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On description of rate-independent behaviour for prestrained solids(*)

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STARTING from a reduced constitutive functional for a simple material, the Cauchy stress tensor is first represented as a polynomial tensor function of the Almansi strain tensor, the rate of deformation tensor and the arc length of the strain path. Then, a general constitutive equation for a class of prestrained plastic materials in finite deformation is derived by applying to the resulting tensor function the restriction of rate-independence of the plastic deformation. The existence of the yield criterion and its general representation are shown as a consequence of the rate-independence requirement imposed on a tensor function representation for the constitutive relation. The inverse relation for the simplified constitutive functional is also derived, and the validity and limitations of the flow theory of classical plasticity is discussed with reference to the present results.

Wychodząc od zredukowanego funkcjonału konstytutywnego dla materiału prostego, przedstawiono najpierw tensor naprężenia Cauchy'ego jako wielomianową funkcję tensorową tensora odkształcenia Almansiego, tensora prędkości odkształceń oraz długości łuku drogi odkształcenia. Następnie wyprowadzono ogólne równanie konstytutywne dla klasy plastycznych materiałów sprężonych przy skończonym odkształceniu przez nałożenie na otrzymaną funkcję tensorową warunku niezależności deformacji plastycznej od prędkości odkształcenia. Istnienie warunku uplastycznienia i jego ogólna postać są konsekwencją warunku niezależności od prędkości, nałożonego na tensorową reprezentację w związku konstytutywnym. Wyprowadzono również zależność odwrotną dla uproszczonego funkcjonału konstytutywnego oraz przedyskutowano zakres ważności teorii płynięcia klasycznej teorii plastyczności w odniesieniu do otrzymanych wyników.

Исходя из приведенного определяющего функционала для простого материала, представлен сначала тензор напряжения Коши, как многочленная тензорная функция тензора деформаций Альманси, тензора скорости деформаций и длины дуги пути деформации. Затем выведены общие определяющие уравнения для класса пластических сжатых материалов, при конечной деформации, путем наложения на полученную тензорную функцию условия независимости пластической деформации от скорости деформации. Существование условия пластичности и его общий вид являются следствием условия независимости от скорости, наложенного на тензорное представление в определяющем соотношении. Выведена тоже обратная зависимость для упрощенного определяющего функционала и обсуждена область справедливости теории течения классической теории пластичности по отношению к полученным результатам.

1. Introduction

AMONG characteristic features of plastic deformation, its history-dependence, rate-independence and yielding seem to be the most essential ones. The classical theories of plasticity may be empirical representations of these phenomena. THOMAS [1, 2], on the other hand, postulating proportionality between the stress and the plastic strain rate tensors and

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expressing the rate-independence in terms of the condition of vanishing one-to-one correspondence between the components of these two tensors, arrived at both a general form of a yield criterion and a stress-strain rate relation for incompressible, isotropic, perfectly plastic materials. Afterwards, the condition was replaced by SAWCZUK and STUTZ [3] by a more stringent requirement that the tensorial relation between plastic strain rate and stress should be homogeneous of zero order with respect to the strain rate. This approach to deriving a yield condition and a stress-strain relation for perfectly plastic materials was applied to the case of initially anisotropic solids by BOEHLER and SAWCZUK [4, 5] and further developed by BOEHLER [6, 7] within the framework of tensor function representations.

The rate-independence of material response is not a specific property of perfect plasticity alone, but is an essential feature to the plastic deformation in general. Therefore, the condition of homogeneity assuring this rate-independence may also be applicable to the general constitutive relation expressed as a functional of the history of deformations.

In the present note a general yield criterion and a general constitutive relation for a class of hardening plastic materials are examined by imposing the rate independence requirement on the material response functional and, eventually, on a differential type of constitutive relations. The discussion throughout the paper is concerned with finite rigid-plastic deformations of materials undergoing deformation-induced hardening or softening, but inclusion of infinitesimal elastic deformation is straightforward. In Sect. 2 a representation for the Cauchy stress tensor as a polynomial isotropic function of the Almansi strain tensor, the rate of deformation tensor, and the arc length of the strain path is derived from a reduced constitutive functional of a simple material. Then, the requirement of the rate-independence is applied in Sect. 3 to the resulting polynomial representation to derive a general yield condition for hardening plastic materials in Sect. 4. Incompressible materials are considered in the next section, whereas in Sect. 6 the constitutive relation is shown to be invertible. For a given prestrain the rate of the deformation tensor is expressed as a polynomial isotropic tensor function of stress, strain and the arc length of the strain path. The validity and the limitation of the flow theory of classical plasticity are also discussed in Sect. 7 with reference to the present results. The note terminates with conclusions and remarks as to advantages and limitations of the tensor function representations approach to plasticity.

2. Approximation of constitutive functional

The history-dependence is generally taken as the first characteristic of plastic deformations and it is a general feature in the mechanical response of materials. For a simple material the Cauchy stress tensor $\mathbf{T}(t)$ at a given material point at a time t may be expressed as a functional of the deformation gradient $\mathbf{F}(t-s)$ of the material point for all past instants:

$$\mathbf{T}(t) = \mathop{\mathbf{f}}_{s=0}^{\infty}[\mathbf{F}(t-s)]. \tag{2.1}$$

In virtue of principles of continuum mechanics the stress state specified by Eq. (2.1) is expressed eventually as follows [8, 9]:

$$\mathbf{T}(t) = \underset{s=0}{\overset{\infty}{\mathbf{g}}}[\mathbf{C}_{(t)}(t-s), \mathbf{B}(t)], \tag{2.2}$$

where $\mathbf{C}_{(t)}(t-s)$ and $\mathbf{B}(t)$ are the relative right and the left Cauchy-Green tensor. By introducing the Green and the Almansi strain tensor defined by

$$\mathbf{G} = \frac{1}{2}(\mathbf{C}-\mathbf{I}), \quad \mathbf{P} = \frac{1}{2}(\mathbf{I}-\mathbf{B}^{-1}), \tag{2.3}$$

Eq. (2.2) leads to

$$\mathbf{T}(t) = \underset{s=0}{\overset{\infty}{\mathbf{h}}}[\mathbf{G}_{(t)}(t-s), \mathbf{P}(t)], \tag{2.4}$$

where the symbol $\mathbf{I}$ denotes the unit tensor.

