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Some aspects of the mathematical modelling of long nonlinear waves

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THIS PAPER begins with a brief review of the notions of far fields and long waves, and indicates why the study of equations of KdV and KdVB type are important to the modelling of unidirectional long nonlinear waves. An asymptotic solution is then developed for the shock wave solution for the KdVB equation that applies when dissipative effects predominate over dispersive effects. The sensitivity of this solution to the matching condition used at the origin is demonstrated. Some numerical experiments are then described concerning the propagation of one or more KdV solitons in the presence of noise. It appears from these that noise retards the speed of propagation of the solutions and that the KdV equation introduces correlations into the noise that were not present initially, thereby modifying the noise spectrum. Finally, a new time regularised long wave (TRLW) equation is proposed which is conservative, possesses travelling wave solutions and conservation laws and is capable of characterising bidirectional wave propagation. It is shown that when the TRLW equation is modified to include dissipation, as in Burgers'equation, a change of parameter in the KdVB shock wave asymptotic solution yields the new shock solution.

Praca rozpoczyna się krótkim przeglądem oznaczeń pól dalekiego oddziaływania i długich fal. Wskazano na znaczenie skalarnych fal typu KdV (Korteweg-de Vries) oraz KdVB (Korteweg-de Vries-Burgers) przy modelowaniu jednowymiarowych długich nieliniowych fal. Opracowano następnie rozwiązanie asymptotyczne dla fali uderzeniowej opisanej przez równanie KdVB słuszne w przypadku, kiedy efekty dysypacji dominują nad efektami dyspersji. Zbadano stopień wrażliwości tego rozwiązania w zależności od warunków dopasowania rozwiązania na początku. Opisane są dalej pewne eksperymenty numeryczne dctyczące propagacji jednego lub więcej rozwiązań KdV przy występowaniu szumu. Z analizy tej wynika, że zakłócenia zmniejszają prędkość propagacji fal oraz że równanie KdV wprowadza nieistniejącą początkowo poprawkę na szum, zmieniając tym spektrum szumu. Wreszcie, zaproponowano nowe równanie (TRLW) opisujące wygładzone przez czas długie fale. Równanie to jest zachowawcze, posiada rozwiązania w postaci biegnącej fali i jest w stanie scharakteryzować propagacje dwuwymiarowych fal. Wykazano, że po dokonaniu modyfikacji równania TRLW w celu wprowadzenia dysypacji, tak jak w równaniu Burgersa, zmiana parametrów w rozwiązaniu asymptotycznym fali uderzeniowej dla równania KdVB prowadzi do nowego rozwiązania z falą uderzeniową.

Работа начинается кратким обозрением обозначений полей далекого взаимодействия и длинных волн. Указано на значение скалярных волн типа КдВ (Кортевег-де Вриз), а также КдВБ (Кортевег-де Вриз-Бургерс), при моделировании одномерных длинных нелинейных волн. Затем разработано асимптотическое решение для ударной волны, описанной уравнением КдВБ, справедливо в случае, когда эффекты диссипации преобладают над эффектами дисперсии. Исследована степень чувствительности этого решения в зависимости от условий согласования решения в начальный момент. Описаны далее некоторые численные эксперименты, касающиеся распространения одного, или большего количества решений КдВ при выступании шума. Из этого анализа следует, что возмущения уменьшают скорость распространения волн и что уравнение КдВ вводит несуществующую вначале поправку на шум, изменяя таким образом спектр шума. Наконец, предположено новое уравнение (ТРЛВ), описывающее выглаженные временем длинные волны. Это уравнение консервативно, имеет решения в виде бегущей волны и в состоянии описать распространение двумерных волн. Показано, что если провести модификацию уравнения ТРЛВ с целью введения диссипации, так как в уравнении Бургерса, то изменение параметров в асимптотическом решении ударной волны для уравнения КдВБ приводит к новому решению с ударной волной.

1. Long waves and far fields

THE MODELLING of long nonlinear waves in a continuum is of considerable importance and it arises in connection with topics as diverse as the study of gravity waves in fluids, plasma waves, anharmonic lattice waves, longitudinal dispersive waves in elastic rods and also in a variety of other circumstances. Accounts of the way in which these waves occur and of many of their properties, together with extensive bibliographies, are to be found in the review papers by JEFFREY and KAKUTANI [1] and SCOTT, CHU and MCLAUGHLIN [2].

These seemingly different topics have as unifying features the facts that they each involve systems of partial differential equations, their long wave behaviour is determined as the result of a perturbation argument, and yet this behaviour of a system is in each case governed only by a scalar quasi-linear partial differential equation. Depending on the physical attributes of the problem involved, and to some extent on the mathematical modelling philosophy that is adopted, so will depend the precise form of the scalar equation that occurs. Aside from the scalar equation derived from the Boussinesq equations that was used first by PEREGRINE [3] to describe an undular bore, and studied later in great detail by BENJAMIN, BONA and MAHONY [4], the prototype equation that usually results is a variant of the Korteweg–de Vries–Burgers' (KdVB) equation

$$u_t + uu_x - \nu u_{xx} + \mu u_{xxx} = 0, \quad \nu \geqslant 0, \quad \mu \geqslant 0. \tag{1.1}$$

This equation characterises unidirectional wave propagation since it contains only a first order time derivative, and it is so called because when $\mu = 0$ it reduces to Burgers' equation, and when $\nu = 0$ to the Korteweg–de Vries (KdV) equation. The dissipative effect in the KdVB equation is provided by the term $-\nu u_{xx}$, in which the condition $\nu > 0$ in Eq. (1.1) is required to ensure that "energy" is dissipated and not added. The dispersive effect is provided by the term μu_{xxx}, and for convenience we take $\mu > 0$ in Eq. (1.1) to conform with the convention usually adopted when working with the KdV equation.

