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Geometrically nonlinear analysis of plastic toroidal shells with an open profile(*)

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A THIN-WALLED, plastic toroidal shell with an open meridional cross-section, subject to simultaneous in-plane bending and external pressure, is considered. The theory of large deflections and rotations but small strains is applied to describe a rotationally — symmetric deformation of the whole shell (allowing for a change of the toroidal angle, $k = d\vartheta/d\Theta - 1$) and axially-symmetric deflection of the flexible profile. Depending on prescribed meridional boundary conditions (free edges, elastic support, pin-joined support), the effects of geometric softening or/and hardening of the structure during postcritical deformation of the shell, as well as various collapse mechanisms, were observed.

Rozważono cienkościenną, plastyczną powłokę toroidalną z otwartym przekrojem południkowym, poddaną jednoczesnemu działaniu zginania w płaszczyźnie i ciśnienia zewnętrznego. Zastosowano teorię dużych ugięć i obrotów lecz małych odkształceń do opisu obrotowo-symetrycznych deformacji powłoki (z dopuszczeniem zmiany kąta równoleżnikowego, $k = d\vartheta/d\Theta - 1$) oraz osiowo-symetrycznego ugięcia podatnego profilu. Zależnie od przyjętych warunków brzegowych dla południka (swobodny brzeg, sprężyste podparcie, nieprzesuwne przegubowe podparcie), obserwowano efekty geometrycznego osłabienia bądź umocnienia konstrukcji w trakcie pokrytycznej deformacji powłoki, jak również zbadano różne mechanizmy wyczerpania nośności.

Рассмотрена тонкостенная, пластическая тороидальная оболочка, с открытым меридиональным сечением, подвергнутая одновременному действию изгиба в плоскости и внешнего давления. Применена теория больших прогибов и вращений, но малых деформаций для описания вращательно-симметричных деформаций оболочки (с допуском изменения угла параллели $k = d\vartheta/d\Theta - 1$), а также осесимметричного прогиба податливого профиля. В зависимости от принятых граничных условий для меридиана (свободная граница, упругое опирание, непередвижное шарнирное опирание) наблюдались эффекты геометрического ослабления или упрочнения конструкции в процессе докритической деформации оболочки, как тоже исследованы разные механизмы исчерпания несущей способности.

1. Introduction

GEOMETRIC effects in the thin-walled toroidal shells with a closed meridional profile (curved tubes), subject to bending or surface loadings, have been examined by many authors. The geometrically nonlinear theory of elastic curved tubes has been formulated by E. REISSNER and R. A. CLARK [1–4] and developed by J. T. BOYLE, J. SPENCE [5], M. HAMADA, T. NAKATANI [6] and E. REISSNER [7, 8, 9]. On the other hand, toroidal shells subject to external loads, additionally require the analysis of both symmetric and nonsymmetric deformations of a profile.

A general theory of both geometrically and physically nonlinear toroidal shells with

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a closed meridional cross-section was developed by: J. SKRZYPEK and P. G. HODGE [10] J. SKRZYPEK and M. ŻYCZKOWSKI [11–14] (Hencky–Ilyushin or Nadai–Davis equations; finite deflections — small or large strains theory), A. MUC and J. SKRZYPEK [15, 16], J. BIELSKI and J. SKRZYPEK [17] (Prandtl–Reuss equations; finite deflections — small strains theory).

In the case of toroidal shells with an open meridional cross-section, the geometrically nonlinear analysis of the flexibility of the profile is even more important than for shells with a closed one. The main applications of such shells are connected with the design and analysis of deformations of elastic expansion bellows of various shape (U, S, Ω) loaded with axial force and/or internal pressure, e.g.: N. C. DAHL [18], Y. I. BERLINER and Y. L. VIKHMAN [19], C. R. CALLADINE [20], G. E. FINDLEY, J. SPENCE [21], M. HAMADA, S. TAKEZONO [22–23]. Reissner's „small finite deflection" theory [7] was applied to the numerical analysis of corrugated diaphragms and U-shaped bellows by M. HAMADA, Y. SEGUCHI [24]. A general survey of theories used to analyse bellows was done by J. F. WILSON [25]. Other questions arise when toroidal shells with open profiles are used as elements of mine gallery linings. Shell arches with flexible profiles applied there are usually loaded with external pressure and in-plane bending moment. The result is a change of both: a unit toroidal angle and a shape of profile.

In the present paper we apply the general theory and the method of solution developed in [10–14] to the problem of symmetric deformation of a plastic shell-arch with an open profile. The influence of both geometric effects — of the whole shell-arch as well as of the flexible profile — on the limit states and collapse modes of the shell is examined. Various boundary conditions prescribed along circumferential edges of the shell are considered.

