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Displacement description of dislocation lines
I. Cyclic functions

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THE NOTION of cyclic functions is introduced. Construction of a certain class of cyclic functions is presented in a n-dimensional metric space in which the second-order mixed derivatives do not commute. Particular forms of cyclic functions are given in a three-dimensional metric space and in a four-dimensional Minkowskian space.

Wprowadzono pojęcie funkcji cyklicznych. Podano konstrukcję pewnej klasy funkcji cyklicznych w n-wymiarowej przestrzeni metrycznej spełniającej dany warunek komutacyjny dla drugich pochodnych mieszanych. Podano wyrażenia na konkretne funkcje cykliczne w trójwymarowej przestrzeni euklidesowej i czterowymiarowej przestrzeni Minkowskiego.

Введено понятие циклических функций. Дается построение некоторого класса циклических функций в n-мерном метрическом пространстве, удовлетворяющих заданному условию коммутации для вторых смещанных производных. Даются выражения для конкретных циклических функций в трехмерном евклидовом пространстве и в четырехмерном пространстве Минковского.

1. Introduction

THE MATHEMATICAL theory of dislocations in a linear elastic medium is well developed, although there exists a certain dualism within this theory. The dislocation is known to be a linear defect; in most of the papers, however, the surface model of a dislocation is used. This is due to the impossibility of a direct application of the Burgers condition defining the dislocation line. This condition states that the loop D is a dislocation line if, for each closed curve called the Burgers circuit embracing only once the dislocation line D, the total displacement increment equals the constant vector $\mathbf{b}$ called the Burgers vector,

$$\oint_B d\mathbf{u} = \mathbf{b}. \tag{1.1}$$

In the papers using the surface model of dislocation the starting point is usually either the condition for the displacement jump at the dislocation surface $[\![\mathbf{u}]\!]_S = \mathbf{b}$, or the initial distortion $\overset{\circ}{\beta}$ defined as a singular distribution with a support at the dislocation surface $S(\partial S = D)$.

The only consistent approach to a dislocation treated as a linear defect may be found in a paper by E. KOSSECKA [1]; however, no displacement field can be attributed to that dislocation. The dislocation might be connected with a multivalued displacement field, as it was demonstrated by ROGULA [2], and such an approach is very close to that proposed in the present paper.

Let us analyse what difficulties are encountered in the displacement description of dislocations treated as linear defects. To this end let us write, quite formally, the Burgers condition

$$\oint_B \nabla_i u_l dx^i = b_l. \tag{1.2}$$

The differential counterpart of that condition has the form

$$\in^{kij}\nabla_i\nabla_j u_l = b_l \oint_D d\zeta^k \delta(\mathbf{x}-\boldsymbol{\zeta}). \tag{1.3}$$

This condition is not satisfied by any function or distribution since it assumes that the mixed second derivatives are not equal to each other.

Let us, furthermore, make the purely formal assumption that

$$\mathbf{u} = \mathbf{b}\Omega. \tag{1.4}$$

After substitution in Eq. (1.3) we obtain

$$\in^{kij}\nabla_i\nabla_j\Omega = \oint_D d\zeta^k \delta(\mathbf{x}-\boldsymbol{\zeta}). \tag{1.5}$$

If we had at our disposal the pseudo-functions satisfying the condition (1.5), the displacement description of a dislocation line would be possible. The construction of such pseudo-functions, to be called cyclic functions, is the aim of the present paper. From Eq. (1.3) it is evident that the first derivatives of cyclic functions must be distributions.

The applications of cyclic functions to the description of dislocations in a three-dimensional space, the time constituting a parameter, and to the four-dimensional description in the time-space will be given in [3].

2. Cyclic functions

Let a closed smooth line D, without double points or loops, be given in E^3. Denote by $\mathcal{S}_D$ the set of all smooth surfaces S based on the line D,

$$\mathcal{S}_D = \{S : \partial S = D\}.$$

Each pair of surfaces $S_i, S_j \in \mathcal{S}_D$ bounds a region V_{ij}

$$V_{ij} = \{\mathbf{x} \in E^3 : \partial V_{ij} = S_i \cup S_j\}.$$

Let $X(D, a)$ denote the set of all functions

$$f_S : E^3 \to R$$

such that for each function f_S there exists exactly one surface $S \in \mathcal{S}_D$ at which the function f_S suffers a jump $a \in R$, $[\![f_S]\!]_S = a$, and at the remaining points the functions are of class C^2. Moreover, at the surface S the function f_S assumes such values $f_S(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in S$, that

$$\lim_{E^3\setminus S \ni \mathbf{x}_n \to \boldsymbol{\xi}} |f_S(\mathbf{x}_n) - f_E(\boldsymbol{\xi})| = \left|\frac{a}{2}\right|.$$

Within the set $X(D, a)$ let us introduce the equivalence relation $\mathscr{R}$ defined by the formula

$$f_{S_i}\mathscr{R}f_{S_j} \Leftrightarrow |f_{S_i}-f_{S_j}| = \begin{cases} 0 & \text{for} \quad \mathbf{x} \in E^3\setminus V_{ij}, \\ |a| & \text{for} \quad \mathbf{x} \in V_{ij}, \\ \left|\dfrac{a}{2}\right| & \text{for} \quad \mathbf{x} \in \partial V_{ij}. \end{cases}$$

It will be shown that the relation $\mathscr{R}$ introduced here is the equivalence relation, i.e. it is reflexive, symmetric and transitive.

(a) *Reflexivity*;

$$|f_{S_i}-f_{S_i}| = \begin{cases} 0 & \text{for} \quad \mathbf{x} \in (E^3\setminus V_{ii}) = E^3, \\ |a| & \text{for} \quad \mathbf{x} \in V_{ii} = \emptyset, \\ \left|\dfrac{a}{2}\right| & \text{for} \quad \mathbf{x} \in \partial V_{ii} = \emptyset; \end{cases}$$

since $V_{ii} = \emptyset$, and $|f_{S_i}-f_{S_i}| = 0$ for $\mathbf{x} \in E^3$.