Let us now outline the reason for the relevance of an equation such as Eq. (1.1) to the study of long nonlinear waves governed by a system of n quasi-linear equations with order not less than two. We consider a general system

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + \left[\sum_{\beta=0}^{s}\prod_{\alpha=1}^{p}\left\{H_\alpha^\beta\frac{\partial}{\partial t} + K_\alpha^\beta\frac{\partial}{\partial x}\right\}\right]U = 0, \quad p \geqslant 2, \tag{1.2}$$

where U is an n element vector with the components $u_1, u_2, \ldots, u_n$ and A, H_α^β, K_α^β are $n\times n$ matrices with arbitrarily differentiable elements that depend only on U. Then, following a method developed by TANIUTI and WEI [5], it has been shown [1] how, when the system (1.2) satisfies certain conditions, a general reductive perturbation method may be developed to find the behaviour of small but finite perturbations relative to some constant solution U_0 of the system.

In brief, we shall suppose that the first order system obtained from Eq. (1.2) by neglecting the terms in square brackets is hyperbolic (see JEFFREY [6]), that $U \to U_0$ as $x \to -\infty$, and that U may be expanded in the form

$$U = \sum_{j=0}^{\infty} \varepsilon^j U_j, \tag{1.3}$$

with corresponding expansions being valid for any other functions of U that arise. Then, for a disturbance wave associated with any one of the n real eigenvalues $\lambda^{(\alpha)}$ of A, it can be shown [1] that the first order perturbation U_1 has the form

$$U_1 = r_0^{(\alpha)} u, \tag{1.4}$$

where $r_0^{(\alpha)} = r^{(\alpha)}(U_0)$ is a suitably normalised eigenvector of the matrix $A(U_0)$ corresponding to the eigenvalue $\lambda_0^{(\alpha)} = \lambda^{(\alpha)}(U_0)$, and that the scalar u satisfies an equation similar to Eq. (1.1). When deriving Eq. (1.4) a coordinate-stretching is involved of the form

$$\xi = \varepsilon^a(x - \lambda_0^{(\alpha)} t), \quad \tau = \varepsilon^{a+1} t, \quad a = 1/(p-1), \tag{1.5}$$

which automatically directs attention to the stretched time $\varepsilon^{a+1} t$, and hence to large times and correspondingly large distances from the origin. For this reason such solutions are often called far fields. Thus, in general, the system (1.2) will have associated with it n distinct far fields corresponding to each of the n real eigenvalues of A, and each will be governed by a scalar equation similar to Eq. (1.1). These equations describe long waves in the sense that the wavelengths involved are large in relation to the magnitude of the class of perturbations that is to be considered.

This situation is well illustrated by the KdV equation governing long waves in shallow water [1, 7] which can be written (see Fig. 1)

$$v_T + \sqrt{gh_0}\left[1 + \frac{3}{2}(v/h_0)\right] v_X + \frac{1}{6}\sqrt{gh_0}\, h_0^2 v_{XXX} = 0, \tag{1.6}$$

with h_0 the equilibrium depth, $v(X, T)$ the local surface elevation relative to the equilibrium level, g the acceleration due to gravity, X the horizontal distance and T the time. If the maximum amplitude and wavelength of the disturbances are δ and Λ, respectively, then, setting

$$\eta = \delta/h_0 \quad \text{and} \quad \mu = \frac{1}{6}(h_0/\Lambda)^2,$$

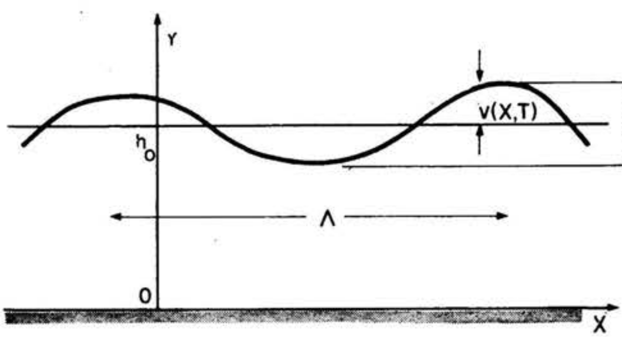

FIG. 1. Long waves in shallow water.

Eq. (1.6) will hold when η, μ are small. In terms of the non-dimensional variables

$$x = X/\Lambda, \quad t = T\sqrt{gh_0}/\Lambda, \quad u = \frac{3}{2}v/(\eta h_0),$$

the KdV equation (1.6) becomes

$$u_t + u_x + \eta u u_x + \mu u_{xxx} = 0. \tag{1.7}$$

If desired, this may be expressed in terms of the same canonical form of the KdV equation that is implied by Eq. (1.1) with $\nu = 0$ by making the variable changes $x \to x - t$ and $u \to u/\eta$.

For further discussion of long waves in the context of far fields and for an account of the relevance to their study of the so-called nonlinear Schrödinger equation

$$iu_t + \frac{1}{2}u_{xx} + a|u|^2 u = 0, \tag{1.8}$$

we refer to the papers by TANIUTI [8], JEFFREY [9] and DAVEY [10]. We mention here only the fact that the far fields of the purely hyperbolic system derived from Eq. (1.2) by neglecting the terms in square brackets are merely simple waves. As neither dispersive nor dissipative effects act to prevent the steepening of waves in the purely hyperbolic case, the simple wave far fields will only exist until such time as shocks form.

It is important to recognise that while the hyperbolic equation derived from Eq. (1.1) by neglecting the dissipative and dispersive terms has simple wave solutions, it has no travelling wave solutions. Such solutions are, however, possessed by Burgers' equation, the KdV equation and the KdVB equation. This comes about because of a balance that occurs between the steepening effect due to the nonlinear term uu_x, and the smoothing effect produced by dissipation and dispersion. These travelling wave solutions are of considerable importance mathematically, and different aspects of them will concern us throughout the remainder of this paper.