2. Statement of the problem

2.1. Assumptions

An incomplete, thin-walled toroidal shell of an originally semicircular meridional cross-section is analysed. The concept of substitutive sandwich section is applied as an approximation of uniform section of the wall. A core is assumed as perfectly rigid in normal direction and perfectly flexible for bending (Fig. 1). A theory of finite displacements and rotations but small strains is used. Deformation of the element of the shell is governed by the classical Love–Kirchhoff hypothesis of straight and inextensible normals. The deformation of an overall meridional (radial) cross-section obeys the Bernoulli hypothesis: radial cross-sections of the shell remain plane and normal to the deformed axis of the torus. Moreover, only rotationally symmetric deformation of the whole shell, as well as only axially symmetric deformation of the meridional cross-section with respect to the initial plane of symmetry of the shell, are allowed for. In other words, this means that the deformation of each radial cross-section of the shell is identical although the unit toroidal angle changes. Nonsymmetric deformation modes, which may appear as a result of violence of either rotational or axial symmetry, are not considered in this paper.

The shell is loaded with the uniform symmetric external surface loading and in-plane

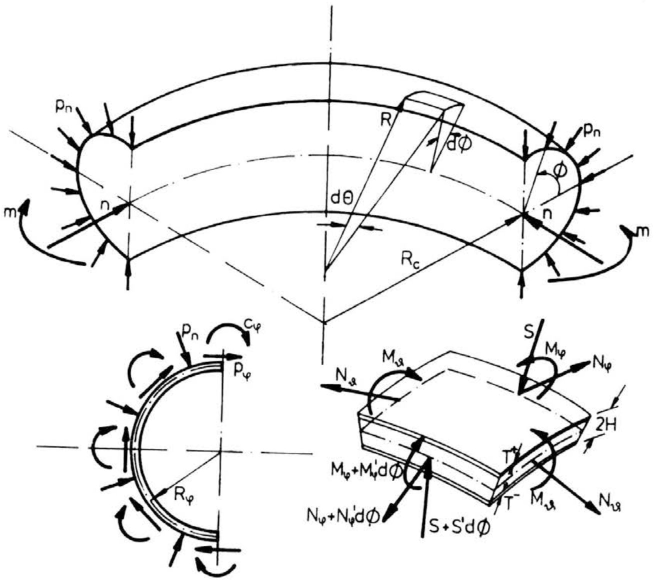

FIG. 1.

bending moment. Working layers of the wall are assumed as rigid/perfectly plastic; the Hencky–Ilyushin deformation theory and the Huber–Mises–Hencky yield condition are applied for the plane stress state in outer (+) and inner (−) layers.

2.2. Notation

Φ, φ	initial and current slope with respect to the symmetry plane,
Θ, ϑ	initial and current angular circumferential coordinates,
$k = d\vartheta/d\Theta - 1$	change of unit toroidal angle,
k_1, k_2	coefficients of elastic constraints,
Δ	specific deflection of meridional cross-section,
$2\hat{H}$	thickness of the substitutive sandwich section,
$\hat{T}$	thickness of the working layers,
$R_\varphi, R_\vartheta = (\hat{R}_\varphi, \hat{R}_\vartheta)/\hat{H}$	radii of curvature in original configuration,
$R_c = \hat{R}_c/\hat{H}$	radius of the center of the undeformed semicircular profile,
$R = R_\vartheta \cos\Phi$	distance betwen a middle surface point and the axis of the shell,
$U_r, U_z = (\hat{U}_r, \hat{U}_z)/\hat{H}$	radial and axial components of displacement of the middle surface point,
$\varepsilon_\varphi^\pm, \varepsilon_\vartheta^\pm$	meridional and circumferential strains of the working layers points,
e_φ, e_ϑ	elongations of the middle surface,
$\varkappa_\varphi, \varkappa_\vartheta = (\hat{\varkappa}_\vartheta, \hat{\varkappa}_\varphi)\hat{H}$	increments of the middle surface curvatures,
$\hat{\sigma}_0$	tensile yield-point stress of the working layers,
$\sigma_\varphi^\pm = \hat{\sigma}_\varphi^\pm/\hat{\sigma}_0 = (2/\sqrt{3})\sin(\omega^\pm + \pi/3)$	meridional stress in the working layers,
$\sigma_\vartheta^\pm = \hat{\sigma}_\vartheta^\pm/\hat{\sigma}_0 = (2/\sqrt{3})\sin(\omega^\pm)$	circumferential stress in the working layers,
$\hat{N}_0 = 2\hat{\sigma}_0\hat{T}$	maximal value of the direct stress per unit length,
$\hat{M}_0 = 2\hat{\sigma}_0\hat{H}\hat{T}$	maximal value of uniaxial bending moment per unit length,
$S = \hat{S}/\hat{N}_0$	resultant shearing force,
$p_n, p_\varphi = (\hat{p}_n, \hat{p}_\varphi)\hat{H}/\hat{N}_0$	true, normal and tangential components of pressure applied to the shell surface,

$c_\varphi = \hat{c}_\varphi \hat{H}/\hat{M}_0$ couple applied to the shell surface,
$N_\varphi, N_\vartheta = (\hat{N}_\varphi, \hat{N}_\vartheta)/\hat{N}_0$
$M_\varphi, M_\vartheta = (\hat{M}_\varphi, \hat{M}_\vartheta)/\hat{M}_0$ generalized stresses referred to the undeformed element of the shell.

