Arch. Mech., 40, 5–6, pp. 857–869, Warszawa 1988

Homogeneous solutions and energy of a linear anisotropic elastic strip

R. WOJNAR (WARSZAWA)

HOMOGENEOUS solutions for a linear anisotropic infinite elastic strip are given when the axis of anisotropy of the strip is not parallel to the boundary line. The solutions are then used to obtain the Airy stress function for a rectangular anisotropic disc subject to tension.

Przedstawiono rozwiązania jednorodne dla liniowego anizotropowego nieskończonego pasma sprężystego, w którym oś anizotropii nie jest równoległa do brzegu. Rozwiązanie to wykorzystano następnie do otrzymania funkcji naprężeń dla prostokątnej tarczy anizotropowej poddanej rozciąganiu.

Представлены однородные решения для линейной анизотропной бесконечной упругой полосы, в которой ось анизотропии не параллельна границе. Это решение использовано затем для получения функций напряжений для прямоугольного анизотропного диска, подвергнутого растяжению.

1. Introduction

CONSIDER a classical problem of linear elastostatics for an infinite anisotropic strip when the axis of the strip does not coincide with the axis of anisotropy ("off axis" case). The boundaries of the strip are traction-free and loads are applied to the strip at infinity.

To analyse the stresses in the strip let us use the principle of complementary energy in which the functional is expressed in terms of the so-called homogeneous solutions f_s (FADLE [1], PAPKOVITSCH [2], KHACHATRIAN [3]). The solutions meet the required boundary conditions as well as the compatibility equation. Similar solutions were employed by CHOI and HORGAN [4], to study an anisotropic strip in which the geometrical axis of the strip coincided with one of the anisotropy axis ("in axis" case). It turns out that, in our case, the energy functional depends explicitly on the three compliance constants only,

$$b_{11} = C_{1111}, \quad b_{12} = C_{1122}, \quad b_{22} = C_{2222}.$$

This observation allows us to work out simple formulae for stresses in a rectangular disc.

2. Basic relations

The field equations describing the behaviour of the linear elastic homogeneous anisotropic body in the absence of body forces and in the plane state of stress are as follows (cf. [5] and Appendix A):

$$E_x = u_{,x}, \quad E_y = v_{,y}, \quad 2E_{xy} = u_{,y}+v_{,x}, \tag{2.1}$$

$$S_{x,x}+S_{xy,y} = 0, \quad S_{xy,x}+S_{y,y} = 0, \tag{2.2}$$

$$\begin{aligned} E_x &= b_{11}S_x+b_{12}S_y+b_{16}S_{xy}, \\ E_y &= b_{12}S_x+b_{22}S_y+b_{26}S_{xy}, \\ 2E_{xy} &= b_{16}S_x+b_{26}S_y+b_{66}S_{xy}, \end{aligned} \tag{2.3}$$

where u, v are components of the displacement in x and y directions, respectively, E_x, E_y, E_{xy} are components of the deformation and S_x, S_y, S_{xy} are the stress components. Coefficients b_{11}, b_{12}, b_{22}, b_{16}, b_{26}, b_{66} are components of the compliance tensor.

Between the deformation components the following compatibility relation takes place

$$E_{x,yy}+E_{y,xx} = 2E_{xy,xy} \tag{2.4}$$

which, expressed in terms of the stresses, is of the form

$$bS_{x,xx}-2b_{16}S_{x,xy}+b_{11}S_{x,yy}+b_{22}S_{y,xx}-2b_{26}S_{y,xy} = 0, \tag{2.5}$$

where

$$b = 2b_{12}+b_{66}. \tag{2.6}$$

Expressing the stress components by a stress function $\Phi = \Phi(x, y)$ according to

$$S_x = \Phi_{,yy}, \quad S_y = \Phi_{,xx}, \quad S_{xy} = -\Phi_{,xy}, \tag{2.7}$$

the following form of compatibility relation is obtained

$$b_{22}\frac{\partial^4\Phi}{\partial x^4}-2b_{26}\frac{\partial^4\Phi}{\partial x^3\partial y}+b\frac{\partial^4\Phi}{\partial x^2\partial y^2}-2b_{16}\frac{\partial^4\Phi}{\partial x\partial y^3}+b_{11}\frac{\partial^4\Phi}{\partial y^4} = 0. \tag{2.8}$$

Let the stress function be of the form

$$\Phi = \Phi(x, y) = \sum_{s=0}^{\infty} e^{-\lambda_s x} f_s(y). \tag{2.9}$$

Functions f_s satisfy the following differential equation

$$b_{11}f_s^{\mathrm{IV}}+2b_{16}\lambda_s f_s'''+b\lambda_s^2 f_s''+2b_{26}\lambda_s^3 f_s'+b_{22}\lambda_s^4 f_s = 0. \tag{2.10}$$

