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The plastic spin concept and the theory of finite plastic deformations with induced anisotropy

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THE AIM of the paper is to discuss the theoretical framework of the constitutive description of finite plastic deformations with strain-induced anisotropy. The anisotropy is modelled by means of the combined isotropic-kinematic hardening. The important role in the formulation of the objective rates plays the concept of plastic spin. An approximate representation of the general constitutive equation for plastic spin is considered.

Celem pracy jest dyskusja teoretycznych podstaw opisu konstytuwnego skończonych deformacji plastycznych z anizotropią indukowaną odkształceniem plastycznym. Wzmocnienie anizotropowe modeluje się przy pomocy połączonego wzmocnienia izotropowo-kinematycznego. Ważną rolę w tym opisie odgrywa koncepcja spinu plastycznego, którą uwzględniono przy formułowaniu prędkości obiektywnych. Rozważono aproksymacje reprezentacji ogólnego równania na spin plastyczny.

Целью работы является обсуждение теоретических основ определяющего описания конечных пластических деформаций с анизотропией индуцированной пластической деформацией. Анизотропное упрочнение моделируется при помощи соединенного изотропно-кинематического упрочнения. Важную роль в этом описании играет концепция пластического спина, которая учтена при формулировке объективных скоростей. Рассмотрены аппроксимации представлений общего уравнения на пластический спин.

1. Introduction

COMPUTATIONAL modelling for ductile fracture, metal forming and strain localization produces an increasing demand for the adequate constitutive description of inelastic behaviour of engineering materials. The models of finite plastic deformations with induced anisotropy have been developed. The constitutive equations for small elastic and finite plastic deformations with combined isotropic-kinematic hardening have been implemented in finite element programmes (cf. e.g. HUGHES [1]).

The introduction of kinematic hardening is related to the formulation of objective rate-type constitutive equations. The application of the Zaremba–Jaumann rate can lead to the non-adequate prediction of the material reaction while the finite shearing with pertinent large rotations of the principal axes of the back stress tensor plays the dominant role.

DIENES [2] demonstrated that the solution of the problem of simple shear leading to large deformations of hypo-elastic material with the Zaremba–Jaumann stress rate predicts stresses which oscillate as the shear strain increases. Considering a similar problem, DUBEY [3] and METZGER and DUBEY [4] proposed to use the principal axes technique developed by BIOT [5] and HILL [6] applied to finite deformations of isotropic elastic-plastic solids. This approach can provide promising results in the prediction of material behaviour, e.g. by the calculations of simple shear problem [4] or buckling stress in an elastic-plastic plate under uniaxial compression [7].

The anisotropic hardening was considered by LEHMANN [8], who found the unwanted oscillatory stresses generated by finite simple shear in plastic materials with kinematic hardening and the Zaremba–Jaumann stress rate. Nearly ten years later, however, similar observation of NAGTEGAAL and DE JONG [9] has been met with broader interest of the mechanics community. LEE *et al.* [10], ONAT [11] and DAFALIAS [12] discussed appropriate modifications of the corotational rates of the Cauchy stress and the kinematic hardening tensor (back stress). DAFALIAS [13, 14], LORET [15] and FRESSENGEAS and MOLINARI [16], as well as VAN DER GIESSEN [17] and BAMMANN and AIFANTIS [18] studied this problem applying the Mandel's concept of the director vectors with related isoclinic configuration, and considered special forms of the constitutive equations for plastic spin. DOGUI and SIDOROFF [19], HUGHES [1] and BOUKADIA *et al.* [20] considered similar question suggesting that the equations should be formulated in a certain rotating frame, rotations of which are to be determined (cf. PAULUN and PĘCHERSKI [21, 22] where the critical discussion of the related studies is provided).

The relatively high value of strain at which the instable reaction occurs might lead to the conclusion that the problem is rather of academic character and can be disregarded in practical applications. However, recent studies show that the more realistic description of the stress-strain curve can lead to the oscillatory solution of the simple shear problem at lower strains (cf. Fig. 3 in [15] and Fig. 5 in [22]). Similar results can also be observed in [23], where it has been shown, on the example of uniaxial tension followed by simple shearing with fixed axial strain, that even at small strains for non-proportional deformation paths the application of the Zaremba–Jaumann rate can lead to erroneous prediction of the material reaction.