3. Fundamental equations

The basic system of equations governing the deformation of the rigid perfectly/plastic toroidal shell, with an arbitrary sufficiently smooth and convex generating curve was derived in [11, 14]. We adopt these formulae for the case of small $1/R_j$ terms (thin shells) and small principal elongations e_j (small strains), when compared to 1. Thus we write the basic equations in the dimensionless form:

geometric relations

$$\begin{aligned}
&\varphi = \operatorname{arctg}\frac{\sin\Phi - U_r'/R_\varphi}{\cos\Phi + U_z'/R_\varphi},\\
&\varepsilon_\varphi^\pm = e_\varphi \pm \varkappa_\varphi, \qquad \varepsilon_\vartheta^\pm = e_\vartheta \pm \varkappa_\vartheta,\\
&e_\varphi = -(U_r'/R_\varphi)\sin\varphi + (U_z'/R_\varphi)\cos\varphi + \cos(\varphi - \Phi) - 1,\\
&e_\vartheta = (U_r/R)(1+k) + k,\\
&\varkappa_\varphi = (\varphi' - 1)/R_\varphi, \qquad \varkappa_\vartheta = \frac{(1+k)\cos\varphi - \cos\Phi}{R},
\end{aligned} \tag{3.1}$$

equations of equilibrium

$$\begin{aligned}
(RN_\varphi)' + (1+k)R_\varphi N_\vartheta \sin\varphi + \varphi' RS &= -RR_\varphi p_\varphi,\\
(RS)' - (1+k)R_\varphi N_\vartheta \cos\varphi - \varphi' RN_\varphi &= -RR_\varphi p_n,\\
(RM_\varphi)' + (1+k)R_\varphi M_\vartheta \sin\varphi + RR_\varphi S &= -RR_\varphi c_\varphi,
\end{aligned} \tag{3.2}$$

physical equations for plane stress state

$$\begin{aligned}
\varepsilon_\varphi^\pm &= \psi^\pm(2\sigma_\varphi^\pm - \sigma_\vartheta^\pm),\\
\varepsilon_\vartheta^\pm &= \psi^\pm(2\sigma_\vartheta^\pm - \sigma_\varphi^\pm),
\end{aligned} \tag{3.3}$$

$$(\sigma_\varphi^\pm)^2 - \sigma_\varphi^\pm \sigma_\vartheta^\pm + (\sigma_\vartheta^\pm)^2 = 1, \tag{3.4}$$

where the dimensionless generalized stresses, N_j and M_j, and the functions which parametrize the HMH yield condition (3.4) are defined as:

$$\begin{aligned}
N_\varphi &= (\sigma_\varphi^+ + \sigma_\varphi^-)/2, \qquad N_\vartheta = (\sigma_\vartheta^+ + \sigma_\vartheta^-)/2,\\
M_\varphi &= (-\sigma_\varphi^+ + \sigma_\varphi^-)/2, \qquad M_\vartheta = (-\sigma_\vartheta^+ + \sigma_\vartheta^-)/2,
\end{aligned} \tag{3.5}$$

$$\sigma_\varphi^\pm = (2/\sqrt{3})\sin(\omega^\pm + \pi/3), \qquad \sigma_\vartheta^\pm = (2/\sqrt{3})\sin(\omega^\pm). \tag{3.6}$$

Considering three kinematic quantities, φ, U_r, U_z, and three static ones, ω^+, ω^-, S, as the basic unknown functions, the following system of six coupled nonlinear ordinary differential equations was derived in [11]:

$$\begin{aligned}
U_r' &= R_\varphi(\sin\Phi - \sin\varphi) + (R_\varphi/R)(F^+ + F^-)\sin\varphi,\\
\varphi' &= 1 - (R_\varphi/R)(F^+ - F^-),
\end{aligned} \tag{3.7}$$

(3.7) [cont.]
$$U_z' = R_\varphi(\cos\varphi - \cos\Phi) - (R_\varphi/R)(F^+ + F^-)\cos\varphi;$$

$$\omega^{+\prime} = -[1/\cos(\omega^+ + \pi/3)]\{(R'/R)\sin(\omega^+ + \pi/3) + (R_\varphi/R)(1+k)\sin\varphi\sin\omega^+ + (\sqrt{3}/2)[(\varphi' - R_\varphi)S + R_\varphi(p_\varphi - c_\varphi)]\},$$

(3.8)
$$\omega^{-\prime} = -[1/\cos(\omega^- + \pi/3)]\{(R'/R)\sin(\omega^- + \pi/3) + (R_\varphi/R)(1+k)\sin\varphi\sin\omega^- + (\sqrt{3}/2)[(\varphi' + R_\varphi)S + R_\varphi(p_\varphi + c_\varphi)]\},$$

$$S' = (\varphi'/\sqrt{3})[\sin(\omega^+ + \pi/3) + \sin(\omega^- + \pi/3)] - (1/R)[R'S - (R_\varphi\cos\varphi/\sqrt{3})(1+k)(\sin\omega^+ + \sin\omega^-)] - R_\varphi p_n,$$

where

(3.9)
$$F^\pm = \{U_r(1+k) + Rk \pm [(1+k)\cos\varphi - \cos\Phi]\}[\cos\omega^\pm/(\cos\omega^\pm - \sqrt{3}\sin\omega^\pm)].$$

For the sake of generality, all types of surface loadings — normal and tangential components of pressure p_n, p_φ as well as couple c_φ — are retained here.