(b) *Symmetry*; the proof is trivial.

(c) *Transitivity*

ASSUMPTIONS

$$|f_{S_i}-f_{S_j}| = \begin{cases} 0 & \text{for} \quad \mathbf{x} \in E^3\setminus V_{ij}, \\ |a| & \text{for} \quad \mathbf{x} \in V_{ij}, \\ \left|\dfrac{a}{2}\right| & \text{for} \quad \mathbf{x} \in \partial V_{ij} = S_i \cup S_j; \end{cases}$$

$$|f_{S_j}-f_{S_k}| = \begin{cases} 0 & \text{for} \quad \mathbf{x} \in E^3\setminus V_{jk}, \\ |a| & \text{for} \quad \mathbf{x} \in V_{jk}, \\ \left|\dfrac{a}{2}\right| & \text{for} \quad \mathbf{x} \in \partial V_{jk} = S_j \cup S_k. \end{cases}$$

THEOREM

$$|f_{S_i}-f_{S_k}| = \begin{cases} 0 & \text{for} \quad \mathbf{x} \in E^3\setminus V_{ik}, \\ |a| & \text{for} \quad \mathbf{x} \in V_{ik}, \\ \left|\dfrac{a}{2}\right| & \text{for} \quad \mathbf{x} \in \partial V_{ik} = S_i \cup S_k. \end{cases}$$

Proof

$$|f_{S_i}-f_{S_k}| = |(f_{S_i}-f_{S_j})+(f_{S_j}-f_{S_k})| = \begin{cases} 0 & \text{for} \quad \mathbf{x} \in E^3\setminus (V_{ij}\cup V_{ik}), \\ |a| & \text{for} \quad \mathbf{x} \in (V_{ij}\cup V_{jk})\setminus (V_{ij}\cap V_{jk}), \\ \left|\dfrac{a}{2}\right| & \text{for} \quad \mathbf{x} \in S_i \cup S_k, \\ 2|a| & \text{or} \quad 0 \quad \text{for} \quad \mathbf{x} \in V_{ij}\cap V_{jk}, \\ 2\left|\dfrac{a}{2}\right| & \text{or} \quad 0 \quad \text{for} \quad \mathbf{x} \in S_j. \end{cases}$$

Thus it should be proved that

$$|(f_{S_i}-f_{S_j})+(f_{S_j}-f_{S_k})| = 0 \quad \text{for} \quad \mathbf{x} \in V_{ij}\cap V_{jk} \quad \text{and} \quad \text{for} \quad \mathbf{x} \in S_j.$$

Let us confine our considerations to the case

$$f_{S_j}-f_{S_k} = a \quad \text{for} \quad \mathbf{x} \in V_{jk}. \tag{2.1}$$

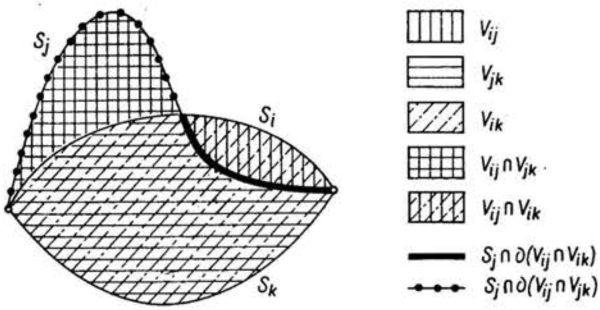

FIG. 1.

From the definition it follows that $f_{S_i}, f_{S_j}, f_{S_k}$ satisfy the relation

$$f_{S_i} = f_{S_j} = f_{S_k} \quad \text{for} \quad \mathbf{x} \in E^3 \setminus (V_{ij} \cup V_{jk}). \tag{2.2}$$

Under the assumption (2.1), due to the jump properties of f_{S_α} and the relations

$$f_{S_i} \mathscr{R} f_{S_j} \wedge f_{S_j} \mathscr{R} f_{S_k}$$

we obtain

$$f_{S_j} - f_{S_k} = a \quad \text{for} \quad \mathbf{x} \in V_{ij} \cap V_{jk}. \tag{2.3}$$

Equations (2.2) and (2.3) yield

$$f_{S_i} - f_{S_k} = 0 \quad \text{for} \quad \mathbf{x} \in [V_{jk} \cap (V_{ij} \cap V_{jk})] = V_{ij} \cap V_{jk},$$

$$S_j = [S_j \cap \partial(V_{ij} \cap V_{jk})] \cup [S_j \cap \partial(V_{ij} \cap V_{ik})].$$

Let us consider the points

$$\mathbf{x} \in S_j \cap \partial(V_{ij} \cap V_{jk}).$$

From the assumption (2.1) and the properties of f_{S_α} at the surfaces S_α it follows that

$$\left.\begin{aligned} f_{S_j}(\boldsymbol{\xi}) - f_{S_k}(\boldsymbol{\xi}) &= \frac{a}{2} \\ f_{S_j}(\boldsymbol{\xi}) - f_{S_i}(\boldsymbol{\xi}) &= \frac{a}{2} \end{aligned}\right\} \quad \text{for} \quad \boldsymbol{\xi} \in S_j \cap \partial(V_{ij} \cap V_{jk}),$$

that is,

$$f_{S_i} - f_{S_k} = 0 \quad \text{for} \quad \boldsymbol{\xi} \in S_j \cap \partial(V_{ij} \cap V_{jk}).$$

Consider now the remaining part of S_j, i.e.

$$\mathbf{x} \in S_j \cap \partial(V_{ij} \cap V_{ik}).$$

Here we have

$$\left.\begin{aligned} f_{S_j}(\boldsymbol{\xi}) - f_{S_i}(\boldsymbol{\xi}) &= -\frac{a}{2} \\ f_{S_j}(\boldsymbol{\xi}) - f_{S_k}(\boldsymbol{\xi}) &= -\frac{a}{2} \end{aligned}\right\} \quad \text{for} \quad \boldsymbol{\xi} \in S_j \cap \partial(V_{ij} \cap V_{ik})$$

whence

$$f_{S_i}(\boldsymbol{\xi}) - f_{S_k}(\boldsymbol{\xi}) = 0 \quad \text{for} \quad \boldsymbol{\xi} \in S_j \cap \partial(V_{ij} \cap V_{ik})$$

or, finally,

$$f_{S_i}(\xi) - f_{S_k}(\xi) = 0 \quad \text{for} \quad \xi \in S_j$$

what concludes the proof.

