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Equations of linear thermoconsolidation

W. DERSKI and S. J. KOWALSKI (POZNAŃ)

A COMPLETE set of linear equations has been derived for a porous medium filled with a liquid, taking into account thermal effects. The considerations are based on thermodynamics of irreversible processes with the usual assumptions of the theory of consolidation and an additional assumption that the local temperature of the two components is the same. A complete thermomechanical interpretation of the coefficients introduced is given. It is shown that the existing approaches to the thermoconsolidation problem are incorrect [2, 8].

W pracy wyprowadzono pełny zestaw równań liniowych dla ośrodka porowatego wypełnionego cieczą z uwzględnieniem efektów termicznych. Rozważania prowadzono na gruncie termodynamiki procesów nieodwracalnych, przy założeniach powszechnie stosowanych w teorii konsolidacji oraz dodatkowo, że temperatura składników lokalnie jest taka sama. W pracy podano pełną termomechaniczną interpretację wyprowadzonych współczynników oraz wykazano niepoprawność dotychczasowych podejść do zagadnienia termokonsolidacji, [2, 8].

В работе выведена полная система линейных уравнений для пористой среды, заполненной жидкостью, с учетом термических эффектов. Рассуждения проводились на грунте термодинамики необратимых процессов, при предположениях обычно применяемых в теории консолидации, а также дополнительно, что температура компонентов локально одинакова. В работе дается полная термомеханическая интерпретация введенных коэффициентов, а также показана некорректность существующих до сих пор подходов к задаче термоконсолидации [2, 8].

Principal symbols

x_i [m] Cartesian coordinates,

t [s] time,

$\mathbf{u}(u_i)$ [m] displacement vector of the skeleton,

$\mathbf{U}(U_i)$ [m] displacement vector of the liquid,

$\mathbf{v}(v_i) = \dfrac{d\mathbf{u}}{dt}$ [m/s] velocity vector of the skeleton,

$\mathbf{w}(w_i) = \dfrac{d\mathbf{U}}{dt}$ [m/s] velocity vector of the liquid,

A [m²] area,

A_p [m²] area of pores in A,

Ω [m³] volume,

Ω_c [m³] volume of liquid in Ω,

Ω_s [m³] volume of skeleton in Ω,

f_A [1] coefficient of surface porosity; $f_A = \dfrac{A_p}{A}$,

f_Ω [1] coefficient of volume porosity; $f_\Omega = \dfrac{\Omega_p}{\Omega}$,

ϱ [kg/m³] density of the two-component medium,

ϱ_c [kg/m³] real density of the liquid,

ϱ_f [kg/m³] density of the liquid as referred to the total volume,

$\mathbf{F}$ [N/m³] diffusion force per unit volume,

σ_{ij} [N/m²] coordinates of the stress tensor in the skeleton per unit area of the total surface,

$\sigma = -pf_A$ [N/m²] liquid stress per unit area of the total surface,

p [N/m²] liquid pressure,

ε_{ij} [1] coordinates of the strain tensor of the skeleton,

ε [1] volume expansion of the skeleton,

θ [1] volume strain of the liquid.

F [J] Helmholtz free energy,

G [J] Gibbs free enthalpy,

K [J] kinetic energy,

$\dot{L}_z$ [J] power of the external forces,

Q [J] quantity of heat,

$\mathbf{q}(q_i) \left[\dfrac{J}{m^2 s} \right]$ vector of heat flux,

$S \left[\dfrac{J}{°K} \right]$ entropy,

U [J] internal energy,

T [°K] absolute temperature,

$\mathbf{X}(x_i) \left[\dfrac{N}{kg} \right]$ body force, per unit mass,

$\vartheta = T - T_0$ [°K] relative temperature T_0 (with reference to the temperature T_0 of the natural state),

$\lambda \left[\dfrac{J}{ms°K} \right]$ overall global coefficient of heat conduction,

$c_{\varrho f} \left[\dfrac{J}{kg°K} \right]$ specific heat of the liquid with constant volume of the components, per unit mass,

$c_{\varrho s} \left[\dfrac{J}{kg°K} \right]$ specific heat of the skeleton with constant volume of the components per unit mass,

$c_\sigma \left[\dfrac{J}{m^3 °K} \right]$ specific heat of the liquid and the skeleton as a whole, with constant pressure, per unit volume,

$c_\Omega \left[\dfrac{J}{m^3 °K} \right]$ specific heat of the liquid and the skeleton as a whole, with constant volume of the components, per unit volume,

$c_{\Omega c} \left[\dfrac{J}{m^3 °K} \right]$ specific heat of the liquid, with constant volume of the components, per unit volume,

$c_{\Omega s} \left[\dfrac{J}{m^3 °K} \right]$ specific heat of the skeleton, with constant volume of the components, per unit volume,

$c_{\sigma s} \left[\dfrac{J}{m^3 °K} \right]$ specific heat of the skeleton, with constant pressure, per unit volume,

$c_{\sigma c} \left[\dfrac{J}{m^3 °K} \right]$ specific heat of the liquid, with constant pressure, per unit volume.

