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Generalized Whittaker's equations for a nonholonomic system

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In this paper the energy integral is derived for a conservative nonholonomic system whose position is defined by group variables. This integral is then used for the reduction of the order of the system. Finally, equations of motion of the reduced system, the so-called Whittaker's equations, are applied to solve the problem of a rolling hoop.

W pracy wyprowadzona jest całka energetyczna dla konserwatywnego układu nieholonomicznego, którego położenie zdefiniowane jest za pomocą zmiennych grupowych. Całka ta jest następnie wykorzystana do obniżenia rzędu układu. Wreszcie, równania ruchu zredukowanego układu, tzw. równania Whittakera, zastosowane są do rozwiązania problemu toczącej się obręczy.

В работе выведен энергетический интеграл для консервативной неголономной системы, положение которой определяется при помощи групповых переменных. Этот интеграл затем используется для снижения порядка системы. Наконец, уравнения движения приведенной системы, т. наз. уравнения Уиттакера, применяются для решения задачи катающегося обруча.

1. Introduction

In [4] Whittaker has shown that the energy integral can be used for the reduction of a conservative holonomic dynamical system with n degrees of freedom to another system with only $n-1$ degrees of freedom. Equations of motion of the reduced system are the so-called Whittaker's equations. In the derivation of these equations, a generalised coordinate plays the role of time as an independent variable.

In [1] Whittaker's equations have been generalised to the case of a linear nonholonomic conservative system whose governing equations of motion are the so-called Hamel-Boltzmann equations. In this paper the author presents the equations of motion and integral of energy after taking constraints into account. However, he has subsequently ignored constraints altogether except at the end of the theoretical part of the paper. This creates an anomalous situation and leads to the pertinent question whether the constraints should be taken into account from the very beginning or not?

The aim of the present paper is to extend the above mentioned ideas to a nonholonomic system whose position is defined by group variables and which possesses an energy integral. The said anomaly has been dismissed by considering the constraints from the very start.

Let us consider a linear nonholonomic dynamical system whose position is defined by n parameters $x_1, x_2, \dots, x_n$, the so-called group variables and which has l degrees of freedom. It is subject to $n-1$ constraints of the form

$$\eta_\alpha = c_{\alpha i}\eta_i \quad (i = 1, 2, \dots, l,\ \alpha = l+1, \dots, n), \tag{1.1}$$

where $c_{\alpha i}$ are functions of x's only, $\eta_1, \eta_2, \ldots, \eta_n$ are the parameters of real displacement of the system without constraints and summation over a repeated suffix is understood. If $X_1, X_2, \ldots, X_n$ are the displacement operators corresponding to these parameters, then the following relations are satisfied:

$$X_e = \xi_e^h \frac{\partial}{\partial x_h} \quad (e, h = 1, 2, \ldots, n),$$

$$(X_e, X_h) = X_e X_h - X_h X_e = C_{ehf} X_f \quad (f = 1, 2, \ldots, n),$$

where ξ_e^h are functions of $x_1, x_2, \ldots, x_n$ and c'_s are constants of structure of the group of displacement operators. In view of Eq. (1.1) the displacement operators for the nonholonomic system are expressed by

$$Y_i = X_i + c_{\alpha i} X_\alpha \quad (i = 1, 2, \ldots, l, \ \alpha = l+1, \ldots, n)$$

and satisfy the commutation relations

$$(Y_i, Y_j) = K_{ijk} Y_k + K^*_{ij\beta} X_\beta \quad (i, j, k = 1, 2, \ldots, l, \ \beta = l+1, \ldots, n), \tag{1.2}$$

where

$$K_{ijk} = C_{ijk} + c_{\beta j} C_{i\beta k} + c_{\alpha i}(C_{\alpha jk} + c_{\beta j} C_{\alpha\beta k}), \tag{1.3}$$

$$K^*_{ij\beta} = K_{ij\beta} + Y_i(c_{\beta j}) - Y_j(c_{\beta i}) - c_{\beta k} K_{ijk}. \tag{1.4}$$