2. Travelling wave solutions

In two dimensions, travelling wave solutions have the form

$$u(x, t) = \tilde{u}(\zeta), \quad \zeta = x - \lambda t, \quad \lambda = \text{const}, \tag{2.1}$$

and they must satisfy some appropriate boundary conditions at infinity. In general, these will determine the permissible range of vlues of λ. In the case of Burgers' equation and the KdV equation, for both of which all the derivatives of the solutions tend to zero as $|x| \to \infty$, these equations have the following well-known solutions [1, 7] satisfying the stated boundary conditions.

Burgers' shock wave ($\mu = 0$)

$$\tilde{u}(\zeta) = \frac{1}{2}(u_\infty^- + u_\infty^+) - \frac{1}{2}(u_\infty^- - u_\infty^+)\tanh[(u_\infty^- - u_\infty^+)\zeta/4\nu], \tag{2.2}$$

which satisfies the boundary conditions

$$\lim_{|\zeta|\to\infty} \tilde{u}|\zeta| = u_\infty^\pm \quad \text{with} \quad u_\infty^- > u_\infty^+,$$

and has $\lambda = \frac{1}{2}(u_\infty^- + u_\infty^+)$ (see Fig. 2.).

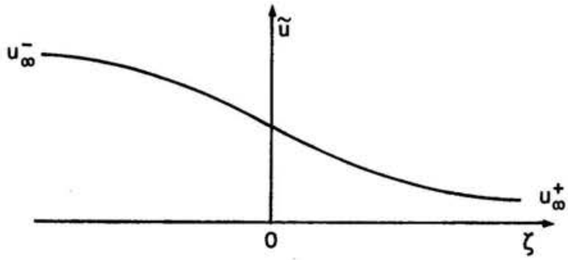

FIG. 2. Burgers' shock wave.

KdV solitary wave ($\nu = 0$)

(2.3) $$\tilde{u}(\zeta) = u_\infty + a\,\mathrm{sech}^2(\zeta \sqrt{a/12\mu}),$$

which satisfies the boundary conditions

$$\lim_{|\zeta|\to\infty} \tilde{u}(\zeta) = u_\infty \quad \text{with} \quad u_\infty \geqslant 0,$$

and has $\lambda = u_\infty + a/3$, (See Fig. 3).

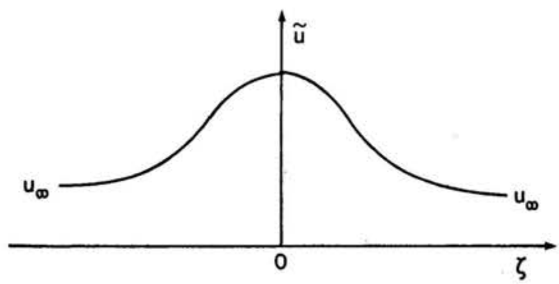

FIG. 3. KdV solitary wave.

The Burgers' shock wave, as the travelling wave (2.2) is called, is seen to propagate with a speed $\lambda = (u_\infty^- + u_\infty^+)/2$ that is uniquely determined by the boundary conditions, but to be invariant with respect to an arbitrary fixed spatial translation. This last result follows because if $\tilde{u}(\zeta)$ is a solution, then so is $\tilde{u}(\zeta+k)$ with $k = \text{const}$. More generally, all solutions $u(x, t)$ of Burgers' equation are invariant with respect to a Galilean transformation. This may be seen by observing that if $u(x, t)$ is a solution, then $u(x+st, t)-s$ with $s = \text{const}$ is also a solution.

The KdV solitary wave, or soliton, is a pulse shaped wave that, relative to the constant value u_∞ at infinity, tends to zero together with all its derivatives as $|\zeta| \to \infty$. Its speed of propagation relative to u_∞ is proportional to the amplitude a, and its width is inversely proportional to the square root of the amplitude. In this travelling wave the speed is not determined uniquely by the boundary conditions, but depends only on the amplitude $a > 0$. Like Burgers' shock wave, the KdV solitary wave is also invariant with respect to a Galilean transformation. It is important to recognise that all solitary waves are similar in the sense that, relative to their respective values at infinity, a translation and scaling of amplitude of such a wave will transform it into any other one.

While these analytical solutions exist for the travelling wave solutions for the Burgers' and KdV equations, no comparable analytical solution exists for the KdVB equation (1.1), and yet this equation is important when modelling waves with a combination of dissipative and dispersive effects. Numerical calculations for Eq. (1.1) carried out by GRAD and HU [11] show that when $\nu^2 < 4\mu$ dispersive effects are the most significant ones and that the solution represents an oscillatory shock wave. For the case $\nu^2 \ll 4\mu$ JOHNSON [12] matched a perturbed solitary wave with a cnoidal wave to obtain an asymptotic solution that exhibited oscillation. However, although the phase plane analysis described by JEFFREY and KAKUTANI [1] and the numerical results of GRAD and HU [11] indicate that when dissipative effects predominate and $\nu^2 > 4\mu$ the solution will behave like a Burgers' shock wave, no detailed analysis of this situation has yet been made. Accordingly, we now outline the details of an asymptotic solution for such a KdVB travelling wave, or shock wave, when $\nu^2 > 4\mu$.

