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On constitutive equations and numerical solution of the multidimensional problems of the dynamics of nonisothermic elastic-plastic media with finite deformations

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A SIMPLE model of nonisothermal elastic-plastic media subject to finite deformations is described. This model is built on the basis of the laws of thermodynamics and the principle of minimum of irreversible forces. It is shown that the plasticity condition enables us to define the dissipation function and to obtain the constitutive equations. The results of numerical calculations of certain axi-symmetrical two-dimensional problems are given.

Opisano prosty model nieizotermicznego ośrodka sprężysto-plastycznego poddanego skończonym odkształceniom. Model ten zbudowany jest na podstawie praw termodynamiki oraz na zasadzie minimum nieodwracalnych sił. Pokazano, że warunek plastyczności pozwala zdefiniować funkcję dysypacji i wyprowadzić równania konstytutywne. Podano wyniki obliczeń numerycznych dotyczące kilku osiowo-symetrycznych zagadnień dwuwymiarowych.

Описана простая модель неизотермической упруго-пластической среды подвергнутой конечным деформациям. Эта модель построена на основе законов термодинамики и на принципе минимума необратимых сил. Показано, что условие пластичности позволяет определить функцию диссипации и вывести определяющие уравнения. Приведены результаты численных расчетов, касающихся нескольких осесимметрических двумерных задач.

Introduction

IN A SERIES of recent works attempts have been made to obtain equations which would reflect the main features of the behaviour of elastic-plastic bodies under the conditions of a dynamic load at finite deformations and would be fit to use in specific calculations [1–7].

A detailed survey of the papers devoted to the numerical solution of the dynamic equations of the elastic-plastic media is given in the paper of the authors [7].

In the present work, a simple enough and convenient for practical applications model of a nonisothermic elastic-plastic medium at finite deformations is given. This model is built on the basis of the laws of thermodynamics and the quasi-thermodynamic principle of the minimal irreversible forces. The formulation of the principle in terms of the problem on the conditional extremum for the rate of dissipation at some restrictions is given. It is shown that setting the plasticity condition permits to define the function of dissipation and to obtain the constructive equations of the medium. At small isothermic deformations the equations produced coincide with the classical equations of Prandtl-Reiss.

A complete system of equations of the considered medium noted in the current Lagrangian and Euler frames of reference is given. The hyperbolicity of the system is proved and the expressions for the rates of propagation of weak disturbances are obtained. We propose to reduce the system of differential equations to a special canonic form containing the derivatives only in the direction of the bi-characteristics.

The notation of the equations in such a form proves to be very convenient for the construction of the stable characteristic finite-difference schemes.

The results of the numerical calculations of some two-dimensional axi-symmetrical problems are given. Some interesting peculiarities in the solution are noted.

1. Kinematics of elastic-plastic deformations

We shall consider the nonsteady flows of the nonstrengthening elastic-plastic bodies, the material of which is assumed to be homogeneous, isotropic and the deformations are considered to be finite. To describe the behaviour of the material medium we shall use, as this is done in the papers of [8], the following frames of reference:

1) a current Lagrangian system $\xi^1\xi^2\xi^3$ with a mobile basis $\hat{\mathfrak{s}}_i = \dfrac{\partial \mathbf{r}}{\partial \xi^i}$, where $\mathbf{r}$ is the radius-vector of the material particle;

2) the initial Lagrangian system $\xi^1\xi^2\xi^3$ with an immovable basis $\mathring{\mathfrak{s}}_i$ which, in the general case, is not the basis coinciding with $\hat{\mathfrak{s}}_i$ in the moment of time $t = 0$ but is often chosen owing to other considerations. For example, when using Lagrangian methods of numerical calculation, it is convenient to choose the basis $\mathring{\mathfrak{s}}_i$ so that the body occupies a simple region with a regular calculating net in the space ξ^k;

3) the Euler frame of reference $x^1x^2x^3$ with a fixed basis $\mathfrak{s}_i$. In the Euler system the coordinates of particles are variable and are defined by the law of motion

$$x^k = x^k(\xi^i, t), \quad i, k = 1, 2, 3, \quad t \geqslant 0.$$

Let $\mathbf{v} = \left(\dfrac{d\mathbf{r}}{dt}\right)_{\xi^i = \text{const}} = v_i \mathfrak{s}^i = \hat{v}_i \hat{\mathfrak{s}}^i$ be the velocity vector of the fixed particle, $\sigma = \sigma_{ij}\mathfrak{s}^i\mathfrak{s}^j = \hat{\sigma}_{ij}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j$ — the Euler stress tensor, $g = g_{ij}\mathfrak{s}^i\mathfrak{s}^j = \hat{g}_{ij}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j$ — the metric tensor, ϱ — the material density, ∇_k — the co-variant derivative over the spatial coordinate x^k, d/dt — the complete derivative over the time. Everywhere, if not specially mentioned, the repeated indices mean summing up along them. As in the paper of [8] we denote the values referring, accordingly, to the initial and current states by the symbols (°) and (^).

For the purpose of describing the kinematic characteristics of an elastic-plastic rigid body, we shall use the approach based on the transformation of the basic vectors accompanying the process of deformation.

Let the transformation of the co-variant basic vectors $\mathring{\mathfrak{s}}_i$ of the initial Lagrangian system into the basis $\hat{\mathfrak{s}}_i$ of the current frame of reference be realized by the matrix $F(\xi^i, t)$,

$$\hat{\mathfrak{s}}_i = F_i^k \mathring{\mathfrak{s}}_k. \tag{1.1}$$

To describe the kinematics of the finite deformations we shall use the function of the matrix F which would be the measure of deformation:

$$\frac{1}{2}(F\mathring{g}F^T - \mathring{g})_{ij} = \frac{1}{2}(\hat{g}_{ij} - \mathring{g}_{ij}),$$

where $\mathring{g}_{ij}$ is the matrix of the metric tensor components of the initial Lagrangian system of reference. In the basis $\hat{\mathfrak{s}}^i$ the whole complex of components obtained will then represent the Almansi tensor of finite deformations. Apart from the tensor of full strains we shall introduce the tensors of elastic and plastic deformations. To describe the two latter tensors besides the initial and the current systems we shall use the Lagrangian frame of reference corresponding to the unloaded state. Let us denote the basis of this frame of reference as $\overset{*}{\mathfrak{s}}_i$, and the metric tensor as $\overset{*}{g} = \overset{*}{g}_{ij}\overset{*}{\mathfrak{s}}^i\overset{*}{\mathfrak{s}}^j$. The asterisk over the values will henceforth denote the unloaded state.

In a number of contemporary works on the theory of plasticity the description of the elastic-plastic media is based on splitting the matrix F up into the product of matrices E and P:

$$F = EP. \tag{1.2}$$

The matrices P and E may be interpreted as those of the transformation of the basic vectors $\mathring{\mathfrak{s}}_i$ into $\overset{*}{\mathfrak{s}}_j$ and $\overset{*}{\mathfrak{s}}_j$ into $\hat{\mathfrak{s}}_m$, correspondingly. Using these vectors it is easy to express the matrix of the tensor $\mathbf{e}$ components. Taking the relation (1.2) into account, we have

$$\hat{e}_{ij} = \frac{1}{2}(EP\mathring{g}P^TE^T - P\mathring{g}P^T + P\mathring{g}P^T - \mathring{g})_{ij} = \frac{1}{2}(\hat{g}_{ij} - \overset{*}{g}_{ij}) + \frac{1}{2}(\overset{*}{g}_{ij} - \mathring{g}_{ij}).$$

From this one can see that it is possible to introduce the tensor of the finite elastic deformation

$$\mathbf{e}^{(e)} = \hat{e}^{(e)}_{ij}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j = \frac{1}{2}(\hat{g}_{ij} - \overset{*}{g}_{ij})\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j \tag{1.3}$$

and the tensor of the finite plastic deformation

$$\mathbf{e}^{(p)} = \hat{e}^{(p)}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j = \frac{1}{2}(\overset{*}{g}_{ij} - \mathring{g}_{ij})\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j. \tag{1.4}$$

For the tensors (1.3)–(1.4) the following tensor equality is valid

$$\mathbf{e} = \mathbf{e}^{(e)} + \mathbf{e}^{(p)}.$$

This equality is carried out for components with any structure of the indices in the space of the metric tensor $\mathbf{g}$.

We shall also note that the tensors of the elastic and plastic deformations introduced by the formulae (1.3)–(1.4) are independent of the arbitrary orthogonal transformations of the unloaded state.

Now we shall define the velocity tensors of elastic, plastic and full deformations in the following way:

$$\begin{aligned} \varepsilon^{(e)} &= \hat{\varepsilon}^{(e)}_{ij}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j = \frac{\partial \hat{e}^{(e)}_{ij}}{\partial t}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j, \\ \varepsilon^{(p)} &= \hat{\varepsilon}^{(p)}_{ij}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j = \frac{\partial \hat{e}^{(p)}_{ij}}{\partial t}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j, \\ \varepsilon &= \hat{\varepsilon}_{ij}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j = \frac{\partial \hat{e}_{ij}}{\partial t}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j. \end{aligned} \tag{1.5}$$

For the velocity tensors of deformation (1.5) the following property holds:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}, \tag{1.6}$$

which is essentially used further for constructing the model of finitely deformed elastic-plastic bodies.

2. Some thermodynamic relations

Let us assume the state of the particle ξ^k to be fully determined if, for this particle, we know the spatial position $x^i(\xi^i, t)$, the stress tensor σ_{ij}, the tensor of the elastic deformation $e_{ij}^{(e)}$, the absolute temperature θ, the specific (per unit mass) inner energy U (or some other thermodynamic potential), the specific entropy η and the density of the medium ϱ. We shall ignore the mass forces, the thermal conduction of the material, and the mass sources of heat.

