Archives of Mechanics • Archiwum Mechaniki Stosowanej • 31, 2, pp. 177-184, Warszawa 1979

Interactions of solitary waves in shallow water theory(*)

A. TEMPERVILLE (GRENOBLE)

We consider wave motions in an incompressible inviscid fluid under the action of gravity and having a free surface. The equations are in Lagrangian form. Shallow water theory is introduced by a distortion of variables with a small parameter. So, it is easy to obtain progressive waves. With the method of strained coordinates, we can obtain standing waves and the reflexion of a solitary wave on a rigid wall. In this last case, we have studied: a) motion of point of maximum amplitude, b) magnitude of phase shift between incident wave and reflected wave, c) maximum amplitude attained by wave during interaction. We compare results with Maxworthy's experiments.

Rozważamy ruch falowy w cieczy nieściśliwej pod działaniem pola sił ciężkości o powierzchni swobodnej. Równania mają postać Lagrange'a. Teorię wody płytkiej wprowadza się przez zakłócenie zmiennych za pomocą małego parametru. Można w ten sposób otrzymać fale postępujące. Stosując metodę odkształcalnych współrzędnych uzyskuje się fale stojące oraz odbicie fali solitonowej od sztywnej ścianki. W tym przypadku przeanalizowano a) ruch punktu maksymalnej amplitudy, b) wielkość przesunięcia fazowego między falą padającą a odbitą, c) wielkość maksymalnej amplitudy uzyskanej przez falę podczas oddziaływania. Wyniki porównano z rezultatami doświadczalnymi Maxworthy'ego.

Рассматриваем волновое движение в несжимаемой жидкости со свободной поверхностью под действием поля сил тяжести. Уравнения имеют вид Лагранжа. Теорию мелкой воды вводится путем возмущения переменных при помощи малого параметра. Таким образом можно получить бегущие волны. Применяя метод деформируемых координат получаются стоячие волны и отражение солитоновой волны от жесткой стенки. В этом случае проанализированы: а) движение точки максимальной амплитуды, б) величина фазового сдвига между падающей и отраженной волнами, в) величина максимальной амплитуды, полученной волной во время взаимодействия. Результаты сравнены с экспериментальными результатами Максворти.

1. Introduction

T. Maxworthy [1] has written a paper where he gives the results of experiments on a collision between solitary waves. In this paper I present the theoretical results and compare them with Maxworthy's results.

2. General equations

The two-dimensional motion of an incompressible, inviscid, irrotational fluid with a free surface will be considered.

The relevant Lagrangian equations are expressed in terms of a fixed coordinate system, such that the axis in the x-direction coincides with the still water surface and the y-axis is vertically downwards (Fig. 1).

(*) Paper presented at the XIII Biennial Fluid Dynamics Symposium, Poland, September 5-10, 1977.

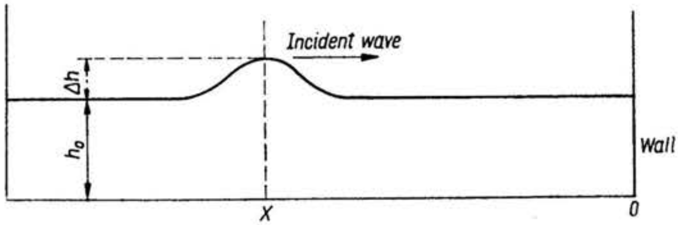

FIG. 1. Wave tank used for experiments.

It will be noted that: h—the bottom ordinate may be presumed to be horizontal, (t) is the time at any instant, and $(\bar{a}, \bar{b})$ are the coordinates of a fluid particle at an initial instant (t_0).

Let the unknown coordinates be $x(\bar{a}, \bar{b}, t)$ and $y(\bar{a}, \bar{b}, t)$, whilst the pressure acting on the particle be $p(\bar{a}, \bar{b}, t)$. The value of (y) for the free surface particles shall be given by $\eta(\bar{a}, t)$.

The equations of motion may thus be expressed as follows:

1) continuity equation

$$\frac{D(x, y)}{D(\bar{a}, \bar{b})} = 1, \tag{2.1}$$

2) momentum equation

$$\frac{D(x, x')}{D(\bar{a}, \bar{b})} + \frac{D(y, y')}{D(\bar{a}, \bar{b})} = F(\bar{a}, \bar{b}), \tag{2.2}$$

where (x') and (y') are the derivatives of (x) and (y), with respect to (t).