Let us consider a travelling wave solution $\tilde{u}(\zeta)$ to the KdVB equation (1.1) with the boundary conditions $\tilde{u}(-\infty) = u_\infty^-$ and $\tilde{u}(\infty) = u_\infty^+$ where, as before, $u_\infty^- > u_\infty^+$, $\zeta = x - \lambda t$. Then $\tilde{u}$ must satisfy the equation

(2.4) $$-\lambda \frac{d\tilde{u}}{d\zeta} + \tilde{u}\frac{d\tilde{u}}{d\zeta} - \nu\frac{d^2\tilde{u}}{d\zeta^2} + \mu\frac{d^3\tilde{u}}{d\zeta^3} = 0.$$

Integrating and using the boundary conditions and the vanishing of derivatives at infinity shows that $\lambda = (u_\infty^- + u_\infty^+)/2$ and so

(2.5) $$\mu\frac{d^2\tilde{u}}{d\zeta^2} - \nu\frac{d\tilde{u}}{d\zeta} + \frac{1}{2}\tilde{u}^2 - \frac{1}{2}(u_\infty^- + u_\infty^+)\tilde{u} + \frac{1}{2}u_\infty^- u_\infty^+ = 0.$$

Making the variabe changes

(2.6) $$v = \frac{\tilde{u} - u_\infty^+}{u_\infty^- - u_\infty^+}, \quad \xi = \frac{(u_\infty^- - u_\infty^+)\zeta}{2\nu} \quad \text{and} \quad \varepsilon = \frac{\mu(u_\infty^- - u_\infty^+)}{2\nu^2}$$

reduces Eq. (2.5) to

(2.7) $$\varepsilon\frac{d^2v}{d\xi^2} - \frac{dv}{d\xi} + v^2 - v = 0$$

with the boundary conditions

(2.8) $$v(-\infty) = 1 \quad \text{and} \quad v(+\infty) = 0.$$

To obtain an asymptotic solution of Eq. (2.7) in the form

(2.9) $$v(\xi) = v_1(\xi) + \varepsilon v_2(\xi) + \varepsilon^2 v_3(\xi) + \ldots,$$

it is necessary to match the asymptotic solution to some feature of the true solution that is important. The natural choice is to match to an appropriate order of ε the value of v

at the point where the curvature of the KdVB shock wave changes sign. Because the KdVB shock wave, like the Burgers' shock wave, is invariant with respect to an arbitrary fixed translation, the origin of ξ may be chosen to be at this point. To determine $v(0)$, and hence $v_1(0)$, $v_2(0)$, ..., we proceed as follows.

Introducing the (v, s)-phase plane with $s = dv/d\xi$ allows Eq. (2.7) to be written as the system

$$\varepsilon \frac{ds}{d\xi} = s - v^2 + v, \tag{2.10}$$

$$\frac{dv}{d\xi} = s. \tag{2.11}$$

This system has critical points at the origin (0, 0) and at (1, 0), with the origin representing a stable node and (1, 0) a saddle point. As these two points correspond to the two boundary conditions (2.8) to be satisfied by the solution to Eq. (2.7) we conclude that the solution corresponding to the trajectory joining these two critical points must be unique. Furthermore, the point P on this trajectory at which $ds/dv = 0$ will correspond to the point where the curvature of the KdVB shock wave changes sign.

To find $v(0)$ we now seek an expansion of s in the form

$$s(v) = f_1(v) + \varepsilon f_2(v) + \varepsilon^2 f_3(v) + \ \dots . \tag{2.12}$$

Using this in the expression for ds/dv obtained by dividing Eq. (2.10) by Eq. (2.11), and equating terms with corresponding powers of ε, shows that the functions f_i are defined recursively and that to first order in ε

$$s = (v^2 - v) + \varepsilon(2v^2 - 3v^2 + v) + \ \dots . \tag{2.13}$$

Again, working to first order in ε, it follows that $ds/dv = 0$ when

$$v(0) = \frac{1}{2} + \frac{\varepsilon}{4}, \tag{2.14}$$

and comparison with Eq. (2.9) then gives as the conditions to be satisfied by $v_1(\xi)$ and $v_2(\xi)$,

$$v_1(0) = \frac{1}{2} \quad \text{and} \quad v_2(0) = \frac{1}{4}. \tag{2.15}$$

The substitution of Eq. (2.9) into Eq. (2.7) followed by a routine calculation, involving equating terms with corresponding powers of ε and integration using the conditions (2.15), finally leads to the result

$$\tilde{u}(\zeta) = \frac{u_\infty^- + u_\infty^+ e^\xi}{1+e^\xi} + \frac{\mu(u_\infty^- - u_\infty^+)^2 e^\xi}{2\nu^2(1-e^\xi)^2}\left[1 + \xi - 2\log\left(\frac{2e^\xi}{1+e^\xi}\right)\right] + 0(\varepsilon^2), \tag{2.16}$$

with

$$\xi = (u_\infty^- - u_\infty^+)\zeta/2\nu, \quad \varepsilon = (u_\infty^- - u_\infty^+)\mu/2\nu^2 \quad \text{and} \quad \nu^2 > 4\mu.$$

It is instructive to compare the results in Eqs. (2.2) and (2.16) and to observe that to the first order in ε the dispersion coefficient μ enters only as a linear factor in the second term. Indeed, using Eqs. (2.11), (2.13) and (2.14) to calculate $dv/d\xi$ when $\xi = 0$ gives

$$(2.17)\qquad \frac{dv}{d\xi} = -\frac{1}{4}+\frac{\varepsilon^2}{16}, \quad \text{or} \quad \frac{d\tilde{u}}{d\zeta} = \frac{(u_\infty^- - u_\infty^+)^2}{2\nu}\left(-\frac{1}{4}+\frac{\varepsilon^2}{16}\right).$$

This shows that the gradient of the KdVB shock wave at $\xi = 0$, or equivalently at $\zeta = 0$, is the same as that of Burgers' shock wave (2.2) to first order in ε.

