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On the incremental collapse criterion accounting for temperature dependence of yield point stress

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THE PAPER presents a systematic derivation of the incremental collapse criterion in the case when the yield point stress is temperature-dependent. The criterion is illustrated by the incremental collapse analysis of a thick-walled tube subject to variations of internal pressure and of temperature field.

W pracy wyprowadzono kryterium zniszczenia przyrostowego dla konstrukcji sprężysto-plastycznych poddanych obciążeniom zmiennym i zmiennemu polu temperatury. Wynik zilustrowano na przykładzie rury grubościennej, poddanej zmiennym: ciśnieniu wewnętrznemu i quasi-stacjonarnemu polu temperatury.

В работе выводится признак прогрессирующего разрушения упруго-пластических конструкций при повторно-переменных нагружениях, учитывая зависимость предела текучести от температуры. Результат иллюстрован примером толстостенной трубы под действием переменного внутренного давления и переменного поля температуры.

1. Formulation of the problem

THE SHAKEDOWN theorems, [1, 2], have been derived initially accounting only for mechanical loads. Their extensions to thermal actions [3, 4, 5, 6, 7] took into consideration not only thermal stresses but also the fact that material constants such as yield point stress vary with temperature. In the case of a static approach [3, 5, 7] this effect as well as the effect of the temperature dependence of elastic moduli can be incorporated relatively easy.

However, more complicated boundary-value problems are to be solved by means of the kinematic approach, especially if incremental collapse is considered. The methods developed [8, 9, 10, 11] allow to find out the critical loads which may cause divergent increments of plastic deformations simply from the analysis of possible mechanisms of those increments, without tedious integration with respect to time as the original theorem required.

The aim of the present note is to clarify the use of kinematic approach in cases in which the temperature variations of yield stress cannot be neglected.

2. Basic relations

Let us adopt the following assumptions:

1. The total strain is the sum of elastic, thermal and plastic terms:

$$\varepsilon_{ij} = \varepsilon^E_{ij} + \varepsilon^T_{ij} + \varepsilon^P_{ij}, \tag{2.1}$$

$$\varepsilon^E_{ij} = E^{-1}_{ijkl}\sigma_{kl}, \quad \varepsilon^T_{ij} = M_{ij}T, \quad \dot{\varepsilon}^P_{ij} = \lambda\frac{\partial f}{\partial\sigma_{ij}}, \quad \lambda \geqslant 0. \tag{2.2}$$

Here ε_{ij}, σ_{ij} denote the strain and stress tensors respectively, T stands for temperature measured from the natural state, E_{ijkl} is the tensor of elastic moduli, M_{ij} — tensor of the thermal expansion coefficients.

2. The form of the yleld condition is as follows:

$$f(\sigma_{ij})-k(\mathbf{x}, T) \leqslant 0, \tag{2.3}$$

where $f(\sigma_{ij})$ is a homogeneous function of order one. The majority of the used yield conditions can be easily transformed into such a form. The domain (2.3) in the stress space is assumed convex whereas

$$k(\mathbf{x}, T) = k_0(\mathbf{x})g(\mathbf{x}, T), \quad g(0) = 1. \tag{2.4}$$

Thus the dissipation function depends not only on the instantaneous plastic strain rate but also on instantaneous temperature:

$$\sigma_{ij}\dot{\varepsilon}^P_{ij} = D(\mathbf{x}, \dot{\varepsilon}^P_{ij}, T) = D_0(\mathbf{x}, \dot{\varepsilon}^P_{ji})g(\mathbf{x}, T), \tag{2.5}$$

where D_0 denotes the value of the dissipation at zero temperature, determined uniquely by the plastic strain rate $\dot{\varepsilon}^P_{ij}$.

In further considerations the function $g(\mathbf{x}, T)$ will be linearized:

$$g(\mathbf{x}, T) = 1-b(\mathbf{x}, T) = 1-AT, \tag{2.6}$$

A being a non-negative material constant.

3. The external actions resulting in some mechanical loads as well as in temperature fields are controlled by a set of load-temperature factors β_s, $s = 1, \ldots, r$, referring to each one of the actions, respectively:

$$P_i(\mathbf{x}, t) = \sum_{s=1}^{r} \beta_s(t)P_i^s(\mathbf{x}), \quad F_i(\mathbf{x}, t) = \sum_{s=1}^{r} \beta_s(t)F_i^s(\mathbf{x}), \tag{2.7}$$

$$T(\mathbf{x}, t) = \sum_{s=1}^{r} \beta_s(t)T^s(\mathbf{x}).$$

Here P_i — surface tractions, F_i — body forces.