The subscript 1 is used to denote quantities per unit volume of the medium. Thus, for instance, S_1 is the entropy per unit volume.

1. Introduction

A SET of linear equations will be derived for a porous medium constituting of an elastic skeleton with its pores filled with a liquid, the action of the temperature field being taken into consideration. This problem has hitherto been considered in two papers. The first was written in 1970 by R. L. SCHIFFMAN [8], who assumed that the temperature of the skeleton and the liquid is the same. The problem formulated in the other paper, which was written in 1973 by H. DERESIEWICZ and C. PECKER [2], was more general and assumed that the two temperatures are different.

Schiffman's paper contains essential notional errors, that of Deresiewicz and Packer gives rise to considerable doubts as regards the heat equation which is established on the formal basis of the Onsager principle. There is no interpretation of the coefficients introduced in the heat equation.

In the present paper the correct equations of motion published in Ref. [3] will be used and the argument will be based on the structure of a state function of the medium. This will enable us to give a clear interpretation of the coefficients.

The approximate assumptions used throughout the paper will be the usual assumptions of the consolidation theory with an additional assumption that the heat exchange between the skeleton and the liquid is perfect, that is, the temperatures of both components are the same at every point of the medium. This is of course a simplification but the error thus introduced is, in our opinion, insignificant.

2. Equation of continuity of mass. Equation of motion

The subject of the present considerations is a porous body, the pores of which are filled with a liquid. It is assumed that the volume porosity f_Ω and the surface porosity f_A are constant. The continuum theory will be used.
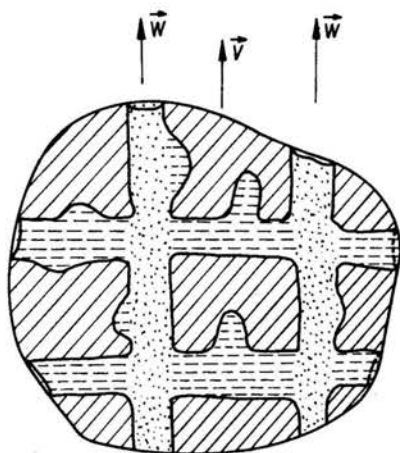

FIG. 1.

The equations of motion have been given for such a medium in Ref. [3]. The results of that reference will be reported here for the sake of lucidity.

From the kinematical point of view the body is composed of two parts:

1) the skeleton, together with the skeleton-bound liquid moving at the same velocity as the skeleton. The density of that component is $(\varrho - \bar{\varrho})$;

2) the free liquid moving at a velocity which is different, in general, from that of the skeleton. Its density will be denoted by $\bar{\varrho}$; the symbol $\varrho = \varrho_s + \varrho_f$ denotes the overall density.

This subdivision of the liquid is illustrated in Fig. 1. The free liquid is marked by dots, the skeleton-bound liquid by dashes. It is seen that the density of the free liquid is a function of the surface porosity f_A. It thus is expressed as

$$(2.1) \qquad \bar{\varrho} = f_A \varrho_c.$$

When the medium moves, the free liquid is carried by the skeleton and performs an additional motion relative to the latter. The part in relative motion produces forces of internal friction. Its density will be denoted by $\bar{\varrho}_w$.

Putting aside the details of the derivation procedure which can be found in Ref. [3], let us quote the equation of continuity of mass in its final form.

1. For the skeleton and the skeleton-bound liquid we have

$$(2.2) \qquad \frac{\partial(\varrho - \bar{\varrho})}{\partial t} + [(\varrho - \bar{\varrho})v_l]_{,l} = 0.$$

2. For the free liquid we have

$$(2.3) \qquad \frac{\partial \bar{\varrho}}{\partial t} + [\bar{\varrho}(w_l - v_l)_{,l}] = -(\bar{\varrho}v_l)_{,l}.$$

3. For the density of the free liquid, which varies during the relative motion, we have

$$(2.4) \qquad \frac{\partial \bar{\varrho}_w}{\partial t} + [\bar{\varrho}_w(w_l - v_l)]_{,l} = 0.$$

Let us also quote after Ref. [3], the equations of motion also without derivation.

1. For the skeleton and the skeleton-bound liquid the equation of motion is

$$\sigma_{lj,j} + (\varrho - \bar{\varrho})X_l = F_i^{sf} + (\varrho - \bar{\varrho})\frac{dv_l}{dt},$$

$$(2.5)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_l}v_l.$$

2. For the free liquid it has the form

$$\sigma_{,j} + \bar{\varrho}X_l = F_i^{fs} + \bar{\varrho}\frac{dw_l}{dt},$$

$$(2.6)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_l}w_l,$$

and

$$F_i^{sf} + F_i^{fs} = 0.$$

3. The first law of thermodynamics

Let us separate mentally from a large region a finite body bounded by a regular surface A, the position of which is determined by the external normal unit vector $\mathbf{n}(n_i)$. The energy balance for this separate body is as follows:

$$(3.1) \qquad \dot{K} + \dot{U} = \dot{L}_z + \dot{Q},$$

which is a mathematical expression of the first law of thermodynamics [7].
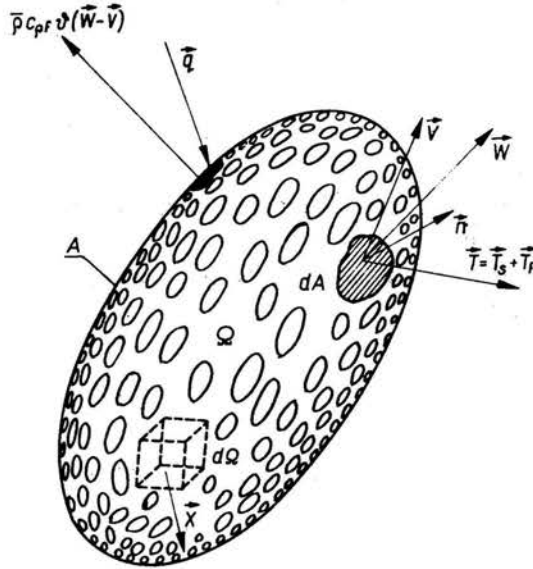
FIG. 2.

The mechanical power is equal to the power of the surface forces acting on the surface A (cf. Fig. 2) and is expressed by the equation

$$(3.2) \qquad \dot{L}_z = \int_A T_{si} v_i dA + \int_\Omega (\varrho - \bar{\varrho}) X_i v_i d\Omega + \int_A T_{fi} w_i dA + \int_\Omega \varrho X_i w_i d\Omega$$

$$= \int_\Omega [\sigma_{ij,j} + (\varrho - \bar{\varrho}) X_i] v_i d\Omega + \int_\Omega [\sigma_{,i} + \bar{\varrho} X_i] w_i d\Omega + \int_\Omega [\sigma_{ij} \dot{\varepsilon}_{ij} + \sigma \dot{\theta}] d\Omega,$$

where $T_{si} = \sigma_{ij} n_j$ is the stress acting on the skeleton per unit area of the total surface, $T_{fi} = \sigma n_i$ is the stress acting on the liquid per unit area of the total surface, $\dot{\varepsilon}_{ij} = \dfrac{1}{2}(v_{i,j} + v_{j,i})$ is the small strain rate of the skeleton, $\dot{\theta} = w_{i,i}$ is the small expansion rate of the liquid. The power of the gravity forces has also been accounted for in Eq. (3.2).

The kinetic energy is described by the equation

$$K = \frac{1}{2} \int_\Omega [(\varrho - \bar{\varrho}) v_i v_i + \bar{\varrho} w_i w_i] d\Omega.$$

Making use of the equations of continuity of mass (2.2) and (2.3), the material derivative of that energy can be written in the form

(3.3)
$$\dot{K} = \int_{\Omega} \left[(\varrho - \bar{\varrho}) v_l \frac{dv_l}{dt} + \bar{\varrho} w_l \frac{dw_l}{dt} \right] d\Omega.$$

The non-mechanical power is equal to the heat supplied to the system in the form of a heat flux q through the surface A and as a result of transport through the same surface of a mass of free liquid.

To evaluate the heat transported by the latter, it is assumed that the mass of the volume Ω is constant (cf. [5]). Such an assumption means that the density of the free liquid in the volume Ω is constant, that is $\partial \bar{\varrho} / \partial t = 0$. Consequently, we find from the equation of continuity of mass that

(3.4) $[\bar{\varrho}(w_l - v_l)]_{,l} = 0, \quad (\bar{\varrho} v_l)_{,l} = 0.$

This condition means that there are differences in velocity $w_l \neq v_l$ but, at the same time, $\bar{\varrho} = \text{const}$ and $\dot{\varepsilon} = \dot{\theta} = 0$.

In agreement with the assumption that the local temperature of the liquid and the skeleton are the same, we evaluate the rate of variation in the quantity of heat in the volume Ω

(3.5) $\dot{Q} = - \int_A q_l n_l dA + \int_A \bar{\varrho} c_{\varrho f} \vartheta (w_l - v_l) n_l dA = - \int_{\Omega} [q_{l,l} - \bar{\varrho} c_{\varrho f} (w_l - v_l) \vartheta_{,l}] d\Omega.$

Let us introduce the rate of internal energy per unit volume

$$\dot{U} = \int_{\Omega} \dot{U}_1 \, d\Omega$$

and, making use of the functions listed above, let us write the local expression of the first law of thermodynamics on the basis of Eq. (3.1):

(3.6) $\dot{U}_1 = [\sigma_{ij} \dot{\varepsilon}_{ij} + \sigma \dot{\theta}] - [q_{l,l} - \bar{\varrho} c_{\varrho f} (w_l - v_l) \vartheta_{,l}] + [F_l^{fs}(w_l - v_l)].$

The last term in the Eq. (3.6) follows from the equations of motion used for the transformation of the global equation (3.1).