The history of the deformation is fully specified by the tensor $\mathbf{G}_{(t)}(t-s)$. However, for the purpose of further specification of the intended material model, we introduce the arc length $L(t)$ of the strain path in the strain space

$$L(t) = \int_{-\infty}^{t} [\operatorname{tr}\dot{\mathbf{G}}^2(\tau)]^{\frac{1}{2}} d\tau, \tag{2.5}$$

where the dot denotes the material time derivative. Including the arc length into the arguments of the functional in Eq. (2.4) explicitly, we represent $\mathbf{T}(t)$ in an alternative form

$$\mathbf{T}(t) = \underset{s=0}{\overset{\infty}{\mathbf{h}}}[\mathbf{G}_{(t)}(t-s), \mathbf{P}(t), L(t)]. \tag{2.6}$$

In specifying further the intended material model we recall the principle of the fading memory [8]. It states that the past strain history has more influence on the present stress the closer the past instant is to the present time. Therefore, if the strain history is smooth enough in the sufficient neighbourhood of the present time t where influence of the strain history on the current stress is significant, the strain history $\mathbf{G}_{(t)}(t-s)$ in Eq. (2.6) may be expanded in the vicinity of $s = 0$:

$$\begin{gathered}\mathbf{G}_{(t)}(t-s) = \sum_{n=0}^{N} \frac{1}{n!} s^n \mathbf{G}^n(t) + \mathbf{R}_N, \\ \mathbf{G}^0(t) = \mathbf{G}_{(t)}(t) = 0, \ \mathbf{G}^n(t) = \frac{d^n}{ds^n}\mathbf{G}_{(t)}(t-s)|_{s=0} \quad (n \geqslant 1).\end{gathered} \tag{2.7}$$

Restricting our considerations to such materials that the remainder term $\mathbf{R}_N$ for a certain integer N can be neglected, and replacing $\mathbf{G}^n(t)$ by the Rivlin-Ericksen tensors $\mathbf{A}_n(t)$ defined by

$$\mathbf{A}_n(t) = \mathbf{C}^n_{(t)}(t) = (-1)^n \frac{d^n}{ds^n}\mathbf{C}_{(t)}(t-s)|_{s=0}, \tag{2.8}$$

the functional on the right-hand side of Eq. (2.6) may be approximated by a tensor function. Hence the Cauchy stress tensor $\mathbf{T}(t)$ is expressed as follows:

$$\mathbf{T}(t) = \mathbf{k}[\mathbf{A}_n(t)(n = 1, 2, \ldots, N), \mathbf{P}(t), L(t)]. \tag{2.9}$$

Equation (2.9) may be obtained also by a small modification of the general constitutive equation for an isotropic material of differential type [8, 10]. This equation constitutes a starting point for the material model to be considered.

In studying plastic deformation, we feel it appropriate to employ the current configuration as the reference one since such quantity as a yield criterion has then a physical meaning. However, in the solution of a certain class of boundary value problems, it is often more convenient to formulate them with respect to the undeformed configuration. The constitutive equation in regard to the undeformed configuration can be derived in a similar way by starting from an appropriate function instead of Eq. (2.2) [8, 9]. A form is then obtained which is similar to that employed by PIPKIN and RIVLIN [11] when they discussed of the mechanics of rate-independent materials.

In the following further analysis will be performed by employing Eq. (2.9) and using representations for tensor-valued tensor functions.

3. Restriction of rate-independence

Since Eq. (2.9) is a tensor-valued function of $N+1$ tensors, its polynomial representation is quite involved. Therefore we restrict our attention to the simplest representation without, however, losing the essential features of the considered model of material response. We assume thus $N = 1$ and include, among the strain parameters, the strain tensor $\mathbf{P}$, the rate of the deformation tensor $\mathbf{D} = 1/2\mathbf{A}_1$ and the arc length of the strain path:

$$\mathbf{T}(t) = \mathbf{k}[\mathbf{D}(t), \mathbf{P}(t); L(t)]. \tag{3.1a}$$

The tensor $\mathbf{D}$ in this equation is expressed in terms of $\mathbf{F}$ as follows:

$$\mathbf{D} = \frac{1}{2}(\mathbf{L}+\mathbf{L}^t), \quad \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{3.1b}$$

We assume that Eq. (3.1) models a class of strain-hardening plastic materials since it relates the actual stress to the state of prestrain. Then, we shall discuss the restrictions imposed on the representation of $\mathbf{T}(t)$ by the rate-independence which is the second characteristic feature of the plastic deformation. By restricting the discussion to the rigid-finite plastic or the infinitesimal elastic-finite plastic deformation of an initially isotropic hardening materials, the plastic deformation can be argued separately [12, 13]. Hence the word "strain" in this paper will stand for the plastic strain.

Since $\mathbf{T}$, $\mathbf{D}$ and $\mathbf{P}$ entering Eq. (3.1) are symmetric tensors, the isotropic tensor function $\mathbf{k}$ may be represented by a polynomial function involving a specific number of tensor generators multiplied by scalar-valued functions of a definite number of invariants constituting the integrity basis. Following the reference [14, 15] we take the representation

$$\mathbf{T} = \eta_0\mathbf{I}+\eta_1\mathbf{P}+\eta_2\mathbf{P}^2+\eta_3\mathbf{D}+\eta_4(\mathbf{PD}+\mathbf{DP})+\eta_5(\mathbf{P}^2\mathbf{D}+\mathbf{DP}^2)$$
$$+\eta_6\mathbf{D}^2+\eta_7(\mathbf{PD}^2+\mathbf{D}^2\mathbf{P})+\eta_8(\mathbf{P}^2\mathbf{D}^2+\mathbf{D}^2\mathbf{P}^2) \tag{3.2}$$

although the irreducible representation does not need the last generator [7, 16, 17]. The coefficients $\eta_0, \eta_1, \ldots, \eta_8$ are here functions of eleven scalars, namely L and ten simultaneous invariants of $\mathbf{P}$ and $\mathbf{D}$ constituting the integrity basis. We prefer to keep L explicitly though it is not an independent invariant in the chosen set. Hence

$$\eta_i = \eta_i(L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{P}^2, \mathrm{tr}\mathbf{P}^3; \mathrm{tr}\mathbf{D}, \mathrm{tr}\mathbf{PD}, \mathrm{tr}\mathbf{P}^2\mathbf{D}, \mathrm{tr}\mathbf{D}^2, \mathrm{tr}\mathbf{PD}^2, \mathrm{tr}\mathbf{P}^2\mathbf{D}^2, \mathrm{tr}\mathbf{D}^3), \quad i = 0, 1, \ldots, 8. \tag{3.3}$$