The equivalence relation $\mathscr{R}$ determines within the set $X(D, a)$ the equivalence classes $X(D, a)/\mathscr{R}$ which will be called *cyclic functions*. The cyclic functions will be denoted by capital Greek letters, and their representants — by the corresponding small Greek letters. The equivalence class is denoted by two vertical lines. In accordance with these notations, the cyclic function Ω is denoted

$$\Omega = \|\omega_S\| = \{\overline{\omega}_S \in X(D, a) \colon \overline{\omega}_S \mathscr{R}\, \omega_S\}.$$

From the above definition it is seen that, similar to the distribution which determines the equivalence class of locally integrable functions differing, at the most, in sets of zero measure, the cyclic function determines the equivalence class of distributions (characterized by jumps at a certain surface) differing, at the most, by a constant in a closed set. Conversely, like a locally integrable function generates exactly one distribution, also each distribution possessing a jump at a certain open surface generates exactly one cyclic function. It will be shown that once the operation of differentiation is introduced, this multivaluedness does not lead to any ambiguities, provided the differentiation algorithm is properly constructed.

Differentiation of cyclic functions

The derivative of a cyclic function Ω is the distribution $\Omega_{,i}$ such that

$$\Omega_{,i} = \frac{\partial \Omega}{\partial x^i} = \frac{\partial \omega}{\partial x^i} \quad \text{when} \quad \mathbf{x} \notin S.$$

This means that $\Omega_{,i}$ is equal to the distributional derivative of such a representant ω_S for which the point $\mathbf{x}$ does not lie on the corresponding jump surface S. From the definition it follows that in order to determine the cyclic function derivative at a point $\mathbf{x}$, the term resulting from the differentiation of the jump at the discontinuity surface must be subtracted from distributional derivative of its representant, i.e.

$$\frac{\partial \Omega}{\partial x^i} = \frac{\partial \omega_S}{\partial x^i} - \tau_i^{\mathrm{III}} a \delta(a^{\mathrm{III}}) E(a^{\mathrm{I}}, a^{\mathrm{II}}). \tag{2.4}$$

Here $a^{\mathrm{I}}, a^{\mathrm{II}}$ are coordinates on the surface S, $E(a^{\mathrm{I}}, a^{\mathrm{II}})$ is the characteristic function of S,

$$E(a^{\mathrm{I}}, a^{\mathrm{II}}) = \begin{cases} 1 & \text{for} \quad x \in S, \\ 0 & \text{for} \quad x \notin S; \end{cases}$$

a^{III} is the coordinate perpendicular to S, τ_i^{III} the i-th component of the unit vector of the a^{III}-axis in the Cartesian frame of reference x_i. The derivative $\partial \omega_S / \partial x^i$ in the formula (2.4) is the distributional derivative.

Consider the cyclic function Ω possessing the property that the distributional derivatives of each of its representant $\omega_S (\Omega = \|\omega_S\|)$ evaluated in the directions normal to $\partial S = D$ and tangent to S are continuous (in the usual sense). In other words, considered are such functions ω_S which suffer no jumps in the process of "sliding down" the surface. Such cyclic functions will be shown to satisfy the equation

$$\in^{ijk}\nabla_j\nabla_k\Omega = a\delta^i(D) = a\oint_D d\zeta^i\delta(\mathbf{x}-\boldsymbol{\zeta}), \tag{2.5}$$

which is equivalent (with accuracy up to the constant a) to Eq. (1.5).

Let us select in the neighbourhood of the surface S such an orthogonal coordinate system $a^L(L = \mathrm{I}, \mathrm{II}, \mathrm{III})$ that one of the surfaces $a^{\mathrm{III}} = \text{const}$ coincides with S, while the coordinate lines a^{III} are orthogonal to the surface S. In addition, one of the coordinate lines a^{I} is assumed to coincide with the boundary $\partial S = D$ of the surface S. From these conditions and from the orthogonality of the coordinate system it follows that the coordinate lines a^{II} lying on the surface S are perpendicular to $\partial S = D$ (Fig. 2).

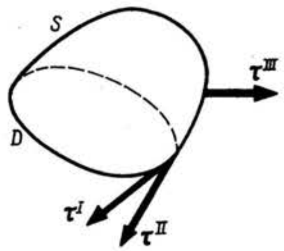

FIG. 2.

Let $\boldsymbol{\tau}^L$ denote the unit vectors tangent to the coordinate lines a^L. According to the differentiation algorithm (2.4) of cyclic functions, we have

$$\frac{\partial\Omega}{\partial x^j} = \frac{\partial\omega_S}{\partial x^j} - a\tau_j^{\mathrm{III}}\delta(a^{\mathrm{III}})E(a^\alpha), \qquad \alpha = \mathrm{I}, \mathrm{II}.$$

Here $E(a^\alpha)$ is the characteristic function of S, and $a = [|\omega|]_S$,

$$\frac{\partial^2\Omega}{\partial x^k\partial x^j} = \frac{\partial^2\omega_S}{\partial x^k\partial x^j} - a\tau_k^L\frac{\partial}{\partial a^L}[\tau_j^{\mathrm{III}}\delta(a^{\mathrm{III}})E(a^\alpha)] = \frac{\partial^2\omega_S}{\partial x^k\partial x^j} - a\left[\frac{\partial\tau_j^{\mathrm{III}}}{\partial x^k}\delta(a^{\mathrm{III}})E(a^\alpha)\right.$$
$$\left. + \tau_k^{\mathrm{III}}\tau_j^{\mathrm{III}}E(a^\alpha)\frac{\partial\delta(a^{\mathrm{III}})}{\partial a^{\mathrm{III}}} + \tau_k^{\mathrm{II}}\tau_j^{\mathrm{III}}\delta(a^{\mathrm{III}})\frac{\partial E(a^\alpha)}{\partial a^{\mathrm{II}}}\right].$$