Using the substitution

$$f_s = C_s e^{\mu_s y}, \tag{2.11}$$

we arrive at the following characteristic equation

$$b_{11}\mu_s^4+2b_{16}\lambda_s\mu_s^3+b\lambda_s^2\mu_s^2+2b_{26}\lambda_s^3\mu_s+b_{22}\lambda_s^4 = 0, \tag{2.12}$$

which, after introduction of a characteristic parameter

$$\omega = \frac{\mu_s}{\lambda_s}, \tag{2.13}$$

takes the form

$$b_{11}\omega^4+2b_{16}\omega^3+b\omega^2+2b_{26}\omega+b_{22} = 0. \tag{2.14}$$

Since Eq. (2.14) has, in general, four different roots, ω_i, $i = 1, \ldots, 4$, functions f_s become

$$f_s = \sum_{i=1}^{4} C_{si} e^{\lambda_s \omega_i y}. \tag{2.15}$$

Constants C_{si} should be determined from the boundary conditions (cf. Appendix B).

3. Boundary conditions

The boundaries $y = \pm 1$ of the strip are free of mechanical tractions; thus

$$S_y(x, y) = 0, \quad S_{xy}(x, y) = 0 \quad \text{for} \quad y = \pm 1, \quad |x| < \infty. \tag{3.1}$$

These conditions can be written in terms of the stress function in the form

$$\Phi_{,xx}(x, y) = 0, \quad \Phi_{,xy}(x, y) = 0 \quad \text{for} \quad y = \pm 1, \quad |x| < \infty, \tag{3.2}$$

or, using Eq. (2.9), in the form

$$f_s(\pm 1) = 0, \quad f_s'(\pm 1) = 0, \tag{3.3}$$

Functions f_s are called homogeneous solutions since they satisfy homogeneous boundary conditions.

4. Quasi-orthogonality of the homogeneous solutions

Multiply Eq. (2.10) by $\lambda_r^2 f_r$, and the same equation taken with index r — by $\lambda_s^2 f_s$. Next, substract the equations and integrate the result over the interval $(-1,1)$. Integrating the first three terms by parts and using the conditions (3.3) we find

$$\int_{-1}^{1} dy[-b_{11}(\lambda_r^2 f_s''' f_r' - \lambda_s^2 f_r''' f_s') + 2b_{16} \lambda_r \lambda_s (\lambda_s f_r'' f_s' - \lambda_r f_s'' f_r') \\ + 2b_{26} \lambda_r^2 \lambda_s^2 (\lambda_s f_s' f_r - \lambda_r f_r' f_s) + b_{22} \lambda_s^2 \lambda_r^2 (\lambda_s^2 - \lambda_r^2) f_r f_s] = 0. \tag{4.1}$$

Next, by virtue of (3.3) we obtain

$$\int_{-1}^{1} dy(f_r f_s' + f_r' f_s) = 0, \tag{4.2}$$

$$\int_{-1}^{1} dy(f_r' f_s'' + f_r'' f_s') = 0. \tag{4.3}$$

Integrating once more the term with coefficient b_{11} in Eq. (4.1) by parts and using (4.2)–(4.3) we arrive at the following relation:

$$\int_{-1}^{1} dy\left(b_{11} f_r'' f_s'' + 2b_{16} \frac{\lambda_r \lambda_s}{\lambda_r - \lambda_s} f_r'' f_s' + 2b_{26} \frac{\lambda_r^2 \lambda_s^2}{\lambda_r - \lambda_s} f_r f_s' - b_{22} \lambda_r^2 \lambda_s^2 f_r f_s\right) = 0, \tag{4.4}$$

where $\lambda_r \neq \lambda_s$.

If $b_{16} = 0$ and $b_{26} = 0$ ("in axis" case), we obtain the quasi-orthogonality relation given by CHOI and HORGAN [4],

$$\int_{-1}^{1} dy(b_{11} f_r'' f_s'' - b_{22} \lambda_r^2 \lambda_s^2 f_r f_s) = 0.$$

In an isotropic case, when $b_{11} = b_{22}$, we get the relations of generalized orthogonality found by Papkovitsch, [2],

$$\int_{-1}^{1} dy(f_r'' f_s'' - \lambda_r^2 \lambda_s^2 f_r f_s) = 0.$$

5. Expression of displacements by homogeneous solutions

Stress components expressed by the homogeneous solutions read

$$S_x = \sum_s e^{-\lambda_s x} f_s'', \quad S_y = \sum_s \lambda_s^2 e^{-\lambda_s x} f_s, \quad S_{xy} = \sum_s \lambda_s e^{-\lambda_s x} f_s'. \tag{5.1}$$