For any function $f(x_1, x_2, \ldots, x_n)$, the derivative df/dt is given by

$$\frac{df}{dt} = \eta_i Y_i(f) \quad (i = 1, 2, \ldots, l). \tag{1.5}$$

Let $\overline{T}$ denote the function obtained from the kinetic energy T after eliminating the dependent $\eta_{,s}$ by means of Eq. (1.1), then

$$T(x_1, x_2, \ldots, x_n; \ \eta_{,s} \ldots, \eta_n) \equiv \overline{T}(x_1, x_2, \ldots, x_n; \ \eta_1, \ldots, \eta_l),$$

where we have assumed that T is independent of time t. Equations of motion of the nonholonomic system as obtained in [2] are

$$\frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial \eta_i}\right) - \eta_j K_{jik} \frac{\partial \overline{T}}{\partial \eta_k} - \eta_j K^*_{ji\beta} \frac{\partial \overline{T}}{\partial \eta_\beta} - Y_i(\overline{T}) = Y_i(U) \tag{1.6}$$

$$(i, j, k = 1, 2, \ldots, l, \ \beta = l+1, \ldots, n),$$

where $U(x_1, x_2, \ldots, x_n)$ is the force function of the system. Equations (1.6) together with

$$\frac{dx_e}{dt} = \eta_i(\xi_i^e + c_{\alpha i} \xi_\alpha^e), \tag{1.7}$$

obtained from Eq. (1.5), form a system of $n+l$ equations to determine $x_1, x_2, \ldots, x_n$, $\eta_1, \ldots, \eta_l$ as functions of time t.

2. Existence of energy integral

In this section it will be proved that the nonholonomic system under consideration admits an energy integral of the form

$$\overline{T} - U = \text{const}. \tag{2.1}$$

Multiplying each of the equations (1.6) by η_i and summing the l expressions so obtained, we get

$$\eta_i \frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial \eta_i}\right) - \eta_i \eta_j K_{jik} \frac{\partial \overline{T}}{\partial \eta_k} - \eta_i \eta_j K^*_{ji\beta} \frac{\partial T}{\partial \eta_\beta} - \eta_i Y_i(\overline{T}) = \eta_i Y_i(U)$$

or

$$(2.1) \qquad \frac{d}{dt}\left(\eta_i \frac{\partial \overline{T}}{\partial \eta_i}\right) - \frac{d\eta_i}{dt} \frac{\partial \overline{T}}{\partial \eta_i} - \eta_i \eta_j K_{jik} \frac{\partial \overline{T}}{\partial \eta_k} - \eta_i \eta_j K^*_{ji\beta} \frac{\partial T}{\partial \eta^\beta} - \eta_i Y_i(\overline{T}) - \eta_i Y_i(U) = 0.$$

Now, using Eqs. (1.1), (1.3) and (1.4) we obtain

$$(2.2) \qquad \eta_i \eta_j K_{jik} = \eta_i \eta_j C_{jik} + \eta_j \eta_\beta C_{j\beta k} + \eta_i \eta_\alpha C_{\alpha ik} + \eta_\alpha \eta_\beta C_{\alpha\beta k}.$$

Since

$$C_{efh} = -C_{feh} \quad (e, f, h = 1, 2, \ldots, n),$$

it follows that the right hand side of Eq. (2.2) vanishes and, consequently,

$$\eta_i \eta_j K_{jik} = 0, \quad \eta_i \eta_j K^*_{ij\beta} = 0.$$

In view of these relations and Eq. (1.4), Eq. (2.1) yields

$$(2.3) \qquad \frac{d}{dt}\left(\eta_i \frac{\partial \overline{T}}{\partial \eta_i}\right) - \frac{d}{dt}(\overline{T} + U) = 0.$$

Since $\overline{T}$ is a homogeneous quadratic form in $\eta_{,s}$ it follows that

$$\eta_i \frac{\partial \overline{T}}{\partial \eta_i} = 2\overline{T},$$

and therefore Eq. (2.3) takes the form

$$\frac{d}{dt}(\overline{T} - U) = 0,$$

which on integration gives the required energy integral

$$(2.4) \qquad \overline{T} - U = \text{const}.$$

Putting $L = T + V$ and $\overline{L} = \overline{T} + U$ the equations of motion (1.6) and the relation (2.4) become respectively

$$(2.5) \qquad \frac{d}{dt}\left(\frac{\partial \overline{L}}{\partial \eta_i}\right) - \eta_j K_{jik} \frac{\partial \overline{L}}{\partial \eta_k} - \eta_j K^*_{ji\beta} \frac{\partial L}{\partial \eta_\beta} - Y_i(\overline{L}) = 0,$$

$$(2.6) \qquad \eta_i \frac{\partial \overline{L}}{\partial \eta_i} - \overline{L} = h,$$

where h is the energy constant.