The local equation (3.6) tells us that the variation in the internal energy is due to the energy supplied by the mechanical work (the first term), by the heat (the second term) and the thermal energy produced as a result of resistance against flow (the third term).

4. The second law of thermodynamics. Dissipation functions. Diffusion forces

The total entropy S of the fragment separated mentally from the medium is composed of a part S_z that can be exchanged with the ambient medium through the surface A and S_w which increases in an irreversible manner. In agreement with the Clausius-Duhem postulate, the variation rates of these entropies must satisfy the inequality

(4.1) $\dot{S}_w = \dot{S} - \dot{S}_z \geq 0.$

The variation rate of the entropy $\dot{S}_z$ will be evaluated from the heat flux $\dot{Q}$ through the surface A, the condition (3.4) being taken into account. The entropy function $\dot{S}$ includes the entropy produced by internal friction. However, friction produces temperature variations, therefore it does not influence the heat flow through the surface A. To determine the entropy S_z exchanged with the ambient medium, it suffices to calculate S^* which is determined by the heat flux through the surface. Thus, according to Eq. (3.5), we calculate the variation rate of the entropy S^* and write

$$(4.2) \qquad S^* = -\int_\Omega \frac{q_{i,i}}{T} d\Omega + \int_\Omega \frac{[\bar{\varrho} c_{\varrho f}(w_i - v_i)\vartheta_{,i}]}{T} d\Omega.$$

Let us transform the integrals in the right-hand member of Eq. (4.2) as follows:

$$S^* = -\int_A \frac{q_i - \bar{\varrho} c_{\varrho f}(w_i - v_i)\vartheta}{T} n_i \, dA - \int_\Omega \frac{[q_i - \bar{\varrho} c_{\varrho f}(w_i - v_i)\vartheta]}{T^2} \vartheta_{,i} \, d\Omega.$$

It is found, therefore, that the entropy exchanged with the ambient medium in the thermal form is

$$(4.3) \qquad \dot{S}_z = -\int_A \frac{q_i - \bar{\varrho} c_{\varrho f}(w_i - v_i)\vartheta}{T} n_i \, dA.$$

The remaining part are irreversible variations in the entropy $\dot{S}_w^*$, the entropy sources being not taken into consideration. We have

$$(4.4) \qquad \dot{S}_w^* = -\int_\Omega \frac{[q_i - \bar{\varrho} c_{\varrho f}(w_i - v_i)\vartheta]\vartheta_{,i}}{T^2} \, d\Omega.$$

The integrand in Eq. (4.4) must satisfy the Clausius-Duhem inequality. Assuming that the relations are linear and the medium thermally isotropic, we derive from this inequality the general law of heat conduction:

$$(4.5) \qquad q_i = -\lambda \vartheta_{,i} + \bar{\varrho} c_{\varrho f}(w_i - v_i)\vartheta, \quad \lambda > 0.$$

It is seen that this law involves the heat transport due to the motion of the free liquid. The coefficient of heat conduction $\lambda > 0$ relates to both components as a whole and, in agreement with the theory of mixtures [1], is equal to

$$(4.6) \qquad \lambda = \frac{\varrho_s \lambda_s}{\varrho_s + \varrho_f \varphi_f} + \frac{\varrho_f \lambda_f}{\varrho_f + \varrho_s \varphi_s}.$$

The coefficients φ_s and φ_f in this equation depend on the physical properties of the components. They are determined by the empirical equation given in Ref. [1]. In particular, it may be assumed as a first approximation that $\varphi_s = \varphi_f = 1$ and

$$(4.7) \qquad \lambda = \frac{1}{\varrho}(\varrho_s \lambda_s + \varrho_f \lambda_f),$$

where λ_s and λ_f are the coefficients of heat conduction for the skeleton and the liquid, respectively.

The total value of the internal entropy produced is determined, taking into considera-
tion its sources, as the difference between the variation in the total entropy $\dot{S} \geqslant \dot{S}^*$ and
the entropy $\dot{S}_z$ which can be written in the form of the local relation

$$(4.8) \qquad \dot{S}_{w1} = \dot{S}_1 + \left[\frac{q_l - \bar{\varrho} c_{\varrho f}(w_l - v_l)\vartheta}{T} \right]_{,l} \geqslant 0.$$

Let us differentiate the second component in the inequality (4.8) bearing in mind
the assumptions with which the heat has been calculated ($\dot{\theta} - \dot{\varepsilon} = 0$, $\bar{\varrho}_{,l} = 0$), using the
first law of thermodynamics in its local form (3.6). We thus rewrite our inequality as

$$(4.9) \qquad \dot{S}_1 - \frac{\dot{U}_1}{T} + [F_l^{fs}(w_l - v_l) + \sigma_{lj}\dot{\varepsilon}_{lj} + \sigma\dot{\theta}]\frac{1}{T} - \frac{[q_l - \bar{\varrho} c_{\varrho f}\vartheta(w_l - v_l)]\vartheta_{,l}}{T^2} \geqslant 0.$$

