve at which two consecutive normals intersect; and, in the case of curves in space, the two normals are drawn in the osculating plane. Now it is to be observed that two consecutive normals imply two consecutive elements, and, therefore, at least three consecutive points; and the plane containing these three points may pass through both these normals, or only *one* of them, or neither of them. Hence we have three distinct cases of curvature to discuss: the first, when the three points and the two consecutive normals are all in one plane, to which, of course, our principles are at once applicable; the second, when the plane passing through the three points contains one normal; these are called "the two cases of normal sections;" and the third, when the plane contains neither of the normal lines drawn at the points through which it passes, which is called the case of oblique section. This distinction of three cases is also hence manifest: in passing along ds, an element of a curve traced on the surface, if the tangent planes drawn at the extremities of

ds are such that their line of intersection is at right angles to ds, then the corresponding consecutive normals intersect; but if the line of intersection be not perpendicular to ds, then the normals will not meet; and if the two consecutive elements are such that the plane containing them is not a normal plane at all, the curvature of the section must be determined in a manner analogous to that whereby we have determined the curvature of a curve in space.

First, then, we propose to determine the curvature of such sections as those in which consecutive normals intersect. The lines on the surface, along which this property holds good, are called *lines of curvature*.

Let $$u = \mathrm{F}(x, y, z) = 0 \qquad (42)$$

be the equation to the surface; and, for the sake of convenience, let us adopt the following symbols,

$$\left.\begin{array}{lll} \left(\frac{d\mathrm{F}}{dx}\right) = \mathrm{U}, & \left(\frac{d\mathrm{F}}{dy}\right) = \mathrm{V}, & \left(\frac{d\mathrm{F}}{dz}\right) = \mathrm{W}, \\ \left(\frac{d^2\mathrm{F}}{dy\,dz}\right) = u', & \left(\frac{d^2\mathrm{F}}{dz\,dx}\right) = v', & \left(\frac{d^2\mathrm{F}}{dx\,dy}\right) = w', \\ \left(\frac{d^2\mathrm{F}}{dx^2}\right) = u, & \left(\frac{d^2\mathrm{F}}{dy^2}\right) = v, & \left(\frac{d^2\mathrm{F}}{dz^2}\right) = w; \end{array}\right\} \qquad (43)$$

$$\mathrm{P}^2 = \mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2.$$

Then the equations to the normal at a point x, y, z are (see Art. 134.)

$$\frac{x-\xi}{\mathrm{U}} = \frac{y-\eta}{\mathrm{V}} = \frac{z-\zeta}{\mathrm{W}} = \Omega; \qquad (44)$$

where $$\Omega = \frac{r}{\mathrm{P}},$$

r being the distance of (x, y, z) from (ξ, η, ζ); hence, then, the equations to the normal at the next consecutive point on the surface are

$$\frac{x+dx-\xi}{\mathrm{U}+d\mathrm{U}} = \frac{y+dy-\eta}{\mathrm{V}+d\mathrm{V}} = \frac{z+dz-\zeta}{\mathrm{W}+d\mathrm{W}} = \Omega + d\Omega. \qquad (45)$$

And if these two normals meet, ξ, η, ζ are the same in (44) and (45); whence, multiplying and subtracting,

$$\left.\begin{aligned} dx &= \text{U}\,d\Omega + d\text{U}\,(\Omega + d\Omega), \\ dy &= \text{V}\,d\Omega + d\text{V}\,(\Omega + d\Omega), \\ dz &= \text{W}\,d\Omega + d\text{W}\,(\Omega + d\Omega); \end{aligned}\right\} \tag{46}$$

whence, eliminating Ω and $d\Omega$, we have

$$(\text{V}\,d\text{W} - \text{W}\,d\text{V})dx + (\text{W}\,d\text{U} - \text{U}\,d\text{W})dy + (\text{U}\,d\text{V} - \text{V}\,d\text{U})dz = 0, \tag{47}$$

which, being independent of ξ, η, ζ, holds good for any point on the surface: hence it is the differential equation to a surface by the intersection of which with the surface represented by Equation (42) a line is formed possessing the property, that the normals to the surface at the extremities of an element always intersect. The geometrical meaning of (47) is, what has been said above, that the line of intersection of two consecutive tangent planes drawn at the extremities of ds is at right angles to the tangent line whose direction cosines are proportional to dx, dy, dz. It is also observable that, since the quantities $d\text{U}$, $d\text{V}$, $d\text{W}$ are linear functions of dx, dy, dz, the Equation (47) will be of the second order with respect to these quantities. Hence there may be, in general, two distinct sets of lines of curvature traced at any point of a surface.

154.] Suppose, now, that the equation of which (47) is the total differential is

$$\mathfrak{L} = 0, \tag{48}$$

of which the differential is

$$\mathfrak{U}\,dx + \mathfrak{V}\,dy + \mathfrak{W}\,dz = 0, \tag{49}$$

$\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$ being quantities analogous to U, V, W, and similarly for the other quantities used in the following articles; and therefore

$$\frac{\text{V}\,d\text{W} - \text{W}\,d\text{V}}{\mathfrak{U}} = \frac{\text{W}\,d\text{U} - \text{U}\,d\text{W}}{\mathfrak{V}} = \frac{\text{U}\,d\text{V} - \text{V}\,d\text{U}}{\mathfrak{W}}; \tag{50}$$

whence, using the multipliers U, V, W, the sum of the numerators = 0, and therefore

$$\mathrm{U}\mathfrak{U} + \mathrm{V}\mathfrak{V} + \mathrm{W}\mathfrak{W} = 0. \qquad (51)$$

Whence we conclude, "If two surfaces cut one another in their lines of curvature, they cut at right angles."

155.] Again, let there be three surfaces whose equations are

$$\mathrm{F} = 0, \quad \mathfrak{F} = 0, \quad F = 0, \qquad (52)$$

and let them cut one another at right angles; the conditions that this may be the case are,

$$\left.\begin{aligned} \mathfrak{U}U + \mathfrak{V}V + \mathfrak{W}W &= 0, \\ U\mathrm{U} + V\mathrm{V} + W\mathrm{W} &= 0, \\ \mathrm{U} \quad + \mathrm{V}\mathfrak{V} + \mathrm{W}\mathfrak{W} &= 0. \end{aligned}\right\} \qquad (53)$$

But from (52)

$$\left.\begin{aligned} \mathrm{U}dx + \mathrm{V}dy + \mathrm{W}dz &= 0, \\ \mathfrak{U}dx + \mathfrak{V}dy + \mathfrak{W}dz &= 0, \\ U dx + V dy + W dz &= 0. \end{aligned}\right\} \qquad (54)$$

From the first and second of which there result

$$\frac{dx}{\mathrm{V}\mathfrak{W} - \mathfrak{V}\mathrm{W}} = \frac{dy}{\mathrm{W}\mathfrak{U} - \mathfrak{W}\mathrm{U}} = \frac{dz}{\mathrm{U}\mathfrak{V} - \mathfrak{U}\mathrm{V}}; \qquad (55)$$

and, using the factors

$$\mathrm{V}d\mathrm{W} - \mathrm{W}d\mathrm{V}, \quad \mathrm{W}d\mathrm{U} - \mathrm{U}d\mathrm{W}, \quad \mathrm{U}d\mathrm{V} - \mathrm{V}d\mathrm{U},$$

the numerator of (55) will be

$$(\mathrm{V}d\mathrm{W} - \mathrm{W}d\mathrm{V})dx + (\mathrm{W}d\mathrm{U} - \mathrm{U}d\mathrm{W})dy + (\mathrm{U}d\mathrm{V} - \mathrm{V}d\mathrm{U})dz, \qquad (56)$$

and the denominator

$$\mathrm{P}^2(\mathfrak{U}d\mathrm{U} + \mathfrak{V}d\mathrm{V} + \mathfrak{W}d\mathrm{W}). \qquad (57)$$

So that each of the ratios (55) is equal to

$$\frac{(\mathrm{V}d\mathrm{W} - \mathrm{W}d\mathrm{V})\dfrac{dx}{\mathrm{P}^2} + (\mathrm{W}d\mathrm{U} - \mathrm{U}d\mathrm{W})\dfrac{dy}{\mathrm{P}^2} + (\mathrm{U}d\mathrm{V} - \mathrm{V}d\mathrm{U})\dfrac{dz}{\mathrm{P}^2}}{\mathfrak{U}d\mathrm{U} + \mathfrak{V}d\mathrm{V} + \mathfrak{W}d\mathrm{W}}. \qquad (58)$$

Similarly, using the factors

$$\mathfrak{V}d\mathfrak{W}-\mathfrak{W}d\mathfrak{V}, \quad \mathfrak{W}d\mathfrak{U}-\mathfrak{U}d\mathfrak{W}, \quad \mathfrak{U}d\mathfrak{V}-\mathfrak{V}d\mathfrak{U},$$

the same ratios will be equal to

$$\frac{(\mathfrak{V}d\mathfrak{W}-\mathfrak{W}d\mathfrak{V})\dfrac{dx}{\mathrm{P}^2}+(\mathfrak{W}d\mathfrak{U}-\mathfrak{U}d\mathfrak{W})\dfrac{dy}{\mathrm{P}^2}+(\mathfrak{U}d\mathfrak{V}-\mathfrak{V}d\mathfrak{U})\dfrac{ds}{\mathrm{P}^2}.}{\mathrm{U}d\mathfrak{U}+\mathrm{V}d\mathfrak{V}+\mathrm{W}d\mathfrak{W}} \tag{59}$$

But, on differentiating the first of (53),

$$\mathrm{U}d\mathfrak{U}+\mathrm{V}d\mathfrak{V}+\mathrm{W}d\mathfrak{W}+\mathfrak{U}d\mathrm{U}+\mathfrak{V}d\mathrm{V}+\mathfrak{W}d\mathrm{W}=0. \tag{60}$$