3. Whittaker's equations

Putting $\eta, dt = d\tau$ and assuming that τ plays the role of time, we have

$$(3.1) \qquad \eta_p = \eta, \eta'_p \quad (p = 2, 3, \ldots, l),$$

where

$$\eta'_p = \eta_p \frac{dt}{d\tau} = \frac{\eta_p}{\eta_l}.$$

Let us assume that we get the function Ω after replacing η_p in $\overline{L}$ by η, η'_p, then we have the relation

$$\overline{L}(x_1, x_2, \dots, x_n; \eta_1, \dots, \eta_l) \equiv \Omega(x_1, x_2, \dots, x_n; \eta_1, \eta_2^1, \dots, \eta_l^1). \tag{3.2}$$

Now, Eq. (3.2) yields on differentiation

$$\frac{\partial \overline{L}}{\partial \eta_1} = \frac{\partial \Omega}{\partial \eta_1} - \frac{\partial \Omega}{\partial \eta_p^1} \frac{\eta_p}{\eta_1^2} \quad (p = 2, 3, \dots, l), \tag{3.3}$$

$$\frac{\partial \overline{L}}{\partial \eta_p} = \frac{\partial \Omega}{\partial \eta_p^1} \frac{1}{\eta}, \tag{3.4}$$

$$Y_i(\overline{L}) = Y_i(\Omega) \quad (i = 1, 2, \dots, l). \tag{3.5}$$

The relations (3.3) and (3.4) give

$$\frac{\partial \Omega}{\partial \eta_1} = \frac{\partial \overline{L}}{\partial \eta_1} + \frac{\partial \overline{L}}{\partial \eta_p} \frac{\eta_p}{\eta_1}. \tag{3.6}$$

Using Eqs. (3.2) and (3.6), the energy equation (2.6) assumes the form

$$\eta_1 \frac{\partial \Omega}{\partial \eta_1} - \Omega(x_1, x_2, \dots, x_n; \eta_1, \eta'_2, \dots, \eta'_l) = h. \tag{3.7}$$

This furnishes η_1 as a function of the variables $x_1, x_2, \dots, x_n; \eta'_2, \dots, \eta'_l$, so that

$$\eta_1 = \eta_1(x_1, x_2, \dots, x_n; \eta'_2, \dots, \eta'_l). \tag{3.8}$$

Substituting for η_1 from Eq. (3.8) in the function $\partial\Omega/\partial\eta_1$, we get the function $\overline{L}$ expressed by

$$\frac{\partial \Omega}{\partial \eta_1} \equiv \overline{L'}(x_1, x_2, \dots, x_n; \eta'_2, \dots, \eta'_l). \tag{3.9}$$

The function $\overline{L}'$ will be called Whittaker's function for the system. Equation (3.7) yields the following relations:

$$\frac{\partial \Omega}{\partial \eta_p^1} = \eta_1 \left(\frac{\partial^2 \Omega}{\partial \eta_1 \partial \eta_p^1} + \frac{\partial^2 \Omega}{\partial \eta_1^2} \frac{\partial \eta_1}{\partial \eta_p^1} \right), \tag{3.10}$$

$$Y_i(\Omega) = \eta_1 \left[Y_i \left(\frac{\partial \Omega}{\partial \eta_1} \right) + \frac{\partial_2 \Omega}{\partial \eta_1^2} Y_i(\eta_1) \right] \tag{3.11}$$

and, similarly, from Eq. (3.9) we get

$$\frac{\partial \overline{L'}}{\partial \eta_p^1} = \frac{\partial^2 \Omega}{\partial \eta_1 \partial \eta_p^1} + \frac{\partial^2 \Omega}{\partial \eta_1^2} \frac{\partial \eta_1}{\partial \eta_p^1}, \tag{3.12}$$

$$Y_i(\overline{L'}) = Y_i \left(\frac{\partial \Omega}{\partial \eta_1} \right) + \frac{\partial_2 \Omega}{\partial \eta_1^2} Y_i(\eta_1). \tag{3.13}$$