The aim of the paper is to present the theoretical framework of the constitutive description of finite plastic deformations with strain-induced anisotropy on the example of combined isotropic-kinematic hardening. An important aspect of gross inelastic behaviour of crystalline materials, i.e. the motion of the continuum relative to the underlying substructure plays the pivotal role in this study. The emerging concept of plastic spin is taken into consideration by formulation of the objective rates.

The physical motivation and interpretations of the concepts of material substructure, director vectors and plastic spin as well as the basic relations describing continuum versus substructural kinematics are discussed. Practical specifications of the general constitutive relation for the plastic spin are considered and the pertinent substructure corotational rate is applied to formulate the equations for elastic-plastic material with combined isotropic-kinematic hardening. Examples, experimental verification and numerical implementation of the theory discussed are presented elsewhere. The problems of finite simple shear and simple shear traction were discussed in [21]. The verification with experimental results of SWIFT [24] and the merits of the present proposal vis-à-vis the existing theories were presented in [22]. The solution of simple shear traction problem for isotropic-kinematic hardening under reverse loading is considered in [25], and the application of the theory of elastic-plastic deformations with combined isotropic-kinematic hardening to the numerical analysis of localization phenomena in porous solids is given in [26].

Tensors are denoted by boldface characters and the following symbolic operations are used: $\mathbf{ab} = a_{ij}b_{jk}$, $\mathbf{L}:\mathbf{b} = L_{ijkl}b_{kl}$, $\mathbf{a}:\mathbf{b} = a_{ij}b_{ij}$ with the summation convention over repeated indices. The rank of the tensor is indicated in the text.

2. Physical motivation

For adequate description of anisotropic hardening at large plastic deformations it is necessary to account properly for the material substructure and its evolution in the deformation process. The substructure description is achieved by introducing in the relation between stress, strain and their rates a set of internal structure variables. The structure variables are usually assumed as scalars, vectors or second-order tensors, although higher-order tensors can also be considered, DAFALIAS [13, 14]. These variables represent macroscopically the effects of microstructural rearrangements. They are defined, however, directly at the macrolevel and are determined in macroscopic experiments.

In the case of a polycrystalline metal the substructure is understood as a collection of grains and subgrains as well as dislocation tangles, walls and subboundaries, that contain or delimit microvolumes with differently oriented crystalline lattices. The process of large plastic strains produces such a rearrangement of these microscopic entities that the residual-type stresses and texture effects are produced and anisotropic hardening on the macrolevel can be observed. At the same time the generation and movement of mobile dislocations in active slip systems within the microvolumes delimited by the impenetrable boundaries produce the plastic flow of material relative to the crystalline lattice and the complex dislocation structure. The dislocation structure itself evolves in the course of plastic deformation process contributing to the isotropic and latent hardening.

3. The concept of plastic spin

The macroscopic internal variable measures an average effect of the microscopic residual stresses. It was recognized by MANDEL [27, 28] and emphasized by DAFALIAS [13, 14] that the structure variables and the stress are supported by the substructure of the medium and not by the continuum itself. This is related with the fundamental assumption, motivated in the single crystal plasticity, that the distinction should be made between the kinematics of the continuum and the kinematics of the underlying substructure. The substructure corotational rates require the definition of the spin tensor which is the difference between the plastic spin and the total material spin. Then it appears necessary to formulate the additional constitutive equations for plastic spin. MANDEL [27, 28] and KRATOCHVIL [29] made first such a proposition which was further corroborated in [13–18, 21, 22], as well as by KLEIBER and RANIECKI [30], AIFANTIS [31], and TOKUDA and YAMADA [32].

Consider a polycrystalline material as a continuous body. By the material point X of this body we understand certain minimum volume V which is sufficient for a valid continuum mechanics description of gross elastic-plastic behaviour. The dominant substructure orientation in the volume V is represented by director vectors — the triad of orthogonal unit vectors attached to the material point X, [27, 28]. Such a triad may be visualized in different manners. In a single crystal the director vectors are determined by the orientation of crystal lattice. The situation appears more complex in polycrystalline metals, for the substructure and its orientation undergoes very complicated evolution during large plastic deformation processes.