The values of the factors β_s belong to a certain set Ω in the r-dimensional space of those parameters. The set Ω defines the range of their prescribed variations.

The total stress tensor in an elastic-plastic body subject to actions (2.7) can be presented as follows:

$$\sigma_{ij} = \sigma^E_{ij}+\varrho_{ij}, \tag{2.8}$$

where σ^E_{ij} is the thermoelastic stress calculated under the assumption $\varepsilon^P_{ij} = 0$ and $k = \infty$, ϱ_{ij} being a self-equilibrated stress state appearing as a result of plastic deformations. This state can be expressed in the form

$$\varrho_{ij}(\mathbf{x}, t) = \int_V G^{kl}_{ij}(\mathbf{x}, \xi)\varepsilon^P_{kl}(\xi, t)dV, \tag{2.9}$$

where $G^{kl}_{ij}(\mathbf{x}, \xi)$ is a two-point Green tensor field depending on the elastic moduli tensor field $E_{ijkl}(\mathbf{x})$ and on the boundary conditions.

The thermoelastic stress can be presented as follows:

$$\sigma_{ij}^{E} = \sigma_{ij}^{EE} + \varrho_{ij}^{T}, \tag{2.10}$$

where σ_{ij}^{EE} is the elastic (not thermoelastic!) stress determined uniquely by the mechanical loads P_i, F_i as given by Eqs. (2.7) whereas the thermal stress ϱ_{ij}^{T} (equilibrating vanishing mechanical loads) is to be calculated from the formula (2.9) by substituting ε_{ij}^{P} with $M_{ij}T$. Thus one can also write the following relationship (M_{ij} denotes the thermal expansion coefficients tensor):

$$\sigma_{ij}^{E}(\mathbf{x}, t) = \sum_{s=1}^{r} \beta_s(t)\sigma_{ij}^{Es} = \sum_{s=1}^{r} \beta_s(t)\{\sigma_{ij}^{EEs}(\mathbf{x}) + \varrho_{ij}^{Ts}(\mathbf{x})\},$$

where ε_{ij}^{Es}, ϱ_{ij}^{Ts} are respective thermoelastic and thermal stress fields associated with unit external actions, σ_{ij}^{EEs} denoting respective mechanical stresses.

3. Incremental collapse criterion

The kinematic shakedown theorem, in the case of both thermal and mechanical actions, reads, [5, 6]:

Shakedown is impossible if, for a certain time period (t_1, t_2) there exist:

1. an external actions history $P_i(\mathbf{x}, t)$, $F_i(\mathbf{x}, t)$, $T(\mathbf{x}, t)$,
2. a plastic strain history $\bar{\varepsilon}_{ij}(\mathbf{x}, t)$, such that

$$\int_{t_1}^{t_2} \left\{ \int_V F_i \dot{\bar{u}}_i dV + \int_S P_i \dot{\bar{u}}_i dS + \int_V M_{ij} T \dot{\hat{\varrho}}_{ij} dV \right\} dt > \int_{t_1}^{t_2} \int_V D(\dot{\bar{\varepsilon}}_{ij}, T) dV dt, \tag{3.1}$$

$$\Delta\bar{\varepsilon}_{ij} = \int_{t_1}^{t_2} \dot{\bar{\varepsilon}}_{ij} dt = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}), \quad \bar{u}_i(t_1) = 0, \tag{3.2}$$

$$\dot{\bar{\varepsilon}}_{ij} + E_{ijkl}^{-1} \dot{\hat{\varrho}}_{kl} = \frac{1}{2}(\dot{\bar{u}}_{i,j} + \dot{\bar{u}}_{j,i}). \tag{3.3}$$

The formula (3.3) denotes that $\hat{\varrho}_{ij}(\mathbf{x}, t)$ is the residual stress field defined by the plastic strain field $\bar{\varepsilon}_{ij}(\mathbf{x}, t)$ through the formula (2.9). However, due to Eq. (3.2), the integral increment field $\Delta\bar{\varepsilon}_{ij}$ is compatible and thus $\hat{\varrho}_{ij}(\mathbf{x}, t_1) - \hat{\varrho}_{ij}(\mathbf{x}, t_2) = 0$.

