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Yielding on inclined planes at the tip of a crack

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A SIMPLE, albeit approximate, solution is given for the anti-plane strain yielding from a crack in an infinite elastic body on two planes inclined to the crack plane. Approximate analytical expressions, similar to those for a coplanar yield model, are obtained for the plastic zone spread and the crack opening displacement as functions of the applied shear stress inducing an anti-plane strain condition and the angle of inclination of the yield planes. It is shown that the range of applicability of the small-scale yielding approximation for crack tip plasticity is substantially reduced. The in-plane component of the external shear stress which is completely without effect on a coplanar yield model is also taken into account. The results are in good qualitative agreement with those obtained by far more complicated exact methods for a body loaded in uniform tension.

Podano proste choć przybliżone rozwiązanie problemu uplastycznienia ośrodka nieskończonego zachodzącego wzdłuż dwóch płaszczyzn nachylonych do płaszczyzny szczeliny w antypłaskim stanie odkształcenia. Otrzymano przybliżone wyrażenia analityczne, podobne do wyrażeń uzyskiwanych w przypadku uplastycznienia w płaszczyźnie szczeliny (model koplanarny), a opisujące propagację strefy plastycznej i rozwieranie się szczeliny w zależności od przyłożonych naprężeń ścinających; wyrażenia te uwzględniają warunki antypłaskiego stanu odkształcenia i kąty nachylenia płaszczyzn, w których zachodzi uplastycznienie. Wskazano na istotną redukcję zakresu stosowalności przybliżenia małych odkształceń plastycznych. Uwzględniono składową styczną zewnętrznych obciążeń ścinających, która nie ma żadnego wpływu w przypadku modelu koplanarnego uplastycznienia. Wyniki są jakościowo zgodne z wynikami uzyskiwanymi za pomocą dużo bardziej złożonych metod ścisłych w przypadku ciał poddanych równomiernemu rozciąganiu.

Приведено простое, хотя приближенное, решение задачи перехода в пластическое состояние бесконечной среды, происходящие вдоль двух плоскостей наклоненных к плоскости щели в антиплоском деформационном состоянии. Получено аналитическое приближенное выражение, аналогичное выражениям полученным в случае перехода в пластическое состояние в плоскости щели (копланарная модель), и описывающее распространение пластической зоны и раскрывание щели в зависимости от приложенных напряжений сдвига; эти выражения учитывают условия антиплоского деформационного состояния и углы наклонения плоскостей, в которых происходит переход в пластическое состояние. Указано на существенную редукцию области применения приближения малых пластических деформаций. Учтена касательная составляющая внешних нагрузок сдвига, которая не имеет никакого влияния в случае копланарной модели перехода в пластическое состояние. Результаты качественно совпадают с результатами полученными при помощи значительно более сложных точных методов в случае тел подвергнутых равномерному растяжению.

1. Introduction

THE PROBLEM of plastic yielding near a crack tip confined to zones coplanar with the crack plane was first studied by DUGDALE [1] for a body loaded in uniform tension (so-called Mode I in the terminology of fracture mechanics) and by BILBY, COTTRELL and SWINDEN [2] for a body under anti-plane shear stress (Mode III). The latter described the plastic zone by an array of screw dislocations coplanar with the crack. LARDNER [3] used a tedious procedure to extend this model to yielding on an inclined slip-plane — a phenomenon often observed in single crystals and in polycrystalline aggregates. Plastic deformation in these materials is usually confined to planes inclined to the crack plane. In single crystals the angle of inclination is determined by their crystallographic structure and in polycrystalline aggregates it often ranges between 60° and 70°.

Plastic yielding under plane-strain tension on slip-planes inclined to the crack plane has been treated approximately by RICE [4], numerically by BILBY and SWINDEN [5] for an angle of inclination of 45° and more exhaustively by VITEK [6] and RIEDEL [7].

The study of a non-coplanar yield model is useful in another sense, too. From the point of studying the fracture characteristics — yield zone length and crack opening displacement — of a solid, attention is normally concentrated on low levels of applied stress when the plastic zones are small — so-called small-scale yielding — and the solution is simple [8]. However, the small-scale yielding approximation [8] has a limited range of applicability as suggested by the finite-element results of LARSSON and CARLSSON [9]. It was shown that the deviations from small-scale yielding approximation at substantially lower levels of applied stress are caused by the portion of the non-singular but non-vanishing stress field at the crack tip that results in a stress T acting parallel to the crack plane. It transpired that the deviation of the plastic zone length and the crack opening displacement is linear in T and not quadratic as had hitherto appeared from solutions of the type proposed by DUGDALE [1] and BILBY, COTTRELL and SWINDEN (BCS) [2]. These planar yield models are completely without effect from T. To take the latter into account it is necessary to let the yield spread off the plane of the crack. However, it is well known that the BCS planar yield model gives, in a rather simple manner, a quantitative estimate of the plastic zone length in plane-strain conditions and crack opening displacement that is perfectly analogous to that obtained by a more involved macroscopic analysis [4].