According to MANDEL [33] and KLEIBER and RANIECKI [30], the director triad may be defined on the macrolevel by means of three orthogonal unit vectors $\mathbf{m}_k$, where the vector $\mathbf{m}_1$ is taken to represent a material line lying in the material plane with normal $\mathbf{m}_2$ and $\mathbf{m}_3 = \mathbf{m}_1 \times \mathbf{m}_2$ (in general neither $\mathbf{m}_2$ nor $\mathbf{m}_3$ represents material lines). The material line lying along $\mathbf{m}_1$ can be related with certain preferred direction characterising the anisotropy generated by previous plastic deformation. The vector $\mathbf{m}_1$ can be also associated with the morphological texture induced by plastic deformation. This can be visualized in the experimental observations of plastic deformation and localization of the sheet, material of which is ferritic-austenitic duplex steel, subjected to the tension test, CARLSON and BIRD [34]. The fiber-like texture produced by the stretched austenite colonies can be represented by the vector $\mathbf{m}_1$ and its orientational changes can be followed in the course of plastic flow.

On the other hand, as it was emphasized by DAFALIAS [35], the discussed idea of the director vectors can be considered only as a conceptual derivation, while the crucial point is the constitutive equation for the plastic spin. The substructural spin results from the substraction of the plastic spin from the material spin, without even being necessary to define explicitly the director vectors. In the same spirit the concept of plastic spin was considered in [21, 22].

4. General formulation

In the plasticity theory of single crystals it is usually assumed that the dislocations traversing a volume element produce a change of its shape but they do not change its lattice orientation. This means that the glide directions and the slip planes remain parallel during the plastic deformation process, provided the effects of lattice misorientation and local relative rotations of material microvolumes are negligible. (These phenomena, important for proper modelling of advanced plastic strains and localization, have been studied elsewhere [36, 37]). The macroscopic counterpart of such a situation in finite deformation plasticity of polycrystals is Mandel's concept of the intermediate (relaxed) configuration, called isoclinic, in which the chosen director triad keeps always the same orientation with respect to the fixed axes. Due to this the unique decomposition of the deformation gradient $\mathbf{F}$ is provided, MANDEL [27, 28, 33], LORET [15] and KLEIBER and RANIECKI [30]:

$$\mathbf{F} = \mathbf{E}\mathbf{P}, \tag{1}$$

where **E** and **P** correspond to elastic and plastic transformations and the following basic kinematical relations hold:

$$\mathbf{L} = \dot{\mathbf{E}}\mathbf{E}^{-1} + \mathbf{E}\dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{E}^{-1}, \tag{2}$$

$$\mathbf{D}^e = (\dot{\mathbf{E}}\mathbf{E}^{-1})_s, \quad \mathbf{D}^p = (\mathbf{E}\dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{E}^{-1})_s, \tag{3}$$

$$\mathbf{W}^e = (\dot{\mathbf{E}}\mathbf{E}^{-1})_a, \quad \mathbf{W}^p = (\mathbf{E}\dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{E}^{-1})_a, \tag{4}$$

$$\mathbf{L} = \mathbf{D}+\mathbf{W}, \quad \mathbf{D} = \mathbf{D}^e+\mathbf{D}^p, \quad \mathbf{W} = \mathbf{W}^e+\mathbf{W}^p. \tag{5}$$

Here the superscripts e and p refer to elastic and plastic, and symbols $(t)_s$ and $(t)_a$ correspond to the symmetric and skew-symmetric parts of the second-order tensor **t**. The elastic spin $\mathbf{W}^e$ arises from the geometric constraints imposed by the boundary and compatibility conditions on slip directions within the microvolumes delimited by the impenetrable boundaries. Therefore $\mathbf{W}^e$ is related with the rotation of substructure.