4. Boundary value problem

Although the basic equations (3.7) and (3.8) are generally nonlinear, they are, however, linear with respect to first derivatives of unknowns. To solve them, we need six boundary conditions. In the considered case of axially symmetric deformation of the meridional cross-section with respect to the symmetry plane of the shell, the numerical integration of Eqs. (3.7) and (3.8) can be performed throughout the domain $0 \leqslant \Phi \leqslant (\pi/2)$. Thus the necessary condition of symmetry, at the end $\Phi = 0$, takes the form

(4.1)
$$\varphi(0) = S(0) = U_z(0) = 0.$$

Three other boundary conditions have to be prescribed at the end $\Phi = \pi/2$, depending on the problem considered. In the present paper we shall discuss three variants of the shell corresponding to various boundary conditions (Fig. 2), that is

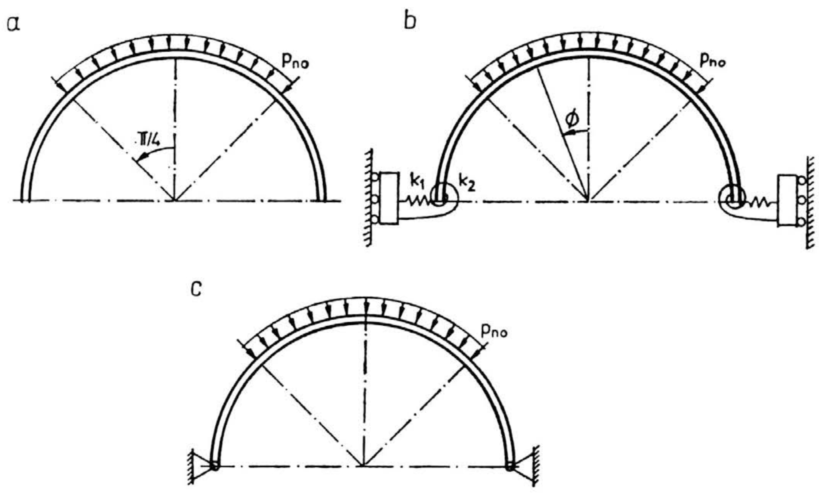

FIG. 2.

a) shell with free edges (no constraints imposed);

b) shell elastically supported (elastic constraints imposed upon axial displacement $U_z(\pi/2)$ and angle of slope $\varphi(\pi/2)$);

c) pin-joined shell (no displacements $U_z(\pi/2)$, $U_r(\pi/2)$ allowed for at the edges).

So the appropriate boundary conditions at the end $\Phi = \pi/2$ must hold:

$$N_\varphi(\pi/2) = M_\varphi(\pi/2) = S(\pi/2) = 0; \tag{4.2$_1$}$$

$$\begin{aligned} N_\varphi(\pi/2) &= k_1 U_z(\pi/2)\cos[\varphi(\pi/2)], \\ S(\pi/2) &= k_1 U_z(\pi/2)\sin[\varphi(\pi/2)], \\ M_\varphi(\pi/2) &= k_2[\varphi(\pi/2)-\pi/2]; \end{aligned} \tag{4.2$_2$}$$

$$U_r(\pi/2) = U_z(\pi/2) = M_\varphi(\pi/2) = 0. \tag{4.2$_3$}$$

The semi-inverse shooting method enables us to change the two-point boundary-value problem to an initial-value one. Three missing initial conditions, for example at the end $\Phi = 0$, that is $U_r(0)$, $N_\varphi(0)$, $M_\varphi(0)$ (and finally $\omega^+(0), \omega^-(0)$), can be determined by using the standard iterative technique, provided that the corresponding conditions (4.2) are satisfied. In a general case, if asymmetric deformation of the initially symmetric toroidal shell is allowed for, the conditions of symmetry at $\Phi = 0$, the condition (4.1), must be replaced by the appropriate conditions at the end $\Phi = -\pi/2$ (analogous to the conditions (4.2)). The Runge–Kutta IV method of numerical integration is used. The Newton procedure for functions of many variables provides the exact solution.