$F(\bar{a}\bar{b})$ will have a value equal to twice as much as the vorticity associated with the particle under consideration and it shall be considered equal to zero.

3) Boundary conditions:

at the bottom

$$y(\bar{a}, h, t) = h, \tag{2.3}$$

at the free surface

$$g\frac{\partial y}{\partial \bar{a}} = \frac{\partial y}{\partial \bar{a}}\frac{\partial^2 y}{\partial t^2} + \frac{\partial x}{\partial \bar{a}}\frac{\partial^2 x}{\partial t^2} \quad \text{if} \quad \bar{b} = \eta(\bar{a}, t_0). \tag{2.4}$$

A priori, the condition at the free surface is difficult to express and its equation is unknown. However, it is known that the Lagrangian variables may be replaced by other variables without altering the equations, provided that the elementary volume is conserved and kept unchanged during this transformation.

By adopting the MICHE coordinates [2], denoted now (a, b), we may verify that this property is maintained, and the bottom surface conditions may be expressed as

$$b = h \quad \text{and} \quad b = 0.$$

By putting $X = x-a$ and $Y = y-b$, the equation above may now be re-written as follows:

$$\frac{\partial X}{\partial a} + \frac{\partial Y}{\partial b} + \frac{\partial X}{\partial a}\frac{\partial Y}{\partial b} - \frac{\partial X}{\partial b}\frac{\partial Y}{\partial a} = 0, \tag{2.5}$$

(2.6) $$\frac{\partial^2 X}{\partial b\,\partial t}\left[1+\frac{\partial X}{\partial a}\right]-\frac{\partial X}{\partial b}\frac{\partial^2 X}{\partial a\,\partial t}+\frac{\partial Y}{\partial a}\frac{\partial^2 Y}{\partial b\partial t}-\frac{\partial^2 Y}{\partial a\partial t}\left[1+\frac{\partial Y}{\partial b}\right]=0,$$

(2.7) $$Y(a,h,t)=0 \quad \text{for} \quad b=h,$$

(2.8) $$g\frac{\partial Y}{\partial a}=\frac{\partial^2 X}{\partial t^2}+\frac{\partial X}{\partial a}\frac{\partial^2 X}{\partial t^2}+\frac{\partial Y}{\partial a}\frac{\partial^2 Y}{\partial t^2} \quad \text{for} \quad b=0.$$

3. Shallow water theory

The variables shall now be re-defined as follows:

$$\tau=\varepsilon\sqrt{gh}\,t, \quad \alpha=\varepsilon a\left[1+\sum_{p=1}^{\infty}a_{2p}\varepsilon^{2p}\right]=\varepsilon af(\varepsilon), \quad \beta=b,$$

where (ε) is a small parameter whose significance will become apparent later on, and a_{2p} a coefficient (to be defined by the classical method of multiscales). In the shallow water case the original equations of motion may now be written as

(3.1) $$\frac{\partial Y}{\partial\beta}+\varepsilon f(\varepsilon)\left[\frac{\partial X}{\partial\alpha}+\frac{\partial X}{\partial\alpha}\frac{\partial Y}{\partial\beta}-\frac{\partial X}{\partial\beta}\frac{\partial Y}{\partial\alpha}\right]=0,$$

(3.2) $$\frac{\partial^2 X}{\partial\beta\partial\tau}+\varepsilon f(\varepsilon)\left[\frac{\partial X}{\partial\alpha}\cdot\frac{\partial^2 X}{\partial\beta\partial\tau}-\frac{\partial X}{\partial\beta}\frac{\partial^2 X}{\partial\alpha\partial\tau}+\frac{\partial Y}{\partial\alpha}\frac{\partial^2 Y}{\partial\beta\partial\tau}-\frac{\partial Y}{\partial\beta}\frac{\partial^2 Y}{\partial\alpha\partial\tau}-\frac{\partial^2 Y}{\partial\alpha\partial\tau}\right]=0,$$

(3.3) $$Y(\alpha,h,\tau)=0 \quad \text{for} \quad \beta=h,$$

(3.4) $$\frac{f(\varepsilon)}{h}\frac{\partial Y}{\partial\alpha}=\varepsilon\frac{\partial^2 X}{\partial\tau^2}+\varepsilon^2 f(\varepsilon)\left[\frac{\partial X}{\partial\alpha}\frac{\partial^2 X}{\partial\tau^2}+\frac{\partial Y}{\partial\alpha}\frac{\partial^2 Y}{\partial\tau^2}\right] \quad \text{for} \quad \beta=0.$$

Suppose that X and Y may be expanded in a series in (ε).