For the sake of brevity and simplicity, isothermal processes will be considered only. The development of thermodynamic theory of finite elastic-plastic deformations can be referred to [15, 27, 30, 33]. Assume the mechanical state variables $(\boldsymbol{\pi}, \mathbf{A})$ corresponding to the isoclinic configuration, where $\boldsymbol{\pi}$ is the second Piola–Kirchoff stress related to the Cauchy stress $\boldsymbol{\sigma}$:

$$\boldsymbol{\pi} = (\det \mathbf{E})\mathbf{E}^{-1}\boldsymbol{\sigma}\mathbf{E}^{-T}. \tag{6}$$

A represents the structural variables. The tensor variable may represent, in the case of kinematic hardening, the back stress $\boldsymbol{\alpha}$ and the scalar variable can correspond to the isotropic hardening parameter $\varkappa$. The other nontrivial specifications are possible (cf. e.g. [37]).

The elastic Green strain $\boldsymbol{\Delta}^e = 1/2\ (\mathbf{E}^T\mathbf{E}-1)$ can be calculated from the free enthalpy function H per unit mass, which may be assumed in the form (cf. [27, 30])

$$H = H(\boldsymbol{\pi}, \mathbf{A}) = H_1(\boldsymbol{\pi})+H_2(\mathbf{A}), \tag{7}$$

$$\boldsymbol{\Delta}^e = -\varrho_k \frac{\partial H}{\partial \boldsymbol{\pi}}, \tag{8}$$

where $H = \Phi - \boldsymbol{\Delta}^e : \dfrac{\boldsymbol{\pi}}{\varrho_k}$ provides the relation with the free energy function per unit mass, Φ, depending, in the case of isothermal processes, on the state variables $(\boldsymbol{\Delta}^e, \mathbf{A})$.

The rate of elastic strain is given by

$$\dot{\boldsymbol{\Delta}}^e = -\varrho_k \frac{\partial^2 H}{\partial \boldsymbol{\tau}\, \partial \boldsymbol{\tau}} : \dot{\boldsymbol{\pi}} = \mathbf{M} : \dot{\boldsymbol{\tau}}, \tag{9}$$

where **M** is the elastic compliance tensor and ϱ_k is the density related with the isoclinic configuration. The transformation of equations (6) and (9) to the current configuration yields

$$\mathbf{D}^e = \mathscr{M} : \check{\boldsymbol{\sigma}}, \tag{10}$$

where

$$\mathscr{M}_{ijkl} = (\det \mathbf{E})\, E^{-1}_{\alpha i} E^{-1}_{\beta j} E^{-1}_{\gamma k} E^{-1}_{\delta l} M_{\alpha\beta\gamma\delta} \tag{11}$$

and

$$\check{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \dot{\mathbf{E}}\mathbf{E}^{-1}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{E}^{-T}\dot{\mathbf{E}} + \boldsymbol{\sigma}\,\mathrm{tr}(\dot{\mathbf{E}}\mathbf{E}^{-1}), \tag{12}$$

or for the Kirchoff stress $\boldsymbol{\tau} = (\det \mathbf{E})\boldsymbol{\sigma}$ (due to (5)

(13) $$\hat{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W}^e\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{W}^e - \mathbf{D}^e\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{D}^e.$$

It is typical for most of the deformed metallic solids that their distortional elastic strains remain small under arbitrary loading conditions, whereas they can undergo large elastic dilatational changes in shape under very high pressure. RANIECKI and NGUYEN [38] have shown, studying thermomechanics of isotropic elastic-plastic solids at finite strain and arbitrary pressure, that the tensor of elastic moduli in Eulerian description can be expressed in terms of derivatives of the free energy as simply as in the case of infinitesimal strains, provided the logarithmic elastic strain $\mathbf{e}^e = \ln \mathbf{V}^e$ is adopted as a state variable, and that the values of the ratios of principal elastic stretches $\mathbf{U}^e$, from the polar decomposition $\mathbf{E} = \mathbf{V}^e\mathbf{R}^e = \mathbf{R}^e\mathbf{U}^e$, belong to the interval [5/6, 7/6]. This condition extends the frequently adopted assumption that $\mathbf{U}^e$ is close to unity and makes it possible to account for large elastic dilatational changes. It has been shown in [38] that the Zaremba–Jaumann type rate of $\mathbf{e}^e$ can be approximated sufficiently close by $\mathbf{D}^e$,