For the purpose of further convenience in studying incompressible motion the deviatoric tensors for $\mathbf{T}$, $\mathbf{P}$ and $\mathbf{D}$ are introduced:

$$\mathbf{S} = \mathbf{T} - \frac{1}{3}(\mathrm{tr}\mathbf{T})\mathbf{I}, \quad \mathbf{Q} = \mathbf{P} - \frac{1}{3}(\mathrm{tr}\mathbf{P})\mathbf{I}, \quad \mathbf{E} = \mathbf{D} - \frac{1}{3}(\mathrm{tr}\mathbf{D})\mathbf{I}. \tag{3.4}$$

Then Eq. (3.2) reduces to the following two functions, a scalar one and a tensor one:

$$\begin{aligned} \mathrm{tr}\mathbf{T} &= \phi_0, \\ \mathbf{S} &= \phi_1\mathbf{Q} + \phi_2\left[\mathbf{Q}^2 - \frac{1}{3}(\mathrm{tr}\mathbf{Q}^2)\mathbf{I}\right] + \phi_3\mathbf{E} + \phi_4\left[(\mathbf{QE}+\mathbf{EQ}) - \frac{2}{3}(\mathrm{tr}\mathbf{QE})\mathbf{I}\right] \\ &\quad + \phi_5\left[(\mathbf{Q}^2\mathbf{E}+\mathbf{EQ}^2) - \frac{2}{3}(\mathrm{tr}\mathbf{Q}^2\mathbf{E})\mathbf{I}\right] + \phi_6\left[\mathbf{E}^2 - \frac{1}{3}(\mathrm{tr}\mathbf{E}^2)\mathbf{I}\right] \\ &\quad + \phi_7\left[(\mathbf{QE}^2+\mathbf{E}^2\mathbf{Q}) - \frac{2}{3}(\mathrm{tr}\mathbf{QE}^2)\mathbf{I}\right] + \phi_8\left[(\mathbf{Q}^2\mathbf{E}^2+\mathbf{E}^2\mathbf{Q}^2) - \frac{2}{3}(\mathrm{tr}\mathbf{Q}^2\mathbf{E}^2)\mathbf{I}\right], \end{aligned} \tag{3.5}$$

where

$$\phi_i = \phi_i(L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; \mathrm{tr}\mathbf{D}, \mathrm{tr}\mathbf{QE}, \mathrm{tr}\mathbf{Q}^2\mathbf{E}, \mathrm{tr}\mathbf{E}^2, \mathrm{tr}\mathbf{QE}^2, \mathrm{tr}\mathbf{Q}^2\mathbf{E}^2, \mathrm{tr}\mathbf{E}^3), \quad i = 0, 1, \ldots, 8. \tag{3.6}$$

The detailed forms of ϕ_i can be calculated readily if the forms of η_i in Eq. (3.2) are given.

The deviatoric strain tensor $\mathbf{Q}$ is not coaxial with the deviator of the rate of the deformation tensor $\mathbf{E}$ in general. Hence, Eq. (3.5) indicates that the principal axes of the deviatoric stress tensor $\mathbf{S}$ do not coincide with those of $\mathbf{E}$ even in initially isotropic materials, and development of the plastic deformation gives rise to the anisotropy in the tensorial relation between $\mathbf{S}$ and $\mathbf{E}$, which is an essential feature of plastic deformations.

The condition of rate-independence of the plastic deformation can be stated as the requirement that the stress tensor $\mathbf{T}$ in Eq. (3.5) should be a homogeneous function of zero order with respect to the rate of the deformation tensor $\mathbf{D}$ [3, 4]. Then, the coefficients $\phi_i (i = 0, 1, \ldots, 8)$ have to be the homogeneous scalar functions with respect to $\mathbf{D}$ of the orders 0, 0, 0, -1, -1, -1, -2, -2, -2, respectively. By introducing new variables

$$\begin{aligned} &E_1 = \mathrm{tr}\mathbf{D}, \quad E_2 = \mathrm{tr}\mathbf{QE}, \quad E_3 = \mathrm{tr}\mathbf{Q}^2\mathbf{E}, \quad E_4 = (\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}, \\ &E_5 = (\mathrm{tr}\mathbf{QE}^2)^{\frac{1}{2}}, \quad E_6 = (\mathrm{tr}\mathbf{Q}^2\mathbf{E}^2)^{\frac{1}{2}}, \quad E_7 = (\mathrm{tr}\mathbf{E}^3)^{\frac{1}{3}} \end{aligned} \tag{3.7}$$

and employing the Euler theorem on homogeneous functions, the condition of homogeneity leads to the following system of differential equations:

$$\frac{\partial\phi_i}{\partial E_j} E_j = n\phi_i, \quad i = 0, 1, \ldots, 8, \quad j = 1, 2, \ldots, 7, \tag{3.8}$$

where

(3.9)
$$n = \begin{cases} 0, & i = 0,1,2, \\ -1, & i = 3,4,5, \\ -2, & i = 6,7,8. \end{cases}$$

The solution of the linear partial differential equations (3.8) are

(3.10)
$$\phi_i = (\mathrm{tr}E^2)^{\frac{n}{2}}\Phi_i, \quad i = 0,1,\ldots,8,$$
$$\Phi_i = \Phi_i[L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; \quad F_j(j = 1,2,\ldots,6)].$$

The integer n in the above equation is as indicated in Eq. (3.9), and F_j are defined as follows:

(3.11)
$$F_1 = \frac{\mathrm{tr}\mathbf{D}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}}, \quad F_2 = \frac{\mathrm{tr}\mathbf{QE}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}}, \quad F_3 = \frac{\mathrm{tr}\mathbf{Q}^2\mathbf{E}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}},$$
$$F_4 = \frac{(\mathrm{tr}\mathbf{QE}^2)^{\frac{1}{2}}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}}, \quad F_5 = \frac{(\mathrm{tr}\mathbf{Q}^2\mathbf{E}^2)^{\frac{1}{2}}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}}, \quad F_6 = \frac{(\mathrm{tr}\mathbf{E}^3)^{\frac{1}{3}}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}}.$$

Finally, the general stress-strain relation for prestrained plastic materials defined by Eq. (3.1) takes the form