Let us introduce the Helmholtz free energy

$$(4.10) \qquad F_1 \overset{\text{df}}{=} U_1 - S_1 T,$$

and rewrite the inequality (4.8) as the function of that energy

$$(4.11) \qquad \dot{F}_1 - S_1 T + F_l^{fs}(w_l - v_l) + \sigma_{lj}\dot{\varepsilon}_{lj} + \sigma\dot{\theta} - \frac{[q_l - \bar{\varrho} c_{\varrho f}\vartheta(w_l - v_l)]\vartheta_{,l}}{T} \geqslant 0.$$

The free energy is a function of the coordinates of the strain tensor ε_{lj}, the expansion
of the liquid θ and the temperature T, that is,

$$(4.12) \qquad F_1 = F_1(\varepsilon_{lj}, \theta, T),$$

therefore, the inequality (4.11) can be written again as follows:

$$(4.13) \qquad \left(\sigma_{lj} - \frac{\partial F_1}{\partial \varepsilon_{lj}} \right)\dot{\varepsilon}_{lj} + \left(\sigma - \frac{\partial F_1}{\partial \theta} \right)\dot{\theta} - \left(S_1 + \frac{\partial F_1}{\partial T} \right)\dot{T} + F_l^{fs}(w_l - v_l)$$
$$- \frac{q_l - \bar{\varrho} c_{\varrho f}\vartheta(w_l - v_l)}{T}\vartheta_{,l} \geqslant 0.$$

This inequality is always satisfied if the equations of state

$$(4.14) \qquad \sigma_{lj} = \frac{\partial F_1}{\partial \varepsilon_{lj}}, \quad \sigma = \frac{\partial F_1}{\partial \theta}, \quad S_1 = -\frac{\partial F_1}{\partial T},$$

and one inequality

$$(4.15) \qquad D = F_l^{fs}(w_l - v_l) - \frac{q_l - \varrho c_{\varrho f}\vartheta(w_l - v_l)}{T}\vartheta_{,l} \geqslant 0,$$

are satisfied, the symbol D denotes the dissipated energy also referred to as a general
dissipation function. The first term refers to the internal sources of heat due to the fric-
tion between the liquid and the skeleton. The second term is the energy dissipated in
irreversible processes due to heat conduction and the transport by the free liquid. It is
known from the previous considerations that this second term satisfies the Clausius-Duhem
inequality and leads to the generalized law of heat conduction (4.5). Thus, to satisfy the
inequality, it must be assumed that

$$F_l^{fs}(w_l - v_l) \geqslant 0.$$

From the inequality just obtained we have the relation

$$F_i^{fs} = \sum_{i=1,3,5\ldots} b_i(w_l - v_l)^i, \quad b_i > 0.$$

Assuming linearity we retain the first term only and write

(4.16) $$F_i^{fs} = b(w_l - v_l), \quad b \geqslant 0.$$

Omitting the body forces and the inertia forces in Eq. (2.6) we arrive at the relation,

(4.17) $$F_i^{fs} = \sigma_{,l} = b(w_l - v_l),$$

therefore, we have obtained the familiar Darcy law of flow, the coefficient b being the Darcy constant.

5. State functions. The constitutive relations

The free energy is a scalar, therefore it must be a function of strain invariants and the temperature T. Since our objective is to develop a linear theory, we take only the first two strain invariants of the skeleton:

$$I_1 = \varepsilon_{kk} = \varepsilon, \quad I_2 = \varepsilon_{ij}\varepsilon_{ij}$$

and write

(5.1) $$F_1 = F_1(I_1, I_2, \theta, T).$$

Let us expand the free energy in Taylor's series in the neighbourhood of the natural state in kinematic variables, retaining terms not higher than the quadratic ones:

(5.2) $$F_1(I_1, I_2, \theta, T) = F_1(0, 0, 0, T) + \frac{\partial F_1(0, 0, 0, T)}{\partial I_1} I_1 + \frac{\partial F_1(0, 0, 0, T)}{\partial I_2} I_2$$
$$+ \frac{\partial F_1(0, 0, 0, T)}{\partial \theta}\theta + \frac{\partial^2 F_1(0, 0, 0, T)}{\partial I_1 \partial\theta} I_1\theta + \frac{1}{2}\frac{\partial^2 F_1(0, 0, 0, T)}{\partial I_1^2} I_1^2$$
$$+ \frac{1}{2}\frac{\partial^2 F_1(0, 0, 0, T)}{\partial\theta^2}\theta^2 + \ldots.$$

From the geometrical interpretation of the expansion of the function F_1 in Taylor's series there follows a mechanical interpretation of its four derivatives:

$$K = \frac{\partial^2 F_1(0, 0, 0, T)}{\partial I_1^2} \quad \text{is the bulk modulus of the skeleton,}$$