Some indication of the sensitivity of this solution to the accuracy with which the matching of $v(0)$ is carried out may be obtained by modifying the conditions used in Eq. (2.15). If, instead of Eq. (2.15) we require $v_1(\xi)$ and $v_2(\xi)$ to satisfy the conditions

$$v_1(0) = \frac{1}{2}(1+\alpha) \quad \text{and} \quad v_2(0) = \frac{1}{4}(1+\beta),$$

where α and β are small, then corresponding to Eq. (2.16) we find the modified result

$$\tilde{u}_m(\zeta) = \frac{(1+\alpha)u_\infty^- + (1-\alpha)u_\infty^+ e^\xi}{(1+\alpha)+(1-\alpha)e^\xi} + \frac{\mu(u_\infty^- - u_\infty^+)^2 e^\xi}{2\nu^2(1+e^\xi)^2}\left[1+\beta+\xi-2\log\left(\frac{2e^\xi}{1+e^\xi}\right)\right] + 0\varepsilon^2).$$

Of the parameters α and β we see, as would be expected, that α is the more significant of the two in its effect on $\tilde{u}_m$. To interpret this quantitatively let us set $u_\infty^+ = 0$ and, working to an accuracy $0(\varepsilon)$, examine the ratio $k(\zeta) = \tilde{u}_m/\tilde{u}$, where

$$k(\zeta) = \frac{(1+\alpha)(1+e^\xi)}{(1+\alpha)+(1-\alpha)e^\xi}, \quad \text{and} \quad \xi = u_\infty^- \zeta/2\nu.$$

Then $k(-\infty) = 1$, but $k(+\infty) = (1+\alpha)/(1-\alpha) \doteqdot 1+2\alpha$, for small α.

Thus, although $\tilde{u}(\zeta) \to 0$ as $\zeta \to +\infty$, the ratio $\tilde{u}_m/\tilde{u} \to 1+2\alpha$. This shows that when the factor multiplying the true value of $v_1(0)$ is $1+\alpha$, instead of unity, this causes $\tilde{u}_m$ to exceed $\tilde{u}$ by a factor $1+2\alpha$ for large positive ζ. Indeed, this error is even significant close to the origin, for setting $\xi = 1$ in the expression for k, which is equivalent to setting $\zeta = 2\nu/u_\infty^-$, shows that $\tilde{u}_m/\tilde{u} = 1+2\alpha e/(1+e)$, so that already $\tilde{u}_m = (1+1.45\alpha)\tilde{u}$.

This analytical demonstration of the sensitivity of the KdVB shock solution to a perturbation at the origin reflects a similar result found computationally by G. I. BARENBLATT [13] for a certain initial valve problem for the KdV equation. Specifically he found, starting from a Burgers' shock wave as initial data, that a small perturbation at the origin caused a totally different solution to evolve from that which arises from the unperturbed initial data.

3. KdV solitions with noise

Interaction between KdV solitons was first studied in the early numerical work carried out by ZABUSKY and KRUSKAL [14], accounts of which are also to be found in references [1, 2]. Since then the understanding of the mechanism of the generation of solitons and the reason for their persistence despite repeated interactions has been advanced by the work of many authors. Notable amongst these are GARDNER, GREENE, KRUSKAL and MIURA [15], LAX [16], HIROTA [17] ZAKHAROV and SHABAT [18] and ABLOWITZ, KAUP, NEWELL and SEGUR [19]. In all of this work smooth initial data was assumed, and only as recently as 1976 in a paper by LAX [20] was initial data considered that comprised a random disturbance superimposed on an otherwise smooth function. However, the result reported by Lax did not relate to solitons but to a special class of periodic solutions of the KdV equation, and it was found by numerical computation that the periodicity of the solution appeared not to be disturbed by the superposed random disturbance.

Just as a long wave is a mathematical idealisation of a physical situation, so also is the assumption that its initial data is smooth. Thus, since there is much general interest in the way solitons interact, and also in their stability, it is appropriate that these situations should be examined when the initial data comprises one or more solitons on which at the initial time a gaussian random disturbance has been superimposed. Accordingly, in the remainder of this section, we describe the results of some numerical experiments designed to examine and quantify the time evolution of soliton solutions to the KdV equation that arise from initial data of this type. Solitons that tend to zero at infinity have been chosen for study ($u_\infty = 0$), and for convenience we henceforth refer to the random disturbance involved as noise.

The finite difference scheme employed by ZABUSKY and KRUSKAL [14] was used for the numerical integration, and the noise was generated by a gaussian random number subroutine designed to produce random numbers with zero mean and a specified standard deviation σ using the direct method suggested by BOX and MULLER [21]. A statistical examination of these random numbers confirmed that their mean and standard deviation had the desired properties to within the variability expected for the sample sizes of 350 numbers that were actually used. The effect of noise on a single soliton was found to be a progressive retardation, or delay, of the disturbed soliton relative to the undisturbed soliton as they advanced with increasing time. This delay δ as a function of time was found to increase as the standard deviation of the noise σ was increased, and the development of the delay is shown in Fig. 4 for different initial standard deviations. The

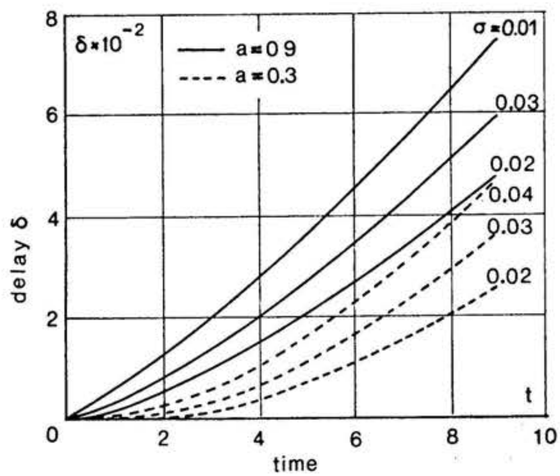

FIG. 4. Single soliton delay δ as a function of soliton amplitude a, initial standard deviation σ and time t.

magnitude of the delay at any given time was found by cross-correlating the soliton solution in the presence of noise with the corresponding steadily progressing smooth soliton solution. The shift of the peak of the cross-correlation function from the origin was interpreted as the spatial delay experienced by the noisy soliton at that time.