(3.12)
$$\mathrm{tr}\mathbf{T} = \Phi_0,$$
$$\mathbf{S} = \Phi_1\mathbf{Q} + \Phi_2\left[\mathbf{Q}^2 - \frac{1}{3}(\mathrm{tr}\mathbf{Q}^2)\mathbf{I}\right] + \Phi_3\frac{\mathbf{E}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}} + \Phi_4\left[\frac{(\mathbf{QE}+\mathbf{EQ})}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}} - \frac{2}{3}F_2\mathbf{I}\right]$$
$$+\Phi_5\left[\frac{(\mathbf{Q}^2\mathbf{E}+\mathbf{EQ}^2)}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}} - \frac{2}{3}F_3\mathbf{I}\right] + \Phi_6\left[\frac{\mathbf{E}^2}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}} - \frac{1}{3}\mathbf{I}\right]$$
$$+\Phi_7\left[\frac{(\mathbf{QE}^2+\mathbf{E}^2\mathbf{Q})}{\mathrm{tr}\mathbf{E}^2} - \frac{2}{3}F_4^2\mathbf{I}\right] + \Phi_8\left[\frac{(\mathbf{Q}^2\mathbf{E}^2+\mathbf{E}^2\mathbf{Q}^2}{\mathrm{tr}\mathbf{E}^2} - \frac{2}{3}F_5^2\mathbf{I}\right],$$

with Φ_i as specified in Eq. (3.10). Moreover, Eq. (3.12) has to satisfy the thermodynamical restrictions

(3.13)
$$\mathrm{tr}\mathbf{T}\,\mathrm{tr}\mathbf{D} \geqslant 0, \quad \mathrm{tr}\mathbf{SE} \geqslant 0.$$

4. Existence of yield criterion

The set of tensors **T** and **P** has ten independent invariants. Selecting seven invariants involving both **T** and **P** or **T** only, and introducing the notation

(4.1)
$$\mathrm{tr}\mathbf{T} = G_0, \quad \mathrm{tr}\mathbf{QS} = G_1, \quad \mathrm{tr}\mathbf{Q}^2\mathbf{S} = G_2, \quad \mathrm{tr}\mathbf{S}^2 = G_3,$$
$$\mathrm{tr}\mathbf{QS}^2 = G_4, \quad \mathrm{tr}\mathbf{Q}^2\mathbf{S}^2 = G_5, \quad \mathrm{tr}\mathbf{S}^3 = G_6,$$

appropriate expressions for $G_k(k = 0,1,\ldots,6)$ can be obtained if Eq. (3.12) is made use of. Using the Cayley-Hamilton theorem extended for two 3×3 matrices [14, 15], we arrive eventually at the following set of relations:

(4.2)
$$G_k = G_k[L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; F_j(j = 1,2,\ldots,6)], \quad k = 0,1,\ldots,6.$$

It is seen in Eq. (4.2) that the invariants G_k ($k = 0, 1, \ldots, 6$) are related to the rate of the deformation tensor only through the six simultaneous invariants F_j ($j = 1, 2, \ldots, 6$). Equation (4.2) may be considered as a transformation from F_j to G_k. If the functions G_k are single-valued, continuous, possess first partial derivatives with respect to F_j and the Jacobian of a set of any six equations among the set (4.2) (say, any six equations of the set (4.2) except the p-th) satisfies the condition

$$(4.3)\qquad J_6 = \left|\frac{\partial G_k}{\partial F_j}\right| \neq 0, \quad j = 1, 2, \ldots, 6; \quad k = 0, 1, \ldots, \hat{p}, \ldots, 6 \quad \text{for any } p \subset (0, 1, \ldots, 6),$$

then, according to the theorem on implicit functions, the six quantities F_j ($j = 1, 2, \ldots, 6$) can be expressed uniquely as functions of G_k ($k = 0, 1, \ldots, \hat{p}, \ldots, 6$) where the symbol ($\hat{\ }$) denotes elimination of the indicated element from the set. Substitution of these F_j into G_p furnishes a scalar-valued function of seven scalars G_k ($k = 0, 1, \ldots, 6$) with L, $\mathrm{tr}\mathbf{P}$, $\mathrm{tr}\mathbf{Q}^2$ and $\mathrm{tr}\mathbf{Q}^3$ as parameters:

$$(4.4)\qquad f = f[L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; G_k(k = 0, 1, \ldots, 6)]$$
$$= f(L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; \mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{QS}, \mathrm{tr}\mathbf{Q}^2\mathbf{S}, \mathrm{tr}\mathbf{S}^2, \mathrm{tr}\mathbf{QS}^2, \mathrm{tr}\mathbf{Q}^2\mathbf{S}^2, \mathrm{tr}\mathbf{S}^3) = 0.$$

A specific form of the function f depends on the forms of Φ_i ($i = 0, 1, \ldots, 8$).

Since Eq. (4.4) is a scalar relation which the stress components have to satisfy in order that plastic deformation may occur in a prestrained material element, it is precisely the yield condition postulated in the classical theory of plasticity. Thus the existence of a yield criterion as the third fundamental feature of plastic deformations has been shown to be the consequence of the rate-independence. The criterion is sensitive to prestraining, hence history-dependent.

Conversely, only when the yield condition (4.4) holds, can the rate of deformation tensors which are independent of the choice of the time scale be determined uniquely for the given values of L, $\mathbf{T}$ and $\mathbf{P}$. This will be demonstrated in Sect. 6. Thus the requirement of rate-independence for plastic deformations is the sufficient condition for the existence of a yield condition for the material defined by Eq. (3.1) whenever the condition (4.3) is satisfied.