(5.3)

$$L = \frac{\partial^2 F_1(0, 0, 0, T)}{\partial I_1 \partial\theta} \quad \text{the coefficient of coupling between the expansion of the liquid and that of the skeleton,}$$

$$M = \frac{\partial^2 F_1(0, 0, 0, T)}{\partial\theta^2} \quad \text{the bulk modulus of the liquid,}$$

$$N = \frac{\partial F_1(0, 0, 0, T)}{\partial I_2} \quad \text{the shear modulus of the skeleton.}$$

Taking into consideration Eqs. (4.14) we can write out the stress-strain relations as

(5.4) $$\sigma_{ij} = 2N\varepsilon_{ij} + \left(K\varepsilon + L\theta + \frac{\partial F_1}{\partial I_1}\right)\delta_{ij}, \quad \sigma = M\theta + L\varepsilon + \frac{\partial F_1}{\partial\theta}.$$

Let us now explain the meaning of the terms $\partial F_1/\partial I_1$ and $\partial F_1/\partial \theta$. We assume that it is only non-mechanical energy that is supplied to the medium. Thus $\sigma_{ij} = \sigma = 0$ and the state of strain denoted by the prime is produced by the temperature field alone, therefore the relations (5.4) are reduced to the form

$$\varepsilon' = -\frac{3}{P}\frac{\partial F_1}{\partial I_1} + \frac{3L}{PM}\frac{\partial F_1}{\partial \theta} = 3\alpha_T \vartheta,$$

(5.5)

$$\theta' = \frac{3L}{PM}\frac{\partial F_1}{\partial I_1} - \frac{2N+3K}{PM}\frac{\partial F_1}{\partial \theta} = 3\bar{\alpha}_T \vartheta, \quad P = 2N+3\left(K - \frac{L^2}{M}\right),$$

where α_T and $\bar{\alpha}_T$ are the coefficients of linear thermal expansion of the skeleton and the liquid, respectively. On solving Eqs. (5.5) for the derivatives of the function F_1, we find

(5.6)
$$-\frac{\partial F_1}{\partial I_1} = \gamma_T \vartheta, \quad -\frac{\partial F_1}{\partial \theta} = \bar{\gamma}_T \vartheta,$$

where

$$\gamma_T = (2N+3K)\alpha_T + 3L\bar{\alpha}_T, \quad \bar{\gamma}_T = 3(L\alpha_T + M\bar{\alpha}_T).$$

The last unknown term of Taylor's expansion of the free energy is $F_1(0, 0, 0, T)$, which can be found from the relation, known from thermodynamics [4], between the specific heat and the entropy

(5.7)
$$c_\Omega = c_{\varepsilon,\theta} = T\left(\frac{\partial S}{\partial T}\right)_{\varepsilon,\theta} = -T\left(\frac{\partial^2 F_1}{\partial T^2}\right)_{\varepsilon,\theta},$$

where c_Ω is a sum of specific heats of the two-component medium per unit volume, that is,

$$c_\Omega = c_{\Omega s} + c_{\Omega c}.$$

By integrating twice Eq. (5.7) with respect to the temperature, we obtain $F_1(0, 0, 0, T)$

(5.8)
$$F_1(0, 0, 0, T) = -\int_{T_0}^{T} dT \int_{T_0}^{T} \frac{c_{\Omega s}}{T} dT - \int_{T_0}^{T} dT \int_{T_0}^{T} \frac{c_{\Omega c}}{T} dT.$$

We have now interpreted all the coefficients of the Taylor expansion of the free energy and the state functions introduced can be written in an explicit manner:

(5.9)
$$F_1 = N\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}K\varepsilon^2 + L\varepsilon\theta + \frac{1}{2}M\theta^2 - \gamma_T\vartheta\varepsilon - \bar{\gamma}_T\vartheta\theta - \int_{T_0}^{T} dT \int_{T_0}^{T} \frac{c_{\Omega s}}{T} dT$$
$$- \int_{T_0}^{T} dT \int_{T_0}^{T} \frac{c_{\Omega c}}{T} dT,$$

(5.10)
$$S_1 = -\frac{\partial F_1}{\partial T} = \gamma_T\varepsilon + \bar{\gamma}_T\theta + c_\Omega \ln\frac{T}{T_0},$$

(5.11)
$$U_1 = F_1 + TS_1 = N\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}K\varepsilon^2 + L\varepsilon\theta + \frac{1}{2}M\theta^2 + (\gamma_T\varepsilon + \bar{\gamma}_T\theta)T_0$$
$$- \int_{T_0}^{T} dT \int_{T_0}^{T} \frac{c_{\Omega s}}{T} dT - \int_{T_0}^{T} dT \int_{T_0}^{T} \frac{c_{\Omega c}}{T} dT + Tc_\Omega \ln\frac{T}{T_0}.$$

These formulae are valid if it is assumed that the physical parameters are material constants independent of the temperature.

