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BRIEF NOTES

Generalization of the concept of Riemann invariants for multidimensional gasdynamics(*)

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RIEMANN invariants, whose total differentials along characteristic rays vanish, are known only in a few cases. In a general case the characteristic relations of gasdynamics are interpreted as a condition of vanishing of a sum of total derivatives taken along non-parallel directions. At a slight expense in complexity, the method of invariants is extended to a general time-dependent multidimensional flow where the Riemann invariants do not exist.

1. Introduction

CALCULATIONS of gas flows using the method of characteristics become progressively more complicated as the number of independent variables increases. At the present time the practical limit is three. The case of four independent variables, three spatial coordinates and time, presents no theoretical difficulties, the pertinent characteristic relations having been cast in various convenient froms by e.g. SHÄFER [3, 1962], RUSANOV [5, 1963], ROESNER [4, 1967] and SAUERWEIN [2, 1967]. The absence of examples of numerical calculations oft hree-dimensional time-dependent flows attests to the practical difficulties of executing such calculations. Believing that the practical difficulties are primarily of a conceptual nature and lead to unnecessarily complicated and uneconomical computer codes, this work is addressed to the problem of simplifying the formulation of the method of characteristics.

The advantages of the method of characteristics are well known. The particular advantage that has a bearing on the economy of computations, namely the use of Riemann invariants, is lost in multidimensional gasdynamics. The problem of existence and construction of Riemann invariants is discussed and reviewed by, e.g. BURNAT [1, 1969]. In this work we shall consider the generalization of the concept of invariants for the purpose of reducing the calculations of multidimensional gas flows to simple integrations of systems of ordinary differential equations. In this manner the conceptual simplicity of the formulation will result in simpler and more efficient computer codes.

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2. Characteristic equations

For a gas with a constant ratio of specific heats γ, neglecting viscosity and heat conduction, the system of characteristic relations for the determination of pressure p, density ϱ, and velocity $\bar{u}$, with $a^2 = \gamma p/\varrho$, is

$$\left[\frac{\partial}{\partial t}+\bar{u}\cdot\nabla\right](p/\varrho^{\gamma}) = 0, \tag{2.1}$$

$$\varrho a\bar{n}_i\cdot\left[\frac{\partial\bar{u}}{\partial t}+\bar{u}\cdot\nabla\bar{u}+a\bar{n}_i\nabla\cdot\bar{u}\right]+\left[\frac{\partial p}{\partial t}+(\bar{u}+a\bar{n}_i)\cdot\nabla p\right]=0, \tag{2.2}$$

where $\bar{n}_i$, $i = 1, \ldots, 4$ are the unit normals, the spatial components of the characteristic normals, $|\bar{n}_i| = 1$.

RUSANOV [5, 1963] has shown that Eqs. (2.1) and (2.2) form a system of five independent relations if the four unit normals $\bar{n}_i$ are chosen so that the end points of the vectors $\bar{n}_i$, which lie on the surface of a unit sphere, do not fall in a common plane. It is always possible to choose four such vectors.

Equation (2.1) defines the entropy function, $R = p/\varrho^{\gamma}$, as a Riemann invariant carried by the fluid particles. Consequently, we concentrate our attention on the remaining four equations in the form of Eq. (2.2), and omit the subscript i.

3. Generalization of invariants

We shall use Rusanov's notation and write the directional derivative taken in the direction of a vector $\bar{W} = \{W_x, W_y, W_z, W_t\}$ as

$$d_W(\ ) = \left(W_x\frac{\partial}{\partial x}+W_y\frac{\partial}{\partial y}+W_z\frac{\partial}{\partial z}+W_t\frac{\partial}{\partial t}\right)(\ ).$$

With $\bar{U} = \{u, v, w, 1\}$, where u, v, w are the Cartesian components of the velocity $\bar{u}$, Eq. (2.1) becomes

$$d_U(p/\varrho^{\gamma}) = 0. \tag{3.1}$$

Using the same notation, Eq. (2.2) may be written as

$$d_{W_1}u+d_{W_2}v+d_{W_3}w+d_V p = 0, \tag{3.2}$$

where

$$\begin{aligned}\bar{V} &= \{u+an_x, v+an_y, w+an_z, 1\},\\ \bar{W}_1 &= \varrho a\{un_x+a, vn_x, wn_x, n_x\},\\ \bar{W}_2 &= \varrho a\{un_y, vn_y+a, wn_y, n_y\},\\ \bar{W}_3 &= \varrho a\{un_z, vz_z, wn_z+a, n_z\}.\end{aligned}$$

The vector $\bar{V}$ is tangent to a characteristic ray or a bi-characteristic. The fact that Eq. (2.2) is in a characteristic form implies that the vectors $\bar{V}$, $\bar{W}_1$, $\bar{W}_2$, $\bar{W}_3$ all lie in a common characteristic hyperplane with a characteristic normal $\bar{N}$, that is,

$$\bar{V}\cdot\bar{N} = \bar{W}_1\cdot\bar{N} = \bar{W}_2\cdot\bar{N} = \bar{W}_3\cdot\bar{N} = 0,$$

where $\bar{N} = \{n_x, n_y, n_z, -(\bar{u}\cdot\bar{n}+a)\}$.