(3.5) $$X=\sum_{n=0}^{\infty}\varepsilon^{2n+1}X_{2n+1}, \quad Y=\sum_{p=1}^{\infty}\varepsilon^{2p}Y_{2p}.$$

It may be easily seen that the relations developed above are in agreement with the general validity of the problem.

By re-arranging the terms we obtain

(3.6) $$\frac{\partial X_1}{\partial\beta}=0 \quad \text{where} \quad X_1=X_1(\alpha,\tau),$$

(3.7) $$Y_2=-(\beta-h)\frac{\partial X_1}{\partial\alpha}.$$

The condition at the free surface now becomes

(3.8) $$\frac{\partial^2 X_1}{\partial\alpha^2}-\frac{\partial^2 X_1}{\partial\tau^2}=0, \quad X_1=f_1(\alpha-\tau)+g_1(\alpha+\tau).$$

Thus we have

(3.9) $$X_3 = -\frac{(\beta-h)^2}{2}\frac{\partial^2 X_1}{\partial\alpha^2} + X_3^*(\alpha, \tau),$$

where X_3^* is the value of X_3 at the bottom

(3.10) $$Y_4 = \frac{(\beta-h)^3}{6}\frac{\partial^3 X_1}{\partial\alpha^3} + (\beta-h)\left[\left(\frac{\partial X_1}{\partial\alpha}\right)^2 - \frac{\partial X_3^*}{\partial\alpha} - a_2\frac{\partial X_1}{\partial\alpha}\right].$$

Likewise, the free surface condition to the fourth order is given by

(3.11) $$\frac{\partial^2 X_3^*}{\partial\alpha^2} - \frac{\partial^2 X_3^*}{\partial\tau^2} = -\frac{h^2}{3}\frac{\partial^4 X_1}{\partial\alpha^4} + 3\frac{\partial X_1}{\partial\alpha}\cdot\frac{\partial^2 X_1}{\partial\alpha^2} - 2a_2\frac{\partial^2 X_1}{\partial\alpha^2}.$$

Before developing these equations any further, the solutions for certain types of movements shall be examined.

4. Progressive waves

We shall now seek solutions in the form $X(\alpha-\tau, \beta)$, such that

(4.1) $$X_n = X_n(\alpha-\tau, \beta) \quad \forall n.$$

Thus we have

(4.2) $$X_1 = f_1(\alpha-\tau),$$

(4.3) $$Y_2 = -(\beta-h)f_1'(\alpha-\tau),$$

(4.4) $$\eta_2 = hf_1'(\alpha-\tau),$$

(4.5) $$X_3 = -\frac{(\beta-h)^2}{2}f_1'' + f_3(\alpha-\tau),$$

(4.6) $$Y_4 = \frac{(\beta-h)^3}{6}f_1''' + (\beta-h)[f_1'^2 - f_3' - a_2 f_1'],$$

(4.7) $$\eta_4 = -\frac{h^3}{6}f_1''' + h[f_3' + a_2 f_1' - f_1'^2].$$

The condition at the water surface, when considering terms up to and including the 4th order only, gives a differential equation enabling f_1 to be calculated:

(4.8) $$L_1(f_1) = -\frac{h^2}{3}f_1^{(4)} + 3f_1'f_1'' - 2a_2 f_1'' = 0.$$

After integration, taking into account that there has been no transport of mass

(4.9) $$f_1(u) = -hAZ\left[\frac{3A}{4h}u, k^2\right],$$

where (Z) is the Jacobi zeta function, K and E are the complete elliptic integrals of the first and second space, h, k^2 and A the three wave parameters.

The coefficient $\varepsilon^2 a_2$ may now be given by

$$\frac{8a_2}{3A^2} = k^2-2+\frac{3E}{K}. \tag{4.10}$$

Special case: when $k \to 1$, the function f_1 tends to the following limit:

$$f_1(u) = -hA\,\mathrm{th}\left[\frac{3A}{4h}u\right], \tag{4.11}$$

which is the equation for a solitary wave and in this case

$$a_2 = -\frac{3A^2}{8}. \tag{4.12}$$

If Eqs. (3.1) and (3.2) are developed further, taking the higher order terms into account the following is obtained:

$$X_5 = X_5^* + \frac{(\beta-h)^2}{2}\left[4f_1'f_1''-f_3''-2a_2f_1''+4\int f_1''^2(\alpha-\tau)\,d\tau\right]+\frac{(\beta-h)^4}{24}f_1^{(4)}, \tag{4.13}$$

$$Y_6 = \frac{(\beta-h)^5}{120}f_1^{(5)}-\frac{(\beta-h)^3}{6}\left[-2f_1''^2+6f_1'f_1''-f_1'''-3a_2f_1'''\right]+(\beta-h)\left[2f_1'f_3'-f_1'^3-\frac{\partial X_5^*}{\partial\alpha}+2a_2f_1'^2-a_2f_3'-a_4f_1'\right]. \tag{4.14}$$

The free surface condition, considering the 6th order terms and taking into account $X_5^* = f_5(\alpha-\tau)$, gives the following differential equation in terms of f_3:

$$\frac{h^2}{3}f_3^{(4)}+2a_2f_3''-3(f_1',f_3')' = \frac{h^2}{30}f_1^{(6)}+f_1''(a_2^2-2a_4)+4a_2f_1'f_1''+\frac{13}{6}h^2f_1''f_1'''+\frac{h^2}{2}f_1'f_1^{(4)}-3f_1''f_1'^2. \tag{4.15}$$

This equation gives a specific solution:

$$f_3' = \frac{2h^2}{3}f_1'''-a_2f_1'+C, \tag{4.16}$$

where C is a coefficient dependent on the parameters h, εA, k^2 as well as a_2 and a_4.

It may be shown, after J-P. GERMAIN [3], that the general solution derived from this equation without the second member may be incorporated in the term $f_1(\alpha-\tau)$ without having to modify the terms of higher order as a result of a small perturbation of one of the parameters.

In order to arrive at a solution in terms of f_3, C can be equated to zero, so that a_4 may be calculated in terms of the parameters h, εA, and k^2.

Thus

$$f_3 = \frac{2h^2}{3}f_1''-a_2f_1. \tag{4.17}$$

5. Standing waves

A solution, periodic in space, such that $X \equiv 0$ for $\alpha = 0$, $\forall \tau$ is examined.

It may be seen that

$$X_1 = f_1(\alpha-\tau)+g_1(\alpha+\tau) \quad \text{with} \quad g_1(u) = -f_1(u). \tag{5.1}$$

This corresponds to the superposition of two progressive waves propagated in the inverse direction

$$\eta_2 = h[f_1'+g_1'], \tag{5.2}$$

$$X_3 = -\frac{(\beta-h)^2}{2}[f_1''+g_1'']+X_3^*. \tag{5.3}$$

with

$$\frac{\partial^2 X_3^*}{\partial \alpha^2} - \frac{\partial X_3^*}{\partial \tau^2} = L_1(f_1)+L_1(g_1)+3(f_1'g_1''+g_1'f_1''), \tag{5.4}$$

where

$$X_3^* = f_3(\alpha-\tau)+g_3(\alpha+\tau)+\frac{3}{4}(f_1g_1'+f_1'g_1) \tag{5.5}$$

and

$$g_3(u) = -f_3(-u). \tag{5.6}$$

From 3rd order considerations, a term expressing the interaction may be seen to have the form: $\frac{3}{4}(f_1g_1'+f_1'g_1)$. If we take for f_1, f_3 and a_2 the expressions in Eqs. (4.9), (4.10) and (4.17), we obtain the solution for a shallow water standing wave [4]. In the case where $k \to 1$, the solution for the reflection of a solitary wave at a rigid wall is obtained. This particular case is treated below and compared to Maxworthy's experimental results.

6. Reflection of a solitary wave

It has been seen that the solution to this problem is given by the following:

$$f_1 = -hA\,\mathrm{th}\left[\frac{3A}{4h}(\alpha-\tau)\right],$$

$$X_1 = f_1(\alpha-\tau)+f_1(\alpha+\tau),$$

$$\eta = \varepsilon^2\eta_2 = \varepsilon^2 h\frac{\partial X_1}{\partial \alpha}.$$

Therefore, we have the equation of the free surface in a parametric form. To have the abscissa of a point of maximum amplitude, we must solve the equation:

$$\frac{\partial \eta}{\partial a} = 0.$$

The following results are thus obtained:

1) If

$$0 \leqslant \sqrt{\frac{g}{h}}\, t \leqslant \frac{2}{3\varepsilon A} \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) = T_0,$$

then $a_{max} = 0$.

2) If

$$\sqrt{\frac{g}{h}}\, t > T_0,$$

then

$$\frac{a_{max}}{h} = \frac{2}{3\varepsilon A} \ln\left(\frac{1+E}{1-E}\right); \quad E = \sqrt{\frac{3m^2-1}{m^2(3-m^2)}}.$$

The abscissa of the maximum position is thus given by

$$\frac{x_{max}}{h} = \frac{a_{max}}{h} + \frac{X(a_{max})}{h}.$$

Figure 2 shows the curve of x_{max} as a function of t; crosses denote Maxworthy's experimental results, when $\Delta h/h = 0.31$, if Δh is the height of the incident wave. The agreement is nearly perfect.

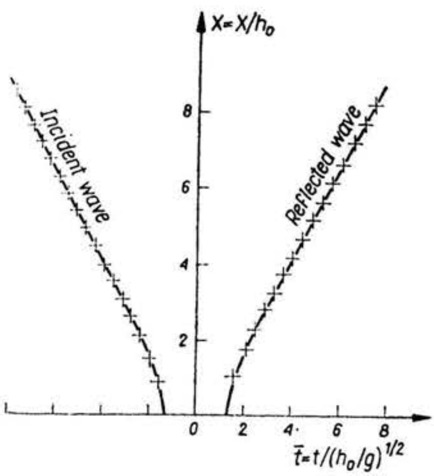

FIG. 2. Motion of point of maximum amplitude.

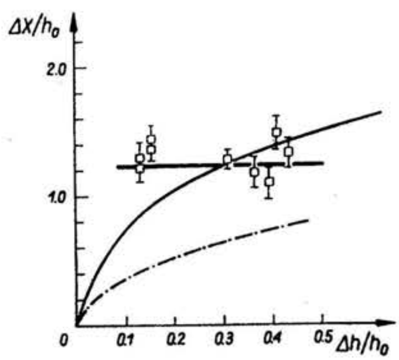

FIG. 3. Phase shift $\Delta X/h_0$ for several values of $\Delta h/h_0$.

□ squares: experimental results of MAXWORTHY,

——— TEMPERVILLE,

—.— OIKAWA and YAJIMA (1973), BYATT-SMITH (1971).

It is also easy to prove that the phase shift (ΔX) between the incident wave and the reflected wave is given by

$$\frac{\Delta X}{h} = \frac{4}{\sqrt{3}} \sqrt{\frac{\Delta h}{h}}.$$

Figure 3 shows the plot of the curve of the phase shift as a function of Δh, which again is in good agreement with Maxworthy's results. But it is difficult to conclude, as Maxworthy did, that there is a constant phase shift. It is my opinion that the BYATT-SMITH's result [5] is not exact.

References

1. T. MAXWORTHY, *Experiments on collisions between solitary waves*, J. Fluid Mech., 76, 1, 177-185, 1976.
2. A. MICHE, *Mouvements ondulatoires de la mer en profondeur constante ou decroissante*, Annales des Ponts et Chaussées, 1944.
3. J-P. GERMAIN, *Contribution à la thèorie des houles en eau peu profonde*, These, Grenoble 1967.
4. A. TEMPERVILLE, 2éme Symposium Franco-Russe d'Océanographie, Annales Hydrographiques, 746 5e série, 5, 1, 1977.
5. J. Q. B. BYATT-SMITH, *An integral equation for unsteady surface waves and a comment on the Boussinesq equation*, J. Fluid Mech., 49, 625-633, 1971.

INSTITUT DE MÉCANIQUE DE GRENOBLE, FRANCE.

Received October 15, 1977.