5. Surface of limit states

Equations (3.7) and (3.8) define six unknown functions φ, U_r, U_z, ω^+, ω^-, S in terms of surface loading components, $p_n(\Phi)$, $p_\varphi(\Phi)$, $c_\varphi(\Phi)$ and curvature parameter k. In the present paper we limit a number of independent loading parameters to two by assuming:

$$\begin{aligned} p_\varphi(\Phi) &\equiv c_\varphi(\Phi) \equiv 0, \quad 0 \leqslant \Phi \leqslant \pi/2, \\ p_n(\Phi) &= \begin{cases} p_{n_0}, & 0 \leqslant \Phi \leqslant \pi/4, \\ 0, & \pi/4 \leqslant \Phi \leqslant \pi/2. \end{cases} \end{aligned} \tag{5.1}$$

Thus, in the simplest case, if p_{n_0} and k are considered as independent control parameters, Eqs. (3.7) and (3.8) with Eqs. (3.5)–(3.6) and (4.1)–(4.2) taken into account, determine the unknown functions: $\varphi = \varphi(\Phi, p_{n_0}, k)$, $U_r = U_r(\Phi, p_{n_0}, k)$, $U_z = U_z(\Phi, p_{n_0}, k)$, $N_\varphi = N_\varphi(\Phi, p_{n_0}, k)$, $M_\varphi = M_\varphi(\Phi, p_{n_0}, k)$, $S = S(\Phi, p_{n_0}, k)$. Two other generalized stresses N_ϑ, M_ϑ can be computed from Eqs. (3.5) and (3.6). The "deformation process", understood here as a sequence of independent solutions obtained on the basis of the Hencky–Ilyushin deformation theory for quasi-statically changing loading parameters, becomes definite after a trajectory $f(p_{n_0}, k) = 0$ has been prescribed. More general cases of the control of the deformation process are discussed in [16].

For the purpose of this paper, it will be convenient to use two geometric parameters which describe: deformation of the whole shell, k, and change of the profile shape, $\Delta =$

$= U_r(0) - U_r(\pi/2)$. In the extended three-dimensional space, a surface of limit states $L = L(p_{n_0}, k, \Delta)$ can be built as a point pattern corresponding to all possible solutions of state equations (3.7) and (3.8) with the appropriate boundary conditions (4.1) and (4.2). To this end we choose p_{n_0} and k as control variables, whereas Δ is to be considered as a functional depending on the state functions in the current configuration:

$$\Delta = F[\varphi(\Phi), U_r(\Phi), U_z(\Phi), \omega^+(\Phi), \omega^-(\Phi), S(\Phi); p_{n_0}, k]. \tag{5.2}$$

Next, we determine, for instance, sections of the surface L, $g_j = g_j\ (p_{n_0}, \Delta)$ at constant curvature parameters (k_j = const). It is inconvenient, however, to use p_{n_0} as a direct control variable (time parameter). A time parameter should change monotonously, whereas p_{n_0} does not. Thus we can choose as a time parameter either one of the unknown components of the vector of initial values, for example $U_r(0)$ or $\varphi(\pi/2)$, or, if none behaves monotonously, we can change control variables during the process. It is also possible to perform integration for increments of both control variables at the tangent direction to the analyzed section of the surface g_j.

6. Discontinuities. Decohesive carrying capacity

The problem of admissible and inadmissible discontinuities for rigid/plastic toroidal shells has been discussed in details in [14]. The assumption of the rotational symmetry implies the continuity of all quantities in the circumferential direction. However, certain discontinuities may be expected in the meridional direction.

Following the Bernoulli hypothesis, the circumferential displacement U_ϑ must be continuous. Discontinuity of the meridional displacement U_φ and slope φ may, in principle, be only considered. We should remember, however, that this kind of discontinuities is regarded as inadmissible and their formation terminates the process (decohesive carrying capacity). A classification of inadmissible kinematic discontinuities, regarded as terminations of the deformation process of rigid-plastic toroidal shells, was proposed by J. SKRZYPEK and M. ŻYCZKOWSKI [14].

On the other hand, from the point of view of a continuous rigid/plastic medium, some discontinuities in static quantities are admissible. The generalized stresses N_φ, M_φ and shear force S have to be continuous but N_ϑ, M_ϑ may suffer from discontinuity. As a consequence the circumferential stresses $\sigma_\vartheta^\pm$ as well as the parameters $\omega^\pm$ may be discontinuous. One can expect discontinuity either in one sheet only or in both sheets (we call them partial or complete discontinuities, respectively). It follows from Eqs. (3.1) that at the point of a static discontinuity the following relations must hold:

(6.1) and/or
$$(\omega^+)^{\mathrm{I}} + (\omega^+)^{\mathrm{II}} = \pi/3$$
$$(\omega^-)^{\mathrm{I}} + (\omega^-)^{\mathrm{II}} = \pi/3,$$

where the superscripts I and II denote left or right-hand side values of these quantities. One can easily prove that at this point, in one or both sheets, ε_ϑ and ε_φ must equal 0.