Hence, writing

$$\left.\begin{aligned}
\mathrm{V}d\mathrm{W}-\mathrm{W}d\mathrm{V}&=\mathrm{P}^2d\mathrm{A}, \quad \mathrm{W}d\mathrm{U}-\mathrm{U}d\mathrm{W}=\mathrm{P}^2d\mathrm{B},\\
\mathrm{U}d\mathrm{V}-\mathrm{V}d\mathrm{U}&=\mathrm{P}^2d\mathrm{C},\\
\mathfrak{V}d\mathfrak{W}-\mathfrak{W}d\mathfrak{V}&=\mathfrak{P}^2d\mathfrak{A}, \quad \mathfrak{W}d\mathfrak{U}-\mathfrak{U}d\mathfrak{W}=\mathfrak{P}^2d\mathfrak{B},\\
\mathfrak{U}d\mathfrak{V}-\quad d\mathfrak{U}&=\mathfrak{P}^2d\mathfrak{C},\\
VdW-WdV&=P^2dA, \quad WdU-UdW=P^2dB,\\
UdV-VdU&=P^2dC,
\end{aligned}\right\} \tag{61}$$

and forming quantities similar to (58) and (59), we should have the system

$$\left.\begin{aligned}
(d\mathfrak{A}+dA)dx+(d\mathfrak{B}+dB)dy+(d\mathfrak{C}+dC)dz&=0,\\
(dA+d\mathrm{A})dx+(dB+d\mathrm{B})dy+(dC+d\mathrm{C})dz&=0,\\
(d\mathrm{A}+d\mathfrak{A})dx+(d\mathrm{B}+d\mathfrak{B})dy+(d\mathrm{C}+d\mathfrak{C})dz&=0,
\end{aligned}\right\} \tag{62}$$

or

$$\left.\begin{aligned}
d\mathrm{A}dx+d\mathrm{B}dy+d\mathrm{C}dz&=0,\\
d\mathfrak{A}dx+d\mathfrak{B}dy+d\mathfrak{C}dz&=0,\\
dAdx+dBdy+dCdz&=0,
\end{aligned}\right\} \tag{63}$$

which are the equations to the lines of curvature on the surfaces F, $\mathfrak{F}$, F, respectively. Hence, "If three surfaces cut one another at right angles, the lines of intersection of any one of the surfaces with the other two are its lines of curvature;" which is Dupin's Theorem.

From this it also follows, that the two lines of curvature on any surface are perpendicular to one another.

156.] There are two cases in which the equation of the lines of curvature is satisfied identically. The first of these is when

$$\frac{dU}{U} = \frac{dV}{V} = \frac{dW}{W};$$

in which case, on account of the linearity of the equations, there is obviously only one line of curvature.

Again: since

$$V\,dW - W\,dV = (Vv' - Ww')\,dx + (Vu' - Wv)\,dy + (Vw - Wu')\,dz; \tag{64}$$

hence, eliminating dx between this and the first of (54), and, for the sake of convenience, writing

$$\left.\begin{aligned} H &= u + \frac{U}{VW}(Uu' - Vv' - Ww'), \\ K &= v + \frac{V}{WU}(Vv' - Ww' - Uu'), \\ L &= w + \frac{W}{UV}(Ww' - Uu' - Vv'), \end{aligned}\right\} \tag{65}$$

$$V\,dW - W\,dV = VL\,dz - WK\,dy; \tag{66}$$

and, modifying the other terms of (47) in the same way as (66), the equation to the lines of curvature becomes

$$U(K - L)\,dy\,dz + V(L - H)\,dz\,dx + W(H - K)\,dx\,dy = 0. \tag{67}$$

Hence, if

$$H = K = L, \tag{68}$$

the equation to the lines of curvature is satisfied independently of dx, dy, dz, and the directions of the lines are consequently indeterminate; and, therefore, where conditions (68) are satisfied, there are an indefinite number of directions along which every two consecutive normals intersect. The two equations (68), together with the equation to the surface, will determine the co-ordinates of the point at which this takes place. But since when either U, V, or W vanishes these conditions become indeterminate, it will be necessary to make some transformations, in order to put them in a determinate form. Suppose, then, that $\text{U} = 0$. Multiplying (68) throughout by UVW, and substituting for H, K, L from (65), it appears that when U vanishes,

$$\text{V}v' - \text{W}w' = 0; \qquad (69)$$

hence, writing for convenience

$$\text{V}v' - \text{W}w' = \text{o}, \qquad (70)$$

(68) becomes

$$u = v + \frac{\text{V}}{\text{UW}}(\text{o} - \text{U}u') = w + \frac{\text{W}}{\text{UV}}(-\text{o} - \text{U}u'); \qquad (71)$$

whence, using the factors $\frac{\text{W}}{\text{V}}, \frac{\text{V}}{\text{W}}$, there results

$$u = \frac{\frac{\text{W}}{\text{V}}v + \frac{\text{V}}{\text{W}}w - 2u'}{\frac{\text{W}}{\text{V}} + \frac{\text{V}}{\text{W}}}. \qquad (72)$$

Hence the conditions become

$$\left.\begin{array}{ll} \text{when U} = 0;\ \text{V}v' - \text{W}w' = 0, & u = \dfrac{\text{V}^2 w - 2\,\text{VW}u' + \text{W}^2 v}{\text{V}^2 + \text{W}^2}, \\ \text{when V} = 0;\ \text{W}w' - \text{U}u' = 0, & v = \dfrac{\text{W}^2 u - 2\,\text{WU}v' + \text{U}^2 w}{\text{W}^2 + \text{U}^2}, \\ \text{when W} = 0;\ \text{U}u' - \text{V}v' = 0, & w = \dfrac{\text{U}^2 v - 2\,\text{UV}w' + \text{V}^2 u}{\text{U}^2 + \text{V}^2}. \end{array}\right\} \qquad (73)$$

The points determined by these equations are called *umbilici*. Hence, to find all the umbilici of a surface, we must determine all the points at which the general conditions (68) are fulfilled, and also find whether there are any points which satisfy any of the special systems (73).

157.] Since, then, there are generally on a surface two lines such that the normals at the points $(x + dx, y + dy, z + dz)$ meet that at the point (x, y, z), we proceed to apply the principles on which we have determined curvature to the case of these lines. Let ρ be the radius of curvature; ξ, η, ζ, the co-ordinates to the centre; x, y, z, to the point on the surface; then we have the following system of equations:

$$\left.\begin{aligned}&(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=\rho^2,\\&(x-\xi)\,dx+(y-\eta)\,dy+(z-\zeta)\,dz=0,\\&(x-\xi)\,d^2x+(y-\eta)\,d^2y+(z-\zeta)\,d^2z=ds^2;\end{aligned}\right\}\quad(74)$$

and, since the centre of curvature is on the normal,

$$\frac{x-\xi}{\text{U}}=\frac{y-\eta}{\text{V}}=\frac{z-\zeta}{\text{W}}=\pm\frac{\rho}{\text{P}},\qquad(75)$$

each of which equalities is, by virtue of the last of (74), equal to

$$\frac{ds^2}{\text{U}d^2x+\text{V}d^2y+\text{W}d^2z};\qquad(76)$$

$$\therefore\quad \frac{\text{P}}{\rho}=\pm\frac{\text{U}d^2x+\text{V}d^2y+\text{W}d^2z}{ds^2}:\qquad(77)$$

but, differentiating the equation to the surface twice, we have

$$\begin{aligned}\text{U}d^2x+\text{V}d^2y+\text{W}d^2z+u\,dx^2+v\,dy^2+w\,dz^2+2\,(u'dydz\\+v'dzdx+w'dxdy)=0;\qquad(78)\end{aligned}$$

whence, writing

$$\frac{dx}{ds}=l,\quad \frac{dy}{ds}=m,\quad \frac{dz}{ds}=n,\qquad(79)$$

we have, neglecting the signs,

$$\frac{\text{P}}{\rho}=ul^2+vm^2+wn^2+2\,(u'mn+v'nl+w'lm).\qquad(80)$$

Also, since

$$m\text{V}+n\text{W}=-l\text{U},\qquad(81)$$

squaring and transposing, and forming similar quantities in V and W, we find the system

$$\left.\begin{aligned} 2\,mn &= \frac{l^2 \mathrm{U}^2 - m^2 \mathrm{V}^2 - n^2 \mathrm{W}^2}{\mathrm{V}\,\mathrm{W}}, \\ 2\,nl &= \frac{m^2 \mathrm{V}^2 - n^2 \mathrm{W}^2 - l^2 \mathrm{U}^2}{\mathrm{W}\,\mathrm{U}}, \\ 2\,lm &= \frac{n^2 \mathrm{W}^2 - l^2 \mathrm{U}^2 - m^2 \mathrm{V}^2}{\mathrm{U}\,\mathrm{V}}, \end{aligned}\right\} \tag{82}$$

and, consequently,

$$\frac{\mathrm{P}}{\rho} = \mathrm{H}\,l^2 + \mathrm{K}m^2 + \mathrm{L}\,n^2. \tag{83}$$

Now, since this radius of curvature is measured along a line of curvature, this condition must be introduced; hence, eliminating U, V, W in turn between (67) and the differential equation of the surface, we find

$$\frac{\mathrm{U}}{l\left(\mathrm{H} - \frac{\mathrm{P}}{\rho}\right)} = \frac{\mathrm{V}}{m\left(\mathrm{K} - \frac{\mathrm{P}}{\rho}\right)} = \frac{\mathrm{W}}{n\left(\mathrm{L} - \frac{\mathrm{P}}{\rho}\right)}; \tag{84}$$

whence

$$\frac{\mathrm{U}^2}{\mathrm{H} - \frac{\mathrm{P}}{\rho}} + \frac{\mathrm{V}^2}{\mathrm{K} - \frac{\mathrm{P}}{\rho}} + \frac{\mathrm{W}^2}{\mathrm{L} - \frac{\mathrm{P}}{\rho}} = 0, \tag{85}$$

a quadratic equation, which will determine the two radii of curvature corresponding to the two lines of curvature.