Hence, according to the constitutive relation (2.3), we have

$$\frac{\partial u}{\partial x} = \sum_s e^{-\lambda_s x}(b_{11} f_s'' + b_{12} \lambda_s^2 f_s + b_{16} \lambda_s f_s'), \tag{5.2}$$

$$\frac{\partial v}{\partial y} = \sum_s e^{-\lambda_s x}(b_{12} f_s'' + b_{22} \lambda_s^2 f_s + b_{26} \lambda_s f_s'), \tag{5.3}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum_s e^{-\lambda_s x}(b_{16} f_s'' + b_{26} \lambda_s^2 f_s + b_{66} \lambda_s f_s'). \tag{5.4}$$

Integrating (5.2) with respect to x we get

$$u = -\sum_s e^{-\lambda_s x}\left(b_{11} \frac{1}{\lambda_s} f_s'' + b_{12} \lambda_s f_s + b_{16} f_s'\right) + g(y), \tag{5.5}$$

where $g(y)$ is an unknown function. Hence, and in virtue of (5.4), we find

$$\frac{\partial v}{\partial x} = \sum_s e^{-\lambda_s x}\left(b_{16} f_s'' + b_{26} \lambda_s^2 f_s + b_{66} \lambda_s f_s' + b_{11} \frac{1}{\lambda_s} f_s''' + b_{12} \lambda_s f_s' + b_{16} f_s''\right) - g'(y), \tag{5.6}$$

and after integration

$$v = -\sum_s e^{-\lambda_s x}\left[b_{11} \frac{1}{\lambda_s^2} f_s''' + 2b_{16} \frac{1}{\lambda_s} f_s'' + (b_{12} + b_{66}) f_s' + b_{26} \lambda_s f_s\right] - xg'(y) + f(y), \tag{5.7}$$

where $f(y)$ is an arbitrary function.

Substituting (5.7) into Eq. (5.3) we obtain

$$-\sum_s e^{-\lambda_s x}\left[b_{11} \frac{1}{\lambda_s^2} f_s^{\mathrm{IV}} + 2b_{16} \frac{1}{\lambda_s} f_s''' + (b_{12} + b_{66}) f_s'' + b_{26} \lambda_s f_s'\right] - xg''(y) + f'(y)$$

$$= \sum_s e^{-\lambda_s x}(b_{12} f_s'' + b_{22} \lambda_s^2 f_s + b_{26} \lambda_s f_s'),$$

or, in virtue of Eq. (2.10)

$$-xg''(y)-f'(y)=0.$$

Hence

$$g''(y)=0, \quad f'(y)=0$$

and

$$g(y)=g_0y+g_1, \quad f(y)=f_0, \tag{5.8}$$

where g_0, g_1 and f_0 are constants.

Therefore the displacements field reads

$$u=-\sum_s e^{-\lambda_s x}\left(b_{11}\frac{1}{\lambda_s}f_s''+b_{16}f_s'+b_{12}\lambda_s f_s\right)+g_0y+g_1, \tag{5.9}$$

$$v=-\sum_s e^{-\lambda_s x}\left[b_{11}\frac{1}{\lambda_s^2}f_s'''+2b_{16}\frac{1}{\lambda_s}f_s''+(b_{12}+b_{66})f_s'+b_{26}\lambda_s f_s\right]-xg_0+f_0. \tag{5.10}$$

5.1. Alternative expression for the displacement component v

Let us integrate both sides of the Eq. (2.10) from $y=-1$ to y. Taking into account the boundary conditions, we get

$$(b_{11}f_s'''+2b_{16}\lambda_s f_s''+b\lambda_s^2 f_s'+2b_{26}\lambda_s^3 f_s)_{(y)}+b_{22}\lambda_s^4\int_{-1}^{y} f_s d\eta=(b_{11}f_s'''+2b_{16}\lambda_s f_s'')_{(y=-1)}.$$

Hence, taking also into account the definition (2.6) of b, we find

$$b_{11}\frac{1}{\lambda_s^2}f_s'''+2b_{16}\frac{1}{\lambda_s}f_s''+(b_{12}+b_{66})f_s'+b_{26}\lambda_s f_s=h_s-b_{12}f_s'-b_{26}\lambda_s f_s-b_{22}\lambda_s^2\int_{-1}^{y} f_s d\eta,$$

where

$$h_s\equiv\left(b_{11}\frac{1}{\lambda_s^2}f_s'''+2b_{16}\frac{1}{\lambda_s}f_s''\right)_{(y=-1)}.$$