It was therefore thought highly desirable to reconsider the BCS model in the light of its above-mentioned deficiencies and to try to modify it in such a way as to allow for the yield to spread off the crack plane and to take into account the T-stress effect without jeopardizing its inherent simplicity. Thus, to avoid the inevitable complicated calculations [3, 6, 7] the screw dislocations representing the inclined slip bands are replaced by superdislocations of appropriate Burgers vectors. It is expected that such a superdislocation approach [10–13] will give an upper bound to the plastic zone length and the crack opening displacement.

2. Theory

Before proceeding with the description of the plastic yielding on inclined planes at the crack tip and the effect of the in-plane stress component (T-stress) it is worth mentioning that the effect of this stress in the anti-plane shear mode has been undetected simply because this non-vanishing but non-singular stress at the crack tip is present only when either the external loads are unsymmetrical relative to the crack plane or the boundary conditions are such as to hinder displacement along the z-axis (i.e. the effect of the finite dimensions of the body).

The situation is much simpler to visualize in the former case, when the loads relative to the crack plane are unsymmetrical. Let the mode III crack (along $y = 0$) contained in an infinitely long (along z-axis) body with a finite rectangular cross-section ($2B \times 2A$) be modelled by a distribution of long straight screw dislocations parallel to the z-axis and lying along the plane $y = 0$. The relative displacement is, of course, in the z-direction. Let the faces of the body $y = \pm A$ be acted upon by stresses σ_{yz}^{A} and σ_{yz}^{-A} respectively, ($\sigma_{yz}^{A} \neq \sigma_{yz}^{-A}$). Note that the external shear stress is not acting at "infinity". Then, by resolving the loading on the two faces, as usual, into a symmetrical part $(\sigma_{yz}^{A} + \sigma_{yz}^{-A})/2 = \sigma$ and an antisymmetrical part $(\sigma_{yz}^{A} - \sigma_{yz}^{-A})/2 = \tau$ and allowing for the moment equilibrium of the latter, it is easy to show that there must exist a uniform shear stress along the faces $x = \pm B$ equal in magnitude to $\sigma_{xz} = \tau A/B = T$. In particular, $T \to 0$ as $B \to \infty$ or $\tau \to 0$.

Now, the external shear stress σ symmetrical relative to the crack plane subjects the screw dislocations to a force in the x-direction. The antisymmetrical forces τ on the faces $y = \pm A$ subject them to equal and opposite forces in the x-direction that cancel each other. The net effect so far as the dislocations in plane $y = 0$ are concerned is that the force in the x-direction due to an unsymmetrical external loading is the same as due to its symmetrical part σ.

It is also well known that an external shear stress $\sigma_{xz} = T$ subjects screw dislocations to a force acting in the y-direction. Thus it is evident that in such a situation a planar yield model is deficient since there is bound to be some spreading of the plasticity off the crack plane. This seems to be in perfect agreement with experimental observations since experiments are, of necessity, performed on specimens with finite dimensions where the boundary conditions play an important role. It is thus clear that the resistance to the motion of the dislocations modelling the slipped region ahead of the crack tip will be altered because of their being subjected to forces in both the x and y directions. It is obvious, therefore, that any realistic model should permit spreading of the plasticity off the crack plane.

As in the original BCS model let the mode III crack of length $2c$ within the region $|x| < c$ be represented by "crack" dislocations of screw orientation each of Burgers vector b. Because of symmetry the crack is slipping freely, and there will be no resistance

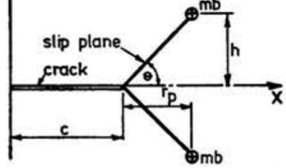

FIG. 1. Schematic picture of the crack and inclined slip planes. The crack extends over $-c \leqslant x \leqslant c$, $-\infty \leqslant z \leqslant \infty$, $y = 0$.