By transforming the inequality (3.1) by means of the Virtual Work Principle and in view of Eqs. (2.10) and (3.3), the following result is obtained:

$$\int_{t_1}^{t_2} \left[\int_V \sigma_{ij}^{EE} \{\dot{\bar{\varepsilon}}_{ij} + E_{ijkl}^{-1} \dot{\hat{\varrho}}_{kl}\} dV + \int_V M_{ij} T \dot{\hat{\varrho}}_{ij} dV \right] dt > \int_{t_1}^{t_2} \int_V D_0(\dot{\bar{\varepsilon}}_{ij}) g(\mathbf{x}, t) dV dt. \tag{3.4}$$

Due to the definition of ϱ_{ij}^{T}, the second integral in the left-hand side of that formula can be presented as follows:

$$\int_V M_{ij} T \dot{\hat{\varrho}}_{ij} dV = -\int_V \dot{\hat{\varrho}}_{ij} E_{ijkl}^{-1} \varrho_{kl}^{T} dV = \int_V \varrho_{ij}^{T} \dot{\bar{\varepsilon}}_{ij} dV. \tag{3.5}$$

The Virtual Work Principle implies also that

$$\int_V \sigma_{ij}^{EE} E_{ijkl}^{-1} \dot{\hat{\varrho}}_{kl} dV = 0. \tag{3.6}$$

Thus finally, the formula (3.4) assumes the following form:

$$\int_{t_1}^{t_2} \int_V \sigma_{ij}^E \dot{\bar{\varepsilon}}_{ij} dV dt > \int_{t_1}^{t_2} \int_V D_0(\dot{\bar{\varepsilon}}_{ij}) g(\mathbf{x}, t) dV dt \tag{3.7}$$

or, equivalently,

$$\int_{t_1}^{t_2} \int_V \{\sigma_{ij}^E \dot{\bar{\varepsilon}}_{ij} + b(\mathbf{x}, T) D_0(\dot{\bar{\varepsilon}}_{ij})\} dV dt > \int_{t_1}^{t_2} \int_V D_0(\dot{\bar{\varepsilon}}_{ij}) dV dt, \tag{3.7'}$$

$b(\mathbf{x}, T)$ being defined by Eq. (2.6).

The most stringent condition of incremental collapse results from a history of loads and temperature defined by a certain history $\bar{\beta}_s(t)$ of the load-temperature factors which maximizes the left-hand side of the inequality (3.7) and for a strain-rate history which minimizes its right-hand side.

To perform this optimization let us notice first the following inequality, generally valid

$$\int_{t_1}^{t_2} D_0(\dot{\varepsilon}_{ij}^p) dt \geqslant D_0(\Delta\varepsilon_{ij}^p), \quad \Delta\varepsilon_{ij}^p = \varepsilon_{ij}^p(t_2) - \varepsilon_{ij}^p(t_1). \tag{3.8}$$

From the following relations the proof yields

$$\begin{gathered} \Delta\varepsilon_{ij}^p = \int_{t_1}^{t_2} \lambda \frac{\partial f}{\partial \sigma_{ij}} dt, \quad \int_{t_1}^{t_2} D_0(\dot{\varepsilon}_{ij}^p) dt = \int_{t_1}^{t_2} \sigma_{ij} \lambda \frac{\partial f}{\partial \sigma_{ij}} dt, \\ D_0(\Delta\varepsilon_{ij}^p) = \bar{\sigma}_{ij} \int_{t_1}^{t_2} \lambda \frac{\partial f}{\partial \sigma_{ij}} dt, \quad \text{where} \quad f(\bar{\sigma}_{ij}) = k_0. \end{gathered} \tag{3.9}$$

Therefore,

$$\int_{t_1}^{t_2} D_0(\dot{\varepsilon}_{ij}^p) dt - D_0(\Delta\varepsilon_{ij}^p) = \int_{t_1}^{t_2} (\sigma_{ij} - \bar{\sigma}_{ij}) \lambda \frac{\partial f}{\partial \sigma_{ij}} dt = \int_{t_1}^{t_2} (\sigma_{ij} - \bar{\sigma}_{ij}) \dot{\varepsilon}_{ij}^p dt \geqslant 0.$$

The last inequality results from the convexity of the yield condition (2.3).