The equation to be solved was taken in the form

$$u_t + uu_x + \mu u_{xxx} = 0, \tag{3.1}$$

which in terms of the parameter $a > 0$ has the soliton solution that vanishes at infinity ($u_\infty = 0$)

$$u(x, t) = a\,\mathrm{sech}^2(kx - \omega t + \delta), \tag{3.2}$$

with $k = \frac{1}{2}\sqrt{(a/3\mu)}$, $\omega = ak/3$ and δ an arbitrary constant. In all of the calculations described in this section the non-dimensional length and time steps used in the integration were, respectively, $h = 0.01$ and $k = 0.0005$. These were chosen so that the stability condition for the finite difference scheme was well satisfied. After selecting a value of the amplitude a, gaussian random numbers drawn from a distribution with zero mean and the desired initial standard deviation $\sigma = 0.02$, 0.03 and 0.04 were added at space-like intervals h to the initial data derived from Eq. (3.2) by setting $t = \delta = 0$ and $\mu = 4.84 \times 10^{-4}$. The results corresponding to $a = 0.3$ are shown as dotted lines in Fig. 4 and those corresponding to $a = 0.9$ are shown as full lines.

As the speed of a soliton is proportional to its amplitude, this retardation must correspond to a progressive reduction in amplitude. Furthermore, as the KdV equation is a conservation equation with an infinite number of conservation laws [1, 2, 7, 14, 22], one of which corresponds to the conservation of energy, the retardation should be accompanied by an increase in the standard deviation of the noise. This was in fact observed, and throughout all calculations the energy invariant remained constant to within 0.3%. The invariant corresponding to the conservation of momentum remained constant to within 0.8%, thereby providing evidence both that the amplitude was changing slowly, and that the integration scheme was conserving the first two invariants satisfactorily in the presence of noise. Since the delays involved were all small, the actual reduction im amplitude of the solitons in the presene of noise was not readily detectable from computer drawn graphs made during the propagation period involved (non-dimensional time interval of length $t = 9$).

The effect of interaction between solitons in the presence of noise is shown in the redrawn computer graphs in Fig. 5. In these computations gaussian noise with zero mean and standard deviation $\sigma = 0.04$ was added to the initial data

$$u(x, 0) = a_2\,\mathrm{sech}^2(k_2 x - \delta_2) + a_1\,\mathrm{sech}^2(k_1 x - \delta_1), \tag{3.3}$$

at space-like intervals $h = 0.01$, where the notation of Eq. (3.2) was adopted with obvious modifications, and

$$a_2 = 0.9, \quad \delta_2 = 5 \quad \text{and} \quad a_1 = 0.3, \quad \delta_1 = 10. \tag{3.4}$$

The time evolution of the noisy solitons is shown as the irregular line in Fig. 5. For purposes of comparison, the noise free analytical solution [7, 17] for the evolution is shown

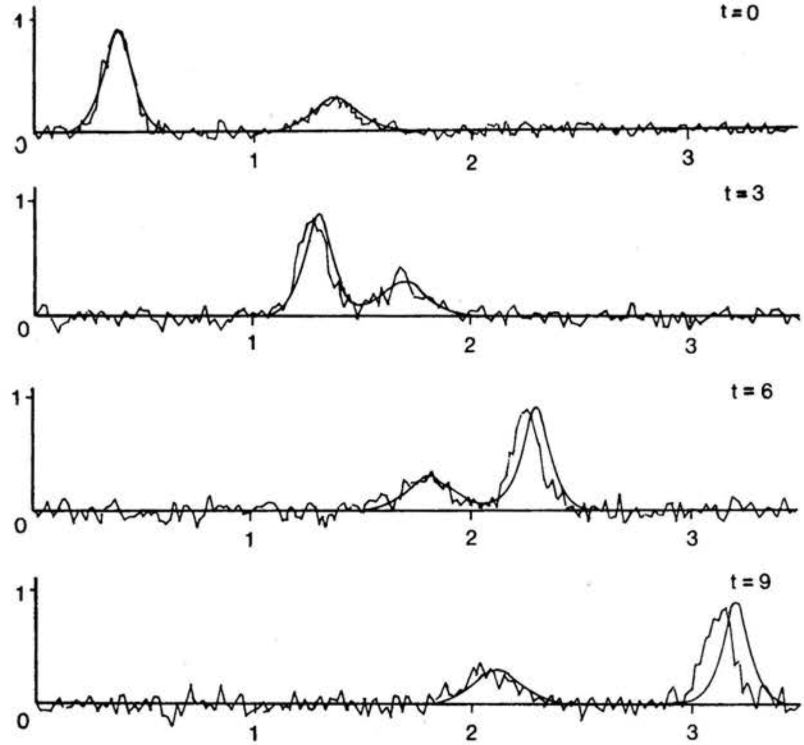

FIG. 5. Interaction of two solitons in the presence of noise with initial standard deviation $\sigma = 0.04$.