If, contrary to the condition (4.3), the Jacobian of degree 6 vanishes identically for six equations of the transformation (4.2), that is, if a relation

$$(4.5)\qquad J_6 = \left|\frac{\partial G_k}{\partial F_j}\right| \equiv 0, \quad j = 1, 2, \ldots, 6;$$
$$k = 0, 1, \ldots, \hat{p}, \ldots, 6 \text{ for certain } \hat{p} \subset (0, 1, \ldots, 6)$$

holds for every set of six G_k, then the number of independent G_k is less than five. The remaining G_k's are therefore expressible in terms of the former ones. Then, the relation analogous to Eq. (4.4) in this case has the form

$$(4.6)\qquad f(L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; G_0, G_1, \ldots, G_{6-p}; F_1, F_2, \ldots, F_q) = 0, \qquad q \leqslant p, p \leqslant 5$$

if the last p equations in the set (4.4), say, are dependent. Though the stress which satisfies the above equation is not influenced by the magnitude of the rate of the deformation tensor $\mathbf{D}$, it depends on the values of F_j and hence on the principal directions of $\mathbf{D}$. It can be thus concluded that, for a given value of the strain tensor $\mathbf{P}$, if deformation were to occur under a set of stress states which have a certain given triplet of principal directions, the rate of deformation tensor should have such principal directions that correspond to the smallest magnitude of the stress tensor in the set of stress states. This may be interpreted also as a kind of yielding. Such phenomena will not be further considered in this note.

A yield criterion disregarding the arguments $\mathrm{tr}\mathbf{P}$ and $\mathrm{tr}\mathbf{T}$ in Eq. (4.4) has been proposed by SHRIVASTAVA, MRÓZ and DUBEY [18] in their discussion of the plastic flow of incompressible materials with anisotropic hardening property in the framework of the classical theory of plasticity. The yield condition was thus taken to be a scalar-valued function of two symmetric deviators and of the arc length of the strain path. In the present note existence of a yield condition has been shown to be the consequence of the rate--independence requirement.

5. Incompressible plastic materials

Since it has been ascertained experimentally for most metals that the volumetric change in plastic deformations is small for the ordinary range of the mean normal stress, theories of plasticity usually postulate incompressibility in the plastic range [19]. Let us specialize the relations derived so far to the case of incompressible plastic materials for which we have the relations [8, 9]:

$$\det(\mathbf{I}-2\mathbf{P}) = 1, \quad \mathrm{tr}\mathbf{D} = 0. \tag{5.1}$$

In an incompressible material the stress is determined by the deformation history only to within an indeterminate isotropic tensor $-p\mathbf{I}$, that is,

$$\mathbf{T} = -p\mathbf{I}+\mathbf{T}_E, \tag{5.2}$$

where $\mathbf{T}_E$ denotes an extra stress which may be determined by the deformation history compatible to the condition (5.1) to within an indeterminate isotropic tensor.

When $\mathbf{T}_E$ is identified with the deviatoric stress $\mathbf{S}$, the indeterminate pressure becomes the mean normal stress, and Eq. (5.2) should satisfy the conditions

$$\left.\begin{aligned} -3p &= \mathrm{tr}\mathbf{T} \\ \mathrm{tr}\mathbf{T}_E &= \mathrm{tr}\mathbf{S} = 0 \end{aligned}\right\}. \tag{5.3}$$

Thus, in virtue of the constraints (5.1), Eq. (3.5) reduces to the form

$$\begin{aligned} \mathrm{tr}\mathbf{T} &= -3p \\ \mathbf{S} &= \overline{\varphi}_1\mathbf{Q}+\overline{\varphi}_2\left[\mathbf{Q}^2-\frac{1}{3}(\mathrm{tr}\mathbf{Q}^2)\mathbf{I}\right]+\overline{\varphi}_3\mathbf{D}+\overline{\varphi}_4\left[(\mathbf{QD}+\mathbf{DQ})-\frac{2}{3}(\mathrm{tr}\mathbf{QD})\mathbf{I}\right] \\ &\quad +\varphi_5\left[(\mathbf{Q}^2\mathbf{D}+\mathbf{DQ}^2)-\frac{2}{3}(\mathrm{tr}\mathbf{Q}^2\mathbf{D})\mathbf{I}\right]+\overline{\varphi}_6\left[\mathbf{D}^2-\frac{1}{3}(\mathrm{tr}\,\mathbf{D}^2)\mathbf{I}\right] \\ &\quad +\overline{\varphi}_7\left[(\mathbf{QD}^2+\mathbf{D}^2\mathbf{Q})-\frac{2}{3}(\mathrm{tr}\mathbf{QD}^2)\mathbf{I}\right]+\overline{\varphi}_8\left[(\mathbf{Q}^2\mathbf{D}^2+\mathbf{D}^2\mathbf{Q}^2)-\frac{2}{3}(\mathrm{tr}\mathbf{Q}^2\mathbf{D}^2)\mathbf{I}\right]; \end{aligned} \tag{5.4}$$

$$\begin{gathered} \overline{\varphi}_i = \overline{\varphi}_i(L;\mathrm{tr}\mathbf{P},\mathrm{tr}\mathbf{Q}^2,\mathrm{tr}\mathbf{Q}^3;\mathrm{tr}\mathbf{QD},\mathrm{tr}\mathbf{Q}^2\mathbf{D},\mathrm{tr}\mathbf{D}^2,\mathrm{tr}\mathbf{QD}^2,\mathrm{tr}\mathbf{Q}^2\mathbf{D}^2), \\ i = 1, 2, \ldots, 8. \end{gathered} \tag{5.5}$$

By applying the condition of the rate-independence, one eventually obtains the following form of the yield condition:

$$(5.6) \qquad \bar{f} = \bar{f}(L,; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; \mathrm{tr}\mathbf{QS}, \mathrm{tr}\mathbf{Q}^2\mathbf{S}, \mathrm{tr}\mathbf{S}^2, \mathrm{tr}\mathbf{QS}^2, \mathrm{tr}\mathbf{Q}^2\mathbf{S}^2, \mathrm{tr}\mathbf{S}^3) = 0$$

for incompressible, hardening plastic materials. The invariants $\mathrm{tr}\mathbf{P}$, $\mathrm{tr}\mathbf{Q}^2$ and $\mathrm{tr}\mathbf{Q}^3$ appearing in Eq. (5.6) have to conform to the constraint imposed by the first equation of the set (5.1).

One notices that Eq. (5.6) does not include the term of the hydrostatic pressure. Though it is not obvious that the hydrostatic pressure does not affect the yield criteria of incompressible plastic materials, its effect has been usually omitted as an a priori assumption [19]. In the theory of representations, in so far as some approaches to the rate-independent motion of materials are concerned, the case of plastic behaviour in incompressible flow was studied by BOEHLER [7]. A particular case of plane plastic flow of initially anisotropic materials was also considered in [7].

In the classical theory of plasticity, consequences of the plastic deformation are interpreted in terms of translation, expansion and rotation of the yield loci. Therefore, in order to establish the plasticity theories capable of describing strain-hardening of incompressible materials insensitive to the hydrostatic pressure, various hardening theories have been proposed so far [20-25]. All of these theories can be derived from the yield criterion (5.6) when specifying appropriately a flow law.