Now, if we are going to investigate the phenomenon of incremental collapse exclusively, not the general case of inadaptation, we can stress the range of the plastic strain histories $\bar{\varepsilon}_{ij}$ to such that their rates are proportional to the total increment $\Delta\bar{\varepsilon}_{ij}$:

$$\begin{gathered} \dot{\bar{\varepsilon}}_{ij}(\mathbf{x}, t) = \Lambda(\mathbf{x}, t) \Delta\bar{\varepsilon}_{ij}(\mathbf{x}), \quad \Lambda(\mathbf{x}, t) \geqslant 0, \\ \int_{t_1}^{t_2} \Lambda(\mathbf{x}, t) dt = 1. \end{gathered} \tag{3.10}$$

The equality in the formula (3.8) takes place in the case of plastic strain rates as defined by Eq. (3.10), i.e. in the case when the vector $\dot{\bar{\varepsilon}}_{ij}$ keeps a constant direction.

Thus the formula (3.7′) can be rewritten as follows:

$$(3.11)\quad \int_{t_1}^{t_2}\int_V \Big\{\sum_{k=1}^{r}\beta_s(t)\sigma_{ij}^{Es}(\mathbf{x})\Lambda(\mathbf{x},t)\Delta\bar{\varepsilon}_{ij}(\mathbf{x})+b(\mathbf{x},T)D_0(\dot{\bar{\varepsilon}}_{ij})\Big\}\,dV\,dt > \int_V D_0(\Delta\bar{\varepsilon}_{ij})\,dV.$$

Now let us analyse the constraints in the problem of optimizing the left-hand side of Eq. (3.11). Those are:

1) factors $\beta_s(t)$ must remain within a given set Ω,

2) the total increments $\Delta\bar{\varepsilon}_{ij}$ are prescribed.

It is easy to see that these constraints are the least stringent if the field $\Lambda(\mathbf{x}, t)$ is selected in such a way that at each instant t it vanishes everywhere with the exception of a certain point $\mathbf{x}_0(t)$, i.e. $\Lambda(\mathbf{x}^1, t)\Lambda(\mathbf{x}^2, t) = 0$ for $\mathbf{x}^1 \neq \mathbf{x}^2$. Then the integrand in the left-hand side of Eq. (3.11) may be optimized at each point separately and the values of $\beta_s(t)$ at instants during which $\dot{\bar{\varepsilon}}_{ij}(\mathbf{x}, t) = 0$ are unimportant. Let us notice that the procedure does not alter the right-hand side of the inequality.

Thus the following optimization problem is to be solved first:

$$(3.12)\qquad L(\mathbf{x}) = \max_{\beta_s\in\Omega}\Big\{\sum_{k=1}^{r}\beta_s\sigma_{ij}^{Es}(\mathbf{x})\Delta\bar{\varepsilon}_{ij}(\mathbf{x})+b\Big(\mathbf{x},\sum_{s=1}^{r}\beta_s T^s(\mathbf{x})\Big)D_0(\Delta\bar{\varepsilon}_{ij}(\mathbf{x})\Big\}$$

and then the load and temperature magnitudes (i.e. the respective values of the factors β_s) which may cause incremental collapse are given by the equation

$$(3.13)\qquad \int_V L(\mathbf{x})\,dV = \int_V D_0(\Delta\bar{\varepsilon}_{ij})\,dV.$$

The problem (3.12), in general, is a complex problem of nonlinear programming. It becomes simpler in the case when domain Ω is a hyperpolyhedron defined by a system of linear inequalities

$$\sum_{k=1}^{r}A_{sk}\beta_s \leqslant d_k,\qquad k = 1, \ldots, m,$$

and if the function $g(T)$ is linear as given by Eq. (2.6). In such a case the domain Ω is uniquely defined by a set of v corners $\beta_s^1, \ldots, \beta_s^v$ and formula (3.12) becomes

$$(3.12')\qquad L(\mathbf{x}) = \max_{j=1,\ldots,v}\sum_{s=1}^{r}\beta_s^j\{\sigma_{ij}^{Es}(\mathbf{x})\Delta\bar{\varepsilon}_{ij}(\mathbf{x})+A(\mathbf{x})T^s(\mathbf{x})D_0\big(\Delta\bar{\varepsilon}_{ij}(\mathbf{x})\big)\}.$$