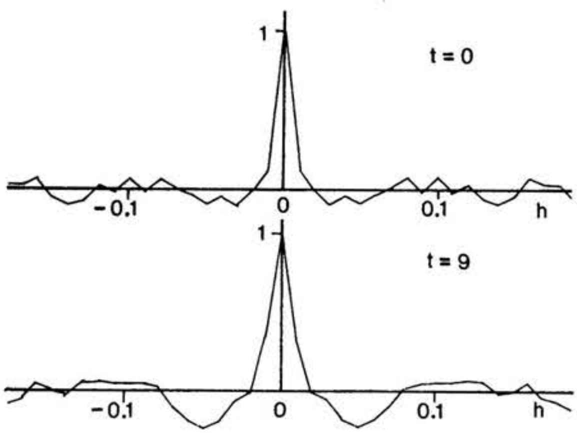

FIG. 6. Autocorrelation functions of the noise associated with the results of FIG. 2.

on the same graph as the smooth line. These graphical results should merely be interpreted as being illustrative, since for practical purposes, due to the graphs being mechanically plotted at small intervals, only alternate points have been graphed. This has led to an apparent exaggeration of the noise and to a certain lack of smoothness in the graph of the theoretical solution, though these effects were not, of course, present in the data actually analysed. Despite these limitations of the graphical display shown in Fig. 5 the soliton delay is seen quite clearly, as is the growth in the size of the noise. When examining these results it should be remembered that in addition to the delay caused by the noise, there is also the phase shift that always accompanies soliton interaction [1, 7, 17].

Figure 6 shows autocorrelation functions of the noise associated with the solitons illustrated in Fig. 5 at the initial time $t = 0$ and the final time $t = 9$. These exhibit a definite change which thereby indicates that correlations have been introduced into the noise as a result of its interaction with the KdV equation during this period. The initial noise may be regarded as an approximation to band limited white noise, and the change in the autocorrelation function shows that the noise spectrum has been modified. The noise at times subsequent to the initial time was estimated by matching the smooth solution to the noisy solution using cross-correlation and then subtracting the determinate part from the soliton solution in the presence of noise.

As these experiments were not related to any specific physical situation, no attempt was made to take account of any relaxation effects that might occur in connection with the noise. This would, of course, influence both the delay mechanism and the time dependence of the autocorrelation function, and through this the time dependence of the noise spectrum.

In addition to the results just described, the evolution of solitons from initial data in the form of a gaussian-shaped positive pulse in the presence of noise was examined. The number of solitons that emerged was still found to follow the asymptotic law derived by BEREZIN and KARPMAN [23], though the solitons experienced delays relative to the corresponding noise free solution.

These results have once again confirmed experimentally the remarkable persistence of KdV solitons. They have, however, also indicated the need both for more accurate numerical experiments to confirm the delay process that has been reported here, and for a theoretical understanding of the precise way in which the noise spectrum is modified with time. A preliminary account of the work described in this Section, without any quantitative results, was first reported in the author's earlier paper [9], while a preliminary version of the present paper was presented at a Symposium in Tallinn [24].

4. The time regularised long wave equation

There is no unique unidirectional long wave equation that characterises nonlinear dispersive systems, and the asymptotic argument used to arrive at a particular long wave equation may also provide equal justification for an alternative equation. Naturally, different equations will have different mathematical properties, so that the choice between equations needs to be determined by the closeness with which their mathematical properties correspond to those of the physical problem that is to be modelled. Since approximations must always be made when modelling physical situations, it is to be expected that no one model is likely to have every one of its mathematical properties in complete agreement with the desired physical criteria.

An excellent illustration of this situation is provided by the work of PEREGRINE [3] who, starting from the Boussinesq equations, derived the following canonical form of equation in connection with his description of the behaviour of an undular bore in water:

$$u_t+u_t+u_x u - u_{xxt} = 0. \tag{4.1}$$

This should be compared with the equivalent canonical form of the KdV equation

$$u_t+u_x+uu_x+u_{xxx} = 0 \tag{4.2}$$

which might also be expected to provide a description of this same phenomenon. In both of these equations the additional term u_x may be removed by an elementary transformation, just as a dispersive parameter μ may be introduced in front of the last term.

Subsequently, as part of a general study of the modelling of long waves in nonlinear dispersive systems by BENJAMIN, BONA and MAHONY [4], attention was focussed on Eq. (4.1) as an alternative to the KdV equation (4.2). In their paper they established the existence, uniqueness and stability of solutions to this equation which they called the regularised long wave equation, though it is now usually known as the BBM equation. As their reasons for preferring it to the KdV equation are directly relevant to what is to follow, we summarise them below. By means of an asymptotic argument they showed, to the same order of approximation involved when deriving the KdV equation (4.2) as a long wave approximation, that $\partial/\partial t \equiv -\partial/\partial x$. When this result is applied once to the last term in Eq. (4.2) it yields the BBM equation (4.1), thereby establishing that the KdV and BBM equations have equal validity asymptotically when describing long waves, though their mathematical properties are somewhat different. Specifically it was argued in [4] that the BBM gives a better description of long waves than does the KdV equation because its linearised dispersion relation

$$\omega = k/(1+k^2), \tag{4.3}$$

has better properties than the equivalent dispersion relation

$$\omega = k-k^3 \tag{4.4}$$

for the KdV equation. The basis of this argument was that the phase velocity ω/k of the KdV equation becomes negative for $|k| > 1$, thereby contradicting the assumption of unidirectional propagation, while the group velocity $d\omega/dk$ has no lower bound, so that there is no limit to the rate at which fine detail may be transmitted in the negative x direction. On the other hand, as the BBM dispersion relation (4.3) is not subject to these objections and also possesses solitary wave solutions, it was suggested that the BBM equation should be preferred for the description of long waves to the asymptotically equivalent KdV equation.

Despite these arguments objections may still be raised to the BBM equation since, although bounded, its group velocity becomes negative for $|k| > 1$. In addition, and perhaps more seriously, as the BBM equation, like the KdV equation, is first order in time, it is only possible to specify u as initial data, whereas in some problems governed by these equations it would be natural to expect to specify both u and $\partial u/\partial t$ as initial data. This would necessitate the equation being second order in time, when it would also become able to characterise bidirectional wave propagation. Furthermore, in the KdV and BBM equations the effect of dispersion is so strong that it induces a stability of solution which precludes the possibility of any "wave breaking" type phenomena occurring. This is physically unrealistic insofar that situations may arise, as with water waves, where both non-breaking and breaking of long waves may be expected to occur under different circumstances. If a model equation is sought which is capable of describing both such phenomena within the one equation, it must obviously be different to the KdV and BBM equations and yet still retain many of their essential features.