Comparison of Eqs. (3.10) and (3.12) gives

$$\frac{\partial \Omega}{\partial \eta_p^1} = \eta_1 \frac{\partial \overline{L'}}{\partial \eta_p^1} \quad (p = 2, 3, \ldots, l), \tag{3.14}$$

and, similarly, Eqs. (3.11) and (3.13) yield

$$Y_i(\Omega) = \eta_1 \overline{Y_i}(L'). \tag{3.15}$$

Combining Eqs. (3.14) and (3.15) with Eqs. (3.4) and (3.5), we obtain

$$\frac{\partial \overline{L}}{\partial \eta_p} = \frac{\partial \overline{L'}}{\partial \eta_p'}, \tag{3.16}$$

$$Y_i(\overline{L}) = \eta_1 Y_i(\overline{L'}) \quad (i = 1, 2, \ldots, l). \tag{3.17}$$

Proceeding in the above manner, it can also be proved that

$$\frac{\partial L}{\partial \eta_\beta} = \frac{\partial L'}{\partial \eta_\beta'} \quad (\beta = l+1, \ldots, n). \tag{3.18}$$

Using Eqs. (3.1), (3.16), (3.17) and (3.18), Eqs. (1.6) and (1.7) assume the form

$$\frac{d}{d\tau}\left(\frac{\partial \overline{L'}}{\partial \eta_p'}\right) - \eta_q' K_{qpr} \frac{\partial \overline{L'}}{\partial \eta_s'} - \frac{\partial L'}{\partial \eta_\beta'} [K^*_{1p\beta} + \eta_q' K^*_{qp\beta}] - Y_p(\overline{L'}) = Q_p, \tag{3.19}$$

$$(p, q, r = 2, 3, \ldots, l;\ \beta = l+1, \ldots, n),$$

$$\frac{dx_e}{d\tau} = \xi_1^e + c_\alpha, \xi_\alpha^e + \eta_p'(\xi_p^e + c_{\alpha p} \xi_\alpha^e), \tag{3.20}$$

where

$$Q_p = K_{,pe} \frac{\partial \overline{L'}}{\partial \eta_p'} + \frac{\partial \overline{L}}{\partial \eta_1} [K_{,p_1} + \eta_q' K_{q_1 p_1}]$$

and $\partial \overline{L}/\partial \eta_1$ denotes the function obtained after substituting for $\eta_1, \ldots, \eta_l$ from Eqs. (3.1) and (3.8).

Now Eqs. (3.19) may be regarded as the equations of motion of a new nonholonomic dynamical system whose Lagrangian is $\overline{L'}$ and for which τ plays the role of time and Q_p are non-conservative forces. Hence the energy integral enables us to reduce a given nonholonomic system with l degrees of freedom to another one with $l-1$ degrees of freedom.

To see how the solution of the system is completed we consider Eqs. (3.19) and (3.20). These form a system of $n+l-1$ equations of the first order in the variables $x_1, \ldots, x_n$, $\eta_2', \ldots, \eta_l'$ whose solution gives these $n+l-1$ quantities as functions of τ in the form

$$\eta_p' = \eta_p'(\tau, h, c_1, c_2, \ldots, c_{n+l-1}) \quad (p = 2, 3, \ldots, l),$$

$$x_e = x_e(\tau, h, c_1, c_2, \ldots, c_{n+l-1}) \quad (e = 1, 2, \ldots, n),$$

where $h, c_1, c_2, \ldots, c_{n+l-1}$ are constants of integration. When the above expressions for η_p' and x_e are substituted in Eq. (3.8), we obtain, after integration, the functional dependence between τ and time t in the form

$$t = \int \frac{d\tau}{\eta_1(\tau, h, c_1, \ldots, c_{n+l-1})} + c_{n+l},$$

where c_{n+l} is another constant of integration. Thus we obtain the complete solution of the equations of motion.

Special cases (i)

As a special case we consider a conservative holonomic system whose position is defined by n generalised coordinates $q_1, \ldots, q_n$. In this case we have the relations

$$x_1 = q_1, \ldots, \quad x_n = q_n, \quad \eta_1 = \dot{q}_1, \ldots, \eta_n = \dot{q}_n, \quad d\tau = dq_1, \quad Y_l = \frac{\partial}{\partial q_l},$$

$$\eta_2' = \frac{dq_l}{dq_1} = q_2', \ldots, \quad \eta_n' = \frac{dq_n}{dq_1} = q_n',$$

therefore all the $C_{,s}$ vanish and, consequently, the $K_{,s}$ and $K^*_{,s}$, also vanish. Hence Eqs. (3.19) give

$$\frac{d}{dq_1}\left(\frac{\partial L'}{\partial q_r'}\right) - \frac{\partial L'}{\partial q_r} = 0 \quad (r = 2, 3, \ldots, n), \tag{3.21}$$

where

$$\overline{L}' = L'(q, \ldots, q_n, q_2', \ldots, q_n', h).$$

Equations (3.21) are identical with those obtained in [5] with the help of the energy integral.