to the motion of the "crack" dislocations. Furthermore, let the plastic bands ahead of the crack tips and lying within the regions $-a < x < -c$ and $c < x < a$ make an angle $\pm\theta$ with the crack plane. These bands are also assumed to be represented by screw dislocations each of Burgers vector b. However, to avoid tedious computations involved in arriving at the influence of the dislocations in the inclined slip planes on the "crack" dislocations, the former are replaced by superdislocations of Burgers vector mb situated at $x = \pm a$ on a parallel plane a distance $\pm h$ from the crack plane $(\tan\theta = h/(a-c))$. Such a replacement overestimates slightly the interaction forces [10–12], but the error involved is insignificant. Likewise, an estimate of the influence of "crack" dislocations on the dislocations representing the inclined slip band is obtained by replacing the former by superdislocations of Burgers vector nb (n is the number of dislocations in the region $0 < x < c$) situated at $x = \pm c$, $y = 0$. It is further assumed that the superdislocations replacing the screw dislocations in the slip bands are of a very short range nature such that their influence is felt only in the slipped regions directly ahead of the crack tips. The simplified yield model thus obtained is a fairly accurate representation of the true picture. It should be mentioned that the yield is assumed to take place according to von Mises' criterion ($\sigma_0^2 = \sigma_{yz}^2 + \sigma_{xz}^2$).

As in the original BCS model let there be $f(x)dx$ dislocations in a distance dx. The aim is to determine $f(x)$ and the relation between c and a as a function of the physical parameters σ, σ_0 and T. The shear stress σ_{yz} at x due to the dislocations at x' is

$$\sigma_{yz}(x) = Af(x')dx'/(x-x'),$$

where the constant A has the value $\mu b/2\pi$, μ being the shear modulus. The requirement that the resultant shear stress on any dislocation in the distribution along $y = 0$ be zero when the system is in equilibrium leads to a singular integral equation

$$\int\limits_D \frac{f(x')dx'}{(x-x')} = P(x)/A, \tag{2.1}$$

where D covers the whole region $-a < x < a$, and $P(x)$ is the resultant external shear stress at x (σ_{yz} for dislocations along $y = 0$). In view of the fact that the crack is slipping freely and the assumption regarding the short-range nature of the superdislocations (this restriction can, however, be relaxed), $P(x)$ can be written as

$$\begin{aligned} P(x) &= \sigma; \quad |x| < c, \\ &= \sigma - \sigma_0 - \frac{2mA(x-a)}{(x-a)^2+h^2}; \quad c < |x| < a, \end{aligned} \tag{2.2}$$

symmetry consideration being taken into account. The variable part in the expression for $P(x)$ within the regions $c < |x| < a$ is the influence of the super-dislocations situated at $|x| = a$, $|y| = h$ on the distributed dislocations within the plastic regions ahead of the crack tips. It is clear that relaxation of the restriction on the short range nature of super-dislocations would introduce a corresponding variable term into the expression for $P(x)$ within the freely slipping crack $|x| < c$.

The general solution of the singular integral equation (2.1) is given by (MUSKHELISHVILI [14])

$$f(x) = -\frac{\sqrt{x^2-a^2}}{\pi^2 A} \int\limits_{-a}^{a} \frac{P(x')dx'}{(x'-x)\sqrt{x'^2-a^2}}, \tag{2.3}$$

subject to the condition

$$\int_{-a}^{a} \frac{P(x)\,dx}{\sqrt{x^2-a^2}} = 0 \tag{2.4}$$

which assures vanishing of the function $f(x)$ (no relative displacement of the crack faces) at $x = \pm a$.

Like the dislocations in the plane $y = 0$, the superdislocation of Burgers vector mb representing the inclined slip band is subject to equilibrium considerations. Furthermore, the stresses at its site must fulfil the yield criterion. The superdislocation is in equilibrium under the action of external stresses σ and T and the repulsive stress from dislocations of like sign in the plane $y = 0$. An approximate estimate of the latter (indeed, the superdislocation technique gives the upper bound) can be obtained by replacing the dislocations in $|x| < c$ and $y = 0$ by a superdislocation of Burgers vector nb situated at $|x| = c$, $y = 0$. If this is done, it is easy to show (WEERTMAN and WEERTMAN [15]) that the repulsive force between two like dislocations of Burgers vectors mb and nb, respectively, situated a distance r ($r = h/\sin\theta$) apart is equal to