Similarly,

$$\frac{\partial^2\Omega}{\partial x^j\partial x^k} = \frac{\partial^2\omega_S}{\partial x^j\partial x^k} - a\left[\frac{\partial\tau_k^{\mathrm{III}}}{\partial x^j}\delta(a^{\mathrm{III}})E(a^\alpha) + \tau_j^{\mathrm{III}}\tau_k^{\mathrm{III}}E(a^\alpha)\frac{\partial\delta(a^{\mathrm{III}})}{\partial a^{\mathrm{III}}} + \tau_j^{\mathrm{II}}\tau_k^{\mathrm{III}}\delta(a^{\mathrm{III}})\frac{\partial E(a^\alpha)}{\partial a^{\mathrm{II}}}\right].$$

Taking into account the equality

$$\frac{\partial^2\omega_S}{\partial x^k\partial x^j} = \frac{\partial^2\omega_S}{\partial x^j\partial x^k}$$

since these are distributional derivatives, we have

$$(\nabla_k\nabla_j - \nabla_j\nabla_k)\Omega = -a(\nabla_k\tau_j^{\mathrm{III}} - \nabla_j\tau_k^{\mathrm{III}})\delta(a^{\mathrm{III}})E(a_\alpha) - a(\tau_k^{\mathrm{II}}\tau_j^{\mathrm{III}} - \tau_j^{\mathrm{II}}\tau_k^{\mathrm{III}})\delta(D).$$

On multiplying both sides by $\in^{ikj}$ we obtain

$$\in^{ikj}\nabla_k\nabla_j\Omega = -a[\varepsilon^{ikj}\nabla_k\tau_j^{\mathrm{III}}\delta(a^{\mathrm{III}})E(a^\alpha)+\in^{ikj}\tau_k^{\mathrm{II}}\tau_j^{\mathrm{III}}]. \tag{2.6}$$

Since

$$\mathrm{rot}\ \boldsymbol{\tau}^L = 0 \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{II}}\times\boldsymbol{\tau}^{\mathrm{III}} = \boldsymbol{\tau}^{\mathrm{I}}$$

we finally have

$$\in^{ijk}\nabla_j\nabla_k\Omega = -at^i, \tag{2.7}$$

where

$$t^i = \oint_D d\zeta^i\delta(\mathbf{x}-\boldsymbol{\zeta}).$$

EXAMPLE

Consider the cyclic function $\Omega = ||\omega_{S_{(k)}}||$, where

$$\omega_{S_{(k)}} = \int_{S_{(k)}} \mathrm{grad}\left(\frac{1}{R}\right)\cdot \mathbf{dS}$$

and select a particular surface S_k based on a closed curve D, so that S_k is the surface of a semi-infinite cylinder with a directrix D and generator $\alpha\,\mathbf{k}$, $\mathbf{k}$ being an arbitrary, fixed unit vector; then

$$\omega_{S_{(k)}} = \int_{S_{(k)}} \mathrm{grad}\left(\frac{1}{R}\right)\cdot \mathbf{dS}_{(k)} = \oint_D dl\int_0^\infty \in^{ijk}l_jk_k\left(-\frac{X_i}{R^3}\right)d\alpha,$$

$$dS_{(k)} = (\mathbf{l}\times\mathbf{k})dld\alpha, \quad \boldsymbol{\zeta}\in D, \quad dll_j = d\zeta_j.$$

$$\omega_{S_{(k)}} = \oint_D dl\in^{ijk}l_jk_k\int_0^\infty \frac{-(x_i\zeta_i-\alpha k_i)d\alpha}{\sqrt{(x_p-\zeta_p-\alpha k_p)(x^p-\zeta^p-\alpha k^p)}^3} \tag{2.8}$$

$$= \oint_D dl\in^{ijk}l_jk_k\int_0^\infty \frac{[\alpha k_i-(x_i-\zeta_i)]d\alpha}{\sqrt{\alpha^2-2\alpha(\mathbf{k}\cdot\mathbf{r})+r^2}^3} = \oint_D dl\frac{\in^{ijk}l_jk_kr_i}{r(r-\mathbf{r}\cdot\mathbf{k})} = \oint_D \frac{(\mathbf{k}\times\mathbf{r})\cdot\mathbf{dl}}{r(r-\mathbf{r}\cdot\mathbf{k})}.$$

The function

$$\omega_{S_{(k)}} = \oint \frac{(\mathbf{k}\times\mathbf{r})\cdot\mathbf{dl}}{r(r-\mathbf{r}\cdot\mathbf{k})}$$

suffers a jump equal to 4π at the surface of the semi-infinite cylinder S_k, and

$$\lim_{S_k\not\ni\mathbf{x}_n\to\xi\in S_k}|\omega_{S_{(k)}}(\mathbf{x}_n)-\omega_{S_{(k)}}(\boldsymbol{\zeta})| = 2\pi.$$

In the particular case of D being an infinite straight line parallel to the x_3-axis and passing through the point $(\xi,\eta,0)$, and $\mathbf{k} = [k_1,k_2,0]$, $k_1^2+k_2^2 = 1$, we obtain

$$\omega_{S_{(k)}} = \int_{-\infty}^{\infty} d\zeta\,\frac{Yk_1-Xk_2}{r(r-\mathbf{r}\cdot\mathbf{k})} = \begin{cases} 2\ \mathrm{sign}(Xk_2-Yk_1)\left(\mathrm{arc\,tg}\,\dfrac{-\boldsymbol{\rho}\cdot\mathbf{k}}{\sqrt{\varrho^2-(\boldsymbol{\rho}\cdot\mathbf{k})^2}}-\dfrac{\pi}{2}\right), \\ \text{for} \quad \mathbf{x}\neq(\xi+\alpha k_1,\eta+\alpha k_2,0), \quad \alpha>0, \\ 0 \quad \text{for} \quad \mathbf{x}=(\xi+\alpha k_1,\eta+\alpha k_2,0), \quad \alpha>0 \end{cases}$$

with the notations $X = x_1-\xi$, $Y = x_2-\eta$, $r^2 = \varrho^2+\zeta^2$, $\varrho^2 = X^2+Y^2$.