Similar classification may be introduced with respect to the discontinuities of kinematic quantities, regarded as inadmissible [10]. Partial kinematic discontinuity is characterized by the infinite increase of the proper meridional strain ε_φ^+ or ε_φ^-. It follows from kinematic considerations that at the point of partial discontinuity the jumps of displacement $U_\varphi]$ and slope $\varphi]$ must occur, with $U_\varphi] = \varphi]$ or $U_\varphi] = -\varphi]$, respectively. However, due to the similarity of deviators, one of the following formulae must hold:

$$\varepsilon_\varphi^\pm = (2\sigma_\varphi^\pm - \sigma_\vartheta^\pm)\frac{\varepsilon_\vartheta^\pm}{2\sigma_\vartheta^\pm - \sigma_\varphi^\pm} \to \infty. \tag{6.2}$$

The above can happen only at these points of the stress ellipse, where $\omega^\pm = \pi/6$ or $\omega^\pm = -5\pi/6$ and where, simultaneously, the denominator in Eq. (6.2) vanishes but $\varepsilon_\vartheta^\pm \neq 0$. If there exists a complete (double) kinematic discontinuity, then the jumps $U_\varphi]$ and $\varphi]$ may be prescribed independently, whereas both ω^+ and ω^- have to be continuous.

7. Results

We discuss the open toroidal shell with initially semicircular meridional cross-section and the following geometric characteristics

$$R_\varphi = 50, \quad R_c = 1000.$$

The uniformly distributed pressure is applied throughout the interval:

$$0 \leqslant \Phi \leqslant \pi/4.$$

Two surfaces of limit states for two types of boundary conditions $(4.2)_1$ and $(4.2)_2$ have been built. For the shell with elastically supported edges, the coefficients of elastic constraints equal

$$k_1 = 0.5, \quad k_2 = 0.9. \tag{7.1}$$

The values of both k_i belong to the interval $\langle 0.1 \rangle$.

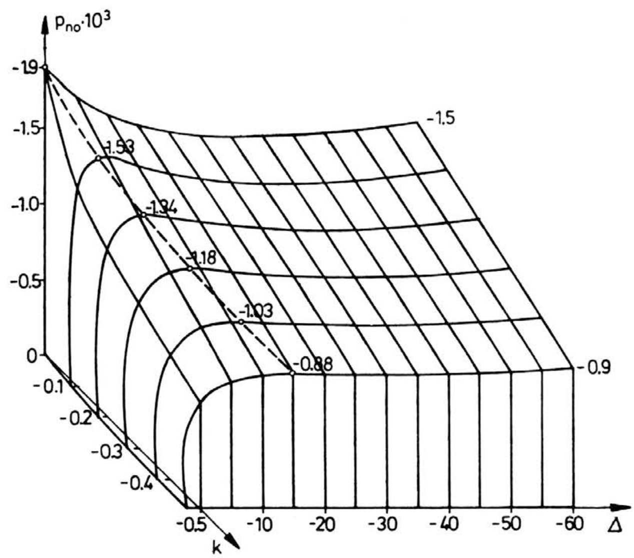

FIG. 3.

The surfaces of limit states are plotted in Figs. 3 and 4. The deformation of the meridional profile with the corresponding strain distributions for chosen points along the lines of $k = -0.01$ for both surfaces are presented in Figs. 5 and 6.

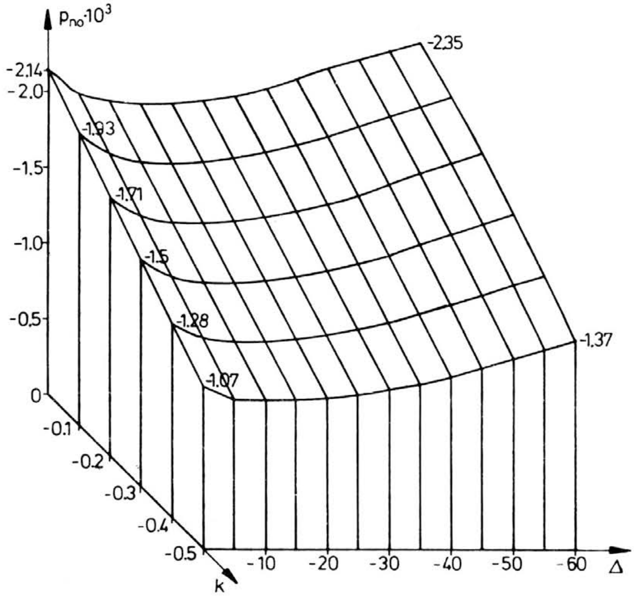

FIG. 4.

In the case of a shell with free edges $(4.2)_1$, a strong effect of geometrical softening ε_s can be noticed. The smaller value of k, the clearer phenomenon of softening, which is due to the unconstrained flattening of the meridian (Table 1). For large values of Δ, when advanced deformation of the profile takes place, a slight effect of hardening ε_h occurs as a result of concavity of the deformed meridian around the point $\Phi = 0$.