Therefore

$$v=\sum_s e^{-\lambda_s x}\left(b_{12}f_s'+b_{22}\lambda_s^2\int_{-1}^{y} f_s d\eta+b_{26}\lambda_s f_s\right)-h-g_0x+f_0, \tag{5.11}$$

where

$$h=h(x)=\sum_s e^{-\lambda_s x}h_s.$$

6. Energy of a rectangular strip element

Elastic energy contained in a strip element between $x=L_1$ and $x=L_2$ (cf. Fig. 1) is equal to

$$E=\frac{1}{2}\int_{L_1}^{L_2} dx\int_{-1}^{1} dy S_{ij}E_{ij}, \quad i,j=1,2, \tag{6.1}$$

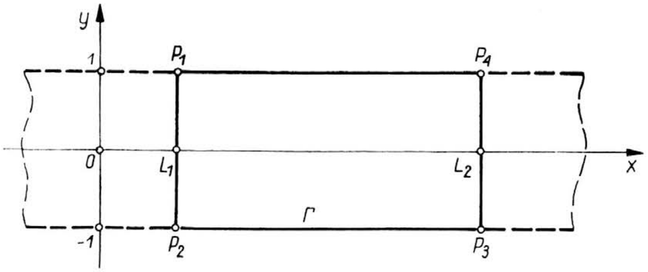

FIG. 1. Rectangular element of the strip with boundary Γ.

or, making use of the divergence theorem,

$$E = \frac{1}{2}\int\limits_{\Gamma} dl\, S_{ij} u_i n_j, \tag{6.2}$$

where n_j represents the outward normal to the contour Γ of the element. Here tensor notation of the components is used and

$$\begin{aligned} &u_1 = u, \quad u_2 = v, \\ &E_{11} = E_x, \quad E_{22} = E_y, \quad E_{12} = E_{xy}, \\ &S_{11} = S_x, \quad S_{22} = S_y, \quad S_{12} = S_{xy}. \end{aligned} \tag{6.3}$$

On the segment P_1P_2 of the contour we have $dl = -dy$ and $\mathbf{n} = (-1, 0)$. Similarly,

$$\begin{aligned} &\text{on} \quad P_2P_3: \quad dl = dx \quad \text{and} \quad \mathbf{n} = (0, -1), \\ &\text{on} \quad P_3P_4: \quad dl = dy \quad \text{and} \quad \mathbf{u} = (1, 0), \\ &\text{on} \quad P_4P_1: \quad dl = -dx \quad \text{and} \quad \mathbf{n} = (0, 1). \end{aligned}$$

Thus the energy expression is of the form

$$E = \frac{1}{2}\Bigg\{\Bigg[\int\limits_{1}^{-1} (-dy) S_{i1} u_i(-1)\Bigg]_{x=L_1} + \Bigg[\int\limits_{L_1}^{L_2} dx\, S_{i2} u_i(-1)\Bigg]_{y=-1}$$

$$+ \Bigg[\int\limits_{-1}^{1} dy\, S_{i1} u_i(+1)\Bigg]_{x=L_2} + \Bigg[\int\limits_{L_2}^{L_1} (-dx) S_{i2} u_i(+1)\Bigg]_{y=1}\Bigg\}.$$

However, according to the boundary conditions (3.1), the integrals taken along the paths P_2P_3 and P_4P_1 vanish and

$$E = \frac{1}{2}\int\limits_{-1}^{1} dy [(S_{i1} u_i)_{x=L_2} - (S_{i1} u_i)_{x=L_1}]. \tag{6.4}$$

Denoting

$$[(\ldots)]_{L_1}^{L_2} = (\ldots)_{x=L_2} - (\ldots)_{x=L_1}, \tag{6.5}$$

we write

$$E = \frac{1}{2} \int_{-1}^{1} dy [S_x u + S_{xy} v]_{L_1}^{L_2}. \tag{6.6}$$

Here

$$S_x = \mathrm{Re} \sum_s e^{-\lambda_s x} f_s'', \tag{6.7}$$

$$S_{xy} = \mathrm{Re} \sum_s \lambda_s e^{-\lambda_s x} f_s', \tag{6.8}$$

and

$$u = -\mathrm{Re} \sum_s e^{-\lambda_s x} \left(b_{11} \frac{1}{\lambda_s} f_s'' + b_{12} \lambda_s f_s + b_{16} f_s' \right) + g_0 y + g_1, \tag{6.9}$$

$$v = -\mathrm{Re} \sum_s e^{-\lambda_s x} \left[b_{11} \frac{1}{\lambda_s^2} f_s''' + 2b_{16} \frac{1}{\lambda_s} f_s'' + (b_{12} + b_{66}) f_s' + b_{26} \lambda_s f_s \right] - g_0 x + f_0, \tag{6.10}$$

or, alternatively,

$$v = \mathrm{Re} \sum_s e^{-\lambda_s x} \left(b_{12} f_s' + b_{22} \lambda_s^2 \int_{-1}^{y} f_s d\eta + b_{26} \lambda_s f_s \right) - \mathrm{Re}\, h(x) - g_0 x + f_0. \tag{6.11}$$