The function $\omega_{S_{(k)}}$ defined by the above formula suffers a jump 4π at the half-plane $k_2X-k_1Y=0$, $k_1X+k_2Y>0$.

Let us now calculate the mixed second distributional derivatives of the function $\omega_{S_{(k)}}$ and the same derivatives of the cyclic function Ω:

$$\frac{\partial\omega_{S_{(k)}}}{\partial x_1} = -2\operatorname{sign}(Xk_2-Yk_1)\,\frac{k_1\varrho^2-X(\boldsymbol{\rho}\cdot\mathbf{k})}{\varrho^2\sqrt{\varrho^2-(\boldsymbol{\rho}\cdot\mathbf{k})^2}}-4\pi H(k_1X+k_2Y)\,\delta(k_1Y-k_2X)$$

$$= -\frac{2Y}{\varrho^2}\underline{-4\pi k_2H(k_1X+k_2Y)\delta(k_1Y-k_2X)},$$

$$\frac{\partial\omega_{S_{(k)}}}{\partial x_2} = \frac{2X}{\varrho^2}\underline{+4\pi k_1H(k_1X+k_2Y)\delta(x_1Y-k_2X)},$$

$$\frac{\partial^2\omega_{S_{(k)}}}{\partial x_2\partial x_1} = 2\frac{X^2-Y^2}{\varrho^4}-\delta(X)\,\delta(Y)\oint_\Gamma\frac{Ydx^1}{\varrho^2}-4\pi k_2^2\delta(k_1X+k_2Y)\,\delta(k_1Y-k_2X)$$

$$+4\pi k_1k_2H(k_1X+k_2Y)\,\delta'(k_1Y-k_2X) = 2\frac{X^2-Y^2}{\varrho^4}+2\pi\delta(X)\,\delta(Y)$$

$$\underline{-4\pi k_2^2\delta(X)\delta(Y)+4\pi k_1k_2H(k_1X+k_2Y)\delta'(k_1Y-k_2X)},$$

$$\frac{\partial^2\omega_{S_{(k)}}}{\partial x_1\partial x_2} = 2\frac{X^2-Y^2}{\varrho^4}-2\pi\delta(X)\,\delta(Y)$$

$$+4\pi k_1^2\,\delta(X)\,\delta(Y)+4\pi k_1k_2H(k_1X+k_2Y)\,\delta'(k_1Y-k_2X),$$

$$X = x_1-\xi,\quad Y = x_2-\eta,\quad \varrho^2 = X^2+Y^2;$$

$H(\mathbf{x})$ is the Heaviside function.

The corresponding derivatives of the cyclic function differ from those written above by the underlined terms which are due to differentiation of the jumps of the distribution $\omega_{S_{(k)}}$ at the surface $S_{(k)}$.

$$\frac{\partial\Omega}{\partial x_1} = -\frac{2Y}{\varrho^2},\quad \frac{\partial\Omega}{\partial x_2} = \frac{2X}{\varrho^2},$$

$$\frac{\partial^2\Omega}{\partial x_2\partial x_1} = 2\frac{X^2-Y^2}{\varrho^4}+2\pi\delta(X)\,\delta(Y),$$

$$\frac{\partial^2\Omega}{\partial x_1\partial x_2} = 2\frac{X^2-Y^2}{\varrho^4}-2\pi\delta(X)\,\delta(Y).$$

These formulae yield the conclusion

$$(\partial_2\partial_1-\partial_1\partial_2)\Omega = 4\pi\delta(x_1-\xi)\,\delta(x_2-\eta)\,1(x_3) = 4\pi\,\delta(D)$$

and so Eqs. (2.6) and (2.7) are satisfied.

In the case of $\mathbf{k}=[1,0,0]$, $\xi=0$, $\eta=0$ we have

$$\omega_{S_{(k)}} = 2(\operatorname{sign}x_2)\left((\operatorname{arctg}\frac{x_1}{|x_2|}+\frac{\pi}{2}\right).$$

3. Cyclic functions in n-dimensional metric spaces

Cyclic functions in n-dimensional spaces are defined like those in a three-dimensional space. The surface S is a $(n-1)$-dimensional open surface with the boundary $\partial S = S_{(n-2)}$ Without going into general considerations let us give the formulae for certain definite cyclic functions.

Let us consider the n-dimensional metric space with a natural topology determined by the metric tensor g. The cyclic function $\Omega_{(n)}$ is given by the formula (according to the remark in Sect. 2 the function is identified with its representant)

$$\Omega_{(n)} = \int\limits_{S_{(n-1)}} dS_{(n-1)} n^{\alpha} \nabla_{\alpha} \Phi_{(n)}, \tag{3.1}$$

where $\Phi_{(n)}$ is a solution (belonging to the class of generalized functions) of the differential equation

$$\nabla^{\alpha}\nabla_{\alpha}\Phi_{(n)} = g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\Phi_{(n)} = -\delta_{(n)}(\mathbf{x}-\boldsymbol{\zeta}) = -\delta(x^1-\zeta^1)\,\delta(x^2-\zeta^2)\,\ldots\,\delta(x^n-\zeta^n). \tag{3.2}$$

It may be shown that the cyclic function defined in this way satisfies identically the equations

$$\in^{\alpha_1\alpha_2\ldots\alpha_n}\nabla_{\alpha_{n-1}}\nabla_{\alpha_n}\Omega_{(n)} = J^{\alpha_1\ldots\alpha_{n-2}}, \qquad (\alpha_1, \alpha_2, \ldots, \alpha_n) = (1, 2, \ldots, n). \tag{3.3}$$