$$F = \frac{A \cdot nb \cdot mb}{2\pi r}(\cos\theta i + \sin\theta j),$$

whence it follows that the stresses induced at mb are

$$\sigma_{yz} = -\frac{An\cos\theta}{r} = -\frac{An(a-c)}{(a-c)^2+h^2},$$

$$\sigma_{xz} = \frac{An\sin\theta}{r} = \frac{Anh}{(a-c)^2+h^2},$$

with $A = \mu b/2\pi$. Moreover, the yield criterion would require

$$\left(\sigma - \frac{An\cos\theta}{r}\right)^2 + \left(T + \frac{An\sin\theta}{r}\right)^2 = \sigma_0^2, \tag{2.5}$$

whence it follows that

$$\frac{An}{r} = \sigma_0\left[\frac{\sigma}{\sigma_0}\cos\theta - \frac{T}{\sigma_0}\sin\theta + \sqrt{1-\left(\frac{\sigma}{\sigma_0}\sin\theta + \frac{T}{\sigma_0}\cos\theta\right)^2}\right]. \tag{2.6}$$

For small-scale yielding, when $\frac{\sigma}{\sigma_0}, \frac{T}{\sigma_0} \ll 1$, Eq. (2.6) is simplified to

$$\frac{An}{r} = \sigma_0\left[1 - \left(\frac{T}{\sigma_0}\sin\theta - \frac{\sigma}{\sigma_0}\cos\theta\right)\right].$$

An approximate relation between h and $(a-c)$ could be established by following arguments from dislocation kinematics (KARIHALOO [11, 12]).

However, without loss of accuracy, it is reasonable to assume that the number of dislocations in the crack, n, and in each of the slip bands, m, is proportional to the length of the crack and the slip band, i.e. $n/m = c/r$. With these remarks in mind, Eq. (2.6) may be rewritten as

$$mA = \frac{An}{r}\frac{(a-c)^2}{c\cos^2\theta}. \tag{2.7}$$

Now, from Eq. (2.2) it is evident that the resistance to the movement of dislocations in the plastic regions ahead of the crack tips reduces progressively as we move from the crack tips $|x| = c$, vanishing ultimately at the plastic zone tips $|x| = a$. In analogy with the BARENBLATT model [16] the cohesive force between the crack faces is a function of the distance from the crack tip. However, SMITH [16] has recently shown that the cohesive zone size and the crack opening-displacement are relatively insensitive to the form of the stress-displacement relationship within the cohesive zone. Thus, within permissible limits of accuracy it is reasonable to replace the variable resistance force by a certain average value. The latter is obtained by equating the actual plastic dissipation energy (area under the variable resistance force curve over the plastic zone) with that due to an average, constant resistance force

$$\sigma_{av} = 2\frac{An}{r}\frac{(a-c)}{c\cos^2\theta}\ln(\sin\theta).$$

From Eq. (2.7) it follows that

$$\sigma_{av} = 2\sigma_0\frac{(a-c)}{c\cos^2\theta}\ln(\sin\theta)\left[\frac{\sigma}{\sigma_0}\cos\theta - \frac{T}{\sigma_0}\sin\theta + \sqrt{1-\left(\frac{\sigma}{\sigma_0}\sin\theta + \frac{T}{\sigma_0}\cos\theta\right)^2}\right]. \tag{2.8}$$

Therefore, Eq. (2.2) may be rewritten as

$$\begin{aligned} P(x) &= \sigma; \quad |x| < c, \\ &= \sigma - \sigma^*; \quad c < |x| < a \end{aligned} \tag{2.9}$$

where, in the absence of T-stress,

$$\begin{aligned} \sigma^* &= \sigma_0 + \sigma_{av}, \\ &= \sigma_0\left\{1 + 2\frac{(a-c)}{c\cos^2\theta}\ln(\sin\theta)\left[\frac{\sigma}{\sigma_0}\cos\theta + \sqrt{1-\left(\frac{\sigma}{\sigma_0}\sin\theta\right)^2}\right]\right\}. \end{aligned} \tag{2.10}$$

Note that the second term within the radical is always negative. It is interesting to make the following two important remarks on Eqs. (2.9) and (2.10). Firstly, the form of the resistance force in the plastic zones is similar to that used in the original BCS model if we replace σ_0 in that model by the modified resistance stress σ^*. Secondly, and what is more important, the resistance encountered by the dislocations in the slipped regions ahead of the crack tips is always less than σ_0 — the yield stress — irrespective of the existence or, otherwise, of the stress T. In fact, by working backwards it is easy to show that if the resistance to the movement of dislocations in the regions $c < |x| < a$ were assumed to be equal to σ_0 as in the original BCS and Dugdale models, the yield criterion would be violated along yield bands not coplanar with the crack plane.