(14) $$\mathring{\mathbf{e}}^e = \dot{\mathbf{e}} - \mathbf{W}^e\mathbf{e}^e + \mathbf{e}^e\mathbf{W}^e = \mathbf{D}^e + O(|\bar{\mathbf{e}}^e|^2)$$

when

$$5/6 \leqslant \frac{U_i^e}{U_j^e} \leqslant 7/6 \quad \text{and} \quad \bar{\mathbf{e}}^e = \mathbf{e}^e - 1/3(\operatorname{tr}\mathbf{e}^e)\mathbf{1}.$$

The assumptions made in [38] that

a) metallic solids are plastically incompressible;

b) an elastic response is not influenced by prior plastic straining;

c) elastic distortional response of metallic solids is linear;

hold also, at least as a first approximation, in the case of elastic-plastic metallic solids with deformation-induced anisotropy. The following constitutive relation of elasticity derived by RANIECKI and NGUYEN [38] can be adopted

(15) $$\mathring{\boldsymbol{\tau}} = \mathscr{L} : \mathbf{D}^e + O(|\bar{\mathbf{e}}^e|^2),$$

where

(16) $$\mathring{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W}^e\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{W}^e$$

and

(17) $$\mathscr{L}_{ijkl} = \varrho_k \frac{\partial^2 \Phi}{\partial e_{ij}^e \partial e_{kl}^e}$$

is the fourth-order tensor of elastic moduli and after specification of the free energy function, [38]:

(18) $$\frac{1}{\beta}\mathscr{L}_{ijkl} = \delta_{ij}\delta_{kl}(K-p) + \mu\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il} - \frac{2}{3}\delta_{ij}\delta_{kl} + \right)\frac{1}{\mu}\frac{\partial}{\partial\beta}(\beta\mu)(\delta_{ij}\bar{\sigma}_{kl} + \delta_{kl}\bar{\sigma}_{ij})$$

which is correct to the order $|\bar{\mathbf{e}}^e|^2$. The last term in (18) can be omitted if the accuracy of the order $|\bar{\mathbf{e}}^e|$ is permissible. The symbol μ denotes the usual isothermal shear modulus and K is the isothermal bulk modulus, $p = -\sigma_{ii}/3$, $\beta = \det(\mathbf{E})$, and $\bar{\sigma}_{ij} = \sigma_{ij} + p\delta_{ij}$ is the deviator of the Cauchy stress tensor.

Assuming that the distortional part of the elastic strain $\bar{\mathbf{e}}^e$ is infinitesimal we arrive, due to (13) and (14), at the following relation

$$\check{\boldsymbol{\tau}} = \mathring{\boldsymbol{\tau}} + O(|\bar{\mathbf{e}}^e|) \tag{19}$$

and the rate $\check{\boldsymbol{\tau}}$ can be approximated by means of the Zaremba–Jaumann type rate $\mathring{\boldsymbol{\tau}}$, that is corotational with the substructure. Observe that Eq. (19) holds for arbitrary large dilatational part of the elastic strain $\mathbf{e}^e$.

The following formulation of constitutive relations was proposed by DAFALIAS [13, 14], within the framework of classical plasticity theory of small elastic and finite plastic deformations and anisotropic hardening with smooth yield surfaces:

yield criterion:

$$f = f(\boldsymbol{\tau}, \mathbf{a}); \tag{20}$$

rate equations:

$$\mathbf{D}^p = \langle \lambda \rangle \mathbf{N}^p(\boldsymbol{\tau}, \mathbf{a}), \tag{21}$$

$$\mathbf{W}^p = \langle \lambda \rangle \boldsymbol{\Omega}^p(\boldsymbol{\tau}, \mathbf{a}), \tag{22}$$

$$\mathring{\mathbf{a}} = \langle \lambda \rangle \bar{\mathbf{A}}(\boldsymbol{\tau}, \mathbf{a}), \tag{23}$$

where $\mathbf{D}^p$ and $\mathbf{W}^p$ denote, respectively, the plastic rate of deformation and spin tensors, $\mathbf{a}$ corresponds to structure variables referred to the current configuration, λ is a properly defined loading index and the brackets $\langle \cdot \rangle$ give $\langle \lambda \rangle = \lambda$ if $\lambda > 0$ and $\langle \lambda \rangle = 0$ if $\lambda \leqslant 0$.