6. Flow law

The constitutive relations of plasticity are usually written so as to specify the rate of the deformation tensor in terms of the stress and other parameters of the strain history. Let us consider a situation when a certain prestrain is attained in a material point. We shall be interested, at the attained state, in the rate of deformation such that the rate independence is satisfied. Thus we want to specify the flow law within the present approach to plastic behaviour. To this end we represent Eq. (3.12) in the form of a set of relations expressing the rate of deformation in terms of the relevant generators of $\mathbf{T}$ and $\mathbf{P}$. Although the outcome can be immediately written, we present the derivations in order to display the requirements involved.

Constructing the six generators of $\mathbf{S}$ and $\mathbf{Q}$, namely $\mathbf{S}$, $\mathbf{QS}+\mathbf{SQ}$, $\mathbf{Q}^2\mathbf{S}+\mathbf{SQ}^2$, $\mathbf{S}^2$, $\mathbf{QS}^2+\mathbf{S}^2\mathbf{Q}$, $\mathbf{Q}^2\mathbf{S}^2+\mathbf{S}^2\mathbf{Q}^2$ from Eq. (3.12) and rearranging them by using the extended Cayley-Hamilton theorem for two 3×3 matrices [14, 15], we have the following relation:

$$(6.1) \qquad \mathbf{J}_l = a_{lm}\mathbf{H}_m, \quad l = 1, 2, \ldots, 6; \quad m = 1, 2, \ldots, 6,$$

where

$$(6.2) \qquad \begin{aligned} &\mathbf{J}_1 = \mathbf{S}+b_{11}\mathbf{I}+b_{12}\mathbf{Q}+b_{13}\mathbf{Q}^2, \quad \mathbf{J}_2 = (\mathbf{QS}+\mathbf{SQ})+b_{21}\mathbf{I}+b_{22}\mathbf{Q}+b_{23}\mathbf{Q}^2,\\ &\mathbf{J}_3 = (\mathbf{Q}^2\mathbf{S}+\mathbf{SQ}^2)+b_{31}\mathbf{I}+b_{32}\mathbf{Q}+b_{33}\mathbf{Q}^2, \quad \mathbf{J}_4 = \mathbf{S}^2+b_{41}\mathbf{I}+b_{42}\mathbf{Q}+b_{43}\mathbf{Q}^3,\\ &\mathbf{J}_5 = (\mathbf{QS}^2+\mathbf{S}^2\mathbf{Q})+b_{51}\mathbf{I}+b_{52}\mathbf{Q}+b_{53}\mathbf{Q}^2,\\ &\mathbf{J}_6 = (\mathbf{Q}^2\mathbf{S}^2+\mathbf{S}^2\mathbf{Q}^2)+b_{61}\mathbf{I}+b_{62}\mathbf{Q}+b_{63}\mathbf{Q}^2; \end{aligned}$$

(6.3)
$$\mathbf{H}_1 = \frac{\mathbf{E}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}}, \quad \mathbf{H}_2 = \frac{\mathbf{QE}+\mathbf{EQ}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}}, \quad \mathbf{H}_3 = \frac{\mathbf{Q}^2\mathbf{E}+\mathbf{EQ}^2}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}},$$
$$\mathbf{H}_4 = \frac{\mathbf{E}^2}{\mathrm{tr}\mathbf{E}^2}, \quad \mathbf{H}_5 = \frac{\mathbf{QE}^2+\mathbf{E}^2\mathbf{Q}}{\mathrm{tr}\mathbf{E}^2}, \quad \mathbf{H}_6 = \frac{\mathbf{Q}^2\mathbf{E}^2+\mathbf{E}^2\mathbf{Q}^2}{\mathrm{tr}\mathbf{E}^2}.$$

The matrix a_{lm} in the relation (6.1) is

(6.4)
$$a_{lm} = a_{lm}[\Phi_i(i = 1, 2, \dots, 8);\ \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3;\ F_j(j = 2, 3, \dots, 6)]$$
$$= a_{lm}[L;\ \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3;\ F_j(j = 1, 2, \dots, 6)].$$

Since we restrict our discussion to the case where Eq. (4.3) holds, $F_j(j = 1, 2, \dots, 6)$ can be expressed unambiguously in terms of $G_k[k = 0, 1, \dots, \hat{p}, \dots, 6$ for any $p \subset (0, 1, \dots, 6)]$. Thus the relation (6.4) can be expressed in an alternative form:

(6.5)
$$a_{lm} = a_{lm}[L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; G_k].$$

The coefficients b_{rs} also have the analogous form

(6.6)
$$b_{rs} = b_{rs}[L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; G_k] \quad r = 1, 2, \dots, 6; \quad s = 1, 2, 3.$$

The a_{lm} and b_{rs} have therefore been expressed in terms of the invariants of $\mathbf{T}$ and the simultaneous invariants of $\mathbf{T}$ and $\mathbf{P}$.

Equation (6.1) is a linear simultaneous equation with real scalar coefficients a_{lm}. Hence, if the determinant

(6.7)
$$D_6 = \det(a_{lm}) \neq 0,$$

then Eq. (6.1) has the unique solution for the rate-independent generators $\mathbf{H}_m$:

(6.8)
$$\mathbf{H}_m = a_{ml}^{-1}\mathbf{J}_l.$$

The first equation of the set (3.12) specifying $\mathrm{tr}\mathbf{T}$ as the function Φ_0 as given in Eq. (3.10) can be written in the form

(6.9)
$$\mathrm{tr}\mathbf{T} = \Phi_0\left[L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; \frac{\mathrm{tr}\mathbf{D}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}}, \quad F_j(j = 2, 3, \dots, 6)\right].$$

Since $F_j(j = 1, 2, \dots, 6)$ can be expressed uniquely in terms of six $G_k(k = 0, 1, \dots, \hat{p}, \dots, 6)$, Eq. (3.12) may be written as

(6.10)
$$\mathrm{tr}\mathbf{T} = \Phi_0\left[L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; \frac{\mathrm{tr}\mathbf{D}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}}, G_k\right]$$

to give eventually

(6.11)
$$\frac{\mathrm{tr}\mathbf{D}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}} = \Phi_0[L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; G_k].$$