After integration by parts and using the free boundary conditions (cf. relations (4.2)–(4.3)) we obtain

$$\begin{aligned} E = \frac{1}{2} \sum_r \sum_s \int_{-1}^{1} dy \Bigg\{ & b_{11} \left[\mathrm{Re}(e^{-\lambda_r x} f_r'') \mathrm{Re} \left(\frac{1}{\lambda_s} e^{-\lambda_s x} f_s'' \right) \right. \\ & \left. + \mathrm{Re}(\lambda_r e^{-\lambda_r x} f_r'') \mathrm{Re} \left(\frac{1}{\lambda_s^2} e^{-\lambda_s x} f_s'' \right) \right] - b_{66} [\mathrm{Re}(e^{-\lambda_r x} \lambda_r f_r') \mathrm{Re}(e^{-\lambda_s x} f_s')] \\ & - 2b_{16} \left[\mathrm{Re}(e^{-\lambda_r x} \lambda_r f_r') \mathrm{Re} \left(e^{-\lambda_s x} \frac{1}{\lambda_s} f_s'' \right) \right] \Bigg\}_{x=L_1}^{x=L_2}, \end{aligned} \tag{6.12}$$

when Eqs. (6.7)–(6.8) and (6.9)–(6.10) are inserted into Eq. (6.6), or alternatively

$$\begin{aligned} E = \frac{1}{2} \sum_r \sum_s \int_{-1}^{1} dy \Bigg\{ & - b_{11} \mathrm{Re}(e^{-\lambda_r x} f_r'') \mathrm{Re} \left(\frac{1}{\lambda_s} e^{-\lambda_s x} f_s'' \right) \\ & + 2b_{12} \mathrm{Re}(e^{-\lambda_r x} f_r') \mathrm{Re}(\lambda_s e^{-\lambda_s x} f_s') - b_{22} \mathrm{Re}(e^{-\lambda_r x} \lambda_r f_r) \mathrm{Re}(\lambda_s^2 e^{-\lambda_s x} f_s) \Bigg\}_{x=L_1}^{x=L_2}, \end{aligned} \tag{6.13}$$

when Eq. (6.10) is replaced with expression (6.11). In both expressions (6.12)–(6.13) only three compliance constants are present explicitly. Moreover, for the special case of anisotropy, i.e. orthotropy, the energy given by Eq. (6.13) is formally the same as that expressed in the principal axis.

7. Complementary energy principle

In order to solve a boundary value problem for the rectangular anisotropic elastic disc shown in Fig. 1, the vertical boundaries of which are subject to prescribed displacements $\hat{u}$ and $\hat{v}$, while its horizontal boundaries are stress-free, we use the principle of complementary energy expressed by the homogeneous solutions introduced in the previous sections.

The complementary energy is given by the formula

$$F = E - E_B, \tag{7.1}$$

where

$$E_B = \int_{\Gamma} S_{ij} n_j \hat{u}_i \, dl$$

or

$$E_B = \int_{-1}^{1} dy [\hat{u} S_x + \hat{v} S_{xy}]_{x=L_1}^{x=L_2},$$

with

$$\left.\begin{aligned} \hat{u}(y) &= u(x, y) \\ \hat{v}(y) &= v(x, y) \end{aligned}\right\} \quad \text{on} \quad x = L_1 \quad \text{or} \quad x = L_2, \quad |y| < 1.$$

Thus

$$E_B = \sum_s \int_{-1}^{1} dy [\hat{u} \operatorname{Re}(e^{-\lambda_s x} f_s'') + \hat{v} \operatorname{Re}(e^{-\lambda_s x} \lambda_s f_s')]_{x=L_1}^{x=L_2}. \tag{7.2}$$

One can show (cf. Appendix B) that

$$f_s = Z_s \Psi_s, \tag{7.3}$$

where Z_s is an unknown constant and Ψ_s is a known function of y. If we introduce the notations

$$\begin{gathered} P_s = P_s(x, y) \equiv e^{-\lambda_s x} \Psi_s(y), \\ Q_s = \frac{1}{\lambda_s} P_s, \qquad R_s = \lambda_s P_s, \qquad S_s = \lambda_s^2 P_s, \end{gathered} \tag{7.4}$$

we get for the stress function (2.9)

$$\Phi = \sum_s Z_s P_s \tag{7.5}$$

and the complementary energy takes the form

$$F = \sum_r \int_{-1}^{1} dy \left\{ \frac{1}{2} \sum_s [-b_{11} \operatorname{Re}(Z_r P_r'') \operatorname{Re}(Z_s Q_s'') + 2b_{12} \operatorname{Re}(Z_r P_r') \operatorname{Re}(Z_s R_s') \right.$$
$$\left. - b_{22} \operatorname{Re}(Z_r R_r) \operatorname{Re}(Z_s S_s)] - [\hat{u} \operatorname{Re}(Z_r P_r'') + \hat{v} \operatorname{Re}(Z_r R_r')] \right\}_{x=L_1}^{x=L_2}.$$