Equation (3.1) integrates immediately to give the Riemann invariant,

$$R = p/\varrho^{\gamma} = \text{constant along } \frac{d\bar{x}}{dt} = \bar{u}.$$

We recognize Eq. (3.2) as a generalization of Eq. (3.1), namely, as a statement that a particular sum of total derivatives of the four dependent variables, taken along four coplanar, but not generally parallel vectors, vanishes. It is not possible either to choose the directions of the normals $\bar{n}$ or to transform the dependent variables in such a way as to render all four vectors $\bar{W}_j$ equal in a multi-dimensional case. However, the form of Eq. (3.2) is particularly well suited to numerical calculations, and the fact that the vectors $\bar{W}_j$ are not parallel introduces only a slight complication. The vectors $\bar{W}_j$ lie outside of the characteristic hypercones the generators of which are the bi-characteristics $\bar{V}$. As a matter of procedure, the data along the vectors $\bar{W}_j$ should be considered as determined by extrapolation from the data on the characteristic hypercones.

4. Method of generalized invariants

In order to retain the advantages of flow calculations using the method of invariants in a general multidimensional case, we shall outline briefly a numerical treatment of Eq. (3.2).

Consider a field point P with the coordinates (x_0, y_0, z_0, t_0) through which we pass the four vectors $\bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{V}$. The data are assumed to be given on a grid of points in the constant time plane, $t = t_0 - \Delta t_0$, where an interpolation formula may be applied to calculate the dependent variables at the points of intersection of the vectors $\bar{W}_j$ with the data plane. We denote these points by P_j. Only P_4 lies at the intersection of the data plane with the characteristics hypercone having a vertex at P. The points P_1, P_2, and P_3 lie outside of such an intersection. The coordinates x_j, y_j, z_j of the intersection points P_j of the vectors $\bar{W}_j$ with the data plane are

$$\begin{aligned}
&x_1 = x_0 - \left(u + \frac{a}{n_x}\right)\Delta t, && y_1 = y_0 - v\Delta t, && z_1 = z_0 - w\Delta t,\\
&x_2 = x_0 - u\Delta t, && y_2 = y_0 - \left(v + \frac{a}{n_y}\right)\Delta t, && z_2 = z_0 - w\Delta t,\\
&x_3 = x_0 - u\Delta t, && y_3 = y_0 - v\Delta t, && z_3 = z_0 - \left(w + \frac{a}{n_z}\right)\Delta t,\\
&x_4 = x_0 - (u + an_x)\Delta t, && y_4 = y_0 - (v + an_y)\Delta t, && z_4 = z_0 - (w + an_z)\Delta z.
\end{aligned} \tag{4.1}$$

Because of the division by the components of the normal $\bar{n}$, the unit normals $\bar{n}$ should be chosen so that none of their components are too small.

With the dependent variables at the points P_j denoted by u_j, v_j, w_j, p_j, Eq. (3.2) may be written approximately as

$$\varrho a n_x (u_1 - u_0) + \varrho a n_y (v_2 - v_0) + \varrho a n_z (w_3 - w_0) + (p_4 - p_0) = 0. \tag{4.2}$$

Writing Eq. (4.2) four times for four independent unit normals $\bar{n}_i$ gives a linear system for the determination of the solution u_0, v_0, w_0, p_0, at the field point P at time $t = t_0$. Since the coefficients in Eq. (4.2) have the meaning of their average values along the vectors $\overline{W}_j$, a second-order accuracy would result from repeating the calculations and using the average values of the coefficients both in Eq. (4.2) and in Eq. (4.1).

A convenient choice of the four unit normals $\bar{n}_i$ is

$$\begin{aligned}\bar{n}_1 &= \{\ \ 1,\ \ \ 1,\ \ \ 1\}/\sqrt{3},\\ \bar{n}_2 &= \{-1,\ \ \ 1,\ \ \ 1\}/\sqrt{3},\\ \bar{n}_3 &= \{\ \ 1,\ -1,\ \ \ 1\}/\sqrt{3},\\ \bar{n}_4 &= \{\ \ 1,\ \ \ 1,\ -1\}/\sqrt{3}.\end{aligned}$$

For an increased accuracy of the iteration process, the rates of change of the normals $\bar{n}_i$ along the rays $\bar{V}$ could be calculated from the equation obtained by VARLEY and CUMBERBATCH [6, 1965] which, in the index notation with summation convention implied, is

$$\frac{dn_i}{dt} = (n_i n_j - \delta_{ij})\left[n_k \frac{\partial u_k}{\partial x_j} + \frac{\partial a}{\partial x_j}\right].$$

5. Conclusions

The main advantage of the method of generalized invariants lies in the simplicity of the task of performing the numerical calculations. The simplicity of Eq. (4.2), which resulted from the approximation of the total derivatives in Eq. (3.2) by finite differences, may be contrasted with the customary treatment of the method of characteristics or the finite difference method applied directly to the Euler Equations. In the latter two methods finite differences are used to approximate partial derivatives, the number of which grows with dimensionality of space.

Further, since a single interpolating function may be used for all five dependent variables and since Eq. (4.2) is to be applied four times for four choices of the normal $\bar{n}_i$, the amount of computer coding is minimal. No transcendental functions are used and all operations reduce to a simple linear algebra and arithmetic. It is believed that the savings in computer time will be significant and that the present approach will place the method of generalized invariants on a competitive footing with the finite difference methods.

The question of accuracy was touched upon by mentioning that accuracy of second order could result from iterating the system once. The necessary condition for stability, namely, the requirement that the domain of dependence of the partial differential equations be contained in the domain of dependence of the approximating system, could be imposed by constructing the interpolation function properly so that it uses the information from all grid points contained in and on the convex hull of the domain of dependence of the partial differential equations for a given size of the time step Δt.

Sample calculations of examples of non-stationary three-dimensional flows will be published in the near future.

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