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Unsteady flow of an elastic-viscous fluid past an infinite porous plate

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IN THE PRESENT work the motion of an elastic-viscous fluid past a two-dimensional unsteady porous plate was studied by using the Laplace transform technique. Expressions for the velocity distribution and the skin friction, for various types of plate motion, have been obtained.

W niniejszej pracy zbadano ruch lepko-sprężystej cieczy wzdłuż dwuwymiarowej, niestacjonarnej porowatej płyty, stosując technikę transformacji Laplace'a. Otrzymano zależności określające rozkład prędkości w cieczy i wartości tarcia powierzchniowego dla różnych przypadków ruchu płyty.

В настоящей работе исследовано движение вязко-упругой жидкости вдоль двумерной, нестационарной пористой плиты, применяя технику преобразования Лапласа. Получены зависимости определяющие распределение скорости в жидкости и значения поверхностного трения для разных случаев движения плиты.

1. Introduction

THE PROBLEM of unsteady motion of a porous plate in an infinite fluid is of practica importance in the analysis of the shaking table of the Fourdrinier paper-making machine. NICOLL *et al.* [1] have studied the laminar motion of a viscous fluid near an oscillating, porous and infinite plane. Their analysis is important because it yields an exact solution of the Navier-Stokes equations of motion. DEBLER and MONTOGOMERY [2] analysed the flow of a viscous liquid over an oscillating, porous plate with suction or with an intermediate film. However, this analysis does not satisfy the initial condition. SHARMA [3] applied Laplace transformation to improve the work of DEBLER and MONTOGOMERY [2] and extended the analysis of NICOLL *et al.* [1] by considering the damped oscillatory motion of a porous, rigid plane in an infinite viscous fluid with suction or blowing.

In the above referred studies, the fluid considered was a Newtonian one. However, a mixture of water and wood-pulp is essentially a non-Newtonian fluid because of the presence of solid constituents in it. Thus the available analyses need to be modified before one can apply them to the actual fluid situation.

In the present paper the analysis of a generalized unsteady motion of a porous, infinite plate in an elastic-viscous fluid has been attempted. Different forms of motion imparted to the plate have been considered. Velocity distribution and skin friction resulting from various modes of plate motion have been obtained. One of the special cases considered can be regarded as a generalization of the analysis due to NICOLL *et al.* [1] and the other as that due to SHARMA [3].

2. Formulation of the problem

The fluid considered is characterized by the constitutive equations

$$s_{ik} = p_{ik} - pg_{ik}, \tag{2.1}$$

$$p^{ik} = 2\int_{-\infty}^{t} \psi(t-t')\frac{\partial x^i}{\partial x'^m}\frac{\partial x^k}{\partial x'^r}\; e^{(1)mr}(x', t')dt', \tag{2.2}$$

where covariant suffixes are written below, contravariant suffixes above, and the usual summation for repeated suffixes is assumed. Here s_{ik} is the stress tensor, p is an arbitrary isotropic pressure, g_{ik} is the metric tensor of a fixed coordinate system x^i, x'^i is the position at time t' of the element which is instantaneously at the point x^i at time t, $e^{(1)ik}$ is the rate of the strain tensor and

$$\psi(t-t') = \int_0^\infty \frac{N(\tau)}{\tau}\exp[-(t-t')/\tau]d\tau,$$

$N(\tau)$ being the distribution function of relaxation times. For fluids with short memory, Eq. (2.2) is simplified to

$$p^{ik} = 2\eta_0 e^{(1)ik} - 2k_0\frac{\delta}{\delta t}e^{(1)ik}, \tag{2.3}$$

where $\eta_0 = \int_0^\infty N(\tau)d\tau$ is the limiting viscosity at small rates of shear, k_0 is the coefficient of elasticity of the fluid,

$$k_0 = \int_0^\infty \tau N(\tau)d\tau,$$

and $\delta/\delta t$ denotes the convected derivative.

Using the simplified equation of state, the equations of motion for the fluid in the Cartesian frame of refenence can be written as

$$\varrho\left(\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} + \eta_0\frac{\partial^2 v_i}{\partial x_k\partial x_k} - k_0\left[\frac{\partial}{\partial t}\left(\frac{\partial^2 v_i}{\partial x_k\partial x_k}\right) + v_m\frac{\partial^3 v_i}{\partial x_m\partial x_k\partial x_k} - \frac{\partial v_i}{\partial x_m}\frac{\partial^2 v_m}{\partial x_k\partial x_k} - 2\frac{\partial^2 v_i}{\partial x_m\partial x_k}\frac{\partial v_m}{\partial x_k}\right], \tag{2.4}$$

where ϱ is the density.

Consider the unsteady flow parallel to an infinite plane surface on which the normal component of velocity takes a given value of $v' = -W$. If x' and y' are measured along and perpendicular to the plane, the velocity components u', v' and pressure are independent of x'. From Eq. (2.4) the governing equation with uniform pressure becomes

$$\frac{\partial u'}{\partial t'} - W\frac{\partial u'}{\partial y'} = \nu\frac{\partial^2 u'}{\partial y'^2} - k_0^*\left(\frac{\partial^3 u'}{\partial y'^2\partial t'} - W\frac{\partial^3 u'}{\partial y'^3}\right), \tag{2.5}$$

where $\nu = \eta_0/\varrho'$ and $k_0^* = k_0/\varrho'$, and the continuity equation for flow is identically satisfied.

The boundary and initial conditions are

$$u'(0, t') = U_0[1+F(t')],$$
$$u'(\infty, t') = 0, \tag{2.6}$$
$$u'(y', 0) = 0,$$

where U_0 is a constant and $F(t')$ is an arbitrary function of time. Let us assume that the fluid velocity in the neighbourhood of the plate is

$$u'(y', t') = U_0[u'_s(y')+u'_f(y', t')], \tag{2.7}$$

where $U_0[u'_s(y')]$ is the velocity at $t = 0$ and $U_0[u'_f(y', t')]$ represents the change in velocity due to $F(t')$. On introducing the non-dimensional parameters

$$\bar{u} = \frac{u'}{U_0}, \quad \bar{y} = \frac{y'W}{\nu}, \quad \bar{k} = \frac{k_0^* W^2}{\nu^2}, \quad \bar{t} = \frac{t'W^2}{4\nu}, \tag{2.8}$$

Eq. (2.5), after dropping the bars, reduces to

$$\frac{1}{4}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - \frac{k}{4}\frac{\partial^3 u}{\partial y^2 \partial t} + k\frac{\partial^3 u}{\partial y^3}. \tag{2.9}$$

The boundary and initial conditions given by Eq. (2.6) become

$$u(0, t) = [1+F(t)],$$
$$u(\infty, t) = 0, \tag{2.10}$$
$$u(y, 0) = 0.$$

The fluid velocity in the neighbourhood of the plate given by Eq. (2.7) becomes

$$u(y, t) = [u_s(y)+u_f(y, t)]. \tag{2.11}$$

Substitution of Eq. (2.11) in Eq. (2.9) gives

$$ku_s''' + u_s'' + u_s' = 0, \tag{2.12}$$

and

$$ku_f''' + u_f'' + u_f' - \frac{1}{4}\frac{\partial u_f}{\partial t} - \frac{k}{4}\frac{\partial u_f''}{\partial t} = 0, \tag{2.13}$$

where the prime denotes differentiation with respect to y. The corresponding boundary conditions are

$$u_s = 1, \quad u_f = F(t) \quad \text{at } y = 0,$$
$$u_s = 0, \quad u_f = 0 \quad \text{as } y \to \infty. \tag{2.14}$$

Following FRATER [4] the exact solution of Eq. (2.12) with corresponding boundary conditions is

$$u_s = \exp\left[-\left(\frac{1-\sqrt{(1-4k)}}{2k}\right)y\right]. \tag{2.15}$$

Equation (2.13) can be solved by applying the Laplace transform defined as

$$\bar{u}(p) = \int_0^\infty \exp(-pt)u(y, t)\,dt,$$

whose inverse is

$$u(y,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{u}(p)\exp(pt)\,dt.$$

Application of the Laplace transform to Eq. (2.13) yields

$$k\bar{u}_f''' + \left(1 - \frac{1}{4}kp\right)\bar{u}_f'' + \bar{u}_f' - \frac{1}{4}p\bar{u}_f = 0, \tag{2.16}$$

where $\bar{u}_f$ is the Laplace transform of u_f. The boundary conditions for Eq. (2.16), represented by Eq. (2.14), reduce to

$$\begin{aligned} \bar{u}_f &= \bar{F} \quad \text{at} \quad y = 0,\\ \bar{u}_f &= 0 \quad \text{as} \quad y \to \infty, \end{aligned} \tag{2.17}$$

where $\bar{F}$ is the Laplace transform of $F(t)$.

Equation (2.16) is a third-order differential equation with two boundary conditions. Following the method proposed by BEARD and WALTERS [5], and SOUNDALGEKAR and PRATAPPURI [6], we assume u_f in the form

$$u_f = u_{f_1} + ku_{f_2} + o(k^2). \tag{2.18}$$

This expansion is justified since the rheological equations are valid only for small values of k.

On substituting Eq. (2.18) into Eq. (2.16) and equating the various powers of k, one obtains

$$\bar{u}_{f_1}'' + \bar{u}_{f_1}' - \frac{p}{4}\bar{u}_{f_1} = 0, \tag{2.19}$$

and

$$\bar{u}_{f_2}'' + \bar{u}_{f_2}' - \frac{p}{4}\bar{u}_{f_2} = -\bar{u}_{f_1}''' + \frac{p}{4}u_{f_1}''. \tag{2.20}$$

The corresponding boundary conditions are

$$\begin{aligned} \bar{u}_{f_1} &= \bar{F}, \quad \bar{u}_{f_2} = 0 \quad \text{at } y = 0,\\ \bar{u}_{f_1} &= 0, \quad \bar{u}_{f_2} = 0 \quad \text{as } y \to \infty. \end{aligned} \tag{2.21}$$

From Eq. (2.18) through (2.21), one obtains

$$\bar{u}_f = \bar{\beta}_1 + k\bar{\beta}_2, \tag{2.22}$$

where

$$\bar{\beta}_1 = \bar{F}\exp(-hy), \tag{2.23}$$

$$\bar{\beta}_2 = \frac{\bar{F}(h^2 + p/4)}{\sqrt{(1+p)}}\,y\exp(-hy), \tag{2.24}$$

$$h = \frac{1}{2}\left[1 + \sqrt{(1+p)}\right]. \tag{2.25}$$

3. Method of solution

The unsteady velocity is found by inverting $\bar{\beta}_1$ and $\bar{\beta}_2$ given by Eq. (2.23) and (2.24). In order to determine the inverse we put $\bar{F} = 1$ and, inverting the values of $\bar{\beta}_1$ and $\bar{\beta}_2$ [7], one gets

$$\beta_1 = L^{-1}(\bar{\beta}_1) = \frac{yH(4t/y^2)}{4\sqrt{\pi}t^{3/2}} e^{-\left(\frac{y}{4\sqrt{t}}+\sqrt{t}\right)^2}, \tag{3.1}$$

$$\beta_2 = L^{-1}(\bar{\beta}_2) = \frac{yH(4t/y^2)}{\sqrt{(\pi t)}}\left[\frac{y^4}{4096t^4}+\frac{y^3}{256t^3}-\frac{3y^2}{256t^3}+\frac{3y^2}{128t^2} - \frac{3y}{32t^2}+\frac{y}{16t}+\frac{3}{64t^2}-\frac{3}{16t}+\frac{1}{16}\right]e^{-\left(\frac{y}{4\sqrt{t}}+\sqrt{t}\right)^2}, \tag{3.2}$$

where $H(4t/y^2)$ is the Heaviside unit step function.

The shear stress in case of an elastic-viscous fluid is given

$$p'_{x'y'} = \eta_0\frac{\partial u'}{\partial y'} - k_0\left(\frac{\partial^2 u'}{\partial y'\partial t'}+v'\frac{\partial^2 u'}{\partial y'^2}\right). \tag{3.3}$$

Non-dimensionalizing Eq. (3.3) with parameters given in Eq. (2.8) and dropping the bars, one gets

$$p_{xy} = \frac{p'_{x'y'}}{\varrho' W U_0} = \frac{\partial u}{\partial y} - \frac{k}{4}\left(\frac{\partial^2 u}{\partial y\partial t} - 4\frac{\partial^2 u}{\partial y^2}\right). \tag{3.4}$$

The skin friction at the plate is

$$p_{xy}]_{y=0} = -1+\alpha_1+k\alpha_2, \tag{3.5}$$

where

$$\alpha_1 = \frac{\partial\beta_1}{\partial y}\bigg]_{y=0}, \qquad \alpha_2 = \frac{\partial\beta_2}{\partial y}\bigg]_{y=0} + \frac{\partial^2\beta_1}{\partial y^2}\bigg]_{y=0} - \frac{1}{4}\frac{\partial^2\beta_1}{\partial y\partial t}\bigg]_{y=0}. \tag{3.6}$$

When $\bar{F}(p) \neq 1$, the values of β_1 and β_2 can be calculated by applying the convolution theorem [8]. Thus

$$\beta_1 = \int_0^t F(t-\lambda)\left[\frac{yH(4\lambda/y^2)}{4\sqrt{\pi}\lambda^{3/2}}e^{-\left(\frac{y}{4\sqrt{\lambda}}+\sqrt{\lambda}\right)^2}\right]d\lambda,$$

$$\beta_2 = \int_0^t F(t-\lambda)\left[\frac{yH(4\lambda/y^2)}{\sqrt{(\pi\lambda)}}\left[\frac{y^4}{4096\lambda^4}+\frac{y^3}{256\lambda^3}-\frac{3y^2}{256\lambda^3}+\frac{3y^2}{128\lambda^2}-\frac{3y}{32\lambda^2} + \frac{y}{16\lambda}+\frac{3}{64\lambda^2}-\frac{3}{16\lambda}+\frac{1}{16}\right]e^{-\left(\frac{y}{4\sqrt{\lambda}}+\sqrt{\lambda}\right)^2}\right]d\lambda, \tag{3.7}$$

where $F(t-\lambda)$ is bounded and integrable over a finite range and $\int_0^t \lambda^{-3/2}F(t-\lambda)e^{-(\lambda+y^2/16)}d\lambda$ is absolutely convergent for time t.

4. Special cases of $F(t)$

Now we consider some special cases of the function $F(t)$, characterizing the motion of the porous plate,

(i) The impulsive velocity field corresponds to

$$F(t) = KH(t), \tag{4.1}$$

where K is constant. From Eqs. (3.7) and (4.1) one obtains

$$\begin{aligned} \beta_1 &= \frac{1}{2}KH(t)\left[e^{-y}\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-\sqrt{t}\right)+\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+\sqrt{t}\right)\right], \\ \beta_2 &= KH(t)\left[\frac{1}{2}ye^{-y}\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-\sqrt{t}\right)+\frac{y}{\sqrt{(\pi t)}}\left\{\frac{y^2}{256t^2}+\frac{y}{16t}-\frac{1}{32t}\right.\right. \\ &\qquad \left.\left.+\frac{7}{16}\right\}e^{-\left(\frac{y}{4\sqrt{t}}+\sqrt{t}\right)^2}\right]. \end{aligned} \tag{4.2}$$

For very large values of t, $\beta_1 = Ke^{-y}$ and $\beta_2 = Kye^{-y}$ so that $u_f = (1+ky)Ke^{-y}$. For this type of motion, Eq. (3.6) gives

$$\begin{aligned} \alpha_1 &= \frac{1}{2}KH(t)\left[1+\operatorname{erf}\sqrt{t}+\frac{e^{-t}}{\sqrt{\pi t}}\right], \\ \alpha_2 &= \frac{KH(t)}{\sqrt{(\pi t)}}\left[\frac{1}{32t}-\frac{1}{16}\right]e^{-t}. \end{aligned} \tag{4.3}$$

The results given by Eq. (4.2) and (4.3) are valid for $t \neq 0$. As $t \to \infty$, $\alpha_1 \to KH(t)$ and $\alpha_2 \to 0$.

(ii) Single acceleration is defined by a function

$$F(t) = KtH(t), \tag{4.4}$$

which is valid for all finite values of t. From Eq. (3.7) and (4.4) one gets

$$\begin{aligned} \beta_1 &= \frac{1}{2}KH(t)\left[\left(t-\frac{y}{4}\right)e^{-y}\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-\sqrt{t}\right)+\left(t+\frac{y}{4}\right)\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+\sqrt{t}\right)\right], \\ \beta_2 &= KH(t)\left[\frac{y(1+8t)}{16\sqrt{(\pi t)}}\exp\left\{-\left(\frac{y}{4\sqrt{t}}+\sqrt{t}\right)^2\right\}+\frac{y}{4}\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+\sqrt{t}\right)\right. \\ &\qquad \left.+y\left(\frac{t}{2}-\frac{1}{4}-\frac{y}{8}\right)\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-\sqrt{t}\right)\right], \end{aligned} \tag{4.5}$$

and from Eq. (3.6) and (4.5) one gets

$$\begin{aligned} \alpha_1 &= \frac{1}{2}KH(t)\left[t(1+\operatorname{erf}\sqrt{t})+\frac{1}{2}\operatorname{erf}\sqrt{t}+e^{-t}\left(\frac{t}{\pi}\right)^{\frac{1}{2}}\right], \\ \alpha_2 &= -\frac{KH(t)}{8}\left[1+\operatorname{erf}\sqrt{t}+\frac{1}{2\sqrt{(\pi t)}}e^{-t}\right]. \end{aligned} \tag{4.6}$$

For large values of t, $\beta_1 \to KH(t)\left(t - \frac{y}{4}\right)e^{-y}$ and $\beta_2 \to KH(t)y\left(t - \frac{1}{2} - \frac{y}{4}\right)e^{-y}$, and $\alpha_1 \to K\left(t + \frac{1}{4}\right)$ and $\alpha_2 \to \left(-\frac{K}{4}\right)$. Thus the skin friction due to $F(t)$ for large t is $\left[-1 + K\left\{t + \frac{1}{4}(1-k)\right\}\right]$. This implies that skin friction reduces on account of the visco-elastic property of the fluid.

(iii) Multiple acceleration is defined by the function

$$F(t) = KtH(t) - K(t-t_0)H(t-t_0), \tag{4.7}$$

which is valid for $t \neq 0$. Let $\beta_1'(y, t)$ and $\beta_2'(y, t)$ denote β_1 and β_2 for the case when $F(t) = KtH(t)$; then, for the present case, one has

$$\begin{aligned} \beta_1 &= \beta_1'(y, t) - \beta_1'(y, t-t_0), \\ \beta_2 &= \beta_2'(y, t) - \beta_2'(y, t-t_0). \end{aligned} \tag{4.8}$$

The values of β_1 and β_2 and α_1 and α_2 can be obtained from Eqs. (4.5), (4.6) and (4.8). For large values of t, $\alpha_1 \to t_0 KH(t)$ and $\alpha_2 \to 0$. It is seen that these values are t_0 times the corresponding values of α_1 and α_2 given by Eq. (4.3). This is because the final increase in fluid velocity for the present case is t_0 times the corresponding increase in fluid velocity when $F(t) = KH(t)$.

(iv) Periodic velocity field is defined by the function

$$F(t) = e^{i\omega t}H(t), \tag{4.9}$$

where ω is the non-dimensional frequency of oscillations given by $\omega = \omega' 4\nu / W^2$. The constant K has been dropped in the further analysis for the sake of covenience. Expressions for β_1 and β_2 are obtained as

$$\begin{aligned} \beta_1 = \frac{1}{2} e^{i\omega t} H(t) \Bigg[& e^{-\frac{1}{2}y(1+\sqrt{(1+i\omega)})} \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} - \sqrt{(1+i\omega)t}\right) \\ & + e^{-\frac{1}{2}y(1-\sqrt{(1+i\omega)})} \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} + \sqrt{(1+i\omega)t}\right)\Bigg], \end{aligned} \tag{4.10}$$

$$\begin{aligned} \beta_2 = e^{i\omega t} y H(t) \Bigg[& \frac{1}{4}\left(\frac{8(1+i\omega)-\omega^2}{8\sqrt{(1+i\omega)}} + \frac{i\omega}{2} + 1\right) e^{-\frac{1}{2}y(1+\sqrt{1+i\omega})} \\ & \times \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} - \sqrt{(1+i\omega)t}\right) + \frac{1}{4}\left(1 + \frac{i\omega}{2} - \frac{8(1+i\omega)-\omega^2}{\sqrt{(1+i\omega)}}\right) \\ & \times e^{-\frac{1}{2}y(1-\sqrt{(1+i\omega)})} \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} + \sqrt{(1+i\omega)t}\right) \\ & + H(t)\frac{y}{\sqrt{(\pi t)}}\left(\frac{y^2}{256t^2} + \frac{y}{16t} - \frac{1}{32t} + \frac{(7+i\omega)}{16}\right) \exp\left[-\left(\frac{y}{4\sqrt{t}} + \sqrt{t}\right)^2\right]. \end{aligned}$$

Similarly, one obtains from Eq. (3.6) and (4.10) the expressions for α_1 and α_2 as

$$(4.11)\quad \begin{aligned} \alpha_1 &= \frac{1}{2}H(t)\left[e^{i\omega t}+\sqrt{(1+i\omega)}\,e^{i\omega t}\operatorname{erf}\sqrt{(1+i\omega)t}+\frac{1}{\sqrt{(\pi t)}}e^{-t}\right],\\ \alpha_2 &= H(t)\left[-\frac{1}{8}i\omega e^{i\omega t}-\frac{(1+i\omega)}{16\sqrt{(\pi t)}}e^{-t}+\frac{e^{-t}}{32t\sqrt{(\pi t)}}\right.\\ &\quad\left.+\left(\frac{\omega^2}{16}-\frac{i\omega}{8}\right)(1+i\omega)^{-\frac{1}{2}}e^{i\omega t}\operatorname{erf}\sqrt{(1+i\omega)t}\right]. \end{aligned}$$

(v) The decaying oscillatory velocity field is represented by the function

$$(4.12)\quad F(t) = H(t)e^{-(\lambda^2-i\omega)t}.$$

For this case the expressions for β_1 and β_2 from Eq. (3.7) and (4.12) are obtained as

$$(4.13)\quad \begin{aligned} \beta_1 &= \frac{1}{2}H(t)e^{-\gamma t}\left[e^{-\frac{1}{2}y(1+\sqrt{(1-\gamma)})}\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-\sqrt{(1-\gamma)t}\right)\right.\\ &\quad\left.+e^{-\frac{1}{2}y(1-\sqrt{(1-\gamma)})}\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+\sqrt{(1-\gamma)t}\right)\right],\\ \beta_2 &= H(t)ye^{-\gamma t}\left[\frac{1}{4}\left(\frac{8(1-\gamma)+\gamma^2}{8\sqrt{(1-\gamma)}}-\frac{\gamma}{2}+1\right)e^{-\frac{1}{2}y(1+\sqrt{(1-\gamma)})}\right.\\ &\quad\times\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-\sqrt{(1-\gamma)t}\right)+\frac{1}{4}\left(1-\frac{\gamma}{2}-\frac{8(1-\gamma)+\gamma^2}{\sqrt{(1-\gamma)}}\right)\\ &\quad\left.\times e^{-\frac{1}{2}y(1-\sqrt{(1-\gamma)})}\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+\sqrt{(1-\gamma)t}\right)\right]\\ &\quad+H(t)\frac{y}{\sqrt{(\pi t)}}\left(\frac{y^2}{256t^2}+\frac{y}{16t}-\frac{1}{32t}+\frac{(7-\gamma)}{16}\right)\exp\left[-\left(\frac{y}{4\sqrt{t}}+\sqrt{t}\right)^2\right], \end{aligned}$$

where $\gamma = \lambda^2-i\omega$.

Similarly, the expressions for α_1 and α_2 are obtained from Eq. (3.6) and (4.13) as

$$(4.14)\quad \begin{aligned} \alpha_1 &= \frac{1}{2}H(t)\left[e^{-\gamma t}+\sqrt{(1-\gamma)}\,e^{-\gamma t}\operatorname{erf}\sqrt{(1-\gamma)t}+\frac{1}{\sqrt{(\pi t)}}e^{-t}\right],\\ \alpha_2 &= H(t)\left[\frac{\gamma}{8}e^{-\gamma t}-\frac{\sqrt{(1-\gamma)}}{16\sqrt{(\pi t)}}e^{-t}+\frac{1}{32t\sqrt{(\pi t)}}e^{-t}+\left(\frac{\gamma}{8}-\frac{\gamma^2}{16}\right)(1-\gamma)^{-\frac{1}{2}}\right.\\ &\quad\left.\times\operatorname{erf}\sqrt{(1-\gamma)t}\right]. \end{aligned}$$

Consider a particular case when $\lambda^2 = 1$. For this case α_1 and α_2 simplify to

$$\alpha_1 = \frac{1}{2} H(t)e^{-t}\left[\sin\omega t + \sqrt{(2\omega)}\left(\sin\omega t S\sqrt{\omega t} + \cos\omega t C\sqrt{\omega t}\right)\right],$$

(4.15)

$$\alpha_2 = H(t)e^{-t}\left[\frac{1}{8}\sin\omega t - \frac{\omega}{8}\cos\omega t - \frac{\omega}{16\sqrt{(\pi t)}} - \frac{(1+\omega^2)}{8\sqrt{(2\omega)}}\left(\cos\omega t S\sqrt{\omega t} - \sin\omega t C\sqrt{\omega t}\right)\right],$$

where $C(\sqrt{\omega t})$ and $S(\sqrt{\omega t})$ are Fresnel integrals.

For the case when $\lambda^2 \neq 1$ and $\omega \gg |1-\lambda^2|$, the expressions for α_1 and α_2 become

$$\alpha_1 = \frac{1}{2} H(t)e^{-\lambda^2 t}\left[\sin\omega t + \sqrt{(2\omega)}\left\{\sin\omega t S\left(\sqrt{\omega t}\right) + \cos\omega t C\left(\sqrt{\omega t}\right)\right\}\right],$$

(4.16)

$$\alpha_2 = H(t)\left[-\frac{\omega}{16\sqrt{(\pi t)}}e^{-t} + e^{-\lambda^2 t}\left\{\frac{1}{8}\sin\omega t - \frac{\omega}{8}\cos\omega t - \frac{(1+\omega^2)}{8\sqrt{(2\omega)}}\left(\cos\omega t S\sqrt{\omega t} - \sin\omega t C\sqrt{\omega t}\right)\right\}\right].$$

5. Conclusions

It may be seen by examining Eqs. (3.1) and (3.2) that β_1 and β_2 depend on the parameter $y/(4\sqrt{t})$. Also, following WATSON [9], it may be noted that a secondary boundary layer is created when the plate velocity is subjected to an instantaneous impulse.

Since the fluid elasticity reduces the skin friction, the power input to fluid at the plate per cycle, given by the expressions

$$P = -\int_0^T p_{xy}\Big]_{y=0} u(0,t)\,dt,$$

where $T = 2\pi/\omega$, is also reduced. Thus the power required to vibrate the shaking-table of the Fourdrinier paper-making machine is less than that estimated by treating the water-wood pulp mixture as a purely viscous liquid.

For the case wherein a damped oscillatory plate velocity occurs about a constant mean, the dominant skin friction term, at large values of oscillation frequency, is α_2. This term fluctuates with a phase lag of $\tan^{-1}\left(S\sqrt{\omega t}/C\sqrt{\omega t}\right)$ with respect to the fluid velocity; and for very large values of ωt, it becomes $\pi/4$. This result is in agreement with the work of STUART [10].

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