Making use of Eqs. (5.6) we rewrite the physical relations (5.4) in which all the constants have already been interpreted in the form

$$\text{(5.12)} \qquad \sigma_{ij} = 2N\varepsilon_{ij} + (K\varepsilon + L\theta - \gamma_T \vartheta)\,\delta_{ij},$$
$$\sigma = L\varepsilon + M\theta - \bar{\gamma}_T \vartheta.$$

Having in view the derivation of a heat equation, we introduce additionally the free enthalpy which is defined by the equation

$$\text{(5.13)} \qquad G_1 = F_1 - \sigma_{ij}\varepsilon_{ij} - \sigma\theta.$$

The free enthalpy will be determined as function of the stress. To this end, the physical relations (5.12) must be solved for the strains and the results must be substituted in Eqs. (5.9) and (5.13).

We shall need the second derivative of the free enthalpy with reference to the temperature with constant pressure. After obvious transformations we have

$$\text{(5.14)} \qquad G_1 = -\frac{1}{4N}\sigma_{ij}\sigma_{ij} + \left[\frac{3}{4N} - \frac{1}{2M} - \frac{3}{2P}\left(1 - \frac{L}{M}\right)^2\right]\sigma^2 - 3(\alpha_T + \bar{\alpha}_T)\vartheta\sigma$$
$$- \frac{3}{2}(\gamma_T \alpha_T + \bar{\gamma}_T \bar{\alpha}_T)\vartheta^2 - \int_{T_0}^{T} dT \int_{T_0}^{T} \frac{c_{\Omega s}}{T}\,dT - \int_{T_0}^{T} dT \int_{T_0}^{T} \frac{c_{\Omega c}}{T}\,dT,$$
$$P = 2N + 3\left(K - \frac{L^2}{M}\right).$$

We shall use the following equation, known from thermodynamics [4], between the specific heat with constant pressure and the entropy

$$c_p = c_\sigma = T\left(\frac{\partial S_1}{\partial T}\right)_\sigma = -T\left(\frac{\partial^2 G_1}{\partial T^2}\right)_\sigma.$$

After some obvious calculation we find the difference between the specific heat with constant volume and constant pressure for the skeleton and the liquid, respectively,

$$\text{(5.15)} \qquad \begin{aligned} c_{\sigma s} - c_{\Omega s} &= 3\gamma_T \alpha_T T, \\ c_{\sigma c} - c_{\Omega c} &= 3\bar{\gamma}_T \bar{\alpha}_T T. \end{aligned}$$

6. The heat equation

To derive the heat equation let us calculate the material derivative of the internal energy (5.11)

$$\text{(6.1)} \qquad \dot{U}_1 = [\sigma_{ij}\dot{\varepsilon}_{ij} + \sigma\dot{\theta}] + [(\gamma_T \dot{\varepsilon} + \bar{\gamma}_T \dot{\theta})T + c_\Omega \dot{T}],$$

the physical relations (5.12) having been taken into account.

The second term is the material derivative of the entropy multiplied by the absolute temperature, that is,

$$T\dot{S}_1 = (\gamma_T\dot{\varepsilon}+\bar{\gamma}_T\dot{\theta})T+c_\Omega\dot{T}.$$

On setting equal the expressions (6.1) and (3.6) for the material derivative of the internal energy, we obtain

(6.2) $\quad T\dot{S}_1 = (\gamma_T\dot{\varepsilon}+\bar{\gamma}_T\dot{\theta})T+c_\Omega\dot{T} = F_l^{fs}(w_l-v_l)-[q_{l,l}-\bar{\varrho}c_{\varrho f}\vartheta_{,l}(w_l-v_l)].$

Making use of the law of heat conduction (4.5), the law of liquid flow (4.17), and substituting the difference of specific heats (5.15), we obtain the requested equation of heat conduction

(6.3) $\qquad\qquad \lambda\nabla^2\vartheta+W_0 = c_\Omega\dot{\vartheta}+\dfrac{c_{\sigma s}-c_{\Omega s}}{3\alpha_T}\dot{\varepsilon}+\dfrac{c_{\sigma c}-c_{\Omega c}}{3\bar{\alpha}_T}\dot{\theta},$

where λ denotes the coefficient of heat conduction which is expressed by Eqs. (4.6), and the source function W_0 includes the heat produced as a result of the flow and, possibly, other sources of heat W

(6.4) $\qquad\qquad W_0 = b(w_l-v_l)(w_l-v_l)+W.$

7. The reduced equations

The unknown quantities in our considerations were:
1) the three density functions $\varrho, \bar{\varrho}, \bar{\varrho}_w$,
2) the three coordinates of the displacement vector of the skeleton, (u_i),
3) the three coordinates of the displacement vector of the liquid, (U_i),
4) the six coordinates of the stress tensor of the skeleton σ_{ij},
5) the stress transmitted by the liquid, σ,
6) the relative temperature, ϑ.

The total number of the unknowns is, therefore, 17. To find them, we have 17 equations:
1) the three equations of continuity of mass (2.2), (2.3), (2.4),
2) the six equations of motion (2.5), (2.6),
3) the seven physical relations (5.12),
4) the heat equation (6.3).