Thus Eq. (6.11) together with the first equation of the set (6.8) provide the final form of a plastic flow law for a prestrained material:

$$\frac{\mathrm{tr}\mathbf{D}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}} = \Psi_0,$$

(6.12)
$$\frac{\mathbf{E}}{(\mathrm{tr}\mathbf{E}^2)^{\frac{1}{2}}} = \Psi_1\mathbf{Q}+\Psi_2\left[\mathbf{Q}^2-\frac{1}{3}(\mathrm{tr}\mathbf{Q}^2)\mathbf{I}\right]+\Psi_3\mathbf{S}+\Psi_4\left[(\mathbf{QS}+\mathbf{SQ})-\frac{2}{3}(\mathrm{tr}\mathbf{QS})\mathbf{I}\right]$$
$$+\Psi_5\left[(\mathbf{Q}^2\mathbf{S}+\mathbf{S}\mathbf{Q}^2)-\frac{2}{3}(\mathrm{tr}\mathbf{Q}^2\mathbf{S})\mathbf{I}\right]+\Psi_6\left[\mathbf{S}^2-\frac{1}{3}(\mathrm{tr}\mathbf{S}^2)\mathbf{I}\right]$$
$$+\Psi_7\left[(\mathbf{QS}^2+\mathbf{S}^2\mathbf{Q})-\frac{2}{3}(\mathrm{tr}\mathbf{QS}^2)\mathbf{I}\right]+\Psi_8\left[(\mathbf{Q}^2\mathbf{S}^2+\mathbf{S}^2\mathbf{Q}^2)-\frac{2}{3}(\mathrm{tr}\mathbf{Q}^2\mathbf{S}^2)\mathbf{I}\right],$$

where

(6.13)
$$\Psi_n = \Psi_n(L; \mathrm{tr}\mathbf{P}, \mathrm{tr}\mathbf{Q}^2, \mathrm{tr}\mathbf{Q}^3; \mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{QS}, \mathrm{tr}\mathbf{Q}^2\mathbf{S}, \mathrm{tr}\mathbf{S}^2, \mathrm{tr}\mathbf{QS}^2, \mathrm{tr}\mathbf{Q}^2\mathbf{S}^2, \mathrm{tr}\mathbf{S}^3),$$
$$n = 0, 1, \ldots, 8.$$

Equation (6.12) together with the thermodynamic restriction (3.13) is the most general flow law for the considered class of prestrained materials defined in Eq. (3.1). Various hardening theories proposed hitherto follow from Eq. (6.12) by specifying the forms of the scalar valued functions Ψ_n properly. The rate of the deformation tensor given by Eq. (6.12) may be different from zero only when the yield condition (4.4) is satisfied.

When the condition (6.7) is not satisfied, $\mathbf{H}_m (m = 1, 2, \ldots, 6)$ are indeterminate, and the constitutive relation (3.12) does not have a unique inversion.

A more compact form of Eq. (6.12) for a not necessarily polynomial representation has been given by Raclin [16]. An appropriate form regarding incompressible instantaneous motion can easily be obtained considering Eq. (5.4).

7. Comparison with the associated flow law

In the classical flow theory of plasticity, as a consequence of Drucker's postulate for stable materials, it is usually assumed that a yield condition is the potential for the plastic strain rate [27]. The associated flow law thus follows. We shall examine the relation of the flow law associated with the general yield criterion (4.4) to the inverted constitutive relation (6.12).

In terms of the variables employed in the present note the flow law associated with the yield condition (4.4) gives the rate of the deformation tensor related to the Cauchy stress as follows:

(7.1)
$$\mathbf{D} = \lambda\frac{\partial f}{\partial \mathbf{T}}.$$

Performing the presoribed differentiation of the yield function (4.4), we obtain

$$(7.2)\quad \mathbf{D} = \lambda\left[\frac{\partial f}{\partial G_J}\frac{\partial G_J}{\partial \mathbf{T}}\right]$$
$$= \lambda\left\{\frac{\partial f}{\partial G_0}\mathbf{I}+\frac{\partial f}{\partial G_1}\mathbf{Q}+\frac{\partial f}{\partial G_2}\left[\mathbf{Q}^2-\frac{1}{3}(\mathrm{tr}\mathbf{Q}^2)\mathbf{I}\right]+2\frac{\partial f}{\partial G_3}\mathbf{S}+\frac{\partial f}{\partial G_4}\left[(\mathbf{QS}+\mathbf{SQ})\right.\right.$$
$$\left.\left.-\frac{2}{3}(\mathrm{tr}\mathbf{QS})\mathbf{I}\right]+\frac{\partial f}{\partial G_5}\left[(\mathbf{Q}^2\mathbf{S}+\mathbf{SQ}^2)-\frac{2}{3}(\mathrm{tr}\mathbf{Q}^2\mathbf{S})\mathbf{I}\right]+3\frac{\partial f}{\partial G_6}\left[\mathbf{S}^2-\frac{1}{3}(\mathrm{tr}\mathbf{S}^2)\mathbf{I}\right].$$

Decomposing the obtained rate of the deformation tensor into the isotropic and the deviatoric part, we finally have the respective relations:

$$(7.3)\quad \mathrm{tr}\mathbf{D} = 3\lambda\left(\frac{\partial f}{\partial G_0}\right),$$
$$\mathbf{E} = \lambda\left\{\frac{\partial f}{\partial G_1}\mathbf{Q}+\frac{\partial f}{\partial G_2}\left[\mathbf{Q}^2-\frac{1}{3}(\mathrm{tr}\mathbf{Q}^2)\mathbf{I}\right]+2\frac{\partial f}{\partial G_3}\mathbf{S}+\frac{\partial f}{\partial G_4}\left[(\mathbf{QS}+\mathbf{SQ})\right.\right.$$
$$\left.-\frac{2}{3}(\mathrm{tr}\mathbf{QS})\mathbf{I}\right]+\frac{\partial f}{\partial G_5}\left[(\mathbf{Q}^2\mathbf{S}+\mathbf{SQ}^2)-\frac{2}{3}(\mathrm{tr}\mathbf{Q}^2\mathbf{S})\mathbf{I}\right]$$
$$\left.+3\frac{\partial f}{\partial G_6}\left[\mathbf{S}^2-\frac{1}{3}(\mathrm{tr}\mathbf{S}^2)\mathbf{I}\right]\right\}.$$

In comparison with Eq. (6.12) the second equation of the set (7.3) does not contain the last two terms of Eq. (6.12). Such difference might be attributable to the fact that Eq. (6.12) should be further restricted by the thermodynamic condition (3.13). However, by applying the restriction (3.13) to Eq. (6.12), we can show that the last term of Eq. (6.12) can be compatible to the condition (3.13). Thus we can conclude that the flow theory of classical plasticity as regarded from the viewpoint of the theory of representations for rate-independent material response furnishes only restricted forms of the stress--strain rate relation even if a general flow potential is assumed. One of the additional terms appearing in Eq. (7.3) may be eliminated when the irreducible set of tensor generators is used [26], but the conclusion still remains that the associated flow law gives a restricted form of rate-independent constitutive equations.

8. Concluding remarks

A general stress-strain relation and the corresponding yield criterion for a class of hardening plastic solids were derived from a reduced constitutive functional for a simple material and the condition of the rate-independence of material response. The induced plastic strain results in an anisotropic stress-strain rate relation. Various theories of anisotropic hardening proposed so far in the framework of classical plasticity were shown to be derivable as special cases of constitutive relations assumed in the form of isotropic tensor functions involving as variables two symmetric second-order tensors, namely those of the rate of deformation and of Almansi strain.

The theory proposed was developed for the model of prestrained plastic solids defined by Eq. (3.1) which is a rough approximation to the general constitutive equation (2.4) but still includes the essential features of plastic response in a more comprehensive manner than theories starting with a yield criterion and developing further associated or non-associated flow law. The approach proposed to study plastic flow within the representations of tensor functions enhances the classical theories of plasticity and furnishes a quite general setting for theories of plasticity. Further specifications are needed as to the incremental presentation of the constitutive equation (3.1) or a more appropriate incremental form of the constitutive relation for a rate-independent material response.

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References

1. T. Y. THOMAS, *Interdependence of the yield condition and the stress-strain relations for plastic flow*, Proc. Nat. Acad. Sci. U. S., **40**, 593-597, 1954.
2. T. Y. THOMAS, *Plastic flow and fracture in solids*, Academic Press, New York 1961.
3. A. SAWCZUK and P. STUTZ, *On formulation of stress-strain relations for soils at failure*, J. Appl. Math. Phys. (ZAMP), **19**, 770–778, 1968.
4. J. P. BOEHLER and A. SAWCZUK, *Equilibre limite des sols anisotropes*, J. de Méc., **9**, 5–33, 1970.
5. J. P. BOEHLER and A. SAWCZUK, *On yielding of oriented solids*, Acta Mech., **27**, 185–206, 1977.
6. J. P. BOEHLER, *A simple derivation of representations for non-polynomial constitutive equations in some case of anisotropy*, ZAMM, **59**, 1979 [in print].
7. J. P. BOEHLER, *Lois de comportement anisotrope des milieux continus*, J. de Méc. **17**, 153–190, 1978.
8. C. TRUESDSLL and W. NOLL, *The non-linear field theories of mechanics*, Encyclopedia of Physics, III/3, Springer, Berlin 1965.
9. D. C. LEIGH, *Nonlinear continuum mechanics*, McGraw-Hill, New York 1968.
10. W. NOLL, *A mathematical theory of the mechanical behaviour of continuous media*, Arch. Rational Mech. Anal., **2**, 197–226, 1958/1969.
11. A. C. PIPKIN and R. S. RIVLIN, *Mechanics of rate-independent materials*, Z. Ang. Math. Phys., **16**, 313–327, 1965.
12. E. H. LEE, *Elastic-plastic deformation at finite strains*, J. Appl. Mech., Trans. ASME, Ser. E, **36**, 1–6, 1969.
13. Y. S. WANG, *A simplified theory of the constitutive equations of metal plasticity at finite deformation*, J. Appl. Mech., Trans. ASME, Ser. E., **40**, 941–947, 1973.
14. R. S. RIVLIN, *Further remarks on the stress-deformation relations for isotropic materials*, J. Rational Mech. Anal., **4**, 681–702, 1955.

15. A. J. M. SPENCER and R. S. RIVLIN, *The theory of matrix polynomials and its application to the mechanics of isotropic continua*, Arch. Rational Mech. Anal., **2**, 309-336, 1958/1959.
16. C.-C. WANG, *A new representation theorem for isotropic functions, Part I, Part II*, Corrigendum, Arch. Rational Mech. Anal., **36**, 166–223 1970; **43**, 392–395, 1971.
17. G. F. SMITH, *On isotropic functions of symmetric tensors, skew-symmetric tensors and vectors*, Int. J. Engng. Sci., **9**, 889–916, 1971.
18. H. P. SHRIVASTAVA, Z. MRÓZ and R. N. DUBEY, *Yield criterion and the hardening rule for a plastic solid*. Z. Ang. Math. Mech., **53**,. 625–633, 1973.
19. R. HILL, *The mathematical theory of plasticity*, The Clarendon Press, Oxford 1950.
20. W. PRAGER, *A new method of analyzing stresses and strains in work-hardening plastic solids*, J. Appl. Mech., **23**, 493–496, 1956.
21. H. ZIEGLER, *A modification of Prager's hardening rule*, Quart. Appl Math., **17**, 55–65, 1959.
22. A. BALTOV and A. SAWCZUK, *A rule of anisotropic hardening*, Acta Mech., **I/2**, 81–92, 1965.
23. G. BACKHAUS, *Zur Fliessgrenze bei allgemeinen Verfestigung*, Z. ang. Math. Mech., **48**, 99-108, 1968.
24. N. L. SVENSSON, *Anisotropy and the Bauschinger effect in cold rolled aluminium*, J. Mech. Eng. Sci., **8**, 162, 1965.
25. Y. YOSHIMURA, *Hypothetical theory of anisotropy and the Bauschinger effect due to plastic strain history*, Aer. Res. Inst. Univ. Tokyo Rep., No. 349, 221–246, 1959.
26. J. RACLIN, *Remarques sur le conditions de plasticité des milieux écroussables*, III Congr. Français de Mécanique (Grenoble, September 1977).
27. P. M. NAGHDI, *Stress-strain relations in plasticity and thermoplasticity*, Plasticity, Proc. 2nd Symp, Naval Struct. Mech., ed. by E. H. Lee and P. Symonds, 121–169, Pergamon, Oxford 1960.

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