For the above values we must observe the local laws of mass conservations:

$$\varrho = \mathring{\varrho} \det||\partial \mathring{x}^i/\partial x^j|| = \mathring{\varrho}(|\mathring{g}_{ik}|/|\hat{g}_{jm}|)^{1/2}, \tag{2.1}$$

momentum:

$$\varrho \frac{\partial \hat{v}^i}{\partial t} = \hat{\nabla}_i \sigma^{ij}, \tag{2.2}$$

energy:

$$\varrho \frac{\partial U}{\partial t} = \hat{\sigma}^{ij} \hat{\varepsilon}_{ij} \tag{2.3}$$

and the second law of thermodynamics or the postulate of the positiveness of the entropy. Note this in the form of the Clausius-Duhem inequality:

$$\varrho \dot{\eta} \geqslant 0. \tag{2.4}$$

We shall consider the set of values $e_{ij}^{(e)}$, θ as independent parameters of state. We shall assume the free energy $F = U - \theta\eta$ to be a smooth enough function of the independent variables of state.

Using the law of the conservation of energy the inequality (2.4) may be written in the following form:

$$-\dot{F} - \eta\dot{\theta} + \frac{1}{\varrho} \hat{\sigma}^{ij} \hat{\varepsilon}_{ij} \geqslant 0$$

whence it follows

$$\left[\frac{1}{\varrho} \hat{\sigma}^{ij} - \frac{\partial F}{\partial \hat{e}_{ij}^{(e)}}\right] \hat{\varepsilon}_{ij}^{(e)} - \left(\eta + \frac{\partial F}{\partial \theta}\right) \dot{\theta} + \frac{1}{\varrho} \hat{\sigma}^{ij} \hat{\varepsilon}_{ij}^{(p)} \geqslant 0. \tag{2.5}$$

The inequality (2.5) leads to the following relations:

$$\hat{\sigma}^{ij} = \varrho \frac{\partial F}{\partial \hat{e}_{ij}^{(e)}} \quad \eta = -\frac{\partial F}{\partial \theta}. \tag{2.6}$$

Taking Eq. (2.6) into account, the Clausius-Duhem inequality changes to the so-called inequality of the internal dissipation:

(2.7)
$$D = \hat{\sigma}^{ij}\hat{\varepsilon}^{(p)}_{ij} \geqslant 0.$$

The further concrete definition of the model discussed is based on the fact that we shall restrict our considerations to the isotropic homogeneous medium, the free energy of which may be represented in the form

(2.8)
$$F = F(I^e_k, \theta), \quad k = 1, 2, 3,$$

where I^e_k are the invariants of the elastic deformation tensor.

Note further that the deviator components of the tensor of elastic deformations are the values of the order of the ratio of the yield strength to the elasticity modulus. This ratio for most of the metals equals 10^{-2}–10^{-3} and the square of it may be ignored as compared to unity.

Let us also introduce the assumption of the plastic incompressibility of the material in the form of

(2.9)
$$\overset{*}{\varrho} = \mathring{\varrho}$$

or

(2.10)
$$\varrho = \mathring{\varrho}(1-2I^e_1+4I^e_2-8I^e_3)^{1/2}.$$

After introducing these assumptions, the relations (2.6) may be written in the form

(2.11)
$$\hat{\sigma}_{ij} = \lambda_1(I^e_1, \theta)I^e_1\hat{g}_{ij}+2\mu_1(\theta)\hat{e}^{(e)}_{ij}-\alpha_1(\theta)\theta\hat{g}_{ij}$$

which is a thermo-elastic relation that takes into account the finality of the volume and plastic deformations of the material with the elastic characteristics independent of the preceding plastic deformations.

In Eq. (2.11) we supposed that I^e_1 has not to influence the connection of the tensor stress deviator with the elastic deformation deviator. The functions $\mu_1(\theta)$ and $\alpha_1(\theta)$ may be defined on the basis of the experimental relations of the shift modulus and of the coefficient of the linear dilatation with temperature. The function $\lambda_1(I^e_1, \theta)$ is restored according to the relation $p = p(\varrho, \theta)$ known for many substances [11].

Further, we shall need the relation (2.11) in the differential form. We shall use the matrix $\partial\hat{\sigma}_{ij}/\partial t$ as a characteristic of the velocity of the stress tensor change. From Eq. (2.11) and with the above accepted assumptions, it follows that

(2.12)
$$\frac{\partial\hat{\sigma}_{ij}}{\partial t} = \lambda_2\dot{I}^{(e)}_1\hat{g}_{ij}+2\mu_2\hat{\varepsilon}_{ij}-2\mu_1\hat{\varepsilon}^{(p)}_{ij}-\alpha_2\dot{\theta}\hat{g}_{ij}+\frac{1}{\mu_2}\frac{\partial\mu_1}{\partial\theta}\dot{\theta}\hat{s}_{ij},$$

where

$$\dot{I}^e_1 = \frac{\partial I^e_1}{\partial t}, \quad s_{ij} = \sigma_{ij}-\frac{1}{3}\sigma^{m}_{\cdot m}g_{ij}, \quad \lambda_2 = \lambda_1+I^e_1\frac{\partial\lambda_1}{\partial I^e_1},$$

$$\mu_2 = \mu_1+\lambda_1 I^e_1-\alpha_1\theta, \quad \alpha_2 = \alpha_1+\theta\frac{\partial\alpha_1}{\partial\theta}-I^e_1\frac{\partial(\lambda_1+2/3\mu_1)}{\partial\theta}$$

and the last term may be neglected if $\dot{\theta}(\partial\mu_1/\partial\theta)$ is the value of the order of the yield strength.

2.1. Plastic domain

Let us note the assumption of the plastic incompressibility of the material (2.9) in the differential form. Differentiate the equality $\det \mathbf{P} = 1$ equivalent to Eq. (2.9) and obtain

$$\operatorname{tr}[\dot{\mathbf{P}}\mathbf{P}^{-1}] = 0. \tag{2.13}$$

From the formula of the non-singular matrix we have

$$\frac{\partial}{\partial t}\det \mathbf{A} = \det \mathbf{A}\cdot \operatorname{tr}(\dot{\mathbf{A}}\mathbf{A}^{-1}).$$

From Eq. (2.13) it follows that $\operatorname{tr} \varepsilon^{(p)}$ determined by the relation

$$\operatorname{tr}\varepsilon^{(p)}_{ii} = \frac{1}{2}\operatorname{tr}(\dot{\mathbf{P}}\mathring{g}\mathbf{P}^T + \mathbf{P}\mathring{g}\dot{\mathbf{P}}^T)$$

turns to zero in the unloaded space $\overset{*}{\mathbf{э}}_i$.

Using the components of the velocity of the plastic deformation $\hat{\varepsilon}^{(p)}_{ij}$ we can write down Eq. (2.13) in the form

$$\hat{d}^{ij}\hat{\varepsilon}^{(p)}_{ij} = 0 \tag{2.14}$$

and express the tensor $\hat{d}^{ij}$ using the stress tensor $\hat{\sigma}_{ij}$. Really, bearing in mind Eqs. (1.3)–(1.5) we may note

$$\hat{d}^{ij}\hat{\varepsilon}^{(p)}_{ij} = \frac{1}{2}\hat{d}^{ij}(\dot{P}^{\alpha}_{\cdot i}P^{\beta}_{\cdot j}\mathring{g}_{\alpha\beta} + P^{\alpha}_{\cdot i}\dot{P}^{\beta}_{\cdot j}\mathring{g}_{\alpha\beta}) = \hat{d}^{ij}P^{\alpha}_{\cdot i}\mathring{g}_{\alpha\beta}P^{\beta}_{\cdot m}P^{-1m}_{\cdot\gamma}\dot{P}^{\gamma}_{\cdot j} = 0.$$

If we define $\hat{d}^{ij}$ so that the equality

$$\hat{d}^{ij}P^{\alpha}_{\cdot i}\mathring{g}_{\alpha\beta}P^{\beta}_{\cdot k} = \hat{d}^{ij}\overset{*}{g}_{ik} = \delta^j_k$$

is valid, then, taking (2.13) into account, this equality may be reduced to the form

$$\hat{d}^{ij}(\hat{g}_{ik} - 2\hat{\epsilon}_{ik}/(1-2/3I^e_1)) = \delta^j_k$$

as

$$P^{\alpha}_{\cdot i}P^{\beta}_{\cdot k}\mathring{g}_{\alpha\beta} = \overset{*}{g}_{ik} = \hat{g}_{ik} - 2\hat{e}^{(e)}_{ik} = (1-2/3\, I^e_1)\hat{g}_{ik} - 2\hat{\epsilon}_{ik},$$

where ϵ_{ik} is the deviator of the elastic deformations.

Convoluting this equality with $\left(\hat{g}^{km} + \dfrac{2\hat{\epsilon}^{km}}{(1-2/3I^e_1)}\right)$ and eliminating the terms of the order of $0(\hat{\epsilon}^{\alpha\beta}\hat{\epsilon}_{\alpha\beta})$, we get

$$\hat{d}^{ij}_{\cdot ii} = \hat{g}^{ij}_{\cdot ii} - \frac{\hat{s}^{ij}}{2\mu}, \qquad \mu = \mu_1(1-2/3I^e_1). \tag{2.15}$$

Let us consider now the thermodynamic inequality of dissipation (2.7). On the basis of Eq. (2.14), Eq. (2.7) may be written as

$$D = (\hat{\sigma}^{ij} - \beta\hat{d}^{ij})\hat{\varepsilon}^{(p)}_{ij} = \hat{q}^{ij}\varepsilon^{(p)}_{ij} \geqslant 0. \tag{2.16}$$

Since Eq. (2.16) is valid for any $\hat{\varepsilon}^{(p)}_{ij}$, it follows that $\hat{g}^{ij}$ is the function of $\hat{\varepsilon}^{(p)}_{ij}$ and, perhaps, of some other parameters of the state, i.e.

$$\hat{q}^{ij} = \hat{\psi}^{ij}(\hat{\varepsilon}^{(p)}_{\alpha\beta}\ldots) \quad \text{at} \quad D \geqslant 0. \tag{2.17}$$

Since the constitutive equations of the elastic-plastic medium should not depend on the scale of time, $\hat{\psi}^{ij}$ are the homogeneous functions of the zero power and, therefore, the six values of $\hat{q}^{ij}$ are the functions of the five independent variables, for example, $\hat{\varepsilon}^p_{ij}/\hat{\varepsilon}^p_{11}$. Hence it follows that there exist some constraints for $\hat{q}^{ij}$, that is, there exists the loading surface for which, in the process of the active loading.

$$f(\hat{q}^{ij}, \theta) = 0, \quad df = 0 \quad \text{at} \quad D > 0. \tag{2.18}$$

At stresses satisfying the condition $f = 0$, $df = 0$, $D = 0$ a neutral loading takes place, and at $f \leqslant 0$ and $df < 0$ an elastic unloading occurs.