Here

$$J^{\alpha_1\alpha_2\ldots\alpha_{n-2}} = \oint\limits_{\partial S_{(n-1)} = S_{(n-2)}} dS_{(n-2)}^{\alpha_1\ldots\alpha_{n-2}}\,\delta_{(n)}(\mathbf{x}-\boldsymbol{\zeta}). \tag{3.4}$$

The surface element $dS_{(n-2)}^{\alpha_1\alpha_2\ldots\alpha_{n-2}}$ is given by the formula

$$dS_{(n-2)}^{\alpha_1\alpha_2\ldots\alpha_{n-2}} = \in^{\alpha_1\alpha_2\ldots\alpha_n} n_{\alpha_{n-1}} m_{\alpha_n} dS_{(n-2)}, \tag{3.5}$$

$\boldsymbol{m}, \mathbf{n}$ being the unit normal vectors of the surface $S_{(n-2)} = \partial S_{(n-1)}$ and $dS_{(n-2)}$ — the scalar element of the surface $S_{(n-2)}$.

In the particular cases of Euclidean and Minkowskian metrics the solutions of Eq. (3.2) to be used later have the following forms:

in Euclidean space

$$\Phi_{(3)} = \frac{1}{4\pi R_{(3)}}, \qquad R_{(3)}^2 = (x^i-\zeta^i)(x_i-\zeta_i), \tag{3.6}$$

in the Minkowskian space $g^{ii} = 1$ for $i < n$, $g^{nn} = -1$, $g^{ij} = 0$ for $i \neq j$,

$$\Phi_{(4)} = \frac{1}{2\pi}\,\delta(-R_{(4)}^2), \qquad R_{(4)}^2 = -(x^4-\zeta^4)^2 + \sum_{1}^{3}(x^i-\zeta^i)^2. \tag{3.7}$$

Let us now pass to a more detailed analysis of the case of Minkowskian space which will make it possible to apply the cyclic functions to the description of moving dislocations.

In compliance with Eqs. (3.1) and (3.7) we have

$$\Omega_{(4)} = \frac{1}{2\pi}\int\limits_{S_{(3)}} dS_{(3)} n^{\alpha}\nabla_{\alpha}\delta(-R_{(4)}^2). \tag{3.8}$$

In the case of $S_{(3)}$ being a space-like surface, and taking from the solution (3.7) only the retarded potential, we obtain from Eq. (3.8) (cf. Eq. (2.8) of paper [4])

$$\Omega_{(4)} = \frac{1}{4\pi} \int\limits_{-\infty}^{t} d\tau \int\limits_{S_{(2)}(\tau)} dS^{i}_{(2)} \left(\nabla_i + \frac{1}{c^2} \dot{\zeta}_i \partial_\tau \right) \frac{\delta(t-\tau-r/c)}{r}. \tag{3.8a}$$

The surface $S_{(3)}$ is constructed in the following manner: consider a closed line D in E^3 and set it into "motion" $D(t)$ in E^3. Let us construct a semi-infinite cylinder with a directrix $D(t)$ and generators parallel to an arbitrary, fixed unit vector $\mathbf{\Lambda}$ ($|\mathbf{\Lambda}|^2 = \pm 1$). The "history of motion" of that cylinder is prescribed by the history of motion of the loop $D(t)$. The three-dimensional surface constructed here is the considered characteristic surface of a representant of the cyclic function given by Eq. (3.8) under the assumption that $x^4 = ct$, c being a positive constant. For the sake of simplicity, the world surface generated by the line $D(t)$ is assumed to be within the null cone. For such a surface $S_{(3)}$ the formula holds true[1]

$$dS_{(3)} n^\alpha = \epsilon^{\alpha\beta\gamma\delta} l_\beta \lambda_\gamma \Lambda_\delta dS_{(2)} d\varepsilon, \tag{3.9}$$

$\mathbf{l} = \mathbf{l}(\boldsymbol{\zeta}(t))$ being a unit vector tangent to the line $D(t)$ in E^3, i.e. $\mathbf{l} = [l_i, 0], \boldsymbol{\lambda} = (1-v^2/c^2)^{-1/2}\left(\frac{1}{c}\dot{\zeta}^i, 1\right)$ is the four-velocity vector of the world surface generated by $D(t)$, $v = |\dot{\boldsymbol{\zeta}}|$, $\dot{\zeta}^i = \dfrac{\partial \zeta^i(l,t)}{\partial t}$, ε varies from zero to infinity, $dS_{(2)}$ is the scalar element of the two-dimensional surface representing the history of motion of $D(t)$.

The position vector of the surface constructed in this manner may be written as

$$\overset{(3)}{\boldsymbol{\zeta}} = \overset{(2)}{\boldsymbol{\zeta}} + \varepsilon \mathbf{\Lambda}.$$

Here $\overset{(3)}{\boldsymbol{\zeta}}$ is the position vector of the surface $S_{(3)}$, while $\overset{(2)}{\boldsymbol{\zeta}}$ — the corresponding vector of the surface $S_{(2)}$. Similarly, for $\mathbf{R} = \mathbf{x} - \boldsymbol{\zeta}$ we have

$$\begin{aligned} \overset{(3)}{\mathbf{R}} &= \overset{(2)}{\mathbf{R}} - \varepsilon\mathbf{\Lambda} = \mathbf{R} - \varepsilon\mathbf{\Lambda}, \\ |\overset{(3)}{\mathbf{R}}|^2 &= \varepsilon^2 \Lambda^2 - 2\varepsilon(\mathbf{\Lambda}\cdot\mathbf{R}) + R^2. \end{aligned} \tag{3.10}$$

From now on the superscripts 2, 3 will be omitted.