Further, if we put

$$Z_s^R = \operatorname{Re} Z_s, \qquad Z_s^I = \operatorname{Im} Z_s$$

and

$$\begin{aligned}(P_s^R, Q_s^R, R_s^R, S_s^R) &= \operatorname{Re}(P_s, Q_s, R_s, S_s),\\ (P_s^I, Q_s^I, R_s^I, S_s^I) &= \operatorname{Im}(P_s, Q_s, R_s, S_s),\end{aligned}$$

and keep in mind that e.g.

$$\operatorname{Re}(Z_q P_q) = Z_q^R P_q^R - Z_q^I P_q^I,$$

$$\ldots,$$

we find that

$$\begin{aligned}F = \sum_r \int_{-1}^{1} dy \Big\{ \frac{1}{2} \sum_s \big[&-b_{11}(Z_r^R P_r^{R''} - Z_r^I P_r^{I''})(Z_s^R Q_s^{R''} - Z_s^I Q_s^{I''})\\ &+ 2b_{12}(Z_r^R P_r^{R'} - Z_r^I P_r^{I'})(Z_s^R R_s^{R'} - Z_s^I R_s^{I'}) - b_{22}(Z_r^R R_r^R - Z_r^I R_r^I)(Z_s^R S_s^R - Z_s^I S_s^I)\big]\\ &- \big[\hat{u}(Z_r^R P_r^{R''} - Z_r^I P_r^{I''}) + \hat{v}(Z_r^R R_r^{R'} - Z_r^I R_r^I)\big]\Big\}_{x=L_1}^{x=L_2},\end{aligned}$$

where primes denote y-derivatives, e.g.

$$P_s^{R'} = \frac{\partial P_s^R}{\partial y}.$$

The complementary energy minimum conditions read

$$\frac{\partial F}{\partial Z_q^R} = 0, \qquad \frac{\partial F}{\partial Z_q^I} = 0, \qquad q = 1, 2, \ldots \tag{7.6}$$

and result in the set of equations

$$\begin{aligned}\sum_s (A_{qs} Z_s^R - B_{qs} Z_s^I) &= E_q,\\ \sum_s (C_{qs} Z_s^R - D_{qs} Z_s^I) &= F_q.\end{aligned} \tag{7.7}$$

Here A_{qs}, B_{qs}, C_{qs} and D_{qs} are prescribed real-valued infinite symmetric matrices, expressed in terms of the four roots of characteristic equation (2.14) resulting from the compatibility equation, and of the infinite sequence λ_s, ($s = 1, 2, \ldots$). Also, E_q and F_q denote prescribed infinite vectors, cf. Appendix C.

If a solution (Z_s^R, Z_s^I) of (7.7) is found, the elastic state in the rectangular elastic disc can be obtained by means of the stress function (7.5).

Acknowledgements

The present paper would have never appeared if the subject would not had been suggested to me by Professor J. P. BOEHLER from the Institute of Mechanics in Grenoble (I.M.G.). I thank him for the suggestion and for his consultations on the subject.

I am also thankful to Professor J. IGNACZAK from the Institute of Fundamental Technological Research (IPPT PAN) in Warsaw for his advices during preparation of this work.

Appendix A. Anisotropic body in a plane state of stress

The following equations describe a static behaviour of three-dimensional linear anisotropic elastic body in the absence of body forces, [6],

$$E_{ij} = \frac{1}{2}(u_{ij}+u_{ji}), \tag{A.1}$$

$$S_{ij,j} = 0, \tag{A.2}$$

$$E_{ij} = C_{ijmn}S_{mn}. \tag{A.3}$$

Here u_i, E_{ij} and S_{ij} are, in this order, components of the displacement, deformation and stress. Moreover,

$$C_{ijmn} = C_{jimn} = C_{ijnm} = C_{mnij} \tag{A.4}$$

are of the components compliance tensor being an inverse of the elasticity tensor(¹).