The rate $\mathring{\mathbf{a}}$ is determined by means of the Zaremba–Jaumann rate $\overset{\triangledown}{\mathbf{a}}$ and the plastic spin:

$$\mathring{\mathbf{a}} = \overset{\triangledown}{\mathbf{a}} + \mathbf{W}^p\mathbf{a} - \mathbf{a}\mathbf{W}^p, \tag{24}$$

where

$$\overset{\triangledown}{\mathbf{a}} = \mathbf{a} - \mathbf{W}\mathbf{a} + \mathbf{a}\mathbf{W}, \tag{25}$$

or equivalently

$$\mathring{\mathbf{a}} = \dot{\mathbf{a}} - (\mathbf{W} - \mathbf{W}^p)\mathbf{a} + \mathbf{a}(\mathbf{W} - \mathbf{W}^p). \tag{26}$$

Symbols f, $\mathbf{N}^p$, $\boldsymbol{\Omega}^p$, $\bar{\mathbf{A}}$ denote isotropic functions of the state variables $\boldsymbol{\tau}$, $\mathbf{a}$ which can be expressed in a general form by means of the representation theory. According to DAFALIAS [12, 13] and LORET [15], the skew-symmetric tensor function $\boldsymbol{\Omega}^p$ takes the form

$$\boldsymbol{\Omega}^p = \eta_1(\mathbf{a}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{a}) + \eta_2(\mathbf{a}^2\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{a}^2) + \eta_3(\mathbf{a}\boldsymbol{\tau}^2 - \boldsymbol{\tau}^2\mathbf{a}) + \eta_4(\mathbf{a}\boldsymbol{\tau}\mathbf{a}^2 - \mathbf{a}^2\boldsymbol{\tau}\mathbf{a}) + \eta_5(\boldsymbol{\tau}\mathbf{a}\boldsymbol{\tau}^2 - \boldsymbol{\tau}^2\mathbf{a}\boldsymbol{\tau}), \tag{27}$$

where η_i are scalar functions of the invariants of $\boldsymbol{\tau}$, $\mathbf{a}$ and any other scalar variable, for example the equivalent plastic strain. Evaluation of these functions is difficult and remains an open question.

The constitutive equations (21) and (23) specified for elastic-plastic material with Huber–Mises yield criterion and combined isotropic-kinematic hardening can be written in the following form:

$$\frac{3}{2}(\boldsymbol{\tau}' - \boldsymbol{\alpha}) : (\boldsymbol{\tau}' - \boldsymbol{\alpha}) = \varkappa^2, \tag{28}$$

$$\mathbf{D}^p = \langle \lambda \rangle (\boldsymbol{\tau}' -), \boldsymbol{\alpha} \tag{29}$$

where λ is determined from the consistency condition.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p = \left[\mathscr{L}^{-1} + \frac{9}{4\varkappa^2 h}(\boldsymbol{\tau}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\tau}' - \boldsymbol{\alpha})\right] : \overset{\circ}{\boldsymbol{\tau}}', \tag{30}$$

$$\dot{\varkappa} = (h - h_\alpha)\dot{\varepsilon}^p, \qquad \dot{\varepsilon}^p = \left(\frac{2}{3} D^p : D^p\right)^{1/2}, \tag{31}$$

$$\overset{\circ}{\boldsymbol{\alpha}} = \frac{2}{3} h_\alpha \mathbf{D}^p. \tag{32}$$

$\boldsymbol{\tau}'$ represents the deviator of the Kirchoff stress $\boldsymbol{\tau}$, the parameter $\varkappa$ corresponds to the "size" of the yield surface (i.e. $\varkappa = \sqrt{(3/2)}\, R$, where R is the radius of the Huber–Mises cylinder). Relation (32) represents the simple case of Prager's kinematic hardening law. The other known equations can be introduced here, and expressed by means of the substructure corotational rate of kinematic hardening parameter, (e.g. the nonlinear laws taking into account the static and dynamic recovery).