Table 1.

k_j	p_{max}	p_{min}	$p(\Delta)$ $\Delta = 60$	$\varepsilon_s = \frac{p_{max} - p_{min}}{p_{max}}$	$\varepsilon_h = \frac{p(\Delta) - p_{min}}{p_{max}}$
0.01	0.0018	0.00142	0.0015	21.4%	4,7%
0.4	0.00103	0.001	0.00103	2.9%	3.4%

In the case of a shell with elastically supported edges $(4.2)_2$, another collapse mechanism takes place. The concavity of the central, loaded part of the meridian is significant. After initial geometrical softening, one can observe an essential geometrical hardening, due to the response of the elastic constraints (Table 2). We notice a significant increase of the meridional strain ε_φ in the most curved parts of the meridian, Fig. 6. According to the previous considerations (Sect. 6) one could even expect the onset of decohesive carrying capacity. In fact, tracing the strain redistribution for more advanced deformation of the

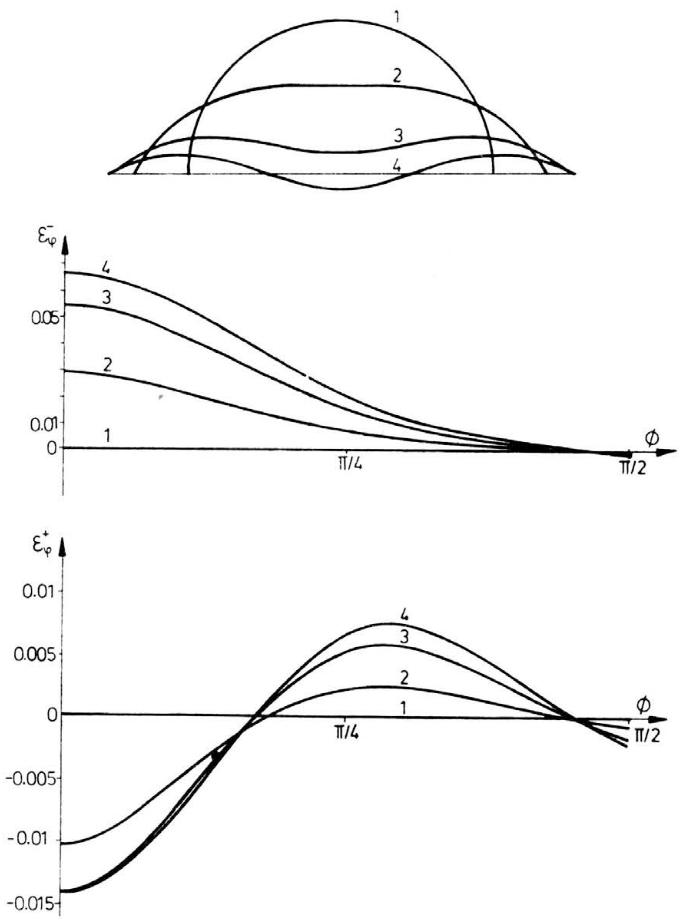

FIG. 5.

Table 2.

k_j	p_{max}	p_{min}	$p(\Delta)$ $\Delta = 60$	$\varepsilon_s = \frac{p_{max} - p_{min}}{p_{max}}$	$\varepsilon_h = \frac{p(\Delta) - p_{min}}{p_{max}}$
0.01	0.00212	0.00189	0.0023	11.1%	19.6%
0.4	0.00128	0.00116	0.00159	9.4%	33.6%

meridional cross-section, we observe a progressive decrease of local maximum of ε_φ. Thus, finally the decohesive carrying capacity is avoided. This effect may be connected with the Hencky–Ilyushin deformation theory used in the present analysis. The accumulation of strains, which is of great importance for local effects, here is not taken into account. The application of an incremental theory of plasticity would probably lead to the phenomenon of termination of the continuous process by the decohesion [14, 17]. A rapid movement of the neutral layer towards the edges of the shell can be observed