Special cases (ii)

Let us consider a linear nonholonomic conservative system whose position is specified by the generalised coordinates $q_1, \ldots, q_n$ and which is subject to constraints

$$a_{\alpha e}\dot{q}_e = 0 \quad (\alpha = l+1, \ldots, n; e = 1, 2, \ldots, n).$$

In this case we have

$$\begin{aligned} x_e &= q_e \quad (e = 1, 2, \ldots, n), \\ \eta_l &= \omega_l = \frac{d\pi_l}{dt} = a_{le}\dot{q}_e, \end{aligned} \tag{3.22}$$

$$\eta_\alpha = \omega_\alpha = \frac{d\pi_\alpha}{dt} = a_{\alpha e}\dot{q}_e = 0 \quad (\alpha = l+1, \ldots, n), \tag{3.23}$$

where a_{eh} are functions of $q_{,s}$ only. Solving Eqs. (3.22) and (3.23) for $\dot{q}_e$ in terms of $\eta_{,s}$, we get

$$\dot{q}_e = b_{el}\eta_l.$$

In view of this the displacement operators for the system are given by

$$Y_l = X_l = b_{el}\frac{\partial}{\partial q_e} = \frac{\partial}{\partial \pi_l} \quad (i = 1, 2, \ldots, l). \tag{3.24}$$

Using the relations (1.2) and (3.24), we have

$$K_{ijk} = b_{ei}b_{hj}\left(\frac{\partial a_{ke}}{\partial q_h} - \frac{\partial a_{kh}}{\partial q_e}\right) = \gamma^k_{ij},$$

$$K^*_{ij\beta} = b_{ei}b_{hj}\left(\frac{\partial a_{\beta e}}{\partial q_h} - \frac{\partial a_{\beta h}}{\partial q_e}\right) = \gamma^\beta_{j,i}.$$

The relation (3.1) gives

$$\eta_p = \eta_1 \eta'_p = \omega_1 \frac{d\pi_p}{d\pi_1} = \omega_1 \pi'_p.$$

In view of the above Eqs. (3.19) assume the form

$$\frac{d}{d\pi_1}\left(\frac{\partial L'}{\partial \pi'_p}\right) - \pi'_q \gamma^s_{pq} \frac{\partial L'}{\partial \pi'_s} - \frac{\partial L'}{\partial \pi_p} = \gamma^s_{p_1} \frac{\partial L'}{\partial \pi'_s} + \pi_1(\gamma'_{p_1} + \pi'_q \gamma'_{pq}),$$

$$(p, q = 2, 3, \dots, l;\ s = 2, 3, \dots, n),$$

where π_1 stands for $\partial L/\partial \pi_1$. These are the same equations as obtained in [1].

4. Example

Let us consider the motion of a heavy circular hoop, of unit mass and radius a, which rolls without sliding on a fixed horizontal plane Ox_1y_1. The centre of inertia G of the hoop is the centre of the figure and the central ellipsoid of inertia is a surface of revolu-

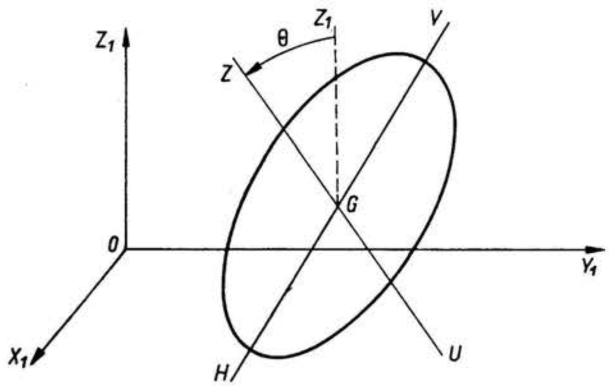

FIG. 1.

tion about the axis GZ of the hoop. As shown in Fig. 1, let H be the point of contact of the hoop with the fixed plane.