Relations describing the linear combination of isotropic-kinematic hardening with the plastic modulus corresponding to the constant slope of the effective stress vs. effective plastic strain curve under radial loading conditions and a parameter determining the proportion of isotropic and kinematic hardening or softening were also discussed by HUGHES [1]. Also LORET [15] and DAFALIAS [12] considered the combined isotropic-kinematic hardening taking into account the saturation effect of the flow strength. A more general form of the equations describing the combined isotropic-kinematic hardening at large plastic strains has been discussed recently by AGAH-TEHRANI *et al.* [39], where the kinematic hardening modulus h_α is considered as a general function depending on the back stress $\boldsymbol{\alpha}$ and on the stress $(\boldsymbol{\tau}' - \boldsymbol{\alpha})$, determining the position on the yield surface. The main difference lies in the formulation of the objective rate of stress.

5. A practical specification of the relation for plastic spin

Consider in (27) the linear terms with respect to the deviatoric stress only and take $\mathbf{a} = \boldsymbol{\alpha}$:

$$\boldsymbol{\Omega}^p = \psi_1(\boldsymbol{\alpha\tau}' - \boldsymbol{\tau}'\boldsymbol{\alpha}) + \psi_2(\boldsymbol{\alpha}^2\boldsymbol{\tau}' - \boldsymbol{\tau}'\boldsymbol{\alpha}^2) + \psi_3(\boldsymbol{\alpha\tau}'\boldsymbol{\alpha}^2 - \boldsymbol{\alpha}^2\boldsymbol{\tau}'\boldsymbol{\alpha}), \tag{31}$$

where the coefficients ψ_i, $i = 1, 2, 3$, are functions of invariants of $\boldsymbol{\alpha}$ and $\boldsymbol{\tau}'$.

Applying the representation of formally similar isotropic function of two tensor variables derived by AGAH-TEHRANI *et al.* [39], relation (31) can be expressed in terms of the generator $(\mathbf{a}\boldsymbol{\tau}' - \boldsymbol{\tau}'\boldsymbol{\alpha})$:

$$\boldsymbol{\Omega}^p = \mathrm{tr}(\mathbf{N})\,(\boldsymbol{\alpha\tau}' - \boldsymbol{\tau}'\boldsymbol{\alpha}) - [\mathbf{N}(\boldsymbol{\alpha\tau}' - \boldsymbol{\tau}'\boldsymbol{\alpha}) + (\boldsymbol{\alpha\tau}' - \boldsymbol{\tau}'\boldsymbol{\alpha})\mathbf{N}], \tag{32}$$

where

$$\mathbf{N} = \psi_1 \mathbf{1} - \psi_2 \boldsymbol{\alpha} + \psi_3\left(\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\alpha}^2)\mathbf{1} - \boldsymbol{\alpha}^2\right).$$

After some transformations (32) leads to

$$\mathbf{\Omega}^p = \left(3\psi_1 - \frac{1}{2}\psi_3 \operatorname{tr}(\boldsymbol{\alpha}^2)\right)(\boldsymbol{\alpha}\boldsymbol{\tau}' - \boldsymbol{\tau}'\boldsymbol{\alpha}) + (\psi_2\boldsymbol{\alpha} + \psi_3\boldsymbol{\alpha}^2)(\boldsymbol{\alpha}\boldsymbol{\tau}' - \boldsymbol{\tau}'\boldsymbol{\alpha}) - (\boldsymbol{\alpha}\boldsymbol{\tau}' - \boldsymbol{\alpha}\boldsymbol{\tau}')(\psi_2\boldsymbol{\alpha} + \psi_3\boldsymbol{\alpha}^2). \tag{33}$$

Due to (22), (29) and (33) the equation for plastic spin can be derived in the form

$$\mathbf{W}^p = \langle\lambda\rangle \mathbf{\Omega}^p(\boldsymbol{\tau}', \boldsymbol{\alpha}) = \left(3\psi_1 - \frac{1}{2}\psi_3 \operatorname{tr}(\boldsymbol{\alpha}^2)\right)(\boldsymbol{\alpha}\mathbf{D}^p - \mathbf{D}^p\boldsymbol{\alpha}) + (\psi_2\boldsymbol{\alpha} + \psi_3\boldsymbol{\alpha}^2)(\boldsymbol{\alpha}\mathbf{D}^p - \mathbf{D}^p\boldsymbol{\alpha}) - (\boldsymbol{\alpha}\mathbf{D}^p - \mathbf{D}^p\boldsymbol{\alpha})(\psi_2\boldsymbol{\alpha} + \psi_3\boldsymbol{\alpha}^2). \tag{34}$$