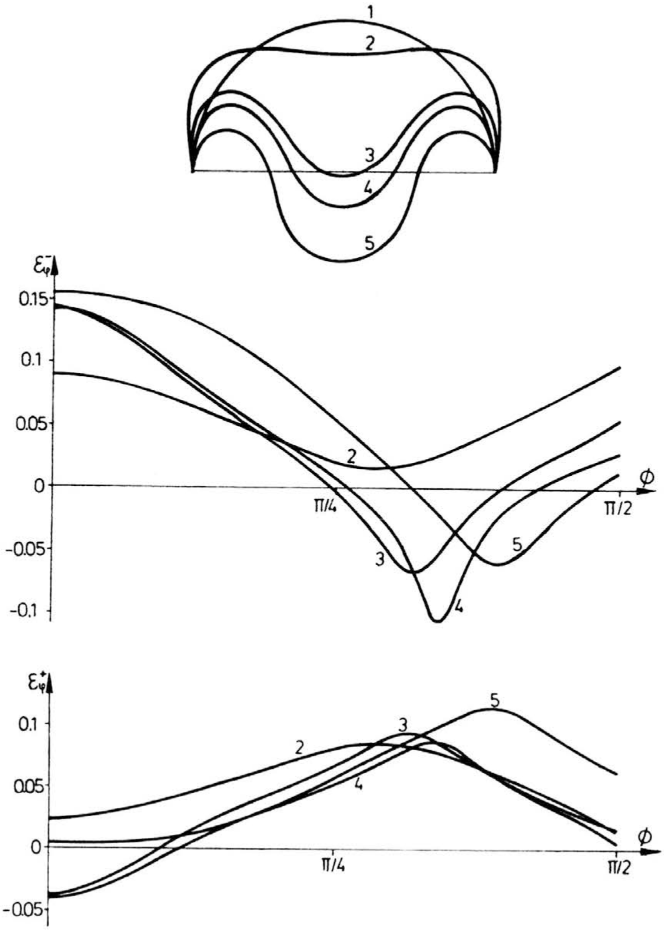

FIG. 6.

in both cases discussed above. It leads to circumferential compression ($\sigma_\vartheta < 0$) through the whole meridional cross-section.

The third case of boundary conditions $(4.2)_3$ has to be discussed separately. The shell with a pin-joined boundary represents an over-rigid structure. For $k = 0.001$, the process of plastic deformation of the cross-section can't be developed because of onset of the local, inadmissible kinematic discontinuity. The formation of three plastic hinges, almost simultaneously in the interior or exterior sheets (depending on the positive or negative meridional curvature) around the point $\Phi = 0$ and in the neighbourhood of $\Phi = \pi/3$ and $\Phi = -\pi/3$ terminates the process, Fig. 7. This effect of an inadmissible kinematic discontinuity can be identified as a plastic decohesion d.c.c.$_p$, according to the classification proposed in [16]. Taking into account elastic strains (which are not considered in this paper) one can expect also other types of d.c.c. On the other hand, the analysis of non-symmetric deformation modes may essentially change these results, particularly in the case of a shell elastically supported or pin-joined.

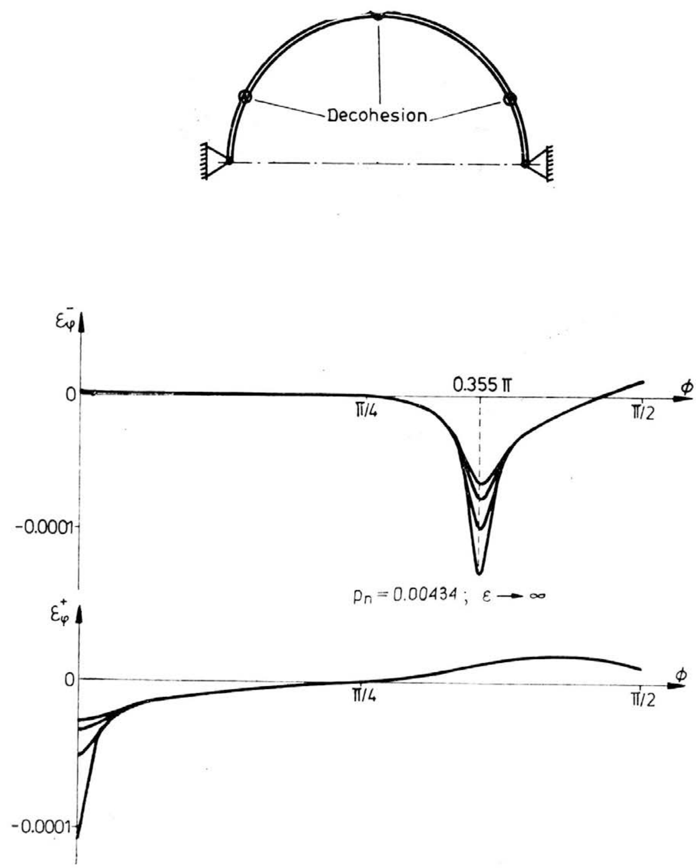

FIG. 7.

8. Conclusions

1. Applying the geometrically nonlinear theory of a rotationally-symmetric plastic toroidal shell, one can build a surface of limit states for any type of meridional boundary conditions.

2. The relaxing of elastic meridional constraints (lower values of k_1, k_2) results in lowering of the whole surface of limit states (lower values of the normal pressure) as well as in the vanishing of the effect of geometrical hardening.

3. On the contrary, in the case of an over-rigid shell the effect of plastic decohesion (due to inadmissible kinematic discontinuity) terminates the process of deformation of the whole structure. The corresponding value of pressure is about two times higher than the maximal pressure for both cases $(4.2)_1$ and $(4.2)_2$.

4. In order to increase the maximal value of pressure $p_{\max}$ as well as to increase the effect of geometric hardening ε_h, one can apply hard springs as a model of elastic support (higher values of k_1, k_2), keeping in mind, however, that the phenomenon of decohesion will terminate the process of deformation.

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