In the case when Ω is a hyperparallelepiped

$$(3.14)\qquad \beta_s^- \leqslant \beta_s \leqslant \beta_s^+,$$

the solution of the optimization problem (3.12) or (3.12′) can be given explicitly. Namely, in this case the incremental collapse criterion (3.13) assumes the following form:

$$(3.15)\qquad \int_V\sum_{s=1}^{r}a_s(\mathbf{x})J_s(\mathbf{x})\,dV = \int_V D_0\big(\Delta\bar{\varepsilon}(\mathbf{x})\big)\,dV,$$

where

$$(3.16)\qquad J_s(\mathbf{x}) = \sigma_{ij}^{Es}(\mathbf{x})\Delta\bar{\varepsilon}_{ij}(\mathbf{x})+A(\mathbf{x})T^s(\mathbf{x})D_0\big(\Delta\bar{\varepsilon}_{ij}(\mathbf{x})\big)$$

and

$$a_s(\mathbf{x}) = \begin{cases} \beta_s^+ & \text{if} \quad J_s(\mathbf{x}) > 0, \\ \beta_s^- & \text{if} \quad J_s(\mathbf{x}) < 0. \end{cases} \tag{3.17}$$

The author does not know of any study in which the incremental collapse criterion could be derived systematically, accounting for the temperature dependence of the yield-point stress. Moreover, it seems that the consequent use of the load-temperature factors makes clear how the elastic stress should be optimized in the basic inequality (3.8). Usually this has been done intuitively by assuming an envelope of thermoelastic stresses. The formulas (3.12), (3.16) and (3.17) show that sometimes this does not need to be correct.

4. Example of application

A thick-walled tube, closed with rigid decks, is subjected to internal pressure p which is allowed to vary within the limits

$$0 \leqslant p \leqslant \bar{p} \tag{4.1}$$

and to a quasi-stationary temperature field

$$T(r) = \theta \frac{\ln(b/r)}{\ln(b/a)}$$

varying independently of the pressure as defined by

$$0 \leqslant \theta \leqslant \bar{\theta}. \tag{4.2}$$

Here a, b — internal and external radii of the tube, θ — internal temperature, provided $T(b) = 0$, r — current radius.

Let us assume that the material of the tube obeys the Tresca yield condition

$$|\sigma_\phi - \sigma_r| \leqslant 2k \tag{4.3}$$

and the plastic constant k decreases linearly with temperature according to the formula (2.6).

The thermoelastic stresses are as below:

$$\begin{aligned} \sigma_r^E(r) &= \frac{pa^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right) - \frac{Ea^2\theta}{2(1-\nu)(b^2-a^2)}\left(1-\frac{b^2}{r^2}+\frac{(b^2-a^2)\ln(r/b)}{a^2\ln(a/b)}\right), \\ \sigma_\phi^E(r) &= \frac{pa^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right) - \frac{Ea^2\theta}{2(1-\nu)(b^2-a^2)}\left(1+\frac{b^2}{r^2}+\frac{(b^2-a^2)\left(1+\ln(r/b)\right)}{a^2\ln(a/b)}\right), \\ \sigma_z^E(r) &= \frac{pa^2}{b^2-a^2}\, \frac{Ea^2\theta}{(1-\nu)(b^2-a^2)}\left(1+\frac{(b^2-a^2)\left(\frac{1}{2}+\ln\frac{r}{b}\right)}{a^2\ln(a/b)}\right). \end{aligned} \tag{4.4}$$

In the case of axial symmetry the only possible mechanism of incremental collapse is given by

$$\dot{u}(r) = \dot{C}/r, \quad \Delta\varepsilon_r = -\Delta C/r^2, \quad \Delta\varepsilon_\phi = \Delta C/r^2 \tag{4.5}$$

associated in this case with the side of the yield condition defined by

$$\sigma_\phi - \sigma_r = 2k(T). \tag{4.6}$$

Thus it is easy to see that

$$D_0 = \sigma_r\left(-\frac{\Delta C}{r^2}\right) + \sigma_\phi\left(\frac{\Delta C}{r^2}\right) = 2k_0\frac{\Delta C}{r^2}. \tag{4.7}$$