A simplified form of the representation of Eq. (27) has been assumed by DAFALIAS [12–14] and LORET [15] in the application to the finite simple shear problem of rigid-plastic material with kinematic hardening, which due to (34) can be expressed as follows:

$$\mathbf{W}^p = 3\psi_1(\boldsymbol{\alpha}\mathbf{D}^p - \mathbf{D}^p\boldsymbol{\alpha}), \tag{35}$$

where

$$\psi_1 = \text{const} \quad \text{and} \quad \psi_2 = \psi_3 = 0.$$

Analysis of the problem of finite simple shear in rigid plastic material with kinematic hardening reveals that retardation of the angular velocity $\dot{\gamma}/2$ pertaining to the material spin $\mathbf{W}$ provides non-oscillatory solution (c.f. PAULUN and PĘCHERSKI [21, 22]). This is consistent with the observation made in [10], as well as in [19, 20] that the constant angular velocity generates the unlimited rotation of the principal directions of the back stress $\boldsymbol{\alpha}$ as $t \to \infty$, resulting in an oscillatory solution. It has been required, consequently, that the spin corresponding to an angular velocity of material elements, which can only rotate by no more than π as $t \to \infty$, should appear in the proper formulation of the modified spin and resulting evolution equation for $\boldsymbol{\alpha}$. Accordingly, LEE *et al.* [10] expressed the modified spin in terms of the angular velocity of continuously changing material lines coinciding instantaneously with the maximum eigenvector of $\boldsymbol{\alpha}$. On the other hand, the angular velocity of a single material element can be used directly to formulate the modified spin. The material element may be associated with a preferred direction characterising the anisotropy generated by previous plastic flow.

According to this, the following equation for plastic spin has been derived by PAULUN and PĘCHERSKI [25] from the analysis of the simple shear problem

$$\mathbf{W}^p = \sqrt{\frac{3}{2}}\,\frac{6\alpha_{eq}}{h_\alpha^2 + 3\alpha_{eq}^2}(\boldsymbol{\alpha}\mathbf{D}^p - \mathbf{D}^p\boldsymbol{\alpha}), \tag{36}$$

where

$$\alpha_{eq} = \sqrt{\frac{3}{2}\,\boldsymbol{\alpha} : \boldsymbol{\alpha}}.$$

This corresponds to the following approximation of the general equation (34):

$$\psi_2 = \psi_3 = 0 \quad \text{and} \quad \psi_1 = \sqrt{\frac{3}{2}}\,\frac{2\alpha_{eq}}{h_\alpha^2 + 3\alpha_{eq}^2}. \tag{37}$$

Equation (36) was applied for the analysis of the simple shear traction problem under reverse loading conditions and provided satisfactory prediction of the material behaviour.

In the case of the aforementioned simple shear the function (37) is related with the difference between the constant angular speed produced by the material spin **W** and the angular velocity of the material line element lying initially perpendicular to the shear direction. The angular velocity of such a material line can be considered as the simplest representative of any material line element in the case of simple shear pertaining to the approximation of an average spin of all material lines.

6. Concluding remarks

The study of corotational rates, PAULUN and PĘCHERSKI [21, 22], has shown that the choice or formulation of appropriate rate in finite deformation problems is not only based on the question how to avoid the unwanted oscillatory stresses in the simple shear problem but refers rather to proper constitutive description of anisotropic hardening at large plastic strains. The nonlinear generalization of the evolution equation for the back stress $\boldsymbol{\alpha}$ should provide more adequate theoretical prediction of material behaviour, e.g. the proper simulation of reversal loading.

According to author's opinion, further studies should be related to the search for nonlinear specifications of the constitutive equation for plastic spin. The derived relation (34) may be helpful in this attempts. Solving this problem could shed more light on the general description of anisotropic hardening in finite deformation plasticity.

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