Inserting Eqs. (3.9) and (3.10) in (3.8) we obtain[2]

$$\begin{aligned} \Omega_{(4)} &= \frac{1}{2\pi} \int\limits_{S_{(3)}} dS_{(3)} n^\alpha \nabla_\alpha \delta(-\overset{(3)}{R^2}) \\ &= \frac{1}{2\pi} \int\limits_{S_{(2)}} dS_{(2)} \epsilon^{\alpha\beta\gamma\delta} l_\beta \lambda_\gamma \Lambda_\delta \nabla_\alpha \int\limits_0^\infty d\varepsilon\, \delta(\varepsilon^2\Lambda^2 - 2\varepsilon(\mathbf{\Lambda}\cdot\mathbf{R}) + R^2) \\ &= \frac{1}{4\pi} \int\limits_{S_{(2)}} dS_{(2)} \epsilon^{\alpha\beta\gamma\delta} l_\beta \lambda_\gamma \Lambda_\delta \nabla_\alpha \frac{[H(\varepsilon_1) + H(\varepsilon_2)]H(-R^2)}{\sqrt{(\mathbf{\Lambda}\cdot\mathbf{R})^2 - \Lambda^2 R^2}}. \end{aligned} \tag{3.11}$$

(1) The range of Greek indices is 1, 2, 3, 4, and of Latin indices — 1, 2, 3.

(2) The integration procedure is discussed in the Appendix.

Let us analyse the latter integral in the case of two particular forms of the vector $\mathbf{\Lambda}$.

(a) $\mathbf{\Lambda} = [0, 0, 0, 1]$; $\mathbf{\Lambda}$ being the time-like vector, $\Lambda^2 = -1$, and Eqs. (3.11) yield then

$$(3.12)\quad \overset{(-)}{\Omega}_{(4)} = \frac{-1}{4\pi}\int\limits_{S_{(2)}} dS_{(2)}\epsilon^{\alpha\beta\gamma 4} l_\beta \lambda_\gamma \nabla_\alpha \frac{H(-R^2)}{r} = \frac{-1}{4\pi}\int\limits_{S_{(2)}} dS_{(2)}\epsilon^{ijk} l_j \lambda_k \nabla_i \frac{H(-R^2)}{r}.$$

(b) $\mathbf{\Lambda} = [k_1, k_2, k_3, 0]$; $\mathbf{\Lambda}$ being the space-like vector, $\Lambda^2 = 1$, and then

$$(3.13)\quad \overset{(+)}{\Omega}_{(4)} = \frac{1}{4\pi}\int\limits_{S_{(2)}} dS_{(2)}\epsilon^{\alpha j\gamma i} l_j \lambda_\gamma k_i \nabla_\alpha \frac{H(-R^2)}{\sqrt{(\mathbf{k}\cdot\mathbf{r})^2 - R^2}}$$

$$= \frac{1}{4\pi}\int\limits_{S_{(2)}} dS_{(2)}\epsilon^{ijk} l_j k_i (\lambda_k \nabla_4 - \lambda_4 \nabla_k) \frac{H(-R^2)}{\sqrt{(\mathbf{k}\cdot\mathbf{r})^2 - R^2}}.$$

Account has been taken of the fact that the line $D(t)$ lies in E^3, and $\mathbf{l} = [l_i, 0]$. Assuming $x^4 = ct, X^4 = c(t-\tau)$ and using the retarded potential only, the formulae (3.12) and (3.13) may be written in the form

$$(3.14)\quad \overset{(-)}{\Omega}_{(4)} = \frac{1}{4\pi}\int\limits_{-\infty}^{t} d\tau \oint\limits_{D(\tau)} dl\epsilon^{ijk} l_j \dot{\zeta}_k \nabla_i \frac{H(t-\tau-r/c)}{r},$$

and

$$(3.15)\quad \overset{(+)}{\Omega}_{(4)} = \frac{c}{4\pi}\int\limits_{-\infty}^{t} d\tau \oint\limits_{D(\tau)} dl\epsilon^{ijk} l_j k_k \left(\nabla_i + \frac{1}{c^2}\dot{\zeta}_i \partial_t\right) \frac{H(t-\tau-r/c)}{\sqrt{c^2(t-\tau)^2 - r^2 + (\mathbf{r}\cdot\mathbf{k})^2}},$$

$$\mathbf{r} = \mathbf{x} - \boldsymbol{\zeta}(l, \tau).$$

Appendix

Formal evaluation of the integral

$$A = \int\limits_0^\infty d\varepsilon\, \delta(\varepsilon^2 \Lambda^2 - 2\varepsilon(\mathbf{\Lambda}\cdot\mathbf{R}) + R^2),$$

$$\Lambda^2 \equiv |\mathbf{\Lambda}|^2$$

occurring in Eq. (3.11) is simple. It may be written as follows:

$$(\mathrm{A}.1)\quad A = \int\limits_0^\infty d\varepsilon\delta[\Lambda^2(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)]$$

$$= \int\limits_0^\infty \left[\frac{\delta(\varepsilon-\varepsilon_1)}{|\varepsilon_1-\varepsilon_2|} + \frac{\delta(\varepsilon-\varepsilon_2)}{|\varepsilon_2-\varepsilon_1|}\right] d\varepsilon = \frac{H(\varepsilon_1)+H(\varepsilon_2)}{|\varepsilon_1-\varepsilon_2|},$$

$\varepsilon_1, \varepsilon_2$ being the roots of the equation

$$\varepsilon^2 \Lambda^2 - 2\varepsilon(\mathbf{\Lambda}\cdot\mathbf{R}) + R^2 = 0,$$

(A.2)
$$\varepsilon_1 = \frac{1}{\Lambda^2}\left[(\mathbf{\Lambda}\cdot\mathbf{R}) - \sqrt{(\mathbf{\Lambda}\cdot\mathbf{R})^2 - \Lambda^2 R^2}\right],$$

$$\varepsilon_2 = \frac{1}{\Lambda^2}\left[(\mathbf{\Lambda}\cdot\mathbf{R}) + \sqrt{(\mathbf{\Lambda}\cdot\mathbf{R})^2 - \Lambda^2 R^2}\right].$$

The problem of integration in Eq. (A.1) is now reduced to the discussion, when

$$\Delta = (\mathbf{\Lambda}\cdot\mathbf{R})^2 - \Lambda^2 R^2 > 0$$

and $\varepsilon_1, \varepsilon_2$ are positive numbers. Since our interest is confined to causal relations, let us discuss the case

$$-R^2 > 0,$$

or

$$(X^4)^2 > (X^1)^2 + (X^2)^2 + (X^3)^2 = X_i X^i = r^2.$$

(a) Discussion of the sign of Δ

Consider the sign of Δ in two separate cases $|\mathbf{\Lambda}|^2 = 1$ and $|\mathbf{\Lambda}|^2 = -1$.