The construction of the general theory of plasticity is possible when the single requirement to meet the condition (2.16) is imposed on the functions $\hat{\psi}^{ij}$.

Nevertheless, it is worthwhile to limit and determine the form of the functions $\hat{\psi}^{ij}$ basing on the quasi-thermodynamic principle of the maximum of the dissipation velocity which, though it does not result from the laws of thermodynamics, seems rather well-grounded and is widely used in constructing the rheological models of the media [13–14].

We shall give the following formulation of this principle:

For a given $\hat{q}^{ij}$ satisfying the condition (2.18) among any possible $\tilde{\varepsilon}^{(p)}_{ij}$, satisfying the condition

$$D(\tilde{\varepsilon}^{(p)}_{ij}) = \hat{\varepsilon}_{ij}\hat{q}^{ij}$$

(when D is some function of $\tilde{\varepsilon}^{(p)}_{ij}$), there exists the unique value $\hat{\varepsilon}^{(p)}_{ij}$, which gives the maximum value to the rate of dissipation

$$\hat{q}^{ij}\tilde{\varepsilon}^{(p)}_{ij}$$

i.e.

$$\max \hat{q}^{ij}\tilde{\varepsilon}^{(p)}_{ij} = \hat{q}^{ij}\hat{\varepsilon}^{(p)}_{ij}.$$

Formulating this principle as the problem of finding the conditional extremum, let us compile the expression

$$D_0 = \hat{q}^{ij}\tilde{\varepsilon}^{(p)}_{ij} + \alpha[D(\tilde{\varepsilon}^{(p)}_{ij}) - \hat{q}^{ij}\hat{\varepsilon}^{(p)}_{ij}],$$

where α is the Lagrangian factor. From the condition $\partial D_0/\partial \tilde{\varepsilon}^{(p)}_{ij} = 0$ we find $\hat{\varepsilon}^{(p)}_{ij}$:

$$\hat{q}^{ij} = \alpha \left. \frac{\partial D(\tilde{\varepsilon}^{(p)}_{mn})}{\partial \tilde{\varepsilon}^{(p)}_{ij}} \right|_{\tilde{\varepsilon}^{(p)}_{mn} = \hat{\varepsilon}^{(p)}_{mn}}. \tag{2.19}$$

Convoluting with $\hat{\varepsilon}^p_{ij}$ we get the equation for defining α:

$$\hat{q}^{ij}\hat{\varepsilon}^{(p)}_{ij} = \alpha \left. \frac{\partial D}{\partial \tilde{\varepsilon}^{(p)}_{ij}} \right|_{\tilde{\varepsilon}^{(p)}_{mn} = \hat{\varepsilon}^{(p)}_{mn}} \hat{\varepsilon}^{(p)}_{ij} = D(\hat{\varepsilon}^{(p)}_{ij}). \tag{2.20}$$

Since the left hand side of Eq. (2.20) is a homogeneous function of the first order, then $D(\hat{\varepsilon}^{(p)}_{ij})$ is also of the same kind and, consequently, $\alpha = 1$.

Now let us turn to the restrictions on the function $D(\hat{\varepsilon}^{(p)}_{ij})$.

From the condition of the existence and similarity of $\hat{\varepsilon}^p_{ij}$ for any given $\hat{q}^{ij}$ satisfying Eq. (2.18), it follows that the surfaces $D(\hat{\varepsilon}^{(p)}_{ij}) = \text{const}$ are closed and convex. From Eq. (2.20) it also follows that $D(\hat{\varepsilon}^{(p)}_{ij}) \geqslant 0$.

Finally, the condition (2.18) also imposes an essential restriction on the form of the function D. Let us consider the case when Eq. (2.18) has the form

$$\hat{q}^{ij}\hat{q}_{ij} = 2k^2(\theta) \tag{2.21}$$

representing the assumption that the intensity of the generalized stresses is constant. The conditions (2.21) and (2.19) impose restrictions on the form of the function D:

$$\frac{\partial D}{\partial \hat{\varepsilon}_{ij}^{(p)}} \frac{\partial D}{\partial \hat{\varepsilon}_{mn}^{(p)}} g^{im} g^{jn} = 2k^2(\theta). \tag{2.22}$$

Let us assume that D is the isotropic function $\hat{\varepsilon}_{ij}^{(p)}$ and depends on the first two invariants

$$x = \hat{\varepsilon}_{ij}^{(p)} \hat{g}^{ij}, \quad y = \hat{\varepsilon}_{ij}^{(p)} \hat{\varepsilon}_{mn}^{(p)} \hat{g}^{im} \hat{g}^{jn}.$$

Equation (2.22) may then be written

$$3\left(\frac{\partial D}{\partial x}\right)^2 + 4x \frac{\partial D}{\partial x} \frac{\partial D}{\partial y} + 4y\left(\frac{\partial D}{\partial y}\right)^2 = 2k^2.$$

Making use of the fact that D is a homogeneous first order function with respect to $\hat{\varepsilon}_{ij}^{(p)}$, let us represent D in the form

$$D = \sqrt{y}\,\Phi\left(\frac{x}{\sqrt{y}}\right),$$

where Φ is a homogeneous function of a zero order with respect to $\hat{\varepsilon}_{ij}^{(p)}$. Introducing the notation $\xi = x/\sqrt{3y}$, we shall get an equation for Φ:

$$(1-\xi^2)\Phi'^2 + \Phi^2 - 2k^2 = 0 \tag{2.23}$$

the solution of which is

$$\Phi = \sqrt{2}k\left(a\sqrt{1-\xi^2} \pm \sqrt{1-a^2}\,\xi\right), \tag{2.24}$$

where a is an arbitrary parameter depending on θ.

Apart from Eq. (2.24), Eq. (2.23) has still another solution which is the envelope of the family (2.24). Excluding ξ from Eq. (2.24) and from Eq. (2.23) we shall come to the solution

$$\Phi = k\sqrt{2}. \tag{2.25}$$

Using the condition of the plastic incompressibility (2.14) and neglecting the values of the order of $(k/\mu_1)^2$, as compared to unity, it is easy to show that for the case of Eq. (2.24) the dissipation function is

$$D = k\sqrt{\frac{2}{3}}\,\hat{\varepsilon}_{ij}^{(k)} \hat{g}^{ij}.$$

To make sure, one should convolute the equation $\hat{q}^{ij} = \partial D/\partial \hat{\varepsilon}_{ij}^{(p)}$ with $\hat{g}_{ij}$ and $\hat{d}_{ij}$, whence we find that $a = 0$. The function $D = k\sqrt{2/3}\hat{\varepsilon}_{ij}^{(p)} \hat{g}^{ij}$ does not satisfy the condition of the existence of the maximum velocity dissipation at arbitrary $\hat{q}^{ij}$ and will not be discussed.

There remains the dissipation function corresponding to Eq. (2.25) which has the form

$$D = k\sqrt{2}\left(\hat{\varepsilon}_{ij}^{(p)} \hat{\varepsilon}_{mn}^{(p)} \hat{g}^{im} \hat{g}^{jn}\right)^{1/2}. \tag{2.26}$$

The relations (2.16) give

$$\hat{q}_{ij} = \hat{\sigma}_{ij} + \beta d_{ij} = \frac{2k^2}{D} \hat{\varepsilon}_{ij}^{(p)}. \tag{2.27}$$

Convoluting Eq. (2.24) with $\hat{d}^{ij}$ and taking into account that $\hat{d}^{ij}\hat{\varepsilon}_{ij}^{(p)} = 0$, we shall obtain

$$\beta = -\hat{\sigma}^{ij}\hat{d}_{ij}/\hat{d}^{km}\hat{d}_{km} = -p-(\mu-p)\hat{s}^{ij}\hat{s}_{ij}/3\mu^2$$

to the precision of the terms $o(k^2/\mu^2)$.

Thus the condition of plasticity (2.18) and the law of the plastic flow has the form

$$\begin{gathered} f = \hat{s}^{ij}\hat{s}_{ij} - \frac{2k^2(\theta)}{(1-\gamma)^2} = 0, \\ \hat{\varepsilon}_{ij}^{p} = \psi\hat{q}_{ij}, \end{gathered} \tag{2.28}$$

where

$$\hat{q}_{ij} = (1-\gamma)\,(\hat{s}_{ij} - \mu\chi^2\hat{g}_{ij}), \quad \gamma = p/\mu, \quad \chi^2 = \hat{s}^{mn}\hat{s}_{mn}/3\mu^2,$$

$$\psi = 0 \quad \text{when} \quad f < 0 \quad \text{or} \quad f = 0, \quad df = \frac{\partial f}{\partial \hat{q}^{mn}} dq_{mn} + \frac{\partial f}{\partial \theta} d\theta < 0,$$

$$\psi = \frac{D}{2k^2} > 0 \quad \text{when} \quad f = 0, \quad df = 0.$$

To formulate the complete system of equations, it is convenient to express the function ψ by means of the stresses and the velocity of the complete deformation. Evidently, for ψ we have

$$\psi = \frac{1}{2k^2} \hat{q}^{ij}\hat{\varepsilon}_{ij}^{(p)}.$$

Since $\hat{\varepsilon}_{ij}^{(p)} = \hat{\varepsilon}_{ij} - \hat{\varepsilon}_{ij}^{(e)}$ and $\hat{e}_{ij}^{(e)} = \frac{1}{2\mu_1}\hat{s}_{ij} + \frac{1}{3} I_1^e \hat{g}_{ij}$ then

$$\hat{q}^{ij}\hat{\varepsilon}_{ij}^{(p)} = (1-\gamma)\hat{s}^{ij}\frac{\partial \hat{e}_{ij}^{(e)}}{\partial t} = \frac{1-\gamma}{2\mu_1}\hat{s}^{ij}\frac{\partial \hat{s}_{ij}}{\partial t} + \frac{2}{3} I_1^e \hat{q}^{ij}\hat{\varepsilon}_{ij}.$$