Due to the formulas (4.4) and (4.5) the functions $J_p(r)$, $J_\theta(r)$ are as follows:

$$J_p(r) = \frac{a^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right)\left(-\frac{\Delta C}{r^2}\right) + \frac{a^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right)\frac{\Delta C}{r^2} = \frac{\Delta C \cdot 2a^2b^2}{r^4(b^2-a^2)},$$

$$\begin{aligned} J_\theta(r) = &-\frac{E\alpha a^2}{2(1-\nu)(b^2-a^2)}\left[1-\frac{b^2}{r^2}+\frac{(b^2-a^2)\ln(b/r)}{a^2\ln(b/a)}\right]\left(-\frac{\Delta C}{r^2}\right) \\ &-\frac{E\alpha a^2}{2(1-\nu)(b^2-a^2)}\left[1+\frac{b^2}{r^2}-\frac{(b^2-a^2)\left(1-\ln(b/r)\right)}{a^2\ln(b/a)}\right]\frac{\Delta C}{r^2} + \Lambda\frac{\ln(b/r)}{\ln(b/a)}2k_0\frac{\Delta C}{r^2} \\ &= \frac{\Delta C}{r^2}\left\{\frac{E\alpha a^2}{2(1-\nu)(b^2-a^2)}\left[\frac{b^2-a^2}{a^2\ln(b/a)}-\frac{2b^2}{r^2}\right]+2\Lambda k_0\frac{\ln(b/r)}{\ln(b/a)}\right\}. \end{aligned} \tag{4.8}$$

Thus

$$\begin{aligned} &J_p(r) > 0 \quad \text{for} \quad a \leqslant r \leqslant b, \\ &J_\theta(r) < 0 \quad \text{for} \quad a \leqslant r < r_0, \\ &J_\theta(r) > 0 \quad \text{for} \quad r_0 < r \leqslant b, \end{aligned} \tag{4.9}$$

where the intermediate radius r_0 is to be calculated from the equation

$$-\frac{2b^2}{r_0^2} + \frac{b^2-a^2}{a^2\ln(b/a)} + \frac{4\Lambda k_0(1-\nu)(b^2-a^2)\ln(b/r)}{E\alpha a^2\ln(b/a)} = 0. \tag{4.10}$$

According to Eq. (3.17), the functions $a_p(r)$, $a_\theta(r)$ are as follows:

$$a_p(r) = \bar{p}, \quad a_\theta(r) = \begin{cases} 0 & \text{for } a \leqslant r < r_0, \\ \bar{\theta} & \text{for } r_0 < r \leqslant b. \end{cases} \tag{4.11}$$

Finally, the incremental collapse condition (3.15) assumes the form

$$\bar{p}\int_a^b J_p(r)r\,dr + \bar{\theta}\int_{r_0}^b J_\theta(r)r\,dr = \int_a^b D_0(r)r\,dr. \tag{4.12}$$

By substituting the respective magnitudes as defined by Eqs. (4.5), (4.7), (4.8) and (4.11) the following formula results:

$$\bar{p} + \bar{\theta}\left\{\frac{Ea^2\alpha}{2(1-\nu)(b^2-a^2)}\left[\frac{b^2-a^2}{a^2\ln(b/a)}\ln(b/r_0) - \frac{b^2}{r_0^2}+1\right] + \frac{\Lambda k_0\left(\ln(b/r_0)\right)^2}{\ln(b/a)}\right\} = 2k_0\ln(b/a). \tag{4.13}$$

It may be useful to introduce the following dimensionless parameters:

$$\beta = \frac{b}{a}, \quad \varrho = \frac{r_0}{a}, \quad \varepsilon = \frac{4\Lambda k_0(1-\nu)}{E\alpha}. \tag{4.14}$$

Then the formula (4.13) becomes

$$\bar{p}+\bar{\theta}\frac{E\alpha}{2(1-\nu)}\left\{\frac{\ln(\beta/\varrho)}{\ln\beta}+\frac{1}{\beta^2-1}-\frac{\beta^2}{\varrho^2(\beta^2-1)}+\varepsilon\frac{\ln(\beta/\varrho)}{\ln\beta}\right\}=2k_0\ln\beta \tag{4.15}$$

and Eq. (4.10) assumes the following form:

$$-2\beta^2/\varrho^2+(\beta^2-1)/\ln\beta+\varepsilon(\beta^2-1)\ln(\beta/\varrho)/\ln\beta = 0. \tag{4.16}$$