(a_1) $|\mathbf{\Lambda}|^2 = 1$

$\Delta = (\mathbf{\Lambda}\cdot\mathbf{R})^2 - \Lambda^2 R^2 > 0$ always, since $(\mathbf{\Lambda}\cdot\mathbf{R})^2 > 0$ and $-R^2 > 0$ from the assumption.

(a_2) $|\mathbf{\Lambda}|^2 = -1$

(A.3)
$$\Delta = (\mathbf{\Lambda}\cdot\mathbf{R})^2 + R^2 = (\Lambda_4 X^4 + \Lambda_i X^i)^2 - (X^4)^2 + r^2$$
$$= (\Lambda_4^2 - 1)(X^4)^2 + 2\Lambda_4 X^4(\Lambda_i X^i) + [(\Lambda_i X^i)^2 + r^2].$$

This expression may be treated as a quadratic trinomial in X^4; its discriminant

(A.4)
$$\overline{\Delta} = (\Lambda_4)^2(\Lambda_i X^i)^2 - ((\Lambda_4)^2 - 1)[(\Lambda_i X^i)^2 + r^2]$$
$$= (\Lambda_i X^i)^2 + (1 - (\Lambda_4)^2) r^2 = -(\Lambda_i \Lambda^i)(X_i X^i) + (\Lambda_i X^i)^2 \leqslant 0$$

use being made of the assumption $|\mathbf{\Lambda}|^2 = -1$; hence

$$|\mathbf{\Lambda}|^2 = -(\Lambda_4)^2 + (\Lambda_i \Lambda^i) = -1, \quad 1 - (\Lambda_4)^2 = -(\Lambda_i \Lambda^i) \leqslant 0.$$

From the fact that $\overline{\Delta} \leqslant 0$ and $[(\Lambda_4)^2) - 1] \geqslant 0$, the inequality below follows:

$$\Delta = (\mathbf{\Lambda}\cdot\mathbf{R})^2 + R^2 \geqslant 0.$$

Thus it is shown that under the assumption $R^2 < 0$ real-valued roots $\varepsilon_1, \varepsilon_2$ always exist.

(b) The sings of $\varepsilon_1, \varepsilon_2$.

(b_1) $\Lambda^2 = 1$

$$\varepsilon_1 = (\mathbf{\Lambda}\cdot\mathbf{R}) - \sqrt{(\mathbf{\Lambda}\cdot\mathbf{R})^2 - R^2} < 0$$

$$\varepsilon_2 = (\mathbf{\Lambda}\cdot\mathbf{R}) + \sqrt{(\mathbf{\Lambda}\cdot\mathbf{R})^2 - R^2} > 0, \quad \text{because} \quad -R^2 > 0$$

and hence for $\Lambda^2 = 1$ the integral (A.1) assumes the value

$$A = \frac{H(\varepsilon_2)H(-R^2)}{2\sqrt{(\mathbf{\Lambda}\cdot\mathbf{R})^2-R^2}} = \frac{H(-R^2)}{2\sqrt{(\mathbf{\Lambda}\cdot\mathbf{R})^2-R^2}}.$$

(b$_2$) $\Lambda^2 = -1$

$$\varepsilon_1\cdot\varepsilon_2 = \frac{R^2}{\Lambda^2} = -R^2 > 0$$

and hence both roots are of the same sign,

$$\varepsilon_1+\varepsilon_2 = -2(\mathbf{\Lambda}\cdot\mathbf{R}).$$

Both roots $\varepsilon_1, \varepsilon_2$ are positive when $(\mathbf{\Lambda}\cdot\mathbf{R}) < 0$,

$$\mathbf{\Lambda}\cdot\mathbf{R} = \Lambda_4 X^4 + \Lambda_i X^i = (\operatorname{sign}\Lambda_4)\sqrt{1+\Lambda_i\Lambda^i}X^4 + \Lambda_i X^i.$$

Since, according to our assumption, $|X^4| > |X^i|$, the sign of the scalar product $\mathbf{\Lambda}\cdot\mathbf{R}$ depends on the sign of the product $(\operatorname{sign}\Lambda_4)X^4$. For the retarded potential $X^4 > 0$, and hence we must obtain $\Lambda_4 < 0$, that is $\Lambda^4 > 0$. For an advanced potential $X^4 < 0 \Rightarrow \Rightarrow \Lambda^4 < 0$. This fact has a simple physical interpretation: assuming for S_2 in Eq. (3.11) the surface describing the history of motion of the dislocation line, the assumption of $\mathbf{\Lambda}$ being directed towards the past ($\Lambda^4 < 0$) would mean that already at time $\tau = -\infty$ the motion of the dislocation loop should be determined for the entire time-interval $(-\infty, t)$.

The final conclusion is now the following: If $|\mathbf{\Lambda}|^2 = -1$, both roots $\varepsilon_1, \varepsilon_2$ are positive provided $\Lambda^4 > 0$. In such a case the integral (A.1) assumes the value

$$A = \frac{H(\varepsilon_1)H(-R^2)}{|\varepsilon_1-\varepsilon_2|} = \frac{H(-R^2)}{2\sqrt{(\mathbf{\Lambda}\cdot\mathbf{R})^2-\Lambda^2R^2}}$$

since the term containing $H(\varepsilon_2)$ corresponds to the advanced potential.

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