Using this relation, the formulae (2.12), differentiating the condition of plasticity (2.18) over time, and taking into account the relation (2.28), after simple enough but bulky calculations we obtain

$$\psi = \frac{1}{2k^2}\left\{\left(1-\frac{2}{3} I_1^e\right)\hat{q}^{ij}\hat{\varepsilon}_{ij} - \frac{k^2}{\mu_1^2}(m\dot{\theta} + n\dot{I}_1^e) - \frac{1}{(1-\gamma)\mu_1}\hat{g}_n^i \hat{g}^{nj}\hat{\varepsilon}_{ij}\right\}, \tag{2.29}$$

where

$$m = (1-\gamma)\frac{k'}{k} - \frac{\alpha_2\mu_1 + p\dfrac{\partial \mu_1}{\partial \theta}}{(1-\gamma)\mu\mu_1},$$

$$n = \frac{1}{\mu}\left[\lambda_1 + \frac{2}{3}\mu_1 + \frac{2}{3}p\left(1-\frac{2}{3}I_1^e\right)^{-1}\right].$$

To describe the changes of the temperature field we shall use the equation for the production of the entropy which, in the case where there is no heat conduction, has the form

$$\theta \frac{\partial \eta}{\partial t} = \frac{1}{\varrho} \hat{q}^{ij} \hat{\varepsilon}_{ij}^{(p)}.$$

Recollecting the earlier obtained relation $\eta = -\partial F/\partial\theta$ and designating $c_e = \theta \frac{\partial \eta}{\partial \theta} = -\theta \frac{\partial^2 F}{\partial \theta^2}$ as the heat capacity at constant strain, after some transformations we get

$$\dot{\theta} = \frac{2k^2\psi}{\varrho c_e}(1-\delta_1) - \frac{\alpha_2}{\varrho c_e} \hat{g}^{ij}\hat{\varepsilon}_{ij} - \frac{\delta_2}{\varrho c_e} q^{ij}\hat{\varepsilon}^{ij}, \tag{2.30}$$

where

$$\delta_1 = \frac{\theta}{\mu} \frac{\partial \mu_1}{\partial \theta}, \qquad \delta_2 = \frac{\theta(\alpha_2 - \partial\mu_1/\partial\theta)}{(1-\gamma)\mu_1}.$$

The value of $\dot{I}_1^e$ entering Eqs. (2.12) and (2.29) can easily be expressed in terms of the first invariant $\hat{\varepsilon}_i^i$. In fact, with the assumptions made

$$\dot{I}_1^e = (dI_1^e/d\varrho)(\partial\varrho/\partial t) = \left(1 - \frac{2}{3} I_1^e\right)\hat{\varepsilon}_i^i. \tag{2.31}$$

Introducing now Eqs. (2.30)–(2.31) into Eq. (2.29) we shall finally obtain (with the accuracy of $0(k^2/\mu_1^2)$),

$$\psi = \frac{1}{2k^2}\{a_1 \hat{s}^{ij}\hat{\varepsilon}_{ij} + a_2 \hat{s}_k^i \hat{s}^{kj}\hat{\varepsilon}_{ij} + a_3 \hat{g}^{ij}\hat{\varepsilon}_{ij}\}, \tag{2.32}$$

where

$$\begin{aligned}
a_1 &= \left[1-\gamma - \frac{k^2 m}{\mu_1}\left(1 - \frac{\theta\alpha_2}{(1-\gamma)\mu_1}\right)\right]\left(1 - \frac{2}{3} I_1^e\right) H(f) H(\dot{f}),\\
a_2 &= -\frac{1-\gamma}{\mu_1} H(f) H(\dot{f}), \qquad H(x) = \begin{cases} 1, & x \geqslant 0,\\ 0, & x < 0,\end{cases}\\
a_3 &= \frac{k^2}{\mu_1(1-\gamma)}\left(\frac{\theta\alpha_2}{\varrho c_e} - n\left(1 - \frac{2}{3} I_1^e\right)\right) H(f) H(\dot{f}),\\
\frac{\partial\theta}{\partial t} &= k_1 \hat{g}^{ij}\hat{\varepsilon}_{ij} + k_2 \hat{s}^{ij}\hat{\varepsilon}_{ij} + k_3 \hat{s}_m^i \hat{s}^{mj}\hat{\varepsilon}_{ij},
\end{aligned} \tag{2.33}$$

where

$$k_1 = [(1-\delta_1)a_2 - \theta\alpha_2]/\varrho c_e,$$
$$k_2 = (1-\gamma)[(1-\delta_1)a_1 - \delta_2]/\varrho c_e, \quad k_3 = (1-\gamma)^2(1-\delta_1)a_3/\varrho c_e.$$

The equation of elasticity in the differential form (2.12) will now be written as

$$
(2.34)\qquad \frac{\partial\hat{\sigma}_{ij}}{\partial t} = \{\Lambda\hat{g}^{\alpha\beta}\hat{g}_{ij}+\mu_2(\delta_i^\alpha\delta_j^\beta+\delta_j^\alpha\delta_i^\beta)-\tilde{\varkappa}^2\hat{s}^{\alpha\beta}\hat{s}_{ij}
-\beta_1\hat{s}_m^\alpha\hat{s}^{m\beta}\hat{s}_{ij}-\beta_2\hat{s}_m^\alpha\hat{s}^{m\beta}\hat{g}_{ij}-\tilde{\omega}_1\hat{g}^{\alpha\beta}\hat{s}_{ij}-\tilde{\omega}_2\hat{s}^{\alpha\beta}\hat{g}_{ij}\}\hat{\nabla}_\alpha\hat{v}^\beta,
$$

where

$$
\Lambda = \lambda_2\left(1-\frac{2}{3}I_1^e\right)-\alpha_2 k_1,\quad \tilde{\varkappa}^2 = \mu_1 a_1/k^2,\quad \beta_1 = \mu_1 a_2/k^2,
$$

$$
\beta_2 = \alpha_2 k_3,\quad \tilde{\omega}_1 = \mu_1 a_3/k^2,\quad \tilde{\omega}_2 = \alpha_2 k_2.
$$

The relation (2.34) represents the equations of state noted in the current frame of reference ξ^i. For the transfer into the Euler system x^i it is necessary to use the relations [8]

$$
\left(\frac{\partial\hat{\sigma}_{ij}}{\partial t}\right) = \left(\frac{d\sigma_{ij}}{dt}+\sigma_{\alpha i}\nabla_j v^\alpha+\sigma_{\alpha j}\nabla_i v^\alpha\right)
$$

after which the equations (2.34) are transformed into

$$
\frac{d\sigma_{ij}}{dt} = \{\Lambda g^{\alpha\beta}g_{ij}+\mu(\delta_i^\alpha\delta_j^\beta+\delta_j^\alpha\delta_i^\beta)-\tilde{\varkappa}^2 s^{\alpha\beta}s_{ij}-\beta_1 s_\gamma^\alpha s^{\gamma\beta}s_{ij}
-\beta_2 s_\gamma^\alpha s^{\gamma\beta}g_{ij}-\tilde{\omega}_1 g^{\alpha\beta}s_{ij}-\omega_2 s^{\alpha\beta}g_{ij}\}\nabla_\alpha v_\beta-s_{\alpha i}\nabla_j v^\alpha-s_{\alpha j}\nabla_i v^\alpha,
$$

where

$$
\mu = \mu_2-p = \mu_1\left(1-\frac{2}{3}I_1^e\right).
$$

3. Complete system of equations

Using the Euler coordinates the complete system of equations may be written in the form

$$
(3.1)\qquad
\begin{aligned}
\frac{dv^i}{dt} &= \frac{1}{\varrho}\nabla_j\sigma^{ij},\\
\frac{d\sigma_{ij}}{dt} &= Q_{ij}^{\alpha\beta}\nabla_\alpha v_\beta-s_{\alpha i}\nabla_j v^\alpha-s_{\alpha j}\nabla_i v^\alpha,\\
\frac{d\theta}{dt} &= (k_1 g^{ij}+k_2 s^{ij}+k_3 s_\alpha^i s^{\alpha j})\nabla_i v_j,\\
\frac{de_{ij}^{(p)}}{dt} &= \psi q_{ij}-e_{i\alpha}^{(p)}\nabla_j v^\alpha-e_{i\alpha}^{(p)}\nabla_i v^\alpha,
\end{aligned}
$$

where

$$
Q_{ij}^{\alpha\beta} = \Lambda g^{\alpha\beta}g_{ij}+\mu(\delta_i^\alpha\delta_j^\beta+\delta_j^\alpha\delta_i^\beta)-\tilde{\varkappa}^2 s^{\alpha\beta}s_{ij}-\beta_1 s_\gamma^\alpha s^{\gamma\beta}s_{ij}-\beta_2 s_\gamma^\alpha s^{\gamma\beta}g_{ij}-\tilde{\omega}_1 g^{\alpha\beta}s_{ij}-\tilde{\omega}_2 s^{\alpha\beta}g_{ij}.
$$

The system (3.1) is a system of quasi-linear equations with coefficients depending on the components of the tensor σ_{ij} and the temperature θ.

When using the numerical methods of integration connected with the Lagrangian nets, it is sometimes preferable instead of Eqs. (3.1) to apply the system derived from Eqs. (3.1) by substituting the derivatives over x^i for the derivatives over the Lagrangian variables ξ^i. Such an approach assures the simplicity of formulating the boundary and other conditions essentially connected with the Lagrangian system and, at the same time, allows the form of the equations to remain simple. This is conditioned by the use of the dependent variables g_{ij}, σ_{ij}, v_i, $e_{ij}^{(p)}$, corresponding to the system x^i.