From the first of the above two remarks it follows that the formulae derived in the original BCS paper are directly applicable to the generalized BCS model if one replaces

σ_0 in these formulae by σ^*. In particular, the expressions for the projection of the plastic zone on the crack plane $r_p = (a-c)$ and the crack opening displacement δ_t, which is directly related to the number of dislocations representing the crack, would take the form

$$r_p = c\left[\sec\frac{\pi}{2}\frac{\sigma}{\sigma^*}-1\right],$$

(2.11)

$$\delta_t = \frac{4c}{\pi\mu}\sigma^*\ln\left(\frac{a}{c}\right).$$

Note, however, that the expression for the projection of the plastic zone on the crack plane, r_p, is an implicit one because σ/σ^* also contains r_p/c. In fact, the first of the two expressions (2.11) can be rewritten as

$$\sigma/\sigma^* = \frac{2}{\pi}\cos^{-1}[1/(1+r_p/c)], \tag{2.12}$$

whence it follows that

$$(\sigma/\sigma_0)^2\left(\frac{1}{A_1^2}-2\frac{A_2}{A_1}\cos\theta+A_2^2\right)-\frac{\sigma}{\sigma_0}\left(\frac{2}{A_1}-2A_2\cos\theta\right)+(1-A_2^2)=0, \tag{2.13}$$

where

$$A_1 = \frac{2}{\pi}\cos^{-1}[1/(1+r_p/c)],$$

$$A_2 = 2\frac{r_p}{c\cos^2\theta}\ln(\sin\theta).$$

Note, if $\theta = 0$ (coplanar yield model), $\sigma/\sigma_0 = A_1$.

Thus the length of the plastic zone is found in an inverse manner whereby, for a given $\theta, r_p/c$ is assumed and the corresponding value of σ/σ_0 is calculated from Eq. (2.13).

3. Conclusions

Figure 2 shows the variation of r_p/c with σ/σ_0 for various values of the angle of inclination of plastic bands, θ.

Although the exact significance of mode III anti-plane shear crack to that of mode I tensile crack is as yet uncertain, some similarities are easily observed as is clear from a comparison of Fig. 2 with Fig. 3 of VITEK [6] and Fig. 2 of RIEDEL [7]. The results obtained by the present superdislocation technique seem to be in good qualitative agreement with those of [6, 7] for small-scale yielding. The results diverge with increasing σ/σ_0. Likewise, the results diverge as θ increases. This may be a consequence of the simplifying assumptions made in the present analysis. However, for the practically important range of the external stress level the results are in reasonable agreement. More importantly, the simple nature of the BCS model is retained.

It should be mentioned that inclusion of the T-stress in the analysis gave results similar to those of RICE [4] and RIEDEL [7]. In particular, the deviation of r_p/c from the planar yield model $\sigma/\sigma_0 = A_1$ was linear in T/σ_0 and not quadratic, as would follow from an expansion of Eq. (2.12) in powers of σ/σ_0 with σ^* replaced by σ_0.

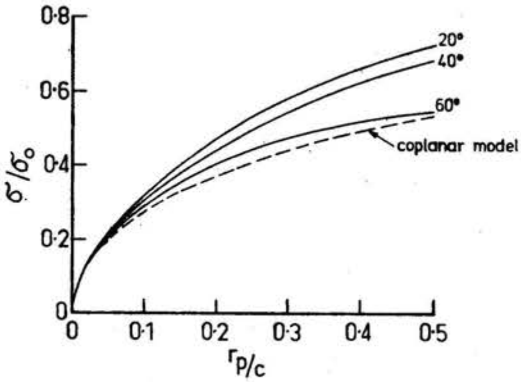

FIG. 2. Variation of the projection of the plastic zone on the crack plane, r_p with the externally applied shear stress, σ for various values of θ, c — half the crack length, σ_0 — yield stress of the material. Dashed line corresponds to the coplanar yield model[2].

Finally, it may be pointed out that the present technique has an added advantage in that it can be used to simulate a stress-displacement law (flow-characteristics) in the plastic regions different from the ideal plasticity relationship ($\sigma^* = \text{const}$) if $P(x)$ is treated as a variable. The resulting analysis would be further complicated. However, such a possibility has important implications inasmuch as the flow characteristics of real materials deviate substantially from the theoretical ideal plasticity approximation.

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