Consider a linear transformation of coordinates (rotation)

$$x'_a = c_{ab}x_b, \qquad \text{where} \qquad c_{ab} = \cos(x'_a, x_b).$$

Then the components of compliance tensor are transformed according to the rule

$$C'_{abcd} = c_{ai}c_{bj}c_{cm}c_{dn}C_{ijmn}.$$

In a plane state of stress parallel to the (x_1, x_2)-plane we have

$$S_{13} = 0, \quad S_{23} = 0, \quad S_{33} = 0 \tag{A.5}$$

and relations (A.3) take the form

$$\begin{aligned}
E_{11} &= C_{1111}S_{11}+C_{1122}S_{22}+2C_{1112}S_{12},\\
E_{22} &= C_{2211}S_{11}+C_{2222}S_{22}+2C_{2212}S_{12},\\
E_{33} &= C_{3311}S_{11}+C_{3322}S_{22}+2C_{3312}S_{12},\\
2E_{23} &= 2C_{2311}S_{11}+2C_{2322}S_{22}+4C_{2312}S_{12},\\
2E_{13} &= 2C_{1311}S_{11}+2C_{1322}S_{22}+4C_{1312}S_{12},\\
2E_{12} &= 2C_{1211}S_{11}+2C_{1222}S_{22}+4C_{1212}S_{12}.
\end{aligned} \tag{A.6}$$

If we are interested in a solution of the (x_1, x_2) in-plane problem only, we ignore Eqs. $(A.6)_{3,4,5}$. Applying the conditions (A.5) to the Eqs. (A.1) and (A.2) we arrive at the set of equations (2.1), (2.2) and (2.3). In these equations notations are used

$$\begin{aligned}
&x = x_1, \qquad y = x_2, \qquad u = u_1, \qquad v = u_2,\\
&E_x = E_{11}, \qquad E_y = E_{22}, \qquad E_{xy} = E_{12},\\
&S_x = S_{11}, \qquad S_y = S_{22}, \qquad S_{xy} = S_{12},
\end{aligned}$$

(¹) In the monograph [6] the compliance tensor is denoted by K_{ijmn}.

and

$$b_{11} = C_{1111}, \quad b_{12} = C_{1122}, \quad b_{22} = C_{2222},$$
$$b_{16} = 2C_{1112}, \quad b_{26} = 2C_{2212}, \quad b_{66} = 4C_{1212}.$$

Appendix B. Explicit form of Ψ_s

Substituting (2.15) into boundary conditions (3.3) we obtain the following set of four equations for every s

$$\begin{aligned} C_1 e^{\mu_1} + C_2 e^{\mu_2} + C_3 e^{\mu_3} + C_4 e^{\mu_4} &= 0, \\ C_1 e^{-\mu_1} + C_2 e^{-\mu_2} + C_3 e^{-\mu_3} + C_4 e^{-\mu_4} &= 0, \\ C_1 \mu_1 e^{\mu_1} + C_2 \mu_2 e^{\mu_2} + C_3 \mu_3 e^{\mu_3} + C_4 \mu_4 e^{\mu_4} &= 0, \\ -C_1 \mu_1 e^{-\mu_1} - C_2 \mu_2 e^{-\mu_2} - C_3 \mu_3 e^{-\mu_3} - C_4 \mu_4 e^{-\mu_4} &= 0. \end{aligned} \tag{B.1}$$

Subscript s is omitted for the sake of simplicity. The set (B.1) admits non-zero solutions if and only if the determinant of C_i, $i = 1, 2, 3, 4$, is identically zero, that is if

$$(\mu_1\mu_3 + \mu_2\mu_4)\operatorname{sh}\mu_{13}\operatorname{sh}\mu_{24} - (\mu_1\mu_2 + \mu_3\mu_4)\operatorname{sh}\mu_{12}\operatorname{sh}\mu_{34} - (\mu_1\mu_4 + \mu_2\mu_3)\operatorname{sh}\mu_{14}\operatorname{sh}\mu_{23} = 0. \tag{B.2}$$

We have denoted here

$$\mu_{ij} = \mu_i - \mu_j, \quad i, j = 1, 2, 3, 4.$$

In view of the relation (2.13) between μ and λ, the transcendental equation (B.2) yields an infinite sequence of values λ_s. We express solutions of (B.1) in terms of C_{s1} obtaining what follows:

$$C_{s2} = \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3} \frac{\operatorname{sh}(\omega_1 - \omega_4)\lambda_s}{\operatorname{sh}(\omega_2 - \omega_4)\lambda_s} C_{s1},$$

$$C_{s3} = \frac{\omega_1 - \omega_2}{\omega_3 - \omega_2} \frac{\operatorname{sh}(\omega_1 - \omega_4)\lambda_s}{\operatorname{sh}(\omega_3 - \omega_4)\lambda_s} C_{s1},$$

$$C_{s4} = \frac{\omega_1 - \omega_2}{\omega_4 - \omega_2} \frac{\operatorname{sh}(\omega_1 - \omega_3)\lambda_s}{\operatorname{sh}(\omega_3 - \omega_4)\lambda_s} C_{s1}.$$