This system of equations has the form

$$
\begin{aligned}
\varrho\partial v^i/\partial t &= (\partial\sigma^{ij}/\partial\xi^m)(\partial\xi^m/\partial x^j)+\sigma^{nj}\Gamma^i_{nj}+\sigma^{ni}\Gamma^j_{nj},\\
\partial\sigma_{ij}/\partial t &= Q_{ij}^{\alpha\beta}(\partial\xi^m/\partial x^\alpha)(\partial v^\beta/\partial\xi^m)-Q_{ij}^{\alpha\beta}v_m\Gamma^m_{\alpha\beta}-(\partial v_\alpha/\partial\xi^m)\\
&\quad\times(s_i^\alpha\partial\xi^m/\partial x^j+s_j^\alpha\partial\xi^m/\partial x^i)-v^\beta(s_{i\alpha}\Gamma^\alpha_{\beta j}+s_{j\alpha}\Gamma^\alpha_{i\beta}),\\
\partial\theta/\partial t &= (k_1g^{ij}+k_2s^{ij}+k_3s^i_\alpha s^{\alpha j})((\partial v_i/\partial\xi^m)(\partial\xi^m/\partial x^j)-v_\gamma\Gamma^\gamma_{ij}),\\
\partial e_{ij}^{(p)}/\partial t &= \psi q_{ij}-(\partial v_\alpha/\partial\xi^m)(e_{i\alpha}^{(p)}\partial\xi^m/\partial x^j+e_{j\alpha}^{(p)}\partial\xi^m/\partial x^i)-v^\beta(e_{i\alpha}^{(p)}\Gamma^\alpha_{\beta j}+e_{j\alpha}^{(p)}\Gamma^\alpha_{\beta i}),
\end{aligned}
\tag{3.2}
$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Cristofell symbols of the Euler frame of reference used.

4. Characteristic properties of the complete system of equations

Let us find the equation of the characteristic surface $\varphi(\xi^i, t) = 0$ of the system (3.2). For this purpose we use its definition according to which the Cauchy problem posed on the characteristic surface has no unique solution. This condition is equivalent to the requirement of the impossibility to define, by means of the considered system of equations, the normal to the surface derivatives, if, for $\varphi = 0$, the values of the functions and the tangential derivatives are known [15].

Let

$$
\nu_i = \frac{\partial\varphi}{\partial\xi^i}\bigg/\left(\hat{g}^{mn}\frac{\partial\varphi}{\partial\xi^m}\frac{\partial\varphi}{\partial\xi^n}\right)^{1/2}, \quad c = -\frac{\partial\varphi}{\partial t}\bigg/\left(\hat{g}^{mn}\frac{\partial\varphi}{\partial\xi^m}\frac{\partial\varphi}{\partial\xi^n}\right)^{1/2}
$$

be the covariant components of the unique spatial normal to the surface $\varphi = 0$ in the system ξ^i and the propagation velocity of the characteristic surface in relation to the medium particles, accordingly.

Let $T_{,N}$ be the partial derivative of a certain value (in particular, the tensor component) in the direction of the normal $\boldsymbol{\nu}$ to $\varphi = 0$, $n_j = \nu_\alpha\dfrac{\partial\xi^\alpha}{\partial x^j}$ be the component of the spatial normal (which is also a unit vector) to the surface $\varphi(x^m, t) = 0$ in the Euler coordinates.

Then, from Eqs. (3.2) there follows the system of equations

$$
\begin{aligned}
&-\varrho c v_{i,N}-\sigma_{i\alpha,N}n^\alpha = \ldots,\\
&-c\sigma_{ij,N}-Q_{ij}^{\alpha\beta}n_\alpha v_{\beta,N}+(s_i^\alpha n_j+s_j^\alpha n_i)v_{\alpha,N} = \ldots,\\
&-c\theta_{,N}-(k_1g^{ij}+k_2s^{ij}+k_3s^i_\alpha s^{\alpha j}{}_i)v_{i,N} = \ldots,\\
&-ce_{ij,N}^{(p)}-\frac{q_{ij}}{2k^2}(a_1s^\alpha+a_2s_\beta s^{\alpha\beta}+a_3n^\alpha)v_{\alpha,N}+(e_{i\alpha}^{(p)}n_j+e_{j\alpha}^{(p)}n_i)v^\alpha_{,N} = \ldots,
\end{aligned}
\tag{4.1}
$$

where the dots mean the terms which do not contain the normal derivatives $T_{,N}$ and $s^\alpha = s^{\alpha\beta}n_\beta$.

The condition of the equality to zero of the matrix determinant of the system coefficients (4.1) is a necessary condition for the existence of the characteristic surfaces for the system (3.2). This condition may be written in the form

$$c^p \det||a\delta^i_j - n^i n_j + \varkappa^2 s^i s_j + \omega_1 n^i s_j + \omega_2 s^i n_j + \varepsilon b^i_j|| = 0, \tag{4.2}$$

where $p = 7$ in the general three-dimensional case and $p = 5$ in the axi-symmetrical and plane-deformed states. Here the following notations are introduced:

$$a = (\varrho c^2 - \mu)/(\Lambda+\mu), \quad \varkappa^2 = \tilde{\varkappa}^2/(\Lambda+\mu), \quad \omega_1 = \tilde{\omega}_1/(\Lambda+\mu),$$
$$\omega_2 = (\tilde{\omega}_2+1)/(\Lambda+\mu), \quad \varepsilon = k/(\Lambda+\mu),$$
$$b^i_j = (s^i_j + \beta_1 s_\alpha s^{\alpha i} s_j + \beta_2 s_\alpha s^{\alpha i} n_j)/k.$$

Using the formula for the determination of the two-matrix sum and withholding the first order terms over ε, Eq. (4.2) may be written as follows:

$$f_0(a) + \varepsilon f_1(a) = 0,$$

where

$$f_0 = \det||a\delta^i_j + n^i n_j - \varkappa^2 s^i s_j - \omega_1 s_j n^i - \omega_2 s^i n_j|| = \det||m^i_j||,$$
$$f_1(a) = \mathrm{tr}||M^i_k b^k_j||,$$

M^i_k is the minor of the matrix element m^k_j.

The solution $f_0(a_0) = 0$ is easy to find in explicit form if, instead of the system (4.1) for $\varepsilon = 0$, we consider an equivalent system:

$$\begin{aligned} &(a_0 - 1 + \omega_1 s_\alpha n^\alpha)N + (\varkappa^2 s_\alpha n^\alpha + \omega_2)S = \ldots, \\ &(-s_\alpha n^\alpha + \omega_1 s_\alpha s^\alpha)N + (a_0 + \varkappa^2 s_\alpha s^\alpha + \omega_2 s_\alpha n^\alpha)S = \ldots, \\ &(-s_\alpha s^\alpha + \omega_1 s^\alpha s^\beta_\alpha s_\beta)N + (\varkappa^2 s^\alpha s^\beta_\alpha s_\beta + \omega_2 s_\alpha s^\alpha)S + a_0 B = \ldots, \end{aligned} \tag{4.3}$$

where

$$N = n^\alpha v_{\alpha,N}, \quad S = s^\alpha v_{\alpha,N}, \quad B = s^\alpha s^\beta_\alpha v_{\beta,N},$$

which is obtained by the convolution of Eq. (4.1) with n^α, s^α and $s^\alpha s^\beta_\alpha$, respectively. The determinant of the system (4.3) turns to zero if

$$a_0 = 0, \quad \varrho c^2 = \mu \tag{4.4}$$

and

$$a_0 = \frac{1}{2}[1 - \varkappa^2 s^\alpha s_\alpha - (\omega_1+\omega_2)s^i n_i] \pm \left\{\frac{1}{4}[\varkappa^2 s_i s^i - (\omega_1+\omega_2)s^i n_i]^2 - (\varkappa^2 - \omega_1\omega_2)[s^i s_i - (s^i n_i)^2]\right\}^{1/2}. \tag{4.5}$$

To find the correction introduced by the term $\varepsilon f_1(a)$, we search for the solution in the form $a = a_0 + \varepsilon a_1$. It is easy to see that $a_1 = f_1(a_0)/f'_a(a_0)$, where a_0 is defined by the formulae (4.3) and (4.4).

5. Reduction of the system to a bi-characteristic form and construction of the difference schemes

For further considerations it is convenient to use the matrix form of the complete system of the differential equations (3.2):

$$\frac{\partial u_\alpha}{\partial t} + {}^k A_{\alpha\beta} \frac{\partial u_\beta}{\partial \xi^k} + N_\alpha = 0, \tag{5.1}$$

where the vector $\mathbf{u} = \{v_i, \sigma_{ij}, \theta, e_{ij}^{(p)}\}$ is composed of the physical components of the vector $\mathbf{v}$, of the tensors $\boldsymbol{\sigma}$, $\mathbf{e}^p$ and θ. We shall further denote these physical components by the former symbols with subscripts. There will be no misunderstanding beacause only physical components are used further. The matrices ${}^k\mathbf{A}$ ($k = 1, 2, 3$) and the vector $\mathbf{N}$, included in the system (5.1), are the nonlinear functions of $\mathbf{u}$ defined by Eqs. (3.2).

It is assumed here and further on that the indices given in Greek letters cover the values from 1 to 16 in the general three-dimensional case and from 1 to 11 in the two-dimensional case and denote the components of matrices and vectors as well. The Latin indices may change from 1 to 3 or from 1 to 2, accordingly.

Now let ν be a certain fixed spatial normal vector to the characteristic surface. Let us denote by $\boldsymbol{\omega}$ the matrix, the lines of which are the left eigenvectors of the matrix $A = {}^k A \nu_k$ of the system (5.1) corresponding to the given $\mathbf{v}$. C_ξ ($\xi = 1, 2, \ldots, 16$) are the eigenvalues equal to the velocities of propagation of the characteristic surface of the system (5.1) in the direction of $\mathbf{v}$. Using the definition $\boldsymbol{\omega}$, it is easy to obtain the relations of compatibility

$$\omega_{\xi\alpha} {}^k A_{\alpha\beta} \nu_k = c_\xi \omega_{\xi\alpha} \delta_{\alpha\beta}, \qquad \Sigma(\xi), \tag{5.2}$$

where the sign Σ means there is no summing up according to the noted indices.

The conditions of compatibility (5.2) which result from the hyperbolicity of the initial system of differential equations (5.1) permit to transform Eq. (5.1) into a system containing the derivatives only in the characteristic plane. Moreover, in using some characteristic planes it is possible to obtain a system in which the derivatives along the bi-characteristics are included only. Choosing, for instance, a plane with the normals $\mathbf{v}^{(q)}$ coinciding with the unit-vectors of the coordinate system ξ^q, convoluting Eq. (5.1) with the matrices ${}^q\omega_{\alpha\beta}$ (the index q indicates that the $\boldsymbol{\omega}$ corresponds to $\mathbf{v}^{(q)}$) and using the relations (5.1), we obtain three systems of equations:

$$\begin{aligned}
&{}^1\omega_{\alpha\beta} \frac{du_\beta}{ds_\alpha^1} + {}^1\omega_{\alpha\beta} {}^a A_{\beta\gamma} \frac{\partial u_\gamma}{\partial \xi^a} + {}^1\omega_{\alpha\beta} N_\beta = 0, \quad a = 2, 3, \quad \Sigma(\alpha),\\
&{}^2\omega_{\alpha\beta} \frac{du_\beta}{ds_\alpha^2} + {}^2\omega_{\alpha\beta} {}^b A_{\beta\gamma} \frac{\partial u_\gamma}{\partial \xi^b} + {}^2\omega_{\alpha\beta} N_\beta = 0, \quad b = 1, 3,\\
&{}^3\omega_{\alpha\beta} \frac{du_\beta}{ds_\alpha^3} + {}^3\omega_{\alpha\beta} {}^m A_{\beta\gamma} \frac{\partial u_\gamma}{\partial \xi^m} + {}^3\omega_{\alpha\beta} N_\beta = 0, \quad m = 1, 2,
\end{aligned} \tag{5.3}$$

where

$$du_\beta/ds_\alpha^q = \partial u_\beta/\partial t + c_\alpha^{(q)} \partial u_\beta/\partial \xi^q, \qquad \Sigma(q),$$

$c_\alpha^{(q)}$ is the velocity of propagation of the characteristic surface with the normal $\mathbf{v}^{(q)}$.

The systems (5.1) and (5.3) in the general case contain 64 equations from which, naturally, only 16 are independent. Let us express the terms ${}^1\mathbf{A}\dfrac{\partial \mathbf{u}}{\partial \xi^1}$, ${}^2\mathbf{A}\dfrac{\partial \mathbf{u}}{\partial \xi^2}$, ${}^3\mathbf{A}\dfrac{\partial \mathbf{u}}{\partial \xi^3}$ from Eqs. (5.3) using the derivatives along the bi-characteristics and insert the attained relations into the initial equation (5.1). After this we obtain the equation

$$\frac{\partial u_\alpha}{\partial t} = \frac{1}{2}\left[{}^q\Omega_{\alpha\beta}{}^q\omega_{\beta\alpha}\frac{du_\gamma}{ds^q_\beta} + N_\alpha\right], \quad q = 1, 2, 3 \tag{5.4}$$

where ${}^q\Omega = [{}^q\omega]^{-1}$. These equations contain the derivatives only along the bi-characteristics.

Using the initial system of equations in the form (5.4), we can propose a rather general method of constructing the finite-difference schemes permitting to make a good approximation of the dependence domain of the differential equations by finite-difference systems. This method was used for solving the problems with small elastic-plastic deformations and was described in detail in [16], therefore, we briefly state the basic idea here.

To calculate $\mathbf{u}$ in the node H on the $n+1$ layer, we have to construct characteristic cones with the top in the point H node up to the cone intersection with the n-layer. By changing the derivatives along the bi-characteristics included into Eq. (5.4) by finite

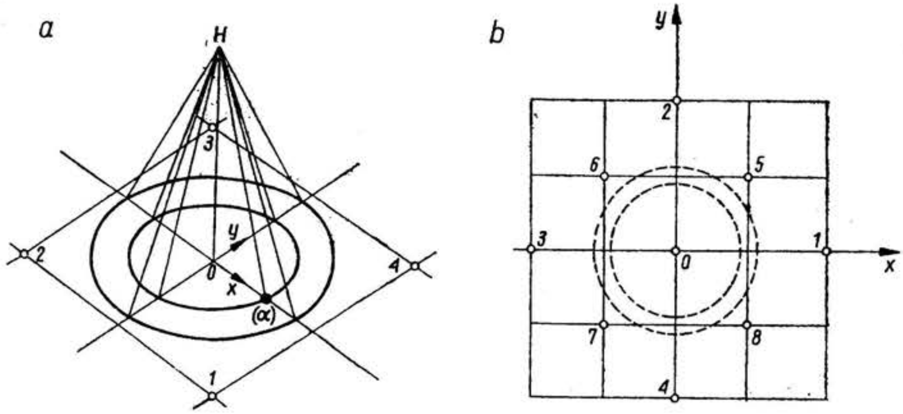


FIG. 1.

differences in $\mathbf{u}^{(H)}$ and $\mathbf{u}^{(M\alpha)}$ in the points M_α of the intersection of the bi-characteristics with the n-layer (Fig. 1a), we obtain the following system of differential equations:

$$u_\lambda^{(H)} = \Omega_{\lambda\alpha}{}^1\omega_{\alpha\beta}{}^1u_\beta^{(M\alpha)} + \Omega_{\lambda\alpha}{}^2\omega_{\alpha\beta}{}^2u_\beta^{(M\alpha)} - u_\lambda^{(\prime)} - \tau N_\lambda. \tag{5.5}$$

The values ${}^k\mathbf{u}^{(M\alpha)}$ are defined by means of the values in the nodes of the net using the interpolation formulae. To obtain the schemes of the first order of accuracy, it is sufficient to take the interpolation formulae along the directions $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ coinciding with the axis of coordinates

$${}^ku_\beta^{(M\alpha)} = u_\beta^{(0)} + \frac{\tau}{4h}c_\alpha^{(k)}\Delta_k u_\beta + \frac{(\tau c_\alpha^{(k)})^2}{8h^2}\Delta_k^2 u_\beta, \tag{5.6}$$

where

$$\Delta_k u_\beta = u_\beta^{(i+1)} - u_\beta^{(i)}, \quad \Delta_k^2 u_\beta = u_\beta^{(i+1)} - 2u_\beta^{(i)} + u_\beta^{(i-1)},$$

and where u_β^i is the value of u_β in the nodal points along the straight line on which there is the point M_α.

In the case of the 4-point pattern (Fig. 1a) we confine our study to two terms in Eq. (5.6) and assume that

$$(5.7) \qquad u_\beta^{(0)} = \frac{1}{4}\sum_{i=1}^{4} u_\beta^{(i)}, \qquad \begin{aligned} \Delta_1 u_\beta &= \frac{1}{2}\,[u_\beta^{(1)} + u_\beta^{(4)} - u_\beta^{(2)} - u_\beta^{(3)}], \\ \Delta_2 u_\beta &= \frac{1}{2}\,[u_\beta^{(3)} + u_\beta^{(4)} - u_\beta^{(1)} - u_\beta^{(2)}]. \end{aligned}$$

We find ${}^k u_\beta^{(M\alpha)}$ by substituting into Eq. (5.6) and, taking into account

$${}^k\Omega_{\xi\alpha}{}^k\omega_{\alpha\beta}\, c_\alpha^{(k)} u_\beta^{(i)} = {}^k\Omega_{\xi\alpha}{}^k\omega_{\alpha\beta}{}^k A_{\beta\gamma} u_\gamma^{(i)} = {}^k A_{\xi\beta} u_\beta^{(i)},$$

after the transformation, we obtain the following difference scheme

$$(5.8) \quad u_\beta^{(H)} = \frac{1}{4}\,(u_\beta^{(1)} + u_\beta^{(2)} + u_\beta^{(3)} + u_\beta^{(4)}) - \frac{\tau}{4h}\,[{}^1A_{\beta\gamma}(u_\gamma^{(1)} + u_\gamma^{(4)} - u_\gamma^{(2)} - u_\gamma^{(3)}) + {}^2A_{\beta\gamma}(u_\gamma^{(3)} + u_\gamma^{(4)} - u_\gamma^{(1)} - u_\gamma^{(2)})] + \tau N_\beta.$$

If we use a 5-point pattern and calculate according to the interpolation formula

$$(5.9) \qquad {}^k u_\beta^{(M\alpha)} = u_\beta^{(0)} - \frac{\tau c_\alpha^{(k)}}{4h}\,\Delta_k u_\beta + \frac{\tau |c_\alpha^{(k)}|}{4h}\,\Delta_k^2 u_\beta,$$

we obtain the following difference scheme:

$$(5.10) \quad u_\alpha^{(H)} = u_\alpha^{(0)} - \frac{\tau}{4h}\Big\{{}^1A_{\alpha\beta}(u_\beta^{(1)} - u_\beta^{(3)}) + {}^2A_{\alpha\beta}(u_\beta^{(2)} - u_\beta^{(4)}) - {}^1\Omega_{\alpha\beta}|c_\beta^{(1)}|{}^1\omega_{\beta\gamma}\Delta_1^2 u_\gamma - {}^2\Omega_{\alpha\beta}|c_\beta^{(2)}|{}^2\omega_{\beta\gamma}\Delta_2^2 u_\gamma\Big\} + \tau N_\alpha.$$

The scheme (5.8) is a generalization of the Courant-Isaacson-Rees scheme for the two-dimensional case.

Using the proposed approach, we can also obtain the schemes of the second order of accuracy. Here it is necessary to use the 9-point pattern (Fig. 1b) and Eq. (5.5) for the four directions of the normal $\mathbf{v}$. In particular, we may obtain, for instance, the scheme of Wendroff-Lax. The characteristic schemes for boundary and angular points are given in the works [7, 16], where the stability of the proposed schemes is also proved.

6. Numerical solution of the problems. Results of calculations

On the basis of the method describing above a series of problems of an elastic-plastic flow occurring at the dynamic loading of the axi-symmetrical bodies were solved. The behaviour of the material was described by the equations given in the present work for the case of an ideal elastic-plastic material at $\mu_1 = \text{const}$, $\alpha_1 = \text{const}$, $c_e = \text{const}$, $\lambda_1 =$

$= \lambda_1(I_1^e)$, $k = k(\theta)$ (the form of these functions is given further). For the calculations the differential scheme of the first order of accuracy (5.8) was used which allowed to perform calculation at the discontinuous boundary conditions without introducing artificial viscosity. The approximate viscosity of the difference scheme automatically assured the "smoothing" of the discontinuities over several spatial cells.

For the estimation of the accuracy of the scheme some problems were solved which showed that the accuracy of the calculations was quite acceptable even in the case of a relatively small number of points. Thus, in calculating the Lamb's problem on the propagation of the harmonic ways in the infinite band of an elastic material, the comparison with the analytical solution showed that the error at the step $h = 1/32$ and the Courant number $\varkappa = \frac{\tau c_0}{h} = 1.0$ was above 3% after 100 steps over t. The comparison with the solution of elastic-plastic one-dimensional problems, obtained by means of the direct one-dimensional method of characteristics, showed that the discrepancies were of the same order. It should be noted that the proposed method is quite economic as the scheme uses a minimum number of points.

6.1. The impact of the round cylinder with a rigid wall

As an example of the non-steady flows with two spatial variables, the problem of the impact with the constant velocity at the end of the elastic-plastic cylinder was considered. This problem is of great interest and has been considered by many authors from various aspects and for different medium models.

Figure 2 presents the relations of the velocity v_z at the speed of the impact $V_0 = 0.01c_0$ $(c_0^2 = (\Lambda(0)+2\mu(0))/\varrho)$ on the axis of the cylinder $r = 0$ at different moments of time

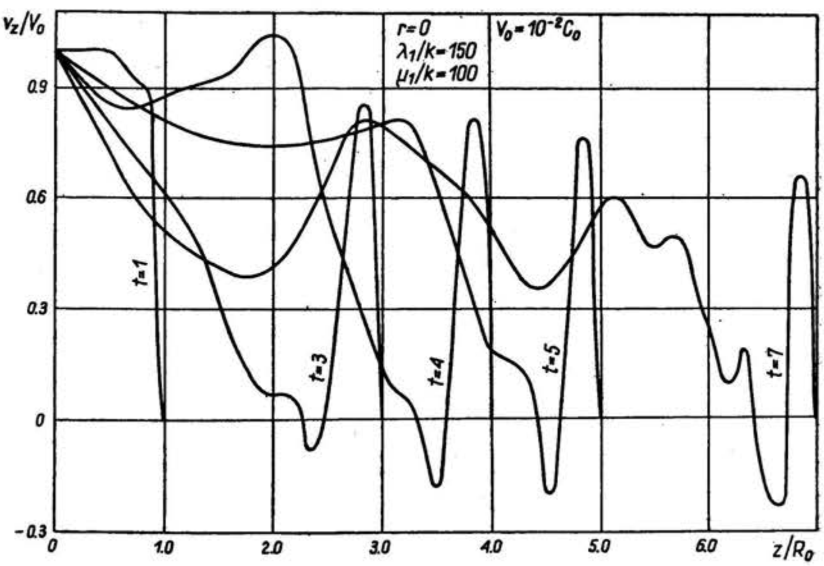


FIG. 2.

t/t_0 ($t_0 = R_0/c_0$). We see that the solution before the arrival of the waves reflected from the lateral surface of the waves has the form of a step function propagating with the speed c_0. Then, after the dilatation waves arrive, the problem becomes more complex. A characteristic feature is the presence of an elastic precursor almost unchanging over the amplitude and moving with the speed c_0. The velocity profile behind it differs much from the corresponding profile in the elementary beam theory and also from the solution of the elastic problem for a cylinder.

An essential difference of an elastic and elastic-plastic solution is also observed when the shock wave is reflected from the free end of the cylinder.

Figure 3 shows the dependence of the axial velocity v_z/V_0 on z for the cylinder of a finite length for $Z_0/R_0 = 4$. For $t/t_0 > 4$ the profile character becomes different. The

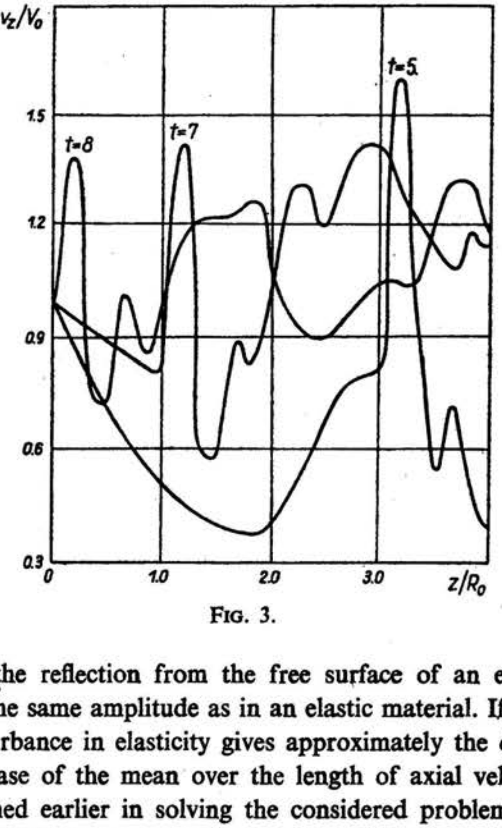


FIG. 3.

axial velocities after the reflection from the free surface of an elastic precursor give a "splash" almost of the same amplitude as in an elastic material. If, however, the reflection of the basic disturbance in elasticity gives approximately the double velocity value in plasticity, the increase of the mean over the length of axial velocity is only 5–10%. This result was obtained earlier in solving the considered problem by means of other numerical methods and for other models in the works [17–19].

In Fig. 4 we can see how the contact stress σ_{zz} on the axis of the bar changes in time. At the beginning it changes from the value corresponding to the plane wave to a certain value near the value for the uniaxial stress state. Relative to this value, oscillations occur with the period of $\sim 4R_0/c_0$, equal to the period of the wave reflected from the free lateral surface of the cylinder. The influence of the wave reflected from the cylinder surface in the vicinity of the axis is rather important because of the concentration in the converging wave. At the increase of the impact velocity the amplitude of the oscillations

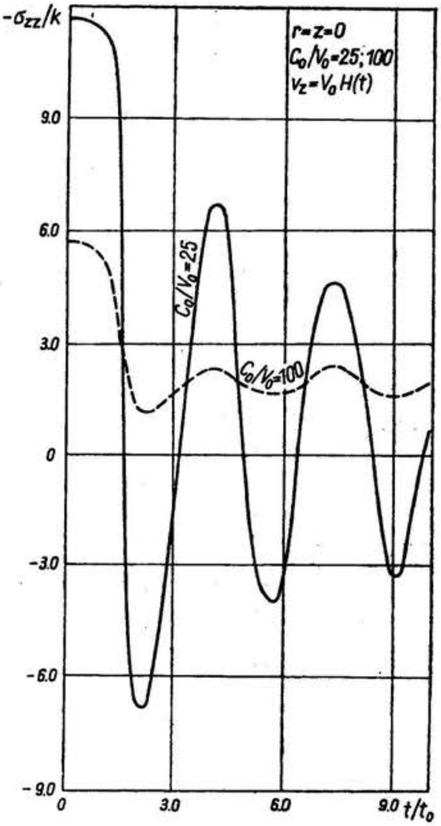


FIG. 4.

increases, so that it exceeds the constant value on which it was superimposed. As a result, much earlier than the shock wave reflected from the free end of the cylinder arrives in the section $z = 0$, there appear regions where the stresses become equal to zero and where the velocity v_z exceeds V_0, that is, the local "jump aside" takes place and it soon starts "stricking" to the wall again; this is repeated with the period $\sim 4R_0/c_0$.

The origin of this phenomenon is conditioned by the fact that at $V_0 \sim 0.1c_0$ the maximum value of the pressure becomes much greater than the components of the stress deviator limited by the yield circle. In this case, owing to the proximity of the stressed state to the spherical one, the radial movement begins to play the same role, as the movement in the axial direction, and the picture differs substantially from the one predicted by the one-dimensional theory.

In the course of time, in the vicinity of the cylinder axis, there appear zones of considerable tension stresses, what indicates the possibility of destruction in these regions.

6.2. Deformating of a circular plate under impulsive loading

Let us consider some results of the calculation of the deformation process of a circular plate (Fig. 5) under the action of pressure impulse. The dimensions of the plate, the

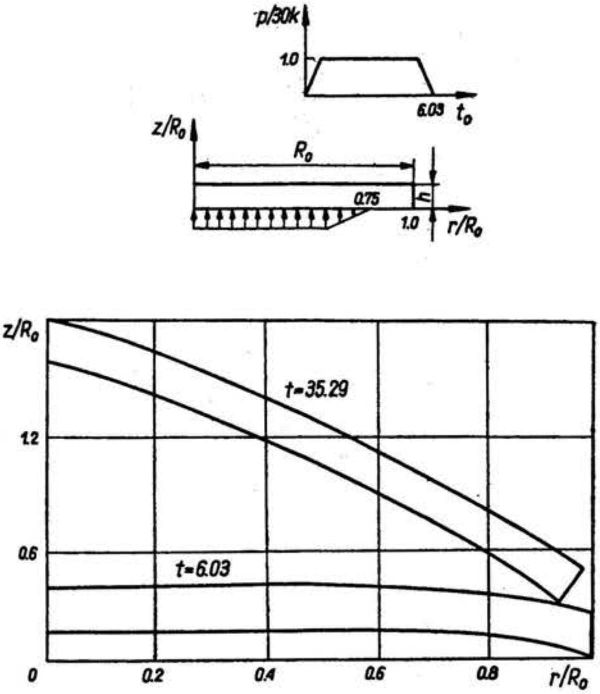


FIG. 5.

distribution of the pressure over r and t are shown in Fig. 5. The modulus values λ_1 and μ_1 corresponded to aluminium and were taken from the work [18]. The relation $k = k(\theta)$ was accepted in the form [20]:

$$k = k_0 \begin{cases} 1 & \text{if} \quad \theta/\theta_0 \leqslant 1.31, \\ 1-(\theta/\theta_0 - 1.31)/5.91 & \text{if} \quad \theta/\theta_0 > 1.31. \end{cases}$$

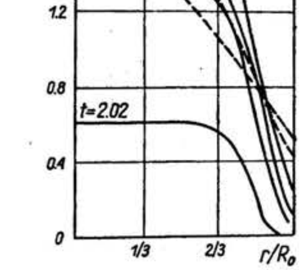
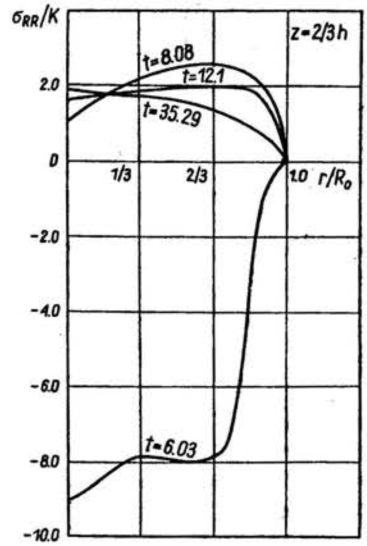
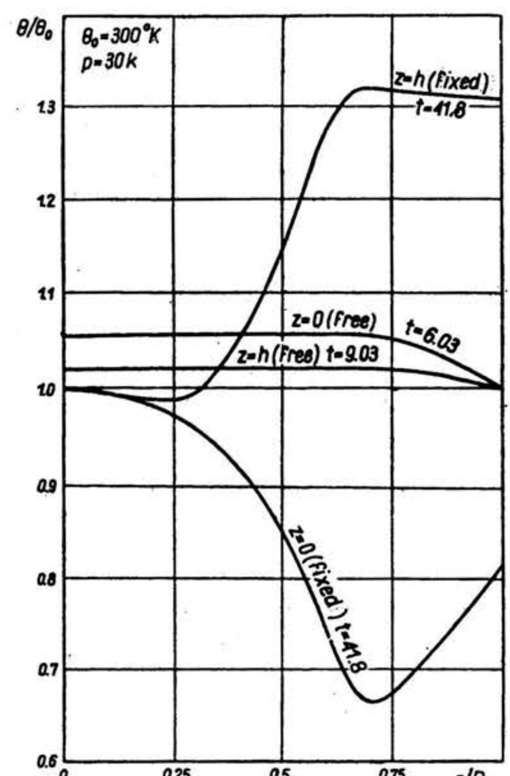


FIG. 6.



σRR/K
z=2/3h
t=8.08
t=12.1
t=35.29
t=6.03
2.0
0
-2.0
-4.0
-6.0
-8.0
-10.0
1/3
2/3
1.0
r/R₀

FIG. 7.



θ/θ₀
θ₀=300°K
p=30k
z=h (fixed)
t=41.8
z=0 (free)
t=6.03
z=h (free) t=9.03
z=0 (fixed) t=41.8
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0
0.25
0.5
0.75
r/R₀

FIG. 8.

In Fig. 5 the shape of the plate is shown at the moment of load relieving (approximately six runs of the sound wave over the thickness) and at a later moment of time ($t/t_0 \simeq \simeq 35$). It can be seen that the character of the deformation at the beginning and at a later stage of motion are considerably different. Up to the moment $t/t_0 \simeq 6$, the movement of the loaded part of the plate is translatory at high speed while the nonloaded part is very slowly attracted into motion as the velocity of the transverse plastic wave is not high; only for $t \geqslant 10t_0$ does the section $r = R_0$ start to move in the direction of the axis z. In the zone of transition from the loaded to the unloaded part of the plate, high gradients of velocity v_z arise (Fig. 6) and, consequently, high velocities of the shear strain as well. In case of high pressures and large mass of the free parts of the plate, the destruction of the plate in this region is possible on account of the shear strain. In the zone of the load application, approximately to the depth 2/3h, a stressed state appears near the state of uniform compression. After the stressed state changes abruptly, the compression turns to stretching (Fig. 7). Figure 6 shows how the distribution of the velocities v_z changes after load relieving.

We can see that the region of high gradients is spread, the form of the deformed plate

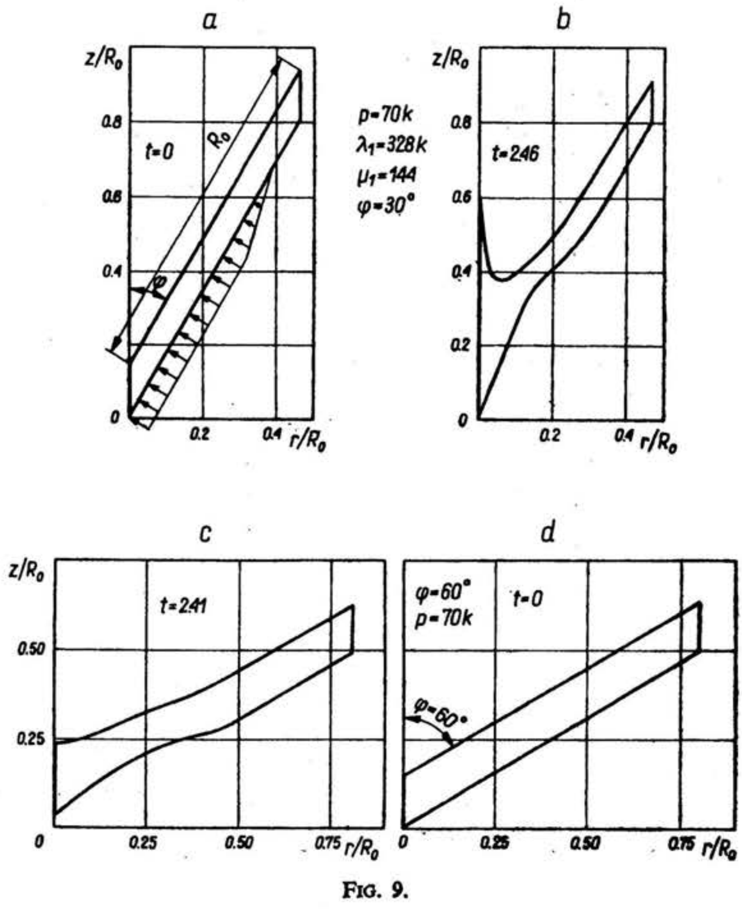


FIG. 9.

changes, see (Fig. 5), the fractures are being smoothed and a great but smooth enough bending of the plate takes place on account of inertial forces.

The calculations made show that at impulsive loading by means of pressure of the order (20–30)k it is impossible to use hydrodynamic approximation. This is so since the behaviour of the material is similar to the ideal fluid only at the stage of high pressure action and in the region where it is applied. In the nonloaded parts of the plate and in the whole plate after pressure relieving the material behaves as an essentially plastic body and this influences the strain of the whole plate.

In Fig. 8 the temperature distribution along the radius is shown.

The temperature change at the pressures considered is seen to be small and does not pronouncedly influence the strain.

6.3. Deformation of a conic thick shell

Consider the numerical results of the problem of the deformation of a thick-walled conic shell by normal pressure suddenly applied to the surface. The geometric dimensions and pressure distribution are shown in Fig. 9.

Unlike the previous problem, here the initial form of the body in the Euler system of coordinates (R, Z) occupies a non-rectangular region, that is why the initial Lagrangian system should be chosen so that in this system the domain is rectangular $0 \leqslant r \leqslant R_0$,

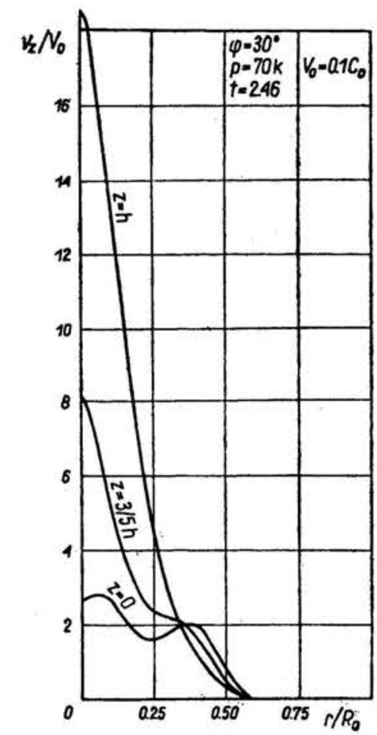


FIG. 10.

$0 \leqslant z \leqslant Z_0$, i.e. at $t = 0$ it is necessary to assume that $R(r, z, 0) = r\sin\varphi$, $Z = z+r\cos\varphi$. Figures 9b and 9d show how the nature of the deformation of a conic shell changes with the angle φ. For this shell with a small opening angle $\varphi = 30°$ near the axis there appears a zone of a great uniform compression. On the inner surface $z = H$ the particle velocities v_Z of the material are very high and in excess of the propagation velocity of the elastic waves. The velocity distribution v_Z over the generator of the cone at a different z is shown in Fig. 10. Moving off the axis, the radial velocity component v_R becomes predominant and the material from the periphery moves to the axis. The stressed state away from the axis differs from the state of uniform compression.

At large angles ($\varphi = 60°$, Fig. 9d) the effect described is much weaker for the load under consideration. Simultaneously, in the region of load application squeezing of the shell takes place, and at small thicknesses and large angles even the "turning out" of the cone is possible.

In the vicinity of the axis, where there are large velocity gradients, there appear large distortions of the net and this leads to computational difficulties. Calculations had to be made at a small integration step.

The accuracy control was effected for the energy balance to be observed. The relative error in the problem of the plate and the gentle cone did not exceed 1%. The relative disbalance in the problem of the cone and strongly pronounced cummulation reached 10%.

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