A model equation that fulfills some of these objectives is provided by the time regularised long wave (TRLW) equation proposed by the author in [9]. It has the same asymptotic validity as the KdV and BBM equations, and is derived from the KdV equation by twice applying the result $\partial/\partial t \equiv -\partial/\partial x$ to the last term in Eq. (4.2) to obtain

$$u_t + u_x + uu_x + u_{xtt} = 0. \tag{4.5}$$

The TRLW equation has the linearised dispersion relation

$$\omega = [-1 \pm (1+5k^2)^{1/2}]/2k, \tag{4.6}$$

a graph of which is shown in Fig. 7a. In Fig. 7b is shown a graph of the function k/ω against ω, since this is of interest in the linearised case as k/ω is analogous to the refractive index in an optically dispersive medium from which the notion of dispersion is derived. Arrows have been used to indicate the direction of wave propagation that is associated with the curves.

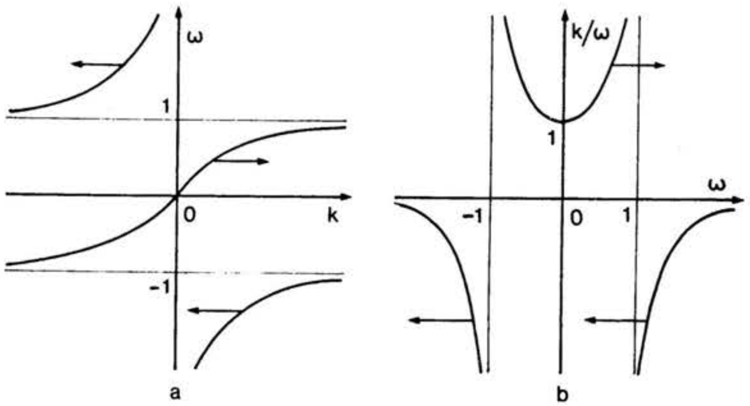

FIG. 7. a. Dispersion relation. b. „Refractive index" versus ω.

It can be seen that, in general, this equation has desirable properties. The wave moving to the right has both phase and group velocities that are positive and remain finite, with each tending to zero as $|k| \to \infty$. The wave moving to the left has a positive group velocity and a negative phase velocity that both tend to zero as $|k| \to \infty$, though for small k the phase and group velocities are unbounded.

Also in favour of the TRLW equation (4.5) is the fact that, like the KdV and BBM equations, it possesses solitary wave solutions. However, unlike the KdV and Burgers' equations, travelling wave solutions to the TRLW equation do not have the property of Galilean invariance. It may be the case that in the loss of Galilean invariance there lies the possibility of a wave breaking mechanism that can operate under certain circumstances, but this has still to be established. In addition, the TRLW equation is conservative in the sense that it may be derived from a Lagrangian density as described by WHITHAM [7]. This density L has the form

$$L = \frac{1}{2}\theta_x^2 + \frac{1}{6}\theta_x^3 + \frac{1}{2}\theta_x\theta_t - \frac{1}{2}\theta_{xt}^2, \tag{4.7}$$

with $u = \theta_x$. The equation also possesses a Hamiltonian density

$$H = \frac{1}{2}u^2 + \frac{1}{6}u^3 + \frac{1}{2}u_t^2, \tag{4.8}$$

and has associated with it quantities which, when integrated over the spatial interval $(-\infty, \infty)$ are time invariant. The first three of these are

$$C_0 = u, \quad C_1 = \frac{1}{2}u^2 - u_x u_t, \quad C_2 = \frac{1}{6}u^3 + u_x u_t + \frac{1}{2}u_t^2, \tag{4.9}$$

and they may be combined to correspond to the conservation of some more familiar quantities. For example, conservation of energy corresponds to the result

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}(C_1 + C_2)dx = 0 \tag{4.10}$$

that follows from Eq. (4.9).

Although the BBM equation possesses classical solitary wave solutions, these do not retain their form after interaction, as do KdV solitons, since a slight change of shape results. It seems probable that a similar change of shape will occur after the interaction of TRLW solitary waves.

An obvious extension of the TRLW equation (4.5) occurs when dissipation is present, as in Burgers' equation. Analogously to Eq. (1.1), we are thus led to consider the dissipative time regularised long wave (DTRLW) equation

$$u_t + uu_x - \nu u_{xx} + \mu u_{xtt} = 0. \tag{4.11}$$

This may be expected to have properties similar to those discussed in Sect. 2 for the KdVB equation and, indeed, a simple calculation shows that the result (2.16) applies to Eq. (4.11) if μ is everywhere replaced by $\mu(u_\infty^- + u_\infty^+)^2/4$.

Although other long wave equations may be derived from the KdV equation using the same operator equivalence $\partial/\partial t \equiv -\partial/\partial x$, none of them has properties that are as satisfactory as those of the TRLW equation. It remains for its properties to be explored systematically, and for ideas like those of Sect. 3 to be extended to the BBM and TRLW equation.

After the completion of this paper it was drawn to the author's attention by L. J. F. BROER [25] that a general paper has already appeared [26] discussing the determination of the pair of canonically conjugate variables that are necessary if Eq. (4.8) is to be used as a proper Hamiltonian. It was also pointed out that Eq. (4.5) has been suggested by JOSEPH and EGRI [27] in an attempt to allow for the specification of the Cauchy data u and $\partial u/\partial t$. However, their objective did not include, as here, the desire to introduce a possible wave breaking mechanism.

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