The parameters defining the position of the hoop are the coordinates x_1, y_1 of G relative to the space fixed system OX, Y, Z, and the Euler angles θ, ψ, ϕ. The coordinate z_1 of G is given by the relation

$$z_1 = a\sin\theta. \tag{4.1}$$

Let p, q, r be the components of instantaneous angular velocity of the hoop, referred to a semi-moving rectangular trihedral $GUVZ$ where the axis GU is perpendicular to the plane ZGZ_1, and the axis GV is directed upwards along the line of greatest slope of the plane of the hoop. Then we have

$$p = \dot{\theta}, \quad q = \dot{\psi}\sin\theta, \quad r = \dot{\psi}\cos\theta + \dot{\phi}.$$

The equations of constraint in terms of p, q, r are

$$\dot{x}_1 = ap\sin\psi\sin\theta - ar\cos\psi,$$
(4.2)
$$\dot{y}_1 = -ap\cos\psi\sin\theta - ar\sin\psi.$$

Let us choose $\theta, \psi, \phi, x_1, y_1, z_1$ as the group variables which specify the position of the system at time t. Due to the holonomic constraint (4.1), the system without constraints (4.2) has five degrees of freedom and therefore we take

$$\eta_1 = p = \dot{\theta}, \quad \eta_2 = q = \dot{\psi}\sin\theta, \quad \eta_3 = r = \dot{\psi}\cos\theta + \phi, \quad \eta_4 = \dot{x}_1, \quad \eta_5 = \dot{y}_1.$$

The corresponding displacement operators X'_s are given by

$$X_1 = \frac{\partial}{\partial\theta} + a\cos\theta\frac{\partial}{\partial z_1},$$

$$X_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} - \cot\theta\frac{\partial}{\partial\phi},$$

$$X_3 = \frac{\partial}{\partial\phi}, \quad X_4 = \frac{\partial}{\partial x_1}, \quad X_5 = \frac{\partial}{\partial y_1}.$$

Evidently, the commutators of all the operators, except (X_1, X_2), vanish and this commutator is expressed by

$$(X_1, X_2) = -\cot\theta X_2 + X_3, \tag{4.3}$$

therefore,

$$C_{122} = -C_{212} = -\cot\theta,$$
(4.4)
$$C_{123} = -C_{213} = 1.$$

The equations of constraint (4.2), when expressed in terms on $\eta_{,s}$, become

$$\eta_4 = a\eta_1\sin\psi\sin\theta - a\eta_3\cos\psi,$$
(4.5)
$$\eta_5 = -a\eta_1\cos\psi\sin\theta - a\eta_3\sin\psi.$$

By using these relations, the displacement operators for the nonholonomic system are given by

$$Y_1 = \frac{\partial}{\partial\theta} + a\sin\psi\sin\theta\frac{\partial}{\partial x_1} - a\cos\psi\sin\theta\frac{\partial}{\partial y_1} + a\cos\theta\frac{\partial}{\partial z_1},$$

(4.6)
$$Y_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} - \cot\theta\frac{\partial}{\partial\phi},$$

$$Y_3 = \frac{\partial}{\partial\phi} - a\cos\psi\frac{\partial}{\partial x_1} - a\sin\psi\frac{\partial}{\partial y_1}.$$

The kinetic and potential energies are expressed respectively by the relations

$$T = \frac{1}{2}\left[\left(\frac{1}{2} + \cos^2\theta\right)a^2\eta_1^2 + \frac{a^2\eta_2^2}{2} + a^2\eta_3^2 + \eta_4^2 + \eta_6^2,\right],$$

$$V = ga\sin\theta.$$

By taking constraint into account, we have

$$\overline{T} = \frac{a^2}{4}(3\eta_1^2+\eta_2^2+4\eta_3^2),$$

(4.7)
$$\overline{L} = \overline{T}-V = \frac{a^2}{4}(3\eta_1^2+\eta_2^2+4\eta_3^2)-ga\sin\theta,$$

$$\Omega = \frac{a^2\eta_1^2}{4}(3+\eta_2^{2\prime}+4\eta_3^{2\prime})-ga\sin\theta.$$

The Whittaker's function L' which is obtained without taking constraints into account is

(4.8)
$$L' = \sqrt{2(h-ga\sin\theta)}\sqrt{a^2\left(\frac{1}{2}+\cos^2\theta\right)+\frac{a^2\eta_2^{2\prime}}{2}+a^2\eta_3^{2\prime}+\eta_4^{2\prime}+\eta_5^{2\prime}}.$$