As the parameter ε is usually small, an approximate solution of this equation may be assumed as follows:

$$\varrho_0=\sqrt{\frac{\beta^2\ln\beta^2}{\beta^2-1}},\quad \varrho=\varrho_0\left(1-\frac{\varepsilon}{2}\ln\frac{\beta}{\varrho_0}\right). \tag{4.17}$$

This solution is sufficiently precise for $b/a < 3$, thus in the majority of practical cases. For $b/a < 1.2$ one can assume $r_0 = (a+b)/2$.

In the case of mild steel one can assume the following material parameters:

$$2k_0 = 2500\ \text{kG/cm}^2,\quad E = 2.1\cdot 10^6\ \text{kG/cm}^2,$$
$$\alpha = 10^{-5}/\text{grad},\quad A = 10^{-3}/\text{grad},$$
$$\nu = 0.3.$$

In the case $b/a = 2$ the formula (4.15) gives

$$\bar{p}+2.6672\frac{\text{kG}}{\text{cm}^2\ \text{grad}}\bar{\theta} = 1732.8675\ \text{kG/cm}^2. \tag{4.18}$$

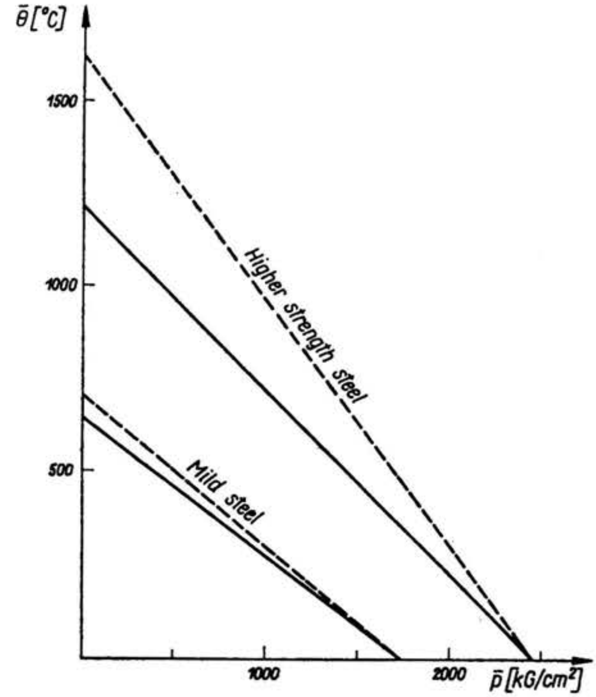

FIG. 1.

For the sake of comparison the same calculation has been repeated for $A = 0$, i.e. without the influence of temperature on yield point stress. In this case the following result has been obtained:

$$\bar{p}+2.5326\,\frac{\text{kG}}{\text{cm}^2\ \text{grad}}\,\bar{\theta} = 1732.8675\ \text{kG/cm}^2. \tag{4.19}$$

Both incremental criteria are presented in Fig. 1, the formula (4.18) — in solid line, the formula (4.19) — in a dashed one.

The influence of the temperature dependence of yield stress becomes more pronounced in the case of steels of higher strength. Figure 1 gives the respective results appropriate for the case:

$$b/a = 1.5, \qquad 2k_0 = 6000\ \text{kG/cm}^2, \qquad E = 2.1\cdot 10^6\ \text{kG/cm}^2,$$
$$\alpha = 10^{-5}/\text{grad}, \qquad A = 2\cdot 10^{-3}/\text{grad}, \qquad \nu = 0.3.$$

In this case the formulas (4.18) and (4.19) are to be substituted, respectively, by the following ones:

$$\bar{p}+1.9988\,\frac{\text{kG}}{\text{cm}^2\ \text{grad}}\,\bar{\theta} = 2432.79\ \text{kG/cm}^2, \tag{4.20}$$

$$\bar{p}+1.5068\,\frac{\text{kG}}{\text{cm}^2\ \text{grad}}\,\bar{\theta} = 2432.79\ \text{kG/cm}^2. \tag{4.21}$$

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