Thus the homogeneous solutions are of the form (7.3)

$$f_s = Z_s \Psi_s,$$

where we have assumed

$$Z_s \equiv C_{s1},$$

$$\Psi_s \equiv \sum_{i=1}^{4} e_{si}^0 e^{\lambda_s \omega_i y},$$

and

$$e_{si}^0 \equiv C_{si}/C_{s1}, \quad i = 1, 2, 3, 4.$$

Appendix C. The infinite matrices

The explicit formulae for matrices appearing in Eqs. (7.7) are as follows:

$$A_{qs} = \int_{-1}^{1} dy[-b_{11}(P_q^{R''}Q_s^{R''} + P_s^{R''}Q_q^{R''}) + 2b_{12}(P_q^{R'}R_s^{R'} + P_s^{R'}R_q^{R'}) - b_{22}(R_q^R S_s^R + R_s^R S_q^R)]_{x=L_1}^{x=L_2},$$

$$B_{qs} = \int_{-1}^{1} dy[-b_{11}(P_q^{R''}Q_s^{I''} + P_s^{I''}Q_q^{I''}) + 2b_{12}(P_q^{R'}R_s^{I'} + P_s^{I'}R_q^{R'}) - b_{22}(R_q^R S_s^I + R_s^I S_q^R)]_{x=L_1}^{x=L_2},$$

$$E_q = \int_{-1}^{1} dy(\hat{u}P_q^{R''} + \hat{v}R_q^{R'})_{x=L_1}^{x=L_2},$$

$$C_{qs} = \int_{-1}^{1} dy[-b_{11}(P_q^{I''}Q_s^{R''} + P_s^{R''}Q_q^{I''}) + \ \dots]_{x=L_1}^{x=L_2},$$

$$D_{qs} = \int_{-1}^{1} dy[-b_{11}(P_q^{I''}Q_s^{I''} + P_s^{I''}Q_q^{I''}) + \ \dots]_{x=L_1}^{x=L_2},$$

$$F_q = \int_{-1}^{1} dy(\hat{u}P_q^{I''} + \hat{v}R_q^{I'})_{x=L_1}^{x=L_2}.$$

We see that expression for C_{qs} can be obtained from that for B_{qs} by interchanging the upper index R with I, and expression for D_{qs} can be obtained from that for A_{qs} by replacing R by I. Now, after performing the integrations we get

$$A_{qs} = \frac{1}{2}\,\mathrm{Re}\,\{\varXi_{qs} + \varXi_{q*s}\},$$

$$B_{qs} = \frac{1}{2}\,\mathrm{Im}\,\{\text{ditto}\},$$

$$C_{qs} = \frac{1}{2}\,\mathrm{Im}\,\{\varXi_{qs} + \varXi_{qs*}\},$$

$$D_{qs} = \frac{1}{2}\,\mathrm{Re}\,\{\varXi_{qs} - \varXi_{q*s}\},$$

where

$$\varXi_{qs} = \lambda_q \lambda_s(\lambda_q + \lambda_s)[e^{-(\lambda_q+\lambda_s)x}]_{x=L_1}^{x=L_2}\,\chi_{qs}$$

and

$$\chi_{qs} = \sum_{i}^{4}\sum_{j}^{4} e_{qi}^{0} e_{sj}^{0} H_{qsij}(-b_{11}\omega_i^2\omega_j^2 + 2b_{12}\omega_i\omega_j - b_{22}),$$

with

$$H_{qsij} = \begin{cases} 2 & \text{if} \quad \lambda_q\omega_i = -\lambda_s\omega_j, \\ \dfrac{2}{\lambda_q\omega_i + \lambda_s\omega_j}\,\mathrm{sh}(\lambda_q\omega_i + \lambda_s\omega_j) & \text{if} \quad \lambda_q\omega_i \neq -\lambda_s\omega_j. \end{cases}$$

Moreover, $\varXi_{q*s}$ is obtained from $\varXi_{qs}$ by replacing λ_q by its conjugate λ_q^*.

References

1. I. FADLE, *Die Selbstspannungs-Eigenwertfunktionen der quadratischen Scheibe*, Ingen. Archiv, **11**, 125–149, 1940.
2. P. F. PAPKOVITSCH, *Ueber eine Form der Loesung des biharmonischen Problems fuer das Rechteck*, DAN SSSR, **27**, 334–338, 1940.
3. SH. M. KHACHATRIAN, *On the determination of stress-strain state of anisotropic strip*, Izv. AN Arm. SSR, **29**, 19–32, 1976.
4. I. CHOI, C. O. HORGAN, *Saint-Venant's principle and end effects in anisotropic elasticity*, Trans. ASME J. Appl. Mech., **44***E*, 424–430, 1977.
5. S. LEKHNITSKII, *Theory of elasticity of an anisotropic elastic body*, Holden Day, San Francisco 1963.
6. M. E. GURTIN, *Linear theory of elasticity*, Hdb. der Phys., ed. S. FLÜGGE, VIa/2, Springer 1972.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received March 4, 1988.