The number of equations can be reduced if the physical relations are substituted into the equations of motion and the familiar geometrical relations for small strains are used. As a result of that, the following reduced equations are found:

$$N\nabla^2 u_i+(N+K)\varepsilon_{,i}+L\theta_{,i}+(\varrho-\bar{\varrho})X_i = \gamma_T\vartheta_{,i}+(\varrho-\bar{\varrho})\frac{dv}{dt}-b(w_l-v_l),$$

$$L\varepsilon_{,i}+M\theta_{,i}+\bar{\varrho}X_i = \bar{\gamma}_T\vartheta_{,i}+\frac{dw_l}{dt}+b(w_l-v_l).$$

The set of equations just written is completed by the heat equation (6.3) and the mass continuity equations.

In the case of small displacements it may be assumed that the material derivatives of the displacement rate are equal to the local derivatives. This simplification enables us to reject the heat sources which are a function of the squares of the differences between the velocity of the liquid and that of the skeleton.

If we assume, in addition, that the variations in the density of the skeleton are negligibly small, the set of linearized reduced equations takes the form

(7.2)
$$N\nabla^2 u + (N+M)\varepsilon_{,i} + L\theta_{,i} + (\varrho - \bar{\varrho})X_i = \gamma_T \vartheta_{,i} + (\varrho - \bar{\varrho})\ddot{u}_i - b(\dot{U}_i - \dot{u}_i),$$
$$L\varepsilon_{,i} + M\theta_{,i} + \bar{\varrho}X_i = \bar{\gamma}_T \vartheta_{,i} + \bar{\varrho}\ddot{U}_i + b(\dot{U}_i - \dot{u}_i),$$
$$\lambda\nabla^2 \vartheta = c_\Omega \dot{\vartheta} + \frac{c_{\sigma s} - c_{\Omega s}}{3\alpha_T}\dot{\varepsilon} + \frac{c_{\sigma c} - c_{\Omega c}}{3\bar{\alpha}_T}\dot{\theta}.$$

These equations are a closed set of seven linear equations with double coupling.

8 Concluding remarks

Our objective was to study fundamental relations, therefore, our considerations were limited to the simplest medium possible, that is, an elastic skeleton with pores filled with a liquid.

It was assumed that the skeleton is linearly elastic, the medium isotropic, the porosity uniform, the temperatures of the liquid and the skeleton locally the same, etc.

There are no obstacles for the formal generalization of our argument to the case of an anisotropic skeleton, for instance, or the case of parameters constituting functions of the temperature, or that of a viscoelastic skeleton. The argument will not be changed in any of these cases.

In the well-known paper of R. L. SCHIFFMAN [8], the assumptions of thermal isotropy, linearity of the thermodynamic relations and equality of the temperatures of the liquid and the skeleton lead to the following heat equation:

(8.1)
$$\frac{k_1}{\tau_0}\nabla^2 \vartheta + \frac{\varrho_0}{T_0} W_0 = \frac{c_e \varrho_0}{T_0}\frac{\partial \vartheta}{\partial t} + c_1 \frac{\partial \varepsilon}{\partial t} + c_2 \frac{\partial \theta}{\partial t}.$$

No sources of heat due to liquid flow are involved (W_0 denotes other sources of heat). The argument was based on Onsager's principle and the author did not explain in a correct manner the coefficients involved in the heat equation, confining himself to the consideration of quasi-stationary states. The present considerations show that such a limitation makes the considerations incorrect.

Three years later C. PECKER and H. DERESIEWICZ published in Acta Mechanica a paper concerning linear thermoconsolidation [2]. The case of different temperatures of the skeleton and the liquid was considered. Some essential errors were committed. First, the influence of the heat produced by mass transport was disregarded. Second, a local equation of conservation of energy was used, which is erroneous, because the heat due to the irreversibility of the liquid flow (an obvious source of heat) was disregarded. The physical equations were written assuming linearity and symmetry. As a result of this,

there is no thermomechanical interpretation of the coefficients introduced. In so far as the heat equations are concerned,

$$(8.2) \quad \begin{aligned} k_s \nabla^2 \dot{\vartheta}_s &= F_{11} \dot{\vartheta}_s + F_{12} \dot{\vartheta}_f + K(T_s - T_f) + R_{11} T_0 \dot{\varepsilon} + R_{21} T_0 \dot{\theta}, \\ k_f \nabla^2 \vartheta_s &= F_{21} \dot{\vartheta}_s + F_{22} \dot{\vartheta}_f - K(T_s - T_f) + R_{12} T_0 \dot{\varepsilon} + R_{22} T_0 \dot{\theta}, \end{aligned}$$

where the indicants s and f refer to the skeleton and the liquid, respectively; the thermal sources due to liquid flow have been rejected as in Schiffman's paper. There is no definition of local entropy and it is not said whether it is a sum of entropies of the liquid and the skeleton at every point or a function of an intermediate temperature.

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