158.] It is also worthy of remark, that

$$\frac{1}{2}\,d\left(\frac{\mathrm{P}}{\rho}\right) = \mathrm{H}\,l\,dl + \mathrm{K}\,m\,dm + \mathrm{L}\,n\,dn; \tag{86}$$

but, taking into account the variations of the radius of curvature corresponding to the variations of l, m, n, we have

$$\left.\begin{aligned} \mathrm{U}\,dl + \mathrm{V}\,dm + \mathrm{W}\,dn &= 0, \\ l\,dl + m\,dm + n\,dn &= 0, \end{aligned}\right\} \tag{87}$$

whence

$$\frac{dl}{\mathrm{V}n - \mathrm{W}m} = \frac{dm}{\mathrm{W}l - \mathrm{U}n} = \frac{dn}{\mathrm{U}m - \mathrm{V}l}; \tag{88}$$

and, using multipliers $\text{H}l$, $\text{K}m$, $\text{L}n$, and adding numerators and denominators, the sum of the denominators = 0, by means of Equation (67), since ds is measured along a line of curvature, therefore

$$\text{H}l\,dl + \text{K}m\,dm + \text{L}n\,dn = 0;$$

and, therefore,

$$d\left(\frac{\text{P}}{\rho}\right) = 0; \qquad (89)$$

that is, on a line of curvature, the radius is a maximum or minimum. Now, the normal sections of greatest and least curvature at any point of the surface are called *the principal sections,* and the corresponding radii of curvature *the principal radii of curvature;* hence the lines of curvature pass through the principal sections of the surface.

159.] Having considered the curvature of the principal normal sections, it remains for us to determine the curvature of any normal section. For this purpose, let $l_1, m_1, n_1, l_2, m_2, n_2$ be the direction cosines of the principal normal planes, and l, m, n those of any other normal section, θ the angle between the latter plane and the first principal normal plane; we then have the relations

$$\left.\begin{aligned} \text{U}l + \text{V}m + \text{W}n &= 0, \\ \text{U}l_1 + \text{V}m_1 + \text{W}n_1 &= 0, \\ \text{U}l_2 + \text{V}m_2 + \text{W}n_2 &= 0, \end{aligned}\right\} \qquad (90)$$

$$\left.\begin{aligned} ll_1 + mm_1 + nn_1 &= \cos\theta, \\ ll_2 + mm_2 + nn_2 &= \sin\theta, \\ l_1l_2 + m_1m_2 + n_1n_2 &= 0, \end{aligned}\right\} \qquad (91)$$

and consequently

$$(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1. \qquad (92)$$

Also, returning to Equations (83) and (84), we find

$$\left.\begin{aligned} \mathrm{H}l^2 + \mathrm{K}m^2 + \mathrm{L}n^2 &= \frac{\mathrm{P}}{\rho}, \\ \mathrm{H}ll_1 + \mathrm{K}mm_1 + \mathrm{L}nn_1 &= \frac{\mathrm{P}}{\rho_1} \cos\theta, \\ \mathrm{H}ll_2 + \mathrm{K}mm_2 + \mathrm{L}nn_2 &= \frac{\mathrm{P}}{\rho_2} \sin\theta; \end{aligned}\right\} \qquad (93)$$

and since, by cross-multiplication from (90),

$$l(m_1n_2 - m_2n_1) + m(n_1l_2 - n_2l_1) + n(l_1m_2 - l_2m_1) = 0, \quad (94)$$

we have, by cross-multiplication from (93),

$$\left.\begin{aligned} &\frac{\mathrm{P}}{\rho}(m_1n_2 - m_2n_1) + \frac{\mathrm{P}}{\rho_1}\cos\theta\,(m_2n - mn_2) + \frac{\mathrm{P}}{\rho_2}\sin\theta\,(mn_1 - m_1n) = 0, \\ &\frac{\mathrm{P}}{\rho}(n_1l_2 - n_2l_1) + \frac{\mathrm{P}}{\rho_1}\cos\theta\,(n_2l - nl_2) + \frac{\mathrm{P}}{\rho_2}\sin\theta\,(nl_1 - n_1l) = 0, \\ &\frac{\mathrm{P}}{\rho}(l_1m_2 - l_2m_1) + \frac{\mathrm{P}}{\rho_1}\cos\theta\,(l_2m - lm_2) + \frac{\mathrm{P}}{\rho_2}\sin\theta\,(lm_1 - l_1m) = 0; \end{aligned}\right\} \quad (95)$$

whence, multiplying these by

$$m_1n_2 - m_2n_1, \qquad n_1l_2 - n_2l_1, \qquad l_1m_2 - l_2m_1$$

respectively, and adding, we find

$$\frac{1}{\rho} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2}, \qquad (96)$$

which formula will give the curvature of any normal section, when the curvatures of the principal normal sections are found.

160.] Having considered the properties of the curvature of normal sections of surfaces, it remains for us to establish Meunier's Theorem, by which we are enabled to calculate the curvature of an oblique section from that of a normal section through the same point. Suppose that the planes of normal and oblique sections pass through the same tangent line, for this may always be effected; then, Equation (96) giving any normal curvature which is required, the direction cosines, l, m, n, will be the same for both sections, although $\frac{dl}{ds}$, $\frac{dm}{ds}$, $\frac{dn}{ds}$, that is, $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, may not be so. Now, if λ, μ, ν be the direction cosines of the radius of curvature (R) of the oblique section, the theory of curves in space gives the following values for λ, μ, ν:

$$\lambda = \mathrm{R}\frac{d^2x}{ds^2}, \quad \mu = \mathrm{R}\frac{d^2y}{ds^2}, \quad \nu = \mathrm{R}\frac{d^2z}{ds^2}. \tag{97}$$

hence, if ϕ be the angle between the two planes of section,

$$\begin{aligned}\cos\phi &= \lambda\frac{\mathrm{U}}{\mathrm{P}} + \mu\frac{\mathrm{V}}{\mathrm{P}} + \nu\frac{\mathrm{W}}{\mathrm{P}} \\ &= \frac{\mathrm{R}}{\mathrm{P}}\left(\mathrm{U}\frac{d^2x}{ds^2} + \mathrm{V}\frac{d^2y}{ds^2} + \mathrm{W}\frac{d^2z}{ds^2}\right) \\ &= -\frac{\mathrm{R}}{\mathrm{P}}\{ul^2 + vm^2 + wn^2 + 2(u'mn + v'nl + w'lm)\} \\ &= \frac{\mathrm{R}}{\rho},\end{aligned} \tag{98}$$

since both lines are on the surface, and must, consequently, satisfy its equation; hence

$$\mathrm{R} = \rho\cos\phi; \tag{99}$$

that is, the radius of curvature of an oblique section of a surface is the projection of the radius of curvature of the normal section, which passes through the same tangent line, upon the plane of the oblique section.

CHAP. XVI.

ON THE PRINCIPLES OF THE INTEGRAL CALCULUS.

161.] THERE are two different modes of considering the Integral Calculus, as there are of the Differential. One, in which the functions on which we operate are considered to be algebraical expressions, and the processes of differentiation and derivation to be analytical artifices, by which one function gives rise to another, and this in its turn becomes the parent of another, and thus a series of algebraical functions are derived one from the other. The other mode of viewing it is that in which we conceive the subject matter of the Differential Calculus to be small quantities or infinitesimals of such magnitudes that it requires an infinity of them, and a particular infinity according to the particular infinitesimal, to be added together to make a finite quantity. It is under this latter point of view that we have considered the Differential Calculus in the preceding pages, and have defined it as in Art. 5., and we may call it the *geometrical* method, in contradistinction to the former, which may be called the *analytical;* as it is under the latter aspect that the methods of the Calculus are applied to problems in Geometry and Mechanics, and under the former to questions of Arithmetical Algebra and abstract functions, as has been done by Lagrange, in the *Calcul des Fonctions,* and other works.

In the analytical method it is plain that every such operation as derivation admits of being reversed, and, as one function gives rise to another by derivation, so would this new function by a reverse process give rise to the former function, and this one again by a reverse process give rise to another function, such that its derived function would be the original one; and thus, by a similar process, we might obtain a series of functions so related to one another, and in such order, that, were the process reversed, each would be the derived of the preceding one. Thus the possibility of the reverse process, which is called *Integration,* is involved in that of derivation; and we employ a symbolisation analogous to that used in the Calculus of Derivation.

Since $\frac{d}{dx}$ is the symbol of operation, such that when performed on $f(x)$ it changes it into $f'(x)$, i. e. since

$$\frac{d}{dx} f(x) = f'(x),$$

the process reverse to derivation must be symbolised by $\left(\frac{d}{dx}\right)^{-1}$, i. e. by $d^{-1}\,dx$, so that the operation symbolised by it being performed on $f'(x)$ may change it into $f(x)$, i. e. so that

$$d^{-1}\,dx f'(x) = f(x).$$

For a reason which will appear hereafter, we symbolise d^{-1} by $\int$, so that we have

$$\int f'(x)\,dx = f(x).$$

Similarly,
$$\int f''(x)\,dx = f'(x),$$

$$\int f'''(x) dx = f''(x),$$

and so on; whereby we have a series of functions formed one from the other by a process exactly the reverse of derivation, viz.

$$f'''(x), \quad f''(x), \quad f'(x), \quad f(x).$$

It is to be observed, that in this mode of considering the subject, the symbol $\int dx$ must be taken as a complex character, and indicative of a certain analytical process to be performed on a certain function.

The other mode of viewing the Differential Calculus being that in which we consider it to be the method of analysing any given function or quantity into its component elements, which are of such a nature that the aggregate of an infinite number must be taken to be equal to the finite quantity which has been analysed, the Integral Calculus, considering such infinitesimals to be already in existence, takes them as its subject matter, operates upon them, and constructs rules and laws for the determination of the finite quantities or functions of which they are the infinitesimal elements. As, then, the Differential Calculus is the mode of resolving a finite quantity or function into its elements, so the Integral Calculus constructs rules to determine the finite quantity or function of which these are the component parts; it takes one of them, which is the general type of all, and finds the sum of them. Thus it is *essentially* a method of summation of series, each term of the series being an infinitesimal, and the difference between any two successive terms being an infinitesimal of a higher order than the terms themselves; and as in the summation of series we can find the sum of any number of terms from the *law* of the series, without knowing any other than general values of the first and last terms, so from knowing any one term which is a type, as we have said, of every term, and therefore gives the law of the series, we can find the general value of the sum of any number of terms. The process by which this general sum is determined is, for an obvious reason, called *Integration*, and the sum is called an *Indefinite Integral*; but, if the first and last terms of the series be given, then the sum of the terms of the series between these two limits can be accurately determined, and the sum is called a *Definite Integral.*

An example will make this plainer. (See fig. 58.)

Let $y = x^2$: then, if x be the variable side of a square, y is the area of it, and, differentiating the above expression,

$$dy = 2x\,dx;$$

that is, we resolve the square into elements, each of which is equal to $2x\,dx$. Let $\text{OM} = \text{MP} = x$, $\text{MN} = dx$: then, from the figure, it is plain, that, as x increases, the area OP, which is equal to x^2, increases by an area $2x\,dx$ (see Ex. 1. Art. 11.), that is, by the gnomon SPN; so that we may consider the whole square to be made up of gnomonic pieces, such as SPN. Thus, then, has the square been resolved by the Differential Calculus. And if $\text{OA} = a$, we should say that the function of x which we are considering, viz. x^2, is finite and continuous for all values of x between $x = a$ and $x = 0$. Hence, then, arises the reverse problem: having given the small quantity $2x\,dx$, with the conditions that it is continuous and finite, and is always increasing or decreasing with x between certain limits, to determine, between the same limits, the finite quantity of which it is the element; in other words, to find the definite integral of $2x\,dx$. In this case, then, we have $2x\,dx$ a type of every term, and we have the first and last terms of the series. Thus, if the limits are $x = a$ and $x = 0$, we have the sum of the series $= a^2$, viz. the square BA; but, if the limits are OA $= a$ and OC $= b$, the sum of the terms of the series would be the gnomon DEC, viz. the sum to the highest limit less the sum to the lowest limit, i. e. $a^2 - b^2$. This demonstrates that, in a geometrical point of view, it is always possible to determine the function or quantity of which the infinitesimal is the component element; and from this it appears, though a rigorous proof will be given hereafter, that, if the indefinite integral be determined, the definite integral is found by subtracting the value of the indefinite integral corresponding to the inferior limit from its value corresponding to the superior limit.

162.] The Symbolisation we adopt is as follows. We use, as in the Differential Calculus, the symbol d to signify the differential or the element of x or of any function of x, thus dx is the element of x; and $\int$ (the long S) to symbolise the *sum* of an infinite number of terms, each of which is an infinitesimal. So that if $\phi(x)dx$, which, it is to be observed, is the algebraical product of two quantities, a finite one $\phi(x)$ and an infinitesimal dx, be the type of the elements, the sum of an infinite number of which it is required to find, the problem is symbolised by

$$\int \phi(x)dx;$$

and if x_1 and x_0 are the limits, x_1 the superior or that corresponding to the last term of the series, and x_0 the inferior or that corresponding to the first term, and if $\mathrm{F}(x)$ is the indefinite integral of $\phi(x)dx$, then we use the following notation,

$$\int_{x_0}^{x_1} \phi(x)dx = \Big[\mathrm{F}(x)\Big]_{x_0}^{x_1} = \mathrm{F}(x_1) - \mathrm{F}(x_0).$$

Subjoined are two integrals determined from first principles, according to the method indicated in the above Article. In both cases suppose x_1 and x_0 to be respectively the superior and inferior limits; and let the difference $x_1 - x_0$ be resolved into n equal parts, each equal to dx; whence

$$x_1 - x_0 = n\,dx, \qquad \therefore\ dx = \frac{x_1 - x_0}{n},$$

and the sum of the series is to be found when dx is very small, that is, when n is very large; the condition of all the dx's being equal, it is convenient to make for simplicity's sake, but it is not necessary for the truth of the result.

Ex. 1.

$$\int_{x_0}^{x_1} x\,dx = x_0 dx + (x_0 + dx)dx + (x_0 + 2dx)dx + \dots$$

$$\dots + \{x_0 + (n-1)dx\}dx$$

$$= \left(nx_0 + \frac{n(n-1)}{2}dx\right)dx$$

$$= \left(nx_0 + \frac{n(n-1)}{2}\frac{x_1 - x_0}{n}\right)\frac{x_1 - x_0}{n}$$

$$= x_0(x_1 - x_0) + \left(1 - \frac{1}{n}\right)\frac{(x_1 - x_0)^2}{2}$$

$$= \frac{x_1^2 - x_0^2}{2}, \qquad \text{when } n = \frac{1}{0};$$

and if, instead of the particular value x_1, the superior limit is x, then

$$\int_{x_0}^{x} x\,dx = \frac{x^2}{2} - \frac{x_0^2}{2} = \frac{x^2}{2} + \text{C};$$

that is, the indefinite integral is $\frac{x^2}{2}$, which corresponds with the result of the Differential Calculus, for $d.\frac{x^2}{2} = x\,dx$; so the definite integral is equal to the value of the indefinite integral corresponding to the superior limit less its value corresponding to the inferior limit.

Ex. 2.

$$\int_{x_0}^{x_1} e^x dx = e^{x_0} dx + e^{x_0+dx} dx + e^{x_0+2dx} dx + \dots + e^{x_0+(n-1)dx} dx$$

$$= e^{x_0}\left(1 + e^{dx} + e^{2dx} + \dots\dots + e^{(n-1)dx}\right) dx$$

$$= e^{x_0} \frac{e^{ndx} - 1}{e^{dx} - 1} dx$$

$$= e^{x_0}(e^{x_1-x_0} - 1) \frac{dx}{e^{dx} - 1};$$

but, when $dx = 0, \frac{dx}{e^{dx} - 1} = 1$, as shown in Art. 19.,

$$\therefore \int_{x_0}^{x_1} e^x dx = e^{x_1} - e^{x_0}.$$

163.] To prove that, in the general case, if $f'(x)dx = d.f(x)$,

$$\int_{x_0}^{x_n} f'(x)dx = \Big[f(x)\Big]_{x_0}^{x_n} = f(x_n) - f(x_0),$$

subject to the condition that $f'(x)$ does not change its sign between the limits x_n and x_0; and, to fix our thoughts, let us suppose $f(x)$ to increase as x increases from x_0 to x_n.

Let us divide the difference $x_n - x_0$ into n parts, $x_1 - x_0$, $x_2 - x_1, \dots\dots x_n - x_{(n-1)}$; then the definite integral is equal to the sum of the products

$$(x_1 - x_0)f'(x_0) + (x_2 - x_1)f'(x_1) + (x_3 - x_2)f'(x_2) + \dots\dots$$
$$\dots + (x_n - x_{n-1})f'(x_{n-1}),$$

each element being multiplied into the function of x corresponding to the beginning of the element; then, since $f'(x_0)$, $f'(x_1), \dots\dots f'(x_{n-1})$ are all quantities of the same sign, by Preliminary Proposition II. the sum of all these products is equal to

$$(x_n - x_0) f'\{x_0 + \theta(x_n - x_0)\},$$

$x_n - x_0$ being the sum of all the elements, and $f'\{x_0 + \theta(x_1 - x_0)\}$ being a mean value of the functions.

Let $x_n - x_0 = h$, in which case the sum of the series becomes

$$hf'(x_0 + \theta h),$$

which, by the theorem enunciated in Art. 51., is equal to

$$f(x_0 + h) - f(x_0) = f(x_n) - f(x_0),$$

$$\therefore \int_{x_0}^{x_n} f'(x)\,dx = f(x_n) - f(x_0);$$

and, therefore, the definite integral of $f'(x)dx$ between the limits x_n and x_0 is the value of the indefinite integral, viz. $f(x)$, when $x = x_n$, less its value when $x = x_0$.

It is also to be observed, that in the series above, which is expressed by the definite integral, the value of $f'(x)$ corresponding to the superior limit is excluded, but that its value corresponding to the inferior limit is included.

If the superior limit x_n were x, x being any value of the variable, subject only to the condition that $f'(x)$ is finite and continuous, and does not change sign between x and x_0, then the integral becomes

$$f(x) - f(x_0);$$

and, as $f(x_0)$ is a constant which would disappear in differentiation, we have the indefinite integral under the form and symbol

$$\int f'(x)\,dx = f(x).$$

If, however, between the limits for which we have to calculate the integral, $f'(x)$ changes sign, we must calculate separately the values of the integral when $f'(x)$ is positive and when it is negative.

Hence, then, the problem of the summation of a series of these infinitesimals resolves itself into this: viz. to determine the function which being differentiated would produce the function which is the type of all the terms of the series which we have to sum; and the process by which we arrive at this new function is called, as we before said, Integration, and the new function the Indefinite Integral. Hence, then, it appears that the process by which this is determined is the reverse process of Differentiation; and, therefore, all the rules for differentiation which have been constructed in Chapter II. may be reversed, and will in that case constitute the rules of the Integral Calculus.

It is unnecessary, then, to add more, as we have brought the subject of definite integration to the point where the ordinary treatises on the Integral Calculus commence; and the rules given in them, and the results arrived at, are as true under the view here taken of the subject as under the other system.

164.] We proceed to give some geometrical applications of the theory of definite integration, as we shall hereby be enabled to complete the subject of the Application of the Calculus to Geometrical Problems.

RECTIFICATION OF CURVES.

A. *Plane Curves referred to Rectangular Co-ordinates.*

Since, by Equation (6) Art. 97.,

$$ds = (dy^2 + dx^2)^{\frac{1}{2}} = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx,$$

if it be required to find the length of the curve between points on the curve corresponding to abscissæ x and x_0, we must determine $\frac{dy}{dx}$ from the equation to the curve in terms of x, and then the length will be equal to

$$\int_{x_0}^{x} \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx.$$

B. *Plane Curves referred to Polar Co-ordinates.*

Since, by Art. 112. Equation (3),

$$ds = (dr^2 + r^2 d\theta^2)^{\frac{1}{2}} = \left(r^2 + \frac{dr^2}{d\theta^2}\right)^{\frac{1}{2}} d\theta,$$

we must, by means of the equation to the curve, express r and $\frac{dr}{d\theta}$ in terms of θ; and if θ and θ_0 be the values of θ corresponding to the extremities of the curve whose length is to be found, the length

$$= \int_{\theta_0}^{\theta} \left(r^2 + \frac{dr^2}{d\theta^2}\right)^{\frac{1}{2}} d\theta.$$

C. *Curves in Space.*

If it be required to determine the Length of a Curve of double Curvature, since, from Art. 139. Equation (17),

$$ds = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}} = \left(1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}\right)^{\frac{1}{2}} dx,$$

if x and x_0 be the values of x corresponding to the extremities of the arc whose length is to be determined, then, having found $\frac{dy}{dx}$ and $\frac{dz}{dx}$, in terms of x, from the equations to the curve, the length of the arc is equal to

$$\int_{x_0}^{x} \left(1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}\right)^{\frac{1}{2}} dx.$$

If it be more convenient to make the general term of the series which we have to sum, viz. ds, a function of any other variable, as e. g. of y, or r, or z, the necessary substitutions must be made, and the same result will be arrived at.

For examples, see Gregory's *Collection*, Integral Calculus, chap. ix. sect. 2.

QUADRATURE OF AREAS.

A. *Plane Areas and Rectangular Co-ordinates.*

165.] The principles of definite integration which we have discussed will also enable us to determine the area of a plane superficies contained between given lines, whether they be straight or curved, that is, to compare it with the area of a square; whence arises the name *quadrature.*

Suppose, now, the area to be resolved into a number of small infinitesimal areas, as in fig. 59. each of which is equal to $dy\, dx$, then the whole area will be equal to the sum of all these between limits which are assigned by the particular form and positions of the boundary lines; that is, if A = the whole area,

$$A = \int\int dy\, dx,$$

integrating first with respect to one variable, and then with respect to the other, the order of integration being usually indifferent. But, as the limits between which these integrations are to be performed depend on the data of each particular problem, we can only give general hints as to the method to be followed.

Suppose, then, it is required to find the area OMABT, fig. 59.

Let the equation to the bounding curve APB be

$$y = f(x),$$

$$\left.\begin{array}{l} OA = a, \\ AB = b, \end{array}\right\} \quad \left.\begin{array}{l} OM = x, \\ ME = y, \end{array}\right\}$$

where x and y refer to E any element of the area; then, if MN $= dx$, UR $= dy$, the area of E $= dx\, dy$, and the area OAB is the sum of all these elements.

Suppose we integrate first with respect to y, considering x to be constant: the effect of this process will be, that all the elements are added together which are contained between the lines PM, QN; and as we have to take account of the whole area PMNQ, we must integrate from $y =$ PM $= f(x)$, which is the superior limit, to $y = 0$, which is the inferior limit. Thus,

$$A = \int\int_0^{f(x)} dy\, dx = \int \Big[y\Big]_0^{f(x)} dx$$

$$= \int f(x)\, dx;$$

which last expression is the value of the differential slice PMNQ, which is the type of every similar slice between O and AB. All these, then, must be summed again from $x = \text{OA} = a$ to $x = 0$; that is, we must integrate as follows

$$\text{A} = \int_0^a f(x)\,dx.$$

Let it not be supposed that any inaccuracy of result arises from the circumstance that the differential slice is an imperfect rectangle at the point P where it meets the curve; for, though the real value of MPQN lies between

$$f(x)\,dx \text{ and } f(x+dx)\,dx,$$

yet the difference between these two, viz.

$$\{f(x+dx)-f(x)\}\,dx = f'(x)\,dx^2,$$

is an infinitesimal of a higher order, and therefore may be neglected.

Had the order of integration been reversed, and we had integrated with respect to x first, we should have calculated the area of a slice such as TSRU, and the superior limit would have been $\text{OA} = a$, and the inferior limit $\text{OK} =$ some function of y dependent on the equation to the curve, say $\phi(y)$; so that

$$\text{A} = \int \{a - \phi(y)\}\,dy,$$

which again must be integrated from $y = b$ to $y = 0$, and the result will give the true value of the area.

Bearing in mind that the process of integration is that of summing a series, of which the function to be integrated is the type of the terms, and the limits are the first and last terms, it is easy to apply the principle to every particular case. As suppose, for instance, it is required to find the area ASBR, in fig. 60. The element $\text{E} = dy\,dx$; integrating which with respect to x, from $x = \text{OL}$ to $x = \text{OK}$, we shall have the area TSRU; and having determined x in terms of y to the curve BRA, say $x = f(y)$, and $x = \text{OK} = \phi(y)$ to the curve BTA, we have

$$\text{A} = \int \{f(y) - \phi(y)\}\,dy;$$

which must be integrated again with respect to y, from the value of y at B to its value at A, which must be determined by the points of intersection of the curve.

If the axes be oblique, inclined to each other at the angle ω, then the element of the area $= dy\,dx \sin\omega$, and the processes of integration must be performed in the manner indicated above.

B. *Plane Areas referred to Polar Co-ordinates.*

166.] Consider r and θ to be the polar co-ordinates to E, any element of the area, fig. 61.; then, if $\text{SE} = r$, $\text{ESX} = \theta$, the element $= r\,dr\,d\theta$, and if the whole area $=$ A,

$$\text{A} = \iint r\,dr\,d\theta;$$

the integration being performed between the assigned limits. In this case we had better integrate first with respect to r, and afterwards with respect to θ; because, although every area may be resolved into triangular elements, such as PSQ, it may not admit of being analysed into circular rings; and, if we have to estimate the area included between S and P, the limits of r will be its particular value at P, say $f(\theta)$, and 0;

$$\therefore \quad \text{A} = \tfrac{1}{2}\int \Big[r^2\Big]_0^{f(\theta)} d\theta.$$

$\therefore \quad \frac{1}{2}\int r^2 d\theta$, between proper limits, is the value of the sectorial area swept out by the radius vector. If, however, we have to estimate an area such as that contained between the two curves TR and PQ, in fig. 62., then the first integration with respect to r must be taken between the limits $r = \text{SP}$ and $r = \text{SR}$.

For examples on these two Articles, see Gregory's *Examples on the Integral Calculus*, chap. ix. section 7.

C. *Curved Surfaces.*

167.] Suppose it is required to find the Area of a Surface such as that drawn in fig. 63. Then, resolving the surface into elements such as PQ, by means of planes parallel to the co-ordinate planes and infinitely near to one another, i. e. at distances dx and dy apart, we thus determine the area of PQ. Its projection on the plane of (xy) is $dy\,dx$; and, since in the limit the inclination of PQ to the plane of (xy) is the same as that of the tangent plane, PQ is inclined to the plane of (xy) at, according to the notation of Art. 133. Equation (13),

$$\cos^{-1} \frac{\left(\frac{d\text{F}}{dz}\right)}{\left\{\left(\frac{d\text{F}}{dx}\right)^2 + \left(\frac{d\text{F}}{dy}\right)^2 + \left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}};$$

$$\therefore \quad \text{area of PQ} = \frac{dx\,dy}{\left(\frac{d\text{F}}{dz}\right)} \left\{\left(\frac{d\text{F}}{dx}\right)^2 + \left(\frac{d\text{F}}{dy}\right)^2 + \left(\frac{d\text{F}}{dz}\right)^2\right\}^{\frac{1}{2}}.$$

And if the equation to the surface be given in the form

$$z = f(x, y),$$

then, see Art. 133.,

$$\text{area of PQ} = \left(1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}\right)^{\frac{1}{2}} dy\,dx;$$

$$\therefore \quad \text{S} = \text{surface} = \iint \left(1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}\right)^{\frac{1}{2}} dy\,dx.$$

The integration being performed in the order and between the limits assigned by the conditions of the problem.

If the Surface be one of Revolution, we can determine its area with greater facility by the following method.

Consider OPB, fig. 64., to be the curve by the revolution of which round OMX the surface is generated. Let $\text{OM} = x$, $\text{MP} = y$, $\text{MN} = dx$, $\text{PQ} = ds$. Then, resolving the surface of revolution into circular annuli, such as that generated by the revolution of PQ round OX, the surface of each annulus is $2\pi y\,ds$, and the whole surface will be the sum of such annuli between the assigned limits. If, therefore, $y = f(x)$ be the equation to the generating curve, $ds = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx$; and if x and x_0 be the abscissæ corresponding to the limits of the surface,

then $$S = \int_{x_0}^{x} 2\pi y \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx.$$

For examples, see Gregory's *Collection*, Integral Calculus, chapter ix. section 4.

CUBATURE OF SOLIDS.

168.] First, let us consider a Solid referred to rectangular co-ordinate Axes, as in fig. 63., and divided into elemental parallelepipedons by planes parallel to the co-ordinate planes. Then, if $x\,y\,z$ are the co-ordinates to any element of the solid, $dx\,dy\,dz$ is the volume of the element; and if the whole volume = V,

$$V = \iiint dx\,dy\,dz,$$

the integration being performed in such order as the problem admits of, and between the limits which it requires. As, for instance, suppose the solid to be such as that delineated in fig. 63. Let us integrate first with respect to z, the superior limit being the value of z to the surface, which is determined by its equation, and the inferior limit being zero. The result of this integration will be the volume of the prism PN, which will be the type of all such similar prisms, into which the solid may be resolved. Then, integrating with respect to y from $y =$ MR to $y = 0$, we shall obtain the value of the whole differential slice SRM, of which the thickness is dx; and, lastly, integrating with respect to x, between the necessary limits, we shall obtain an expression for the content of the solid, which is the sum of all such differential slices, as SRM.

If the elements of the solid are referred to polar co-ordinates, we determine the content of an element as follows. Suppose the solid resolved, as in fig. 65., by planes passing through the axis of z, and inclined to each other at an angle $d\phi$; and then by small planes drawn through the origin at right angles to these planes, and inclined at angles θ and $\theta + d\theta$ to the axis of z.

Let, then, AOQ $= \phi$, COE $= \theta$, OE $= r$; then the content of the element E is $dr.rd\theta.r\sin\theta d\phi$;

$$\therefore \quad \mathrm{V} = \iiint r^2 \sin\theta \, dr \, d\theta \, d\phi;$$

and integrating with respect to r, θ, ϕ in order, and between the assigned limits, we shall be able to determine the volume of the solid.

And if the Solid be one of Revolution, it may be analysed more easily as follows. Suppose the solid to be generated by the revolution of the area (fig. 64.) OBA round OX. Let OM $= x$, MP $= y$, and resolve the solid into circular slices by drawing planes perpendicular to the axis of revolution at distances dx apart. Then the content of each differential slice is $\pi y^2 dx$, where y is the ordinate to the generating curve. If, therefore, $y = f(x)$ be the equation to the curve, and x x_0 are the extreme values of x, i. e. the limits of integration, then

$$\mathrm{V} = \pi \int_{x_0}^{x} \{f(x)\}^2 dx.$$

For examples illustrative of these processes of definite integration, see Gregory's *Collection*, Integral Calculus, chapter ix. section 3.

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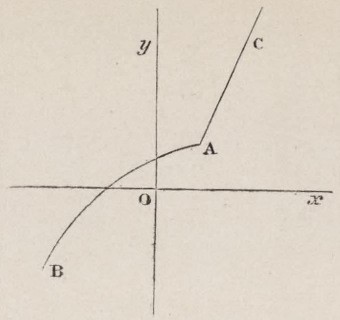

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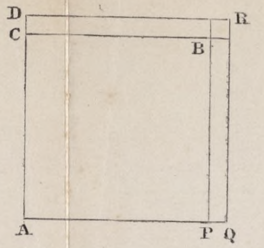

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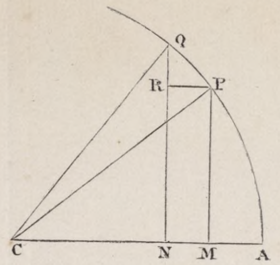

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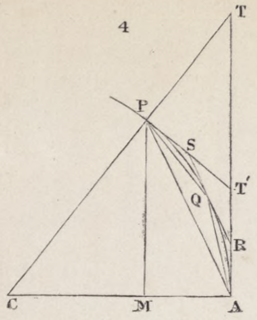

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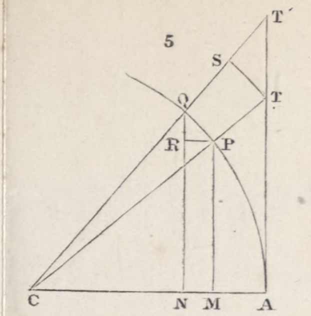

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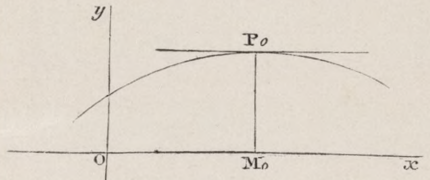

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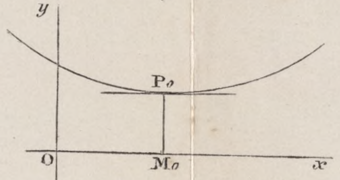

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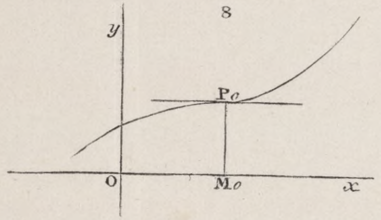

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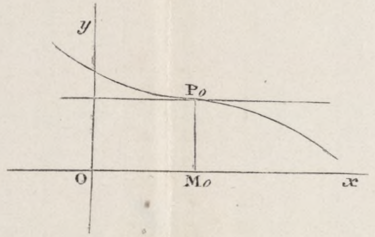

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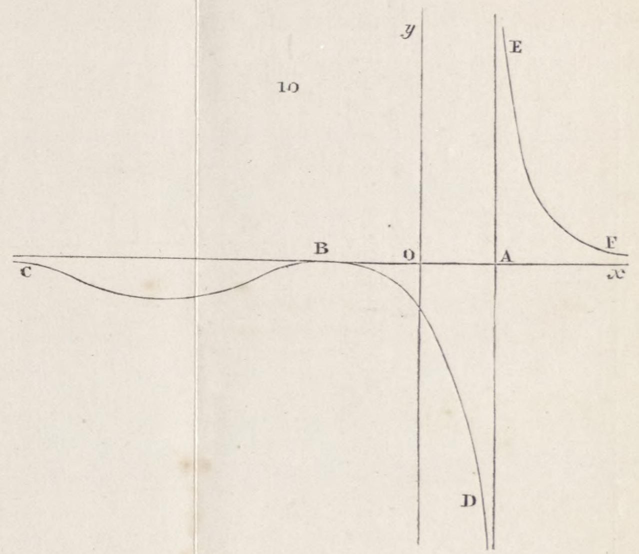

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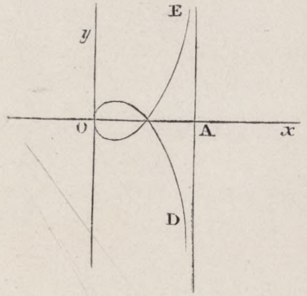

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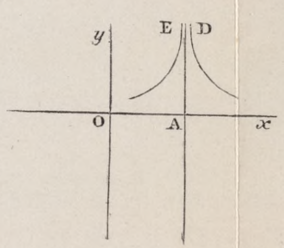

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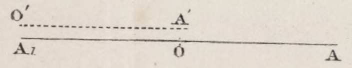

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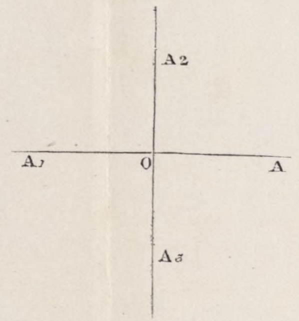

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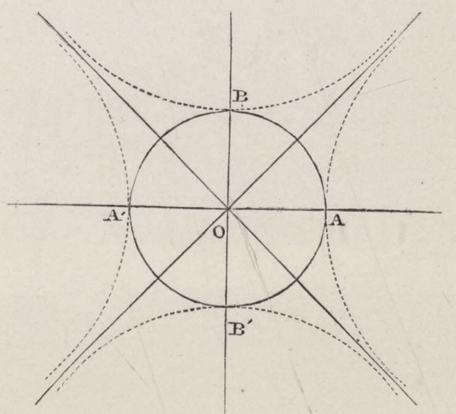

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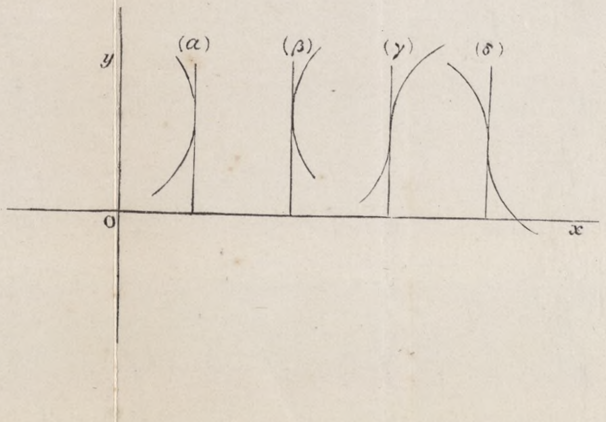

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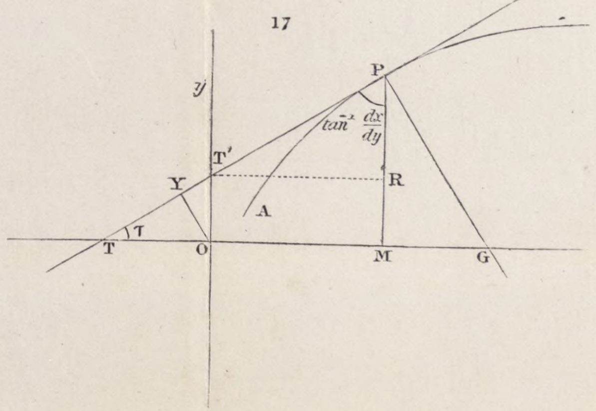

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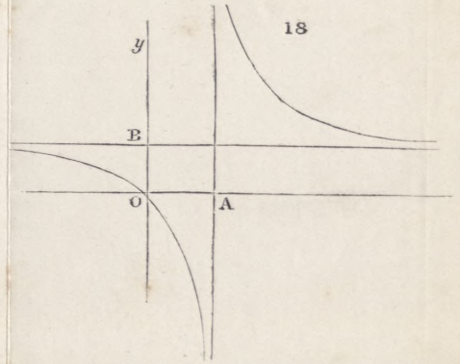

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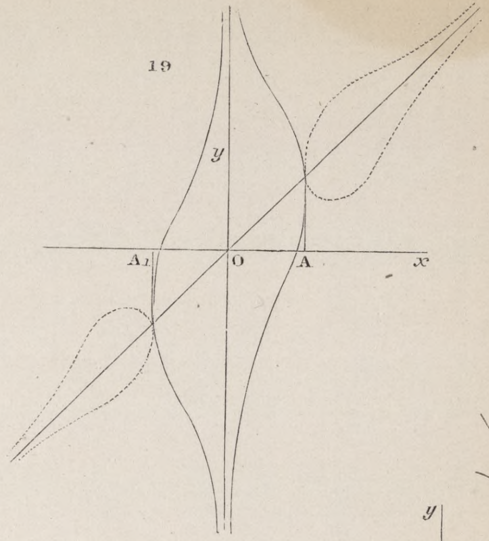

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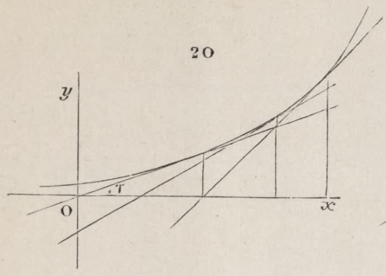

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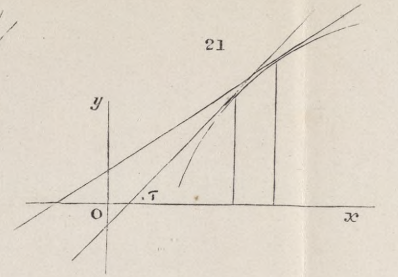

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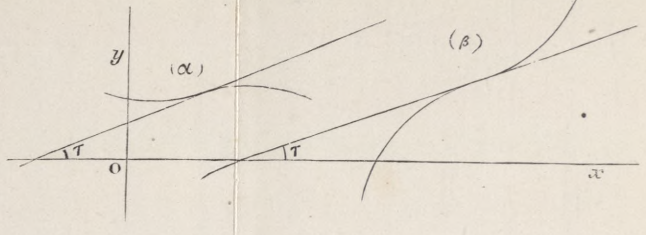

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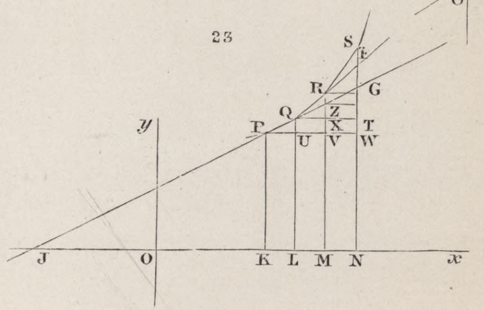

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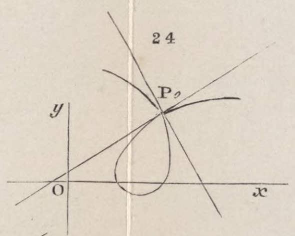

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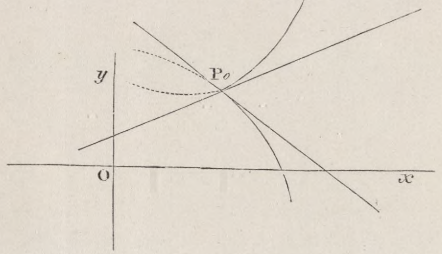

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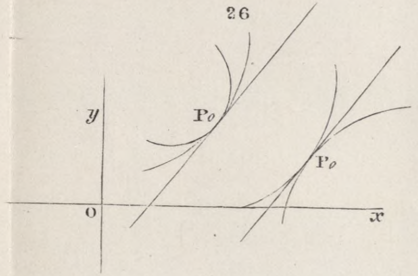

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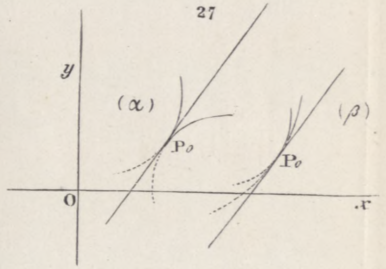

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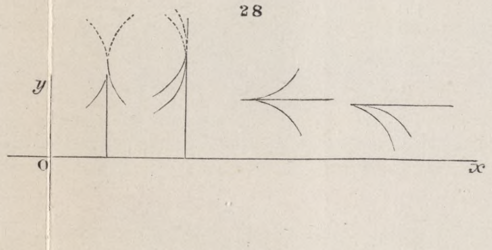

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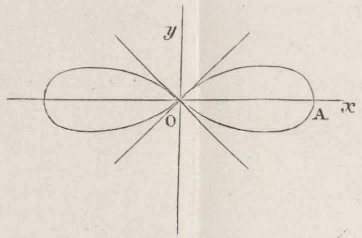

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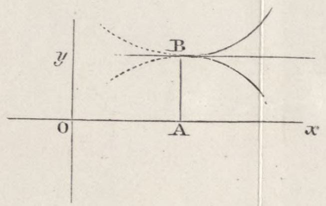

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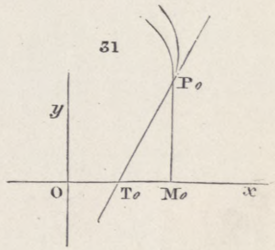

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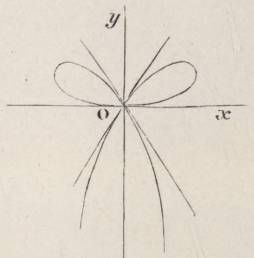

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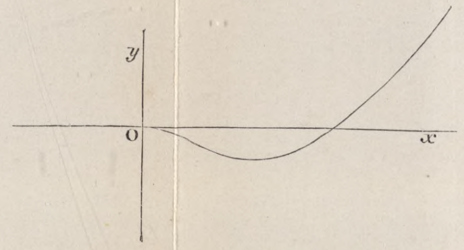

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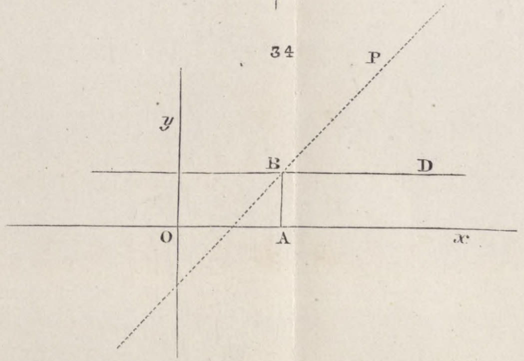

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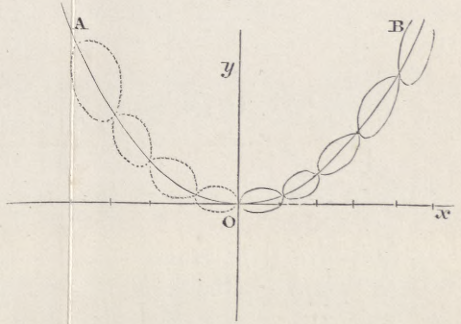

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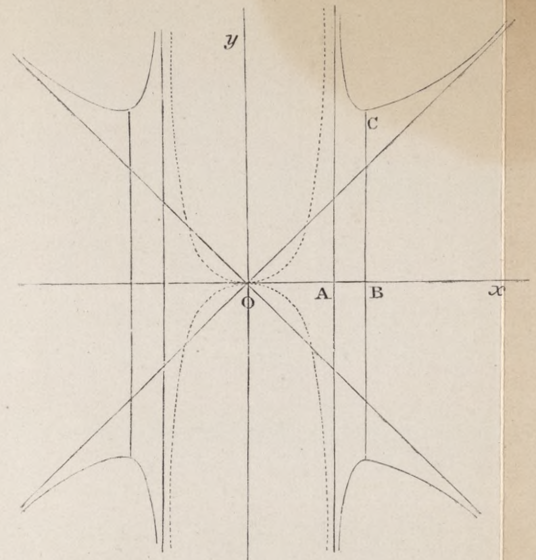

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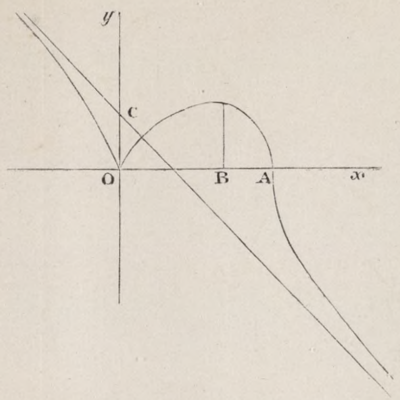

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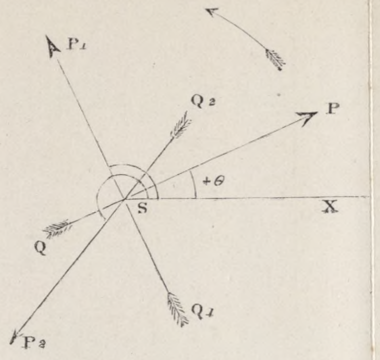

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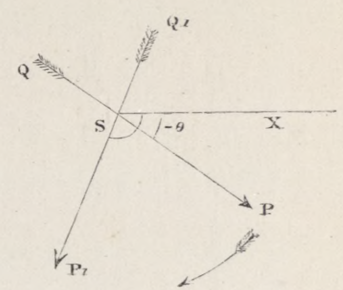

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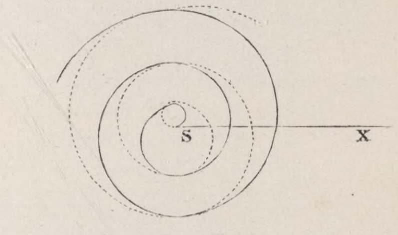

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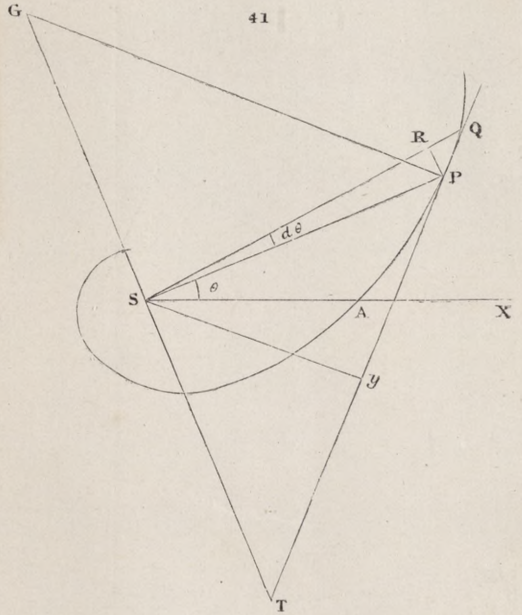

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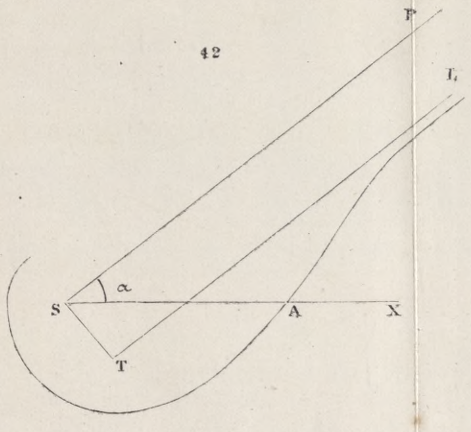

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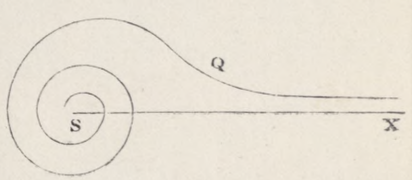

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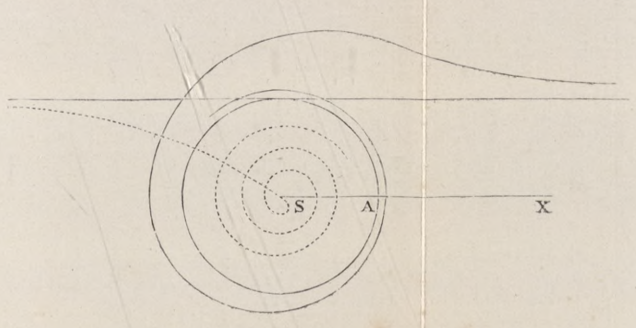

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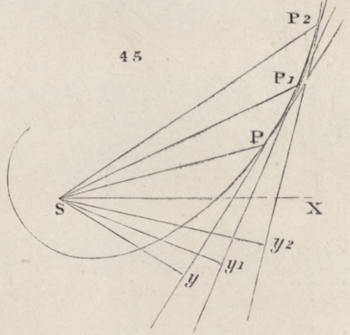

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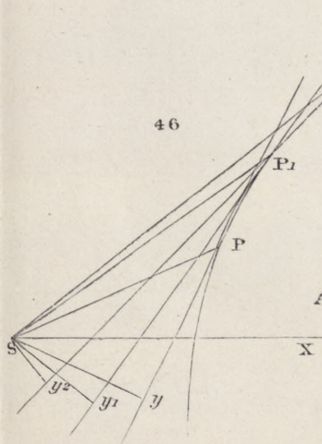

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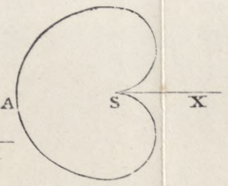

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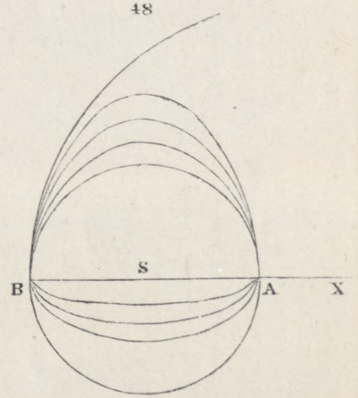

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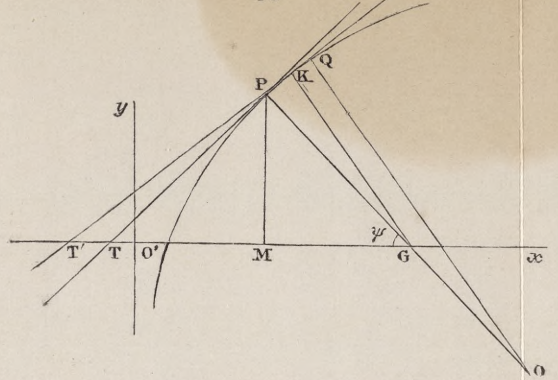

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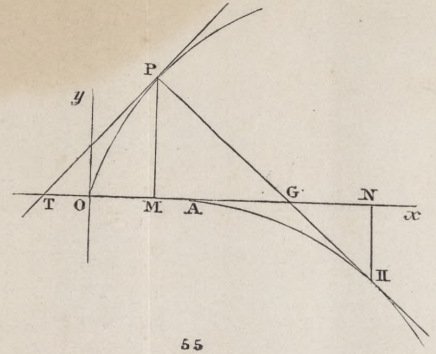

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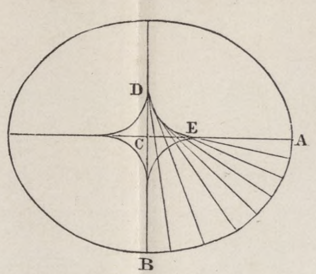

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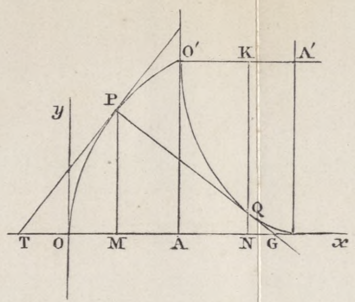

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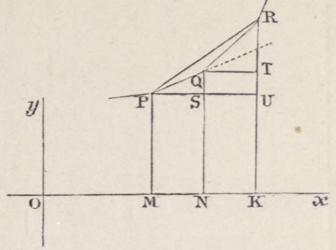

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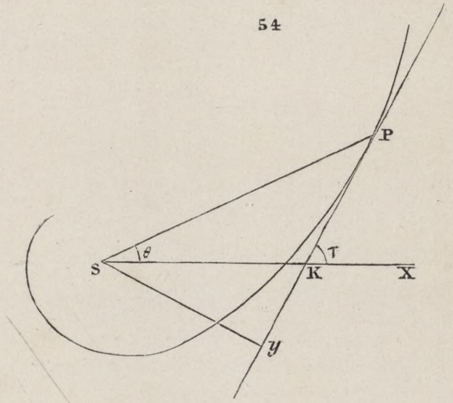

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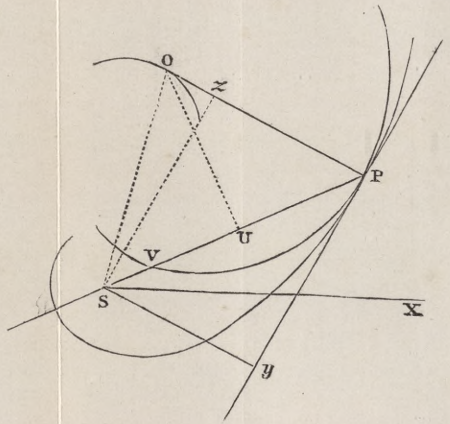

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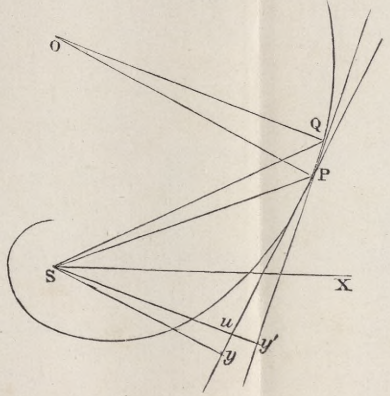

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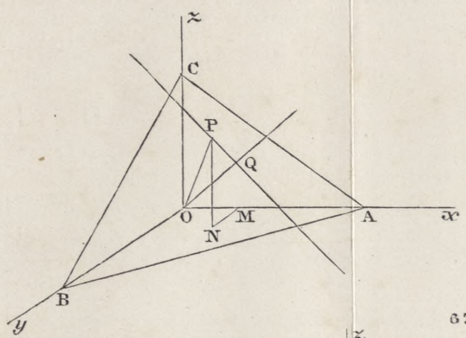

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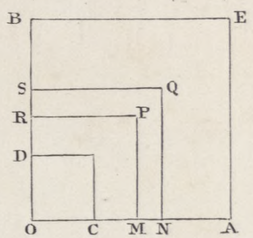

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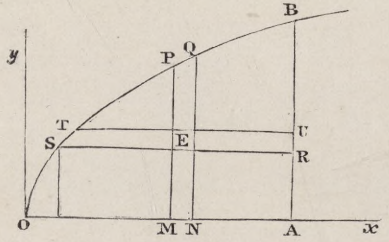

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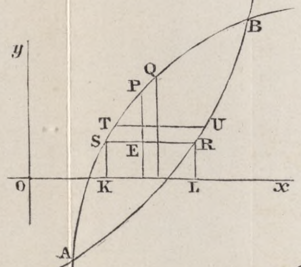

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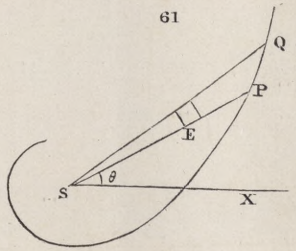

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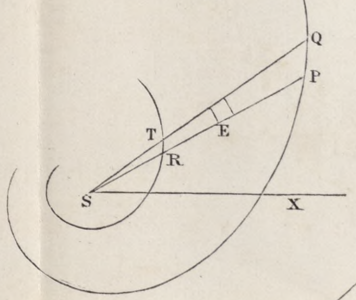

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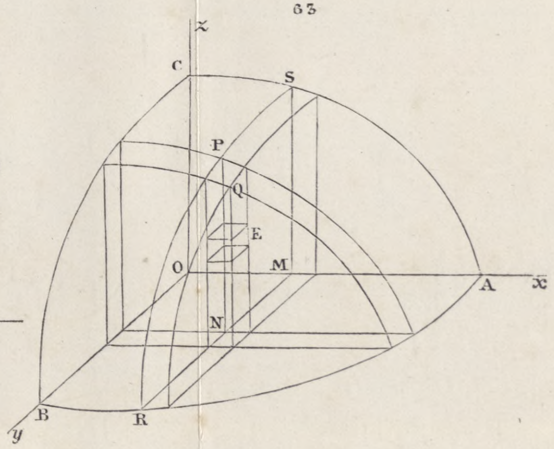

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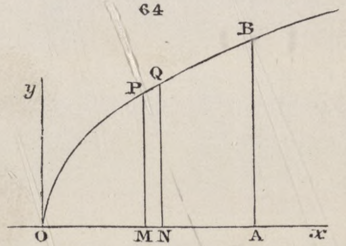

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