The equations of constraint, in terms of $\eta_{,s}$, assume the form

(4.9)
$$\eta_4' = a\sin\psi\sin\theta-a\eta_3'\cos\psi,$$
$$\eta_5' = -a\cos\psi\sin\theta-a\eta_3'\sin\psi.$$

Now, using Eq. (4.9) in Eq. (4.8), the Whittaker's function $\overline{L'}$ for the nonholonomic system is expressed by

(4.10)
$$\overline{L'} = a\sqrt{(h-ga\sin\theta)(3+\eta_2^{2\prime}+4\eta_3^{2\prime})}.$$

In view of Eqs. (4.6) and (4.10), we get

(4.11)
$$Y_2(\overline{L'}) = Y_3(\overline{L'}) = 0.$$

Differentiating Eq. (4.10) with respect to η_2' and η_3' we have

(4.12)
$$\frac{\partial\overline{L'}}{\partial\eta_2'} = a\sqrt{h-ga\sin\theta}\,\frac{\eta_2'}{\sqrt{3+\eta_2^{2\prime}+4\eta_3^{2\prime}}},$$
$$\frac{\partial\overline{L'}}{\partial\eta_3'} = a\sqrt{h-ga\sin\theta}\,\frac{4\eta_3'}{\sqrt{3+\eta_2^{2\prime}+4\eta_3^{2\prime}}}.$$

Differentiating Eq. (4.8) with respect to η_4' and η_5' and using Eq. (4.9) we obtain

(4.13)
$$\frac{\partial\overline{L'}}{\partial\eta_4'} = \frac{2\sqrt{h-ga\sin\theta}(\sin\psi\sin\theta-\eta_3'\cos\psi)}{\sqrt{3+\eta_2^{2\prime}+4\eta_3^{2\prime}}},$$
$$\frac{\partial\overline{L'}}{\partial\eta_5'} = -\frac{2\sqrt{h-ga\sin\theta}(\cos\psi\sin\theta+\eta_3'\sin\psi)}{\sqrt{3+\eta_2^{2\prime}+4\eta_3^{2\prime}}}.$$

Using Eqs. (1.3) (1.4) and (4.4), the non-vanishing $K_{,s}$, $K_{,s}^*$ are given by

(4.14)
$$K_{122} = -\cot\theta, \quad K_{123} = 1, \quad K_{324} = \frac{a\sin\psi}{\sin\theta},$$
$$K_{325} = \frac{a\cos\psi}{\sin\theta}, \quad K_{234}^* = \frac{a\sin\psi}{\sin\theta}, \quad K_{235}^* = -\frac{a\cos\psi}{\sin\theta}.$$

In view of the relations (4.11), (4.12), (4.13) and (4.14), the equations of motion (3.19) and (3.20) yield

$$\frac{d}{d\theta}\left(\eta_2'\sqrt{\frac{h-ga\sin\theta}{3+\eta_2^{2\prime}+4\eta_3^{2\prime}}}\right)=\sqrt{\frac{h-ga\sin\theta}{3+\eta_2^{2\prime}+4\eta_3^{2\prime}}}\,(2\eta_3'-\eta_2'\cot\theta),$$

(4.15)
$$\frac{d}{d\theta}\left(2\eta_3'\sqrt{\frac{h-ga\sin\theta}{3+\eta_2^{2\prime}+4\eta_3^{2\prime}}}\right)=\eta_2'\sqrt{\frac{h-ga\sin\theta}{3+\eta_2^{2\prime}+4\eta_3^{2\prime}}},$$

$$\frac{d\psi}{d\theta}=\eta_2'\operatorname{cosec}\theta,\qquad \frac{d\phi}{d\theta}=\eta_3'-\eta_2'\cot\theta.$$

These equations form a system of four equations for determining the variables ψ, ϕ, η_2', η_3' as functions od θ. After the solution of these equations has been obtained, we use the equation of energy

(4.16)
$$\eta_1=\frac{2}{a}\sqrt{\frac{h-ga\sin\theta}{3+\eta_2^{2\prime}+4\eta_3^{2\prime}}}$$

to determine θ as a function of time t. Thus the solution of the problem is completed.

If we use Eq. (4.16) directly in Eq. (4.15), we get the equations of motion in the form

$$\frac{d\eta_2}{d\theta}=2\eta_3-\eta_2\cot\theta,\qquad 2\frac{d\eta_3}{d\theta}=\eta_2.$$

These equations are similar to those obtained in [3] by